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On the Riesz Representation for Optimal Stopping Problems

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by

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ABSTRACT

On the Riesz representation for optimal stopping problems

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In this thesis we summarize results about optimal stopping problems analyzed with the Riesz representation theorem. Furthermore we consider two examples: Firstly the optimal investment problem with an underlying d-dimensional geometric Brownian motion. We derive formulas for the optimal stopping boundaries for the one- and two-dimensional cases and we find a numerical approximation for the boundary in the two-dimensional problem. After this we change the focus to a space-time one-dimensional geometric Brownian motion with finite time horizon. We use the Riesz representation theorem to approximate the optimal stopping boundaries for three financial options: the American Put option, American Cash-or-Nothing option and the American Asset-or-Nothing option.

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Chapter 1

Introduction

1.1 Literature review

This thesis is based on chapter three of the habilitation treatise "On Stochastic Control and Optimal Stopping in Continuous Time" by Sören Christensen [1] which is common work with Paavo Salminen. In this paper Christensen and Salminen considered an optimal stopping time problem where the value function is bounded. They manage to rewrite the value function with the help of the Riesz representation as an integral over the unknown stopping set. Thus the optimal stopping problem reduces to an integral equation. Furthermore they prove that the solution of this problem characterizes the stopping set uniquely.

1.2 Optimal stopping problems

An optimal stopping problem can be formulated as follows: Find a function V (the value function) and a stopping time τ^* (optimal stopping time) such that

$$V(x) := \sup_{\tau \in \mathcal{M}} \mathbb{E}_x \left(e^{-r\tau} g(X_\tau) \right) = \mathbb{E}_x \left(e^{-r\tau^*} g(X_{\tau^*}) \right),$$

where $(X_t)_{(t \geq 0)}$ is a strong Markov process taking values in $E \subset \mathbb{R}^d$, $x \in E$, $r \geq 0$, $T \in [0, \infty]$ is the time horizon of the problem, $\mathcal{M}$ is the set of all stopping times in the natural filtration of X with values in $[0, T]$ and $g : \mathbb{R}^d_+ \mapsto \mathbb{R}_+$ is the reward function.

Chapter 2

Preliminaries

In this chapter we will review general results about optimal stopping problems including those by Christensen and Salminen.

2.1 r-excessive functions

First, we will consider a general result. For that reason we define the terms r-excessive and lower semicontinuous functions:

Definition 2.1.1. *A non-negative, measurable function u is called r-excessive for a real-valued strong Markov process X if the following two conditions hold:*

$$\mathbb{E}_x(e^{-rt}u(X_t)) \leq u(x) \quad \forall\ t \geq 0,\ x \in E,$$

$$\lim_{t \to 0} \mathbb{E}_x(e^{-rt}u(X_t)) = u(x) \quad \forall\ x \in E.$$

Definition 2.1.2. *A function u is called lower semicontinuous on E if*

$$\liminf_{x \to x_0} u(x) \geq u(x_0) \quad \forall\ x_0 \in E.$$

Theorem 2.1.3 ([4, p. 124])**.** *Let X be a Markov process and $g : \mathbb{R}^n_+ \mapsto \mathbb{R}_+$ be a lower semicontinuous, positive reward function which satisfies the condition*

$$\mathbb{E}_x\left(\sup_{t \geq 0} g(X_t)\right) < \infty.$$

Then the value function V exists and can be characterized as the smallest r-excessive majorant of g. Moreover, the optimal stopping time is the first entrance time into the set

$$S := \{x \in E : g(x) = V(x)\}.$$

So, the question of the existence of the value function is solved, but there is no known explicit solution. The approach of Christensen and Salminen is to describe the value function via the Riesz Representation Theorem for excessive functions. In the following two chapters we examine two examples of processes where their theory can be applied. The first is the multidimensional geometric Brownian motion and the second is the time-space process with limited time.

2.2 The Riesz representation of excessive functions

The general idea of the Riesz representation is to rewrite an r-excessive function as a sum of an r-harmonic function and a potential. Therefore we define harmonic functions.

Definition 2.2.1. *A non-negative measurable function h is called r-harmonic on A if*

$$h(x) = \mathbb{E}_x(e^{-r\tau_A} u(X_{\tau_A})),$$

where $\tau_A := \inf\{t : X_t \notin A\}$.

For the definition of a potential we refer to Kunita and Watanabe [12]. For the following results we use the resolvent kernel G_r of a geometric Brownian motion which is introduced in Section 3.1.

Theorem 2.2.2 ([1])**.** *Let u be a locally integrable r-excessive function for a d-dimensional geometric Brownian motion X. Then u can be represented uniquely as the sum of a r-harmonic function h and an r-potential p. For the potential p there* *exists a unique Radon measure* σ *depending on* u *and* r *on* $\mathbb{R}^d_+$ *such that for all* $x \in \mathbb{R}^d_+$

$$p(x) = \int_{\mathbb{R}^d_+} G_r(x,y)\sigma(dy),$$

Moreover, if u *is additionally bounded, then* $h \equiv 0$.

Chapter 3

Optimal investment problem

In this chapter we look at the optimal investment problem which is one of the most famous optimal stopping problems in continuous time with multidimensional underlying process. First we introduce the underlying stochastic process, which we model with a geometric Brownian motion.

3.1 Model introduction: Multi-dimensional geometric Brownian Motion

To define a d-dimensional geometric Brownian motion, we begin with the d-dimensional Brownian motion

$$W = \left(\left(W_t^{(1)}, \ldots, W_t^{(d)}\right)\right)_{t \geq 0}$$

started at $(0, \ldots, 0)$ such that for $t \geq 0$

$$\mathbb{E}(W_t^{(i)}) = 0,\ \mathbb{E}\left((W_t^{(i)})^2\right) = t,\ \mathbb{E}(W_t^{(i)} W_t^{(j)}) = \sigma_{i,j} t,\ \ i, j = 1, \ldots, d$$

where the covariance matrix $\Sigma := (\sigma_{i,j})_{i,j=1}^d$ with $\sigma_{i,i} := 1$ is non-singular. A d-dimensional geometric Brownian motion is a diffusion X in $\mathbb{R}_+^d$ with the components defined by

$$X_t^{(i)} = X_0^{(i)} \exp\left(a_i W_t^{(i)} + \left(\mu_i - \frac{1}{2} a_i^2\right) t\right),\ \ i = 1, \ldots, d,$$

where $a_i \neq 0$ for $i = 1, \ldots, d$. Note that X is a standard Markov process, see [8, p. 45]. The Green or resolvent kernel for X is given by

$$G_r(x,y) := \int_0^\infty e^{-rt} p(t;x,y)\ dt,$$

where $x, y \in \mathbb{R}_+^d$ and p is a transition density of X.

3.2 Setup

The value function for the optimal investment problem is defined by

$$v_\alpha(x) := \sup_{\tau\in\mathcal{M}} \mathbb{E}_x(e^{-r\tau}(X_\tau^{(0)} - \alpha_1 X_\tau^{(1)} - \ldots - \alpha_d X_\tau^{(d)})^+),\ x \in \mathbb{R}_+^{d+1},$$

where the time horizon is infinite, $r > 0$, $(X_t^{(0)}, \ldots, X_t^{(d)})_{t\geq 0}$ is a $(d+1)$-dimensional geometric Brownian motion and $\alpha \in \mathbb{R}_+^d$ is the weight vector.
With the help of Girsanov's theorem we can set w.l.o.g. $X_t^{(0)} = K$. So the new value function is

$$V(x) = \sup_\tau \mathbb{E}_x\left(e^{-r\tau}\left(K - \sum_{i=1}^d X_\tau^{(i)}\right)^+\right), \tag{3.1}$$

which is an optimal stopping problem with lower semiconinuous reward function $g(x) = (K - \sum_{i=1}^d x_i)^+$. Christensen and Salminen used in [1] the Riesz representation to obtain the following theorem:

Theorem 3.2.1. *For all $x \in \mathbb{R}_+^d$ it holds that*

$$V(x) = \int_S G_r(x,y)\sigma(y)m(dy),$$

where S is the optimal stopping set,

$$\sigma(y) = rK + \sum_{i=1}^d (\mu_i - r)y_i,$$

and m is an absolutely continuous measure with respect to the Lebesgue measure.

A corollary of the latter theorem is the characterization of the the boundary of the optimal stopping set S:

Corollary 3.2.2. *Let g be the reward function of the optimal stopping problem ($g(x) = (K - \sum_{i=1}^{d} x_i)^+$). Then, the integral equation*

$$g(x) = \int_S G_r(x,y)\sigma(y)m(dy) \tag{3.2}$$

holds for all $x \in \partial S$ and characterizes the optimal stopping set uniquely.

3.3 Solution of the optimal investment problem when $d = 1$

Christensen and Salminen derived a formula for the one-dimensional optimal stopping problem for $\mu = r$ using the density for the geometric Brownian motion with respect to the speed measure (see [9, p. 13]). We will now derive the formula for the stopping boundary without restrictions on μ using the density for the geometric Brownian motion with respect to the Lebesgue measure. The proof has the same structure as the one for the two-dimensional case in [1].

Theorem 3.3.1. *The optimal stopping region in the one-dimensional optimal investment problem is $S = (0, x^*]$ where*

$$x^* = \frac{K - (2\hat{r})^{-1/2}\frac{rK}{m_1+\sqrt{2\hat{r}}}}{(2\hat{r})^{-1/2}\frac{\mu-r}{m_1+\sqrt{2\hat{r}}+a} + 1}, \tag{3.3}$$

$m_1 := \frac{\mu-0.5a^2}{a}$ *and* $\hat{r} := r + \frac{1}{2}m_1^2$.

Proof. First notice that Lemma 3.1 in [1] yields that the stopping set is a closed south-west connected subset of $(0, K)$. Hereby south-west connected means that if $x \in S$, then so is y for all $0 < y \leq x$. Therefore S has the form $(0, x^*]$ and the only point on the boundary is x^* and thus left side of (3.2) in x^* is equals $K - x^*$. For the right side of (3.2) consider W_t, the underlying Brownian motion of X_t. Then the density of W_t is given by

$$f_{W_t}(x) = \frac{1}{\sqrt{2\pi t}} \exp\left\{-\frac{x^2}{2t}\right\}.$$

Formula (29) in Erdelyi et al. [2] p.146 yields

$$\int_0^\infty e^{-rt} f_{W_t}(x)dt = \frac{2}{\sqrt{2\pi}}\left(\frac{x^2}{2r}\right)^{1/4} K_{0.5}\left(\sqrt{x^2}\sqrt{2r}\right)$$

$$= \frac{2}{\sqrt{2\pi}}\left(\frac{x^2}{2r}\right)^{1/4} \sqrt{\frac{\pi}{2}}|x|^{-1/2}\sqrt{2r}^{-1/2}\exp(-|x|\sqrt{2r})$$

$$= (2r)^{-1/2}\exp(-|x|\sqrt{2r}),$$

where $K_{0.5}$ is the modified Bessel function of second kind given by (see [10, p. 1021])

$$K_{0.5}(z) = \sqrt{\frac{\pi}{2}} z^{-1/2} e^{-z}, \; z > 0.$$

Introduce $Z_t = W_t + m_1 t$ where $m_1 = \frac{\mu - 0.5a^2}{a}$. Then the density of Z_t is given by

$$f_{Z_t}(x) = \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{(x-m_1t)^2}{2t}\right\} = f_{W_t}(x)\exp\left\{\frac{2xm_1 - m_1^2 t}{2}\right\}.$$

Since $X_t = x_1\exp\{aZ_t\}$, we get for the density of X_t started in x_1

$$f_{X_t}(x) = \frac{\partial}{\partial x}\mathbb{P}(X_t \le x) = \frac{\partial}{\partial x}\mathbb{P}\left(\frac{X_t}{x_1} \le \frac{x}{x_1}\right) = \frac{\partial}{\partial x}\mathbb{P}\left(aZ_t \le \log\frac{x}{x_1}\right)$$

$$= \frac{\partial}{\partial x}\mathbb{P}\left(Z_t \le \frac{1}{a}\log\frac{x}{x_1}\right) = \frac{1}{xa} f_{Z_t}\left(\frac{1}{a}\log\frac{x}{x_1}\right) = \frac{1}{xa} f_{Z_t}\left(\hat{x}\right),$$

where $\hat{x} := \frac{1}{a}\log\frac{x}{x_1}$.

So now we are ready to find the Green function. Notice that for $x_1 > x \Rightarrow \hat{x} < 0$,

$$G_r(x_1, x) = \int_0^\infty e^{-rt} f_{X_t}(x)dt$$

$$= \int_0^\infty e^{-rt}\frac{1}{xa} f_{Z_t}(\hat{x})\; dt$$

$$= \int_0^\infty e^{-rt}\frac{1}{xa}\exp\left\{\frac{2\hat{x}m_1 - m_1^2 t}{2}\right\} f_{W_t}(\hat{x})\; dt$$

$$= \frac{1}{xa}\exp\{\hat{x}m_1\}\int_0^\infty \exp\{-\underbrace{(r + m_1^2/2)}_{=\hat{r}}t\} f_{W_t}(\hat{x})\; dt$$

$$= \frac{1}{xa} \exp\{\hat{x} m_1\} (2\hat{r})^{-1/2} \exp\{-|\hat{x}|\sqrt{2\hat{r}}\}$$

$$\overset{x_1 > x}{=} \frac{1}{xa} (2\hat{r})^{-1/2} \exp\left\{ \frac{m_1 + \sqrt{2\hat{r}}}{a} \log\left(\frac{x}{x_1}\right) \right\}$$

$$= \frac{1}{xa} (2\hat{r})^{-1/2} \left(\frac{x}{x_1}\right)^{\frac{m_1+\sqrt{2\hat{r}}}{a}}$$

Finally, we can calculate the right side of (3.2). Note that $\sigma(y) = rK + (\mu - r)y$.

$$\int_0^{x^*} G_r(x^*, y)\sigma(y)dy = \int_0^{x^*} \frac{1}{ya}(2\hat{r})^{-1/2} \left(\frac{y}{x^*}\right)^{\frac{m_1+\sqrt{2\hat{r}}}{a}} (rK + (\mu - r)y)\ dy$$

$$= \frac{(2\hat{r})^{-1/2}}{a} (x^*)^{-\frac{m_1+\sqrt{2\hat{r}}}{a}} \left[rK \frac{a}{m_1 + \sqrt{2\hat{r}}} y^{\frac{m_1+\sqrt{2\hat{r}}}{a}} + (\mu - r) \frac{a}{m_1 + \sqrt{2\hat{r}} + a} y^{\frac{m_1+\sqrt{2\hat{r}}+a}{a}} \right]_0^{x^*}$$

$$= (2\hat{r})^{-1/2} \left[\frac{rK}{m_1 + \sqrt{2\hat{r}}} + \frac{\mu - r}{m_1 + \sqrt{2\hat{r}} + a} x^* \right].$$

Thus (3.2) yields

$$K - x^* = (2\hat{r})^{-1/2} \left[\frac{rK}{m_1 + \sqrt{2\hat{r}}} + \frac{\mu - r}{m_1 + \sqrt{2\hat{r}} + a} x^* \right]$$

$$\Leftrightarrow \qquad x^* = \frac{K - (2\hat{r})^{-1/2} \frac{rK}{m_1 + \sqrt{2\hat{r}}}}{(2\hat{r})^{-1/2} \frac{\mu - r}{m_1 + \sqrt{2\hat{r}} + a} + 1}.$$

□

Corollary 3.3.2. *The optimal stopping set in the case* $\mu = r$ *is* $S = (0, x^*]$ *where*

$$x^* = \frac{\gamma K}{1 + \gamma} \text{ and } \gamma = \frac{2r}{a^2}.$$

Proof. Note that the denominator of (3.3) is 1 for $\mu = r$ and moreover

$$2\hat{r} = 2r + m_1^2 = \frac{r^2}{a^2} + r + \frac{1}{4}a^2 = \left(\frac{r}{a} + \frac{1}{2}a\right)^2,$$

$$\Rightarrow (2\hat{r})^{-1/2} = \left(\frac{r}{a} + \frac{1}{2}a\right)^{-1} = \frac{a}{r + \frac{1}{2}a^2}.$$

Thus formula (3.3) yields that

$$x^* = K - (2\hat{r})^{-1/2}\frac{rK}{m_1 + \sqrt{2\hat{r}}} = K - \frac{a}{r + \frac{1}{2}a^2}\frac{rK}{\frac{2r}{a}} = K - \frac{K}{1+\gamma} = \frac{\gamma K}{1+\gamma}. \qquad (3.4)$$

□

3.4 The 2-dimensional case

In this section we have a look at the two-dimensional optimal investment problem. Lemma 3.1 in [1] yields that the stopping set S is a closed, convex and south-west connected subset of $\{(x_1, x_2) \in \mathbb{R}_+^2 : K - x_1 - x_2 > 0\}$. By parametrizing the boundary of the stopping set by a curve $x_2 = \gamma(x_1)$ such that $S = \{(x_1, x_2) \in \mathbb{R}_+^2 : x_1 \leq \gamma(x_1)\}$, we can write the characterization (3.2) in the 2-dimensional case by

$$K - x_1 - \gamma(x_1) = \int_0^{x_1^*}\int_0^{\gamma(y_1)} G_r((x_1, \gamma(x_1)), (y_1, y_2))\sigma(y_1, y_2)m(dy). \qquad (3.5)$$

Note that $\gamma(x_1^*) = 0$ (formula (3.3) for x_1^* where $\mu = \mu_1$ and $a = a_1$) and respectively is $\gamma(0) = x_2^*$ (formula (3.3) where $\mu = \mu_2$ and $a = a_2$). Moreover, it is to remark that the formulation of the problem allows μ_1 and μ_2 which are differing from the market interest rate r. Thus we are not allowed to always use formula (3.4) which was indicated in [1].

Christensen and Salminen developed in the same style as in the previous section a formula for the Green function: Let $X_t = (X_t^{(1)}, X_t^{(2)})$ be a geometric Brownian motion which starts in (x_1, x_2), then

$$G_r((x_1, x_2), (u, v)) = \int_0^\infty e^{-rt} f_{X_t}(u, v)\ dt$$

$$= \frac{1}{\pi\sqrt{1-\rho^2}}\frac{1}{a_1 a_2 uv}\exp\left(-\frac{1}{2(1-\rho^2)}A_\rho(\hat{u}, \hat{v}; m_1, m_2)\right)K_0\left(\sqrt{\hat{r}}\sqrt{\frac{2B_\rho(\hat{u}, \hat{v})}{1-\rho^2}}\right),$$

where $\rho := \mathbb{E}\left(W_1^{(1)}W_1^{(2)}\right)$, $m_1 := \left(\mu_1 - \frac{1}{2}a_1^2\right)/a_1$, $m_2 := \left(\mu_2 - \frac{1}{2}a_2^2\right)/a_2$,

$$B_\rho(x, y) := x^2 - 2\rho xy + y^2,$$

$$A_\rho(x,y;m_1,m_2) \coloneqq 2\rho(m_2x+m_1y)-2(m_1x+m_2y),$$

$$\hat{u} \coloneqq \frac{1}{a_1}\log\frac{u}{x_1}, \quad \hat{v} \coloneqq \frac{1}{a_2}\log\frac{v}{x_2}, \quad \hat{r} \coloneqq r+\frac{B_\rho(m_1,m_2)}{2(1-\rho^2)}, \quad \text{and}$$

K_0 is the modified Bessel function of second kind given by (see formula 9.6.21 in [11, p. 376])

$$K_0(u)=\int_0^\infty \frac{\cos(uv)}{\sqrt{1+v^2}}dv, \quad u>0.$$

Since there is no known explicit expression for K_0, we cannot expect to find an explicit solution for the curve γ in (3.5). In the following, we are setting up an numerical approach for solving for this curve. The general idea is to approximate the integrals by a numerical quadrature, specifically the Gauss-Legendre quadrature. This leads to a multidimensional root-finding problem.
Let $w_1,\ldots,w_n$ be the Gauss-Legendre weights and $l_1,\ldots,l_n$ the nodes which are in the interval $(-1,1)$. Then the transformation $b(l_i+1)/2$ brings the nodes into the interval $(0,b)$. Let $t_1,\ldots,t_n$ denote the nodes in $(0,x_1^*)$ and $s_1^{(j)},\ldots,s_n^{(j)}$ the nodes between 0 and $\gamma(t_j)$ where x_1^* is given by (3.3). Then we make the following approximations:

$$\begin{aligned} K-t_j-\gamma(t_j) &= \int_0^{x_1^*}\int_0^{\gamma(y_1)} G_r(t_j,\gamma(t_j),y_1,y_2)\sigma(y_1,y_2)dy \\ &\approx \int_0^{x^*}\frac{\gamma(y_1)}{2}\sum_{i_2=1}^n w_{i_2}G_r(t_j,\gamma(t_j),y_1,s_{i_2}(y_1))\sigma(y_1,s_{i_2}(y_1))dy_1 \\ &\approx \frac{x_1^*}{2}\sum_{i_1=1}^n\frac{\gamma(t_{i_1})}{2}\sum_{i_2=1}^n w_{i_1}w_{i_2}G_r(t_j,\gamma(t_j),t_{i_1},s_{i_2}^{(i_1)})\sigma(t_{i_1},s_{i_2}^{(i_1)}). \end{aligned}$$

So the algorithm for approximating the curve is:

Algorithm 1 Approximating of the 2-dimensional boundary

1: $y_{init} = \dots$ %an arbitrary start curve
2: **for** $j = 1$ **to** n **do**
3: $(F(y))_j = -(K - t_j - y_j) + \sum_{i_1=1}^{n}\sum_{i_2=1}^{n} \frac{x_1^* y_{i_1}}{4} w_{i_1} w_{i_2} G_r(t_j, y_j, t_{i_1}, s_{i_2}^{(i_1)}) \sigma(t_{i_1}, s_{i_2}^{(i_1)})$
4: **end for**
5: solve $F(y) = 0$ with initial guess y_{init}

Algorithm (1) is to be understood as follows. We define in the for-loop a n-dimensional function F and subsequently we try to find a root of F in line 5. For the root-finding you could use the Levenberg-Marquardt method (see [7]) or the Trust-Region-Reflective algorithm (see [5, 6]). We tried both algorithms. The Levenberg-Marquardt algorithm converges faster, but the solution on the left boundary was not monotone decreasing which is contradicting the south-west connected property of the stopping region. So this algorithm converges to a solution which is not the solution of our problem. In order to restrict the solutions to monotone decreasing ones, we introduced boundary conditions for the solutions:

$$y(i) \geq y(i-i), \quad i = 2, \dots, n. \tag{3.6}$$

Since the Levenberg-Marquardt algorithm does not allow boundary conditions, we use the Trust-Region-Reflective algorithm which allows boundary conditions but not such like in (3.6). In order to integrate the boundary conditions we changed the objective function F to F_a where

$$F(y_1, y_2, \dots, y_n) = F_a(y_1, \alpha_2, \dots, \alpha_n) \text{ and}$$

$$\alpha_i = \frac{y_i}{y_{i-1}} \in (0, 1], \quad i = 2, \dots, n.$$

So the first argument of F_a is in the interval $(0, x_2^*)$,where $x_2^* = \gamma(0)$ is the stopping boundary of the one-dimensional case given in (3.3). All the other arguments of F_a have to be between 0 and 1. The implementation in MATLAB can be found in the appendix.

The main problem of this algorithm is its complexity of $\mathcal{O}(n^3)$ per function evaluation of F where n is the number of Gauss-Nodes and therefore the high runtime

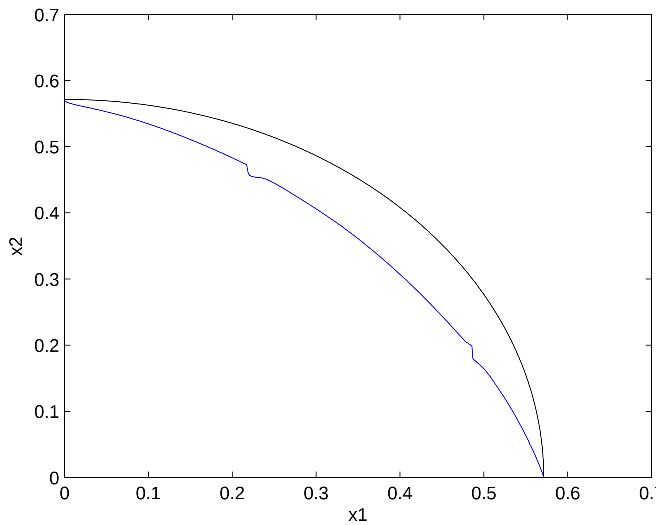

Figure 3.1: Boundary for the 2-D case for parameters $K = 1,\ \mu_1 = \mu_2 = r = 0.06,\ a_1 = a_2 = 0.3, \rho = 0$.

of the root-finding algorithm. For example, for 512 nodes we had a runtime of over one minute for one function evaluation. That poses a problem for the root-finding algorithm, since it usually evaluates the function several thousand times. In order to decrease the runtime, we parallelized the for loop in line 2-4 of Algorithm 1 which decreased the runtime to about 23 seconds with eight workers.

The blue line in Figure 3.1 shows the approximation of the stopping boundary we obtained with Algorithm 1 and the black line is the only ellipsoid which is going through the points $(0, x_2^*)$ and $(x_1^*, 0)$. Note that we stopped the root-finding algorithm after eleven iterations which included 6156 function evaluations and took about 40 hours. The error measured in the Euclidean norm was decreased from 1.31 to 0.32. We observe that the curve is not completely smooth as we expected it to be. The problem might be too few iterations of the solver or the solver itself. Further analysis might be needed here. A second example with different parameters and dependent geometric Brownian motions is shown in Figure 3.2. Again, the black line is the only ellipsoid which is going through the points $(0, x_2^*)$ and $(x_1^*, 0)$. This

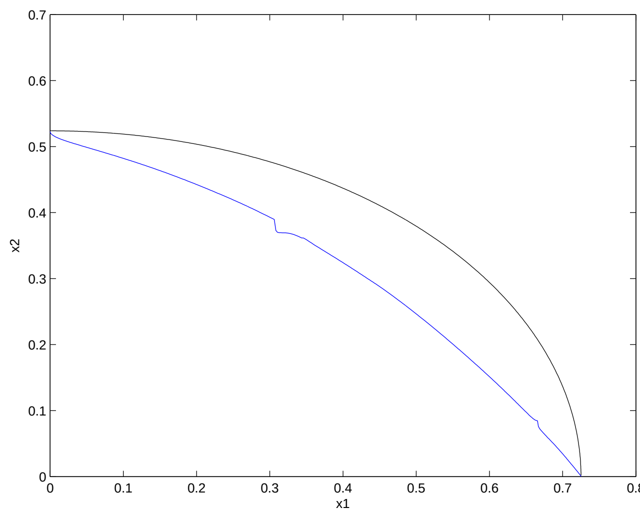

Figure 3.2: Boundary for the 2-D case for parameters $K = 1,\ \mu_1 = 0.05,\ \mu_2 = 0.04,\ r = 0.06,\ a_1 = 0.2,\ a_2 = 0.3,\ \rho = 0.2$.

time we stopped the algorithm after seven iterations which included 4104 function evaluations and took about 28 hours. The error in the Euclidean norm was decreased from 2.02 to 0.60. The plot also shows two areas which are not smooth.

In Figure 3.3 we plot the point-wise absolute error of the objective functions for example 1 and 2 which were shown in Figures 3.1 and 3.2. We observed that the error on the left boundary is a lot higher than for the rest of the x-values and therefore almost all of the error for the whole curve is picked up on the left boundary. Thus, the main concern should be to find a way how to decrease the error in this region. In the following we are suggesting a way which might work, but was not implemented by now. The main idea is to use the symmetry of the problem with respect to the two geometric Brownian motions. By switching the indices of geometric Brownian motion one and two, we can set up a new characterizing equation for the boundary. The resulting boundary function $\gamma^{(s)}$ has to be the inverse of the boundary function of the original problem $\gamma^{(o)}$. These observations led to Algorithm 2.

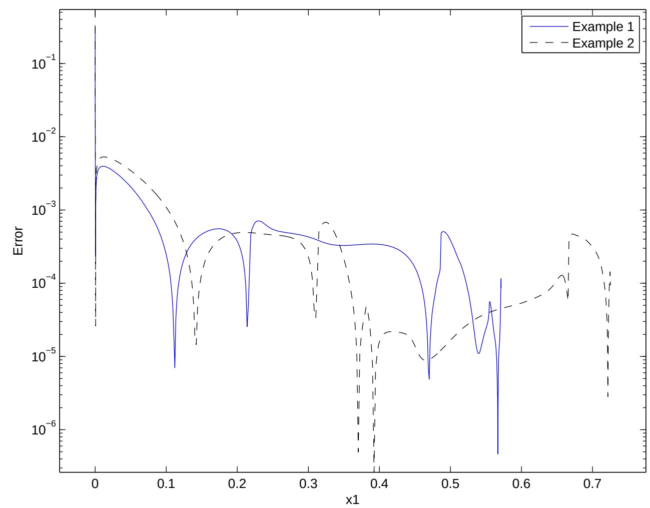

Figure 3.3: Error-plot for the two examples.

Algorithm 2 Improved algorithm for the 2D boundary

1: $y^{(o)} = \ldots$ %start curve for the original problem
2: $y^{(s)} = \ldots$ %start curve for the symmetric problem
3: $F^{(o)} = \ldots$ objective function for the original problem like in Algorithm 1
4: $F^{(s)} = \ldots$ objective function for the symmetric problem like in Algorithm 1
5: **while** Error is bigger than a threshold **do**
6: Do one step with a root-finding algorithm for $F^{(o)}$ with start-value $y^{(o)}$
7: $\Rightarrow$ new $y^{(o)}$
8: Do one step with a root-finding algorithm for $F^{(s)}$ with start-value $y^{(s)}$
9: $\Rightarrow$ new $y^{(s)}$
10: Calculate an "inverse" for $y^{(o)}$ and $y^{(s)}$
11: Set $y^{(o)}$ as weighted sum of $y^{(o)}$ and the "inverse" of $y^{(s)}$
12: Set $y^{(s)}$ as weighted sum of $y^{(s)}$ and the "inverse" of $y^{(o)}$
13: **end while**

Here are some explanations for Algorithm 2:

- Line 5: The term error is to be understand as norm of $F^{(o)}(y^{(o)})$ plus $F^{(s)}(y^{(s)})$.
- Line 10: The "inverse" of $y^{(s)}$ can be calculated by evaluating the nodes $t_1^{(o)}, \ldots, t_n^{(o)}$, which are between 0 and x_1^*, in a spline which is going through

the points $(0, x_2^*), (y_n^{(s)}, t_n^{(s)}), \ldots, (y_n^{(s)}, t_n^{(s)}), (x_1^*, 0)$ where $t_1^{(s)}, \ldots, t_n^{(s)}$ are the Gauss nodes between 0 and x_2^*. Respectively is to be calculated for the "inverse" of $y^{(o)}$.

- Line 11&12: Let w_i denote the weight for node $i \in \{1, \ldots, n\}$. Then $y_i^{(o)}$ is set to $w_i y_i^{(o)} + (1 - w_i)$ times the "inverse" of $y^{(s)}$ evaluated in $t_i^{(o)}$. Hereby is $w_i \in (0, 1)$ small for small i since the error solution on the left boundary is big and respectively is w_i close to one for i close to n.

Some of the formulations have to be specified when coding this algorithm, e.g. the weights w_i and the type of spline which is used for the "inverses".

Chapter 4

The one-dimensional optimal stopping problem with finite time horizon

This chapter concerns the optimal stopping time problem for a space-time process with finite time horizon. First we review general theoretical results, before considering three specific examples: the American Put option, the American Cash-or-Nothing put option and finally the American Asset-or-Nothing put option.

4.1 General theory

A one-dimensional geometric Brownian motion X with volatility σ^2 and drift r (the risk-free interest rate in the market) in space-time is the two-dimensional process $\bar{X} = ((t, X_t))_{0 \le t \le T}$ with the state space $I = [0, T) \times \mathbb{R}_+$. The differential operator associated with $\bar{X}$ is

$$\bar{\mathcal{G}} := \frac{\partial}{\partial t} + \frac{\sigma^2}{2} x^2 \frac{\partial^2}{\partial x^2} + rx \frac{\partial}{\partial x},$$

and its resolvent/Green kernel is

$$\bar{G}_r((s,x),(t,y)) = \begin{cases} e^{-r(t-s)} p(t-s; x, y), & s < t \le T, \\ 0, & t \le s < T, x \ne y, \\ +\infty, & t = s < T, x = y, \end{cases}$$

where $p(t;x,y)$ is the density of X of going in t time steps from x to y.
The following theorem summarizes the results of Christensen and Salminen:

Theorem 4.1.1. *Let u be a bounded r-excessive function for $\bar{X}$ on $(0,T) \times \mathbb{R}_+$ such that $\partial u/\partial t$ and $\partial u/\partial x$ are continuous on $(0,T) \times \mathbb{R}_+$, and $\partial u/\partial x$ is absolutely continuous as a function of the second argument. Then there exists a unique σ-finite measure σ on $(0,T) \times \mathbb{R}_+$ such that*

$$u(s,x) = \int \int_{(0,T)\times\mathbb{R}_+} \bar{G}_r((s,x),(t,y))\sigma(dt,dy).$$

Moreover σ is absolutely continuous with respect to Lebesgue measure on $(0,T) \times \mathbb{R}_+$ and is given by

$$\sigma(ds,dy) = (r - \bar{\mathcal{G}})u(s,y)\ ds\ m(dy),$$

where m is the speed measure.

The value function for an option with payoff $g : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is given by

$$V(s,x) \coloneqq \sup_{\tau \in \mathcal{M}_{s,T}} \hat{\mathbb{E}}_{(s,x)} \left(e^{-r(\tau - s)} g(X_\tau) \right).$$

From the general theory of optimal stopping it is known that V is r-excessive for the space-time process $\bar{X}$ if the reward function is lower semicontinuous and the stopping region consists of the points, where the value is equal to the reward, i.e.,

$$S \coloneqq \{(s,x) : V(s,x) = g(x)\}.$$

We now use Theorem 4.1.1 to formulate a more general version of Theorem 4.1 in [1].

Theorem 4.1.2. *Let $V : (0,T) \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ be the value function for an option with bounded payoff $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ which satisfies the conditions of Theorem 4.1.1 and additionally for $(t,y) \in \text{int}(S)$, let $(r - \bar{\mathcal{G}})g(y) = c$ for some constant c. Then the price of the option at time s when $X_s = x$ has the unique Doob-Meyer decomposition*

$$V(s,x) = c \int_s^T e^{-r(t-s)} \hat{\mathbb{P}}_{s,x}(X_t \in S_t)\ dt + \hat{\mathbb{E}}_{s,x}(e^{-r(T-s)} g(X_T)),$$

where $\hat{\mathbb{P}}$ denotes the martingale measure in the Black-Scholes market and $S_t \coloneqq \{y : (t,y) \in S\}$.

Proof. By using Theorem 4.1.1 we get $\sigma(dt,dy) = (r-\bar{\mathcal{G}})g(y)\ dt\ m(dy) = c\ dt\ m(dy)$ for $(t,y) \in \text{int}(S)$. Since $V(T-,x) = g(x)$, we furthermore have $\sigma(T,dy) = g(y)\ m(dy)$ for $y \in S_T$ and otherwise $\sigma = 0$. That results in

$$\begin{aligned}
V(s,x) &= \iint_{(0,T]\times\mathbb{R}_+} \bar{G}_r((s,x),(t,y))\ \sigma(dt,dy) \\
&= \iint_{\text{int}(S)} \bar{G}_r((s,x),(t,y))(r-\bar{\mathcal{G}})g(y)\ dt\ m(dy) \\
&+ \iint_{\{T\}\times\mathbb{R}_+} \bar{G}_r((s,x),(t,y))g(y)\ \delta_{\{T\}}(dt)\ m(dy) \\
&= c\iint_{\text{int}(S)} \bar{G}_r((s,x),(t,y))\ dt\ m(dy) \\
&+ \int_{\mathbb{R}_+} \bar{G}_r((s,x),(T,y))g(y)\ m(dy) \\
&= c\int_s^T e^{-r(t-s)}\hat{\mathbb{P}}_{s,x}(X_t \in S_t)\ dt + \hat{\mathbb{E}}_{s,x}(e^{-r(T-s)}g(X_T)).
\end{aligned}$$

□

Now we look at three different applications of the Theorem 4.1.2.

4.2 The American put option

The first example is the American put option whose payoff is $g(X_t) = (K-X_t)^+$ at time $t \in [0,T]$ where T is the maturity, K the strike price and X_t the underlying stock which we modeled with an 1-dimensional geometric Brownian motion. Theorem 4.1 in the paper of Christensen and Salminen [1] states that there is an increasing, convex and differentiable function b where $b(T) = K$, $b(0) < K$ and $S := \{(s,x) : x < b(s)\}$ is the optimal stopping set. So the optimal stopping time is $\tau^* := \inf\{t : X_t < b(t)\}$. In addition Theorem 4.1.2 shows that b can be characterized by the following integral equation:

$$K - b(s) = rK\int_s^T e^{-r(t-s)}\hat{\mathbb{P}}_{(s,b(s))}(X_t < b(t))\ dt + \hat{\mathbb{E}}_{(s,b(s))}\left[e^{-r(T-s)}(K-X_T)^+\right], \tag{4.1}$$

In [1], the authors do not provide a way to solve for the unknown curve b. We will now set up a numerical scheme to approximate the unknown function b.
The starting point for the numerical scheme is the characterizing integral equation (4.1). Notice that the second term in (4.1) is exactly the price of a European put option with time to maturity $T - s$, underlying geometric Brownian motion X starting in $b(s)$ and strike K, for which the explicit value is well-known (see [3, p. 291]):

$$\hat{\mathbb{E}}_{(s,b(s))}\left[e^{-r(T-s)}(K - X_T)^+\right] = Ke^{-r(T-s)}\Phi(-d_2(T,s)) - b(s)\Phi(-d_1(T,s)) \tag{4.2}$$

where Φ is the cumulative probability distribution function of the standard normal distribution,

$$d_1(T,s) = \frac{1}{\sigma\sqrt{T-s}}\left[\log\frac{b(s)}{K} + \left(r + \frac{\sigma^2}{2}\right)(T-s)\right] \text{ and}$$

$$d_2(T,s) = \frac{1}{\sigma\sqrt{T-s}}\left[\log\frac{b(s)}{K} + \left(r - \frac{\sigma^2}{2}\right)(T-s)\right] = d_1(T,s) - \sigma\sqrt{T-s}.$$

For the first term in (4.1) it is not possible to find an explicit formula for the integral since its integrand is dependent on the unknown function b. But nevertheless we can find an explicit term for the probability in the integrand:

$$\begin{aligned}
\hat{\mathbb{P}}_{(s,b(s))}(X_t < b(t)) &= \hat{\mathbb{P}}\left(b(s)\exp\left\{\left(r - \frac{\sigma^2}{2}\right)(t-s) + \sigma W_{t-s}\right\} < b(t)\right) \\
&= \hat{\mathbb{P}}\left(\left(r - \frac{\sigma^2}{2}\right)(t-s) + \sigma W_{t-s} < \log\left(\frac{b(t)}{b(s)}\right)\right) \\
&= \hat{\mathbb{P}}\left(\frac{1}{\sqrt{t-s}}W_{t-s} < \frac{1}{\sigma\sqrt{t-s}}\left[\log\left(\frac{b(t)}{b(s)}\right) - \left(r - \frac{\sigma^2}{2}\right)(t-s)\right]\right) \\
&= \Phi(d_3(t,s))
\end{aligned}$$

where

$$d_3(t,s) := \frac{1}{\sigma\sqrt{t-s}}\left[\log\left(\frac{b(t)}{b(s)}\right) - \left(r - \frac{\sigma^2}{2}\right)(t-s)\right]$$

So, (4.1) can be rewritten as

$$K - b(s) = rK \int_s^T e^{-r(t-s)} \Phi(d_3(t,s))\, dt$$
$$+ Ke^{-r(T-s)} \Phi(-d_2(T,s)) - b(s)\Phi(-d_1(T,s)), \quad 0 \le s \le T. \tag{4.3}$$

Notice, that for $s = T$ (4.3) reduces to $K = b(T)$ since $\lim_{s\to T} d_1(T,s) = \lim_{s\to T} d_2(T,s) = 0$.

We are now ready to set up a numerical scheme for approximating b. The only problem in (4.3) is the integral since it depends on the unknown curve b. So, the basic idea now is to approximate the integral by the trapezoidal rule. For that reason let $n \in N$, $h := T/n$ and $t_i := (i-1)h$ for $i = 1, \ldots, n+1$. We use (4.3) with $s = t_n$:

$$K - b(t_n)$$
$$= rK \int_{t_n}^{t_{n+1}} e^{-r(t-t_n)} \Phi(d_3(t,t_n))\, dt + Ke^{-r(t_{n+1}-t_n)} \Phi(-d_2(T,t_n)) - b(t_n)\Phi(-d_1(T,t_n))$$
$$\approx rK \frac{1}{2} h \left[e^{-r(t_{n+1}-t_n)} \Phi(d_3(t_{n+1},t_n)) + e^{-r(t_n-t_n)} \Phi(d_3(t_n,t_n)) \right]$$
$$+ Ke^{-r(t_{n+1}-t_n)} \Phi(-d_2(T,t_n)) - b(t_n)\Phi(-d_1(T,t_n))$$
$$= rK \frac{h}{2} \left[e^{-rh} \Phi(d_3(t_{n+1},t_n)) + \frac{1}{2} \right] + Ke^{-rh} \Phi(-d_2(T,t_n)) - b(t_n)\Phi(-d_1(T,t_n)). \tag{4.4}$$

We interpret the approximation (4.4) as an equation. The only unknown is $b(t_n)$, so we solve this equation for it. An explicit solution is again unrealistic, but we can use a numerical root-finding algorithm for determining $b(t_n)$. Since we face a 1-dimensional problem we can use for example the bisection method.

After finding $b(t_n)$, we can set up a backwards-recursion for approximating all $b(t_i)$: For $i = n-1, n-2, \ldots, 1$

$$K - b(t_i)$$
$$= rK \int_{t_i}^{t_{n+1}} e^{-r(t-t_i)} \Phi(d_3(t,t_i))\, dt + Ke^{-r(T-t_i)} \Phi(-d_2(T,t_i)) - b(t_i)\Phi(-d_1(T,t_i))$$

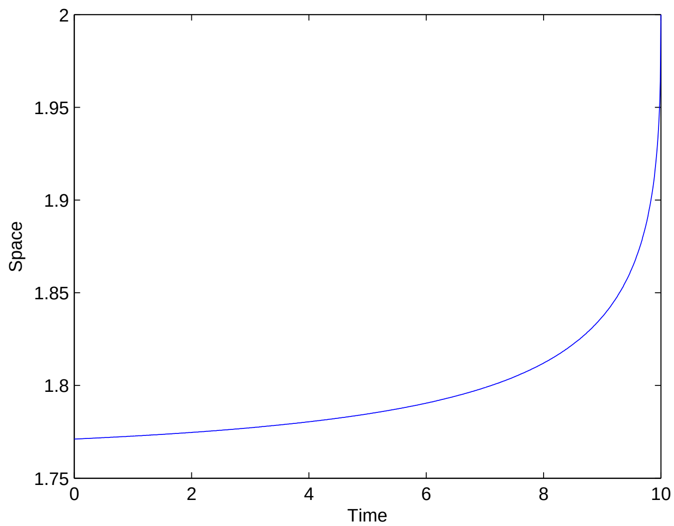

Figure 4.1: Boundary for the American put option for parameters $K = 2, r = 0.03, \sigma = 0.09$ and $T = 10$

$$\approx rK\frac{h}{2}\left[\underbrace{e^{-r\cdot 0}\Phi(d_3(t_i,t_i))}_{=0.5} + 2\sum_{j=i+1}^{n} e^{-rh(j-i)}\Phi(d_3(t_j,t_i)) + e^{-rh(n+1-i)}\Phi(d_3(t_{n+1},t_i))\right]$$

$$+ Ke^{-rh(n+1-i)}\Phi(-d_2(T,t_i)) - b(t_i)\Phi(-d_1(T,t_i)). \tag{4.5}$$

So, in each step we have a one dimensional root finding problem, with the only unknown being $b(t_i)$, which we can solve again with the bisection method. You can find the implementation of this algorithm in MATLAB in the appendix.
Figure 4.1 shows the solution for the optimal stopping curve b for $K = 2, r = 0.03, \sigma = 0.09$ and $T = 10$.

4.3 American Cash-or-Nothing put

The American Cash-or-Nothing put option has a payoff of $\mathbb{1}\{X_t < K\}$. Since we assume that the market interest rate r is equal to the drift of the geometric Brownian motion μ there is no gain in not exercising the option if it is in the money, i.e. $X_t < K$.

So the optimal stopping set is $S = \{(t,x) : 0 < x < K,\ t \in [0,T]\}$. So in contrast to the American put option example we know the optimal stopping boundary. The reason we look at this example is to verify the theory and the numerical approximation introduced in the previous section. In order to do so, we want to use Theorem 4.1.2 to find a characterizing integral equation for the stopping boundary.

We first have to check the assumptions made in Theorem 4.1.2. Since the optimal stopping boundary is equals K^- at every time, the American Cash-or-Nothing Put is the same as a Cash-or-Nothing Barrier option whose value function satisfies the necessary regularity conditions. The last assumption we have to check is that $(r - \hat{\mathcal{G}})g(y)$ is constant for $y < K$. Therefore notice that for $y < K$

$$(r - \hat{\mathcal{G}})\mathbb{1}\{y < K\} = (r - \hat{\mathcal{G}}) \cdot 1 = r.$$

That leads to the result

$$V(s,x) = r\int_s^T e^{-r(t-s)}\hat{\mathbb{P}}_{(s,x)}(X_t < b(t))\ dt + \hat{\mathbb{E}}_{(s,x)}(e^{-r(T-s)}\mathbb{1}\{X_T < K\}).$$

Therefore we get as characterizing equation for the boundary: for any $s > 0$

$$1 = r\int_s^T e^{-r(t-s)}\hat{\mathbb{P}}_{(s,b(s))}(X_t < b(t))\ dt + \hat{\mathbb{E}}_{(s,b(s))}(e^{-r(T-s)}\mathbb{1}\{X_T < K\}). \tag{4.6}$$

Having a closer look at equation (4.6), we notice that the second term is the price of a Cash-or-Nothing put in the Black-Scholes market which is $e^{-r(T-s)}\Phi(-d_2(T,s))$. The first term is just differing by $1/K$ from the first term of the American put option. Therefore we get

$$1 = r\int_s^T e^{-r(t-s)}\Phi(d_3(t,s))\ dt + e^{-r(T-s)}\Phi(-d_2(T,s)).$$

Now, we have two possibilities to check this formula: A numerical approach and checking whether $b(t) = K$ satisfies (4.6). We are going to have a look at both ways. Approaching the problem numerically in the same style as for the American put option gives as the following approximation: for $i = n, \dots, 1$

$$1 \approx r\frac{h}{2}\left[0.5 + 2\sum_{j=i+1}^{n} e^{-rh(j-i)}\Phi(d_3(t_j,t_i)) + e^{-rh(n+1-i)}\Phi(d_3(t_{n+1},t_i))\right]$$

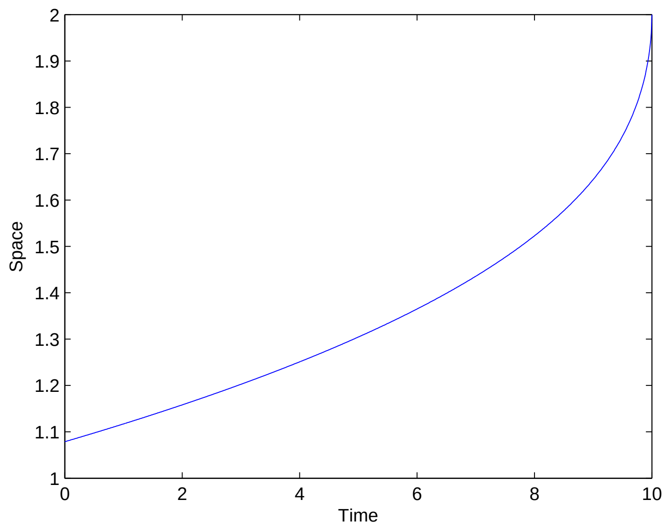

Figure 4.2: Boundary for the American Cash-or-Nothing option for parameters $K = 2, r = 0.03, \sigma = 0.09$ and $T = 10$

$$+ e^{-rh(n+1-i)}\Phi(-d_2(T, t_i)).$$

Sadly, the result is not a constant curve with $b(t) = K$ as we expected. The implementation in MATLAB can be found in the appendix and an illustrative plot is shown in Figure 4.2.

The second approach is to assume that $b(t) = K, \ \forall t \in [0, T]$ and calculate the right side of (4.6) analytically. Notice that $\log(b(t)/K) = \log(b(t)/b(s)) = 0$. So

$$-d_3(T, s) = d_2(T, s) = \frac{1}{\sigma}\left(r - \frac{\sigma^2}{2}\right)\sqrt{T - s} =: -c\sqrt{T - s}.$$

and thus we can simplify (4.6) to

$$1 = r\int_s^T e^{-r(t-s)}\Phi(c\sqrt{t-s})\ dt + e^{-r(T-s)}\Phi(c\sqrt{T-s})$$

$$= \left[-e^{-r(t-s)}\Phi(c\sqrt{t-s})\right]_s^T + \int_s^T e^{-r(t-s)}\phi(c\sqrt{t-s})\frac{c}{2\sqrt{t-s}}\ dt + e^{-r(T-s)}\Phi(c\sqrt{T-s})$$

$$= \frac{1}{2} + \int_s^T e^{-r(t-s)} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{c^2(t-s)}{2}\right) \frac{1}{2} c(t-s)^{-0.5}\ dt$$

$$= \frac{1}{2} + \frac{c}{\sqrt{2\pi}} \int_s^T \frac{1}{2\sqrt{t-s}} \exp\left(-\frac{(c^2+2r)(t-s)}{2}\right)\ dt$$

$$= \frac{1}{2} + \frac{c}{\sqrt{2\pi}} \int_0^{\sqrt{T-s}} \exp\left(-\frac{(c^2+2r)u^2}{2}\right)\ du$$

$$= \frac{1}{2} + \frac{c}{\sqrt{2\pi}} \frac{2}{\sqrt{\pi}} \sqrt{\frac{2}{c^2+2r}}\, \mathrm{erf}\left(\sqrt{\frac{c^2+2r}{2}} \sqrt{T-s}\right)$$

$$= \frac{1}{2} - \frac{4}{\pi} \frac{\sigma^2 - 2r}{\sigma^2 + 2r}\, \mathrm{erf}\left(\sqrt{\frac{c^2+2r}{2}} \sqrt{T-s}\right),$$

where ϕ is the pdf of the standard normal distribution and erf is the error function. Since the error function is monotone increasing, this expression is not constant. Consequently this also contradicts our argument. Although we know that something must be wrong in the argument we were unable to identify the error.

4.4 American Asset-or-Nothing put

The American Asset-or-Nothing put option has a payoff of $X_t \mathbb{1}\{X_t < K\}$. With the same reasoning as for the American Cash-or-Nothing put option we know that we exercise the option as soon as the underlying hits the strike. So the stopping set is $S = \{(s,x) : x < K\}$.

As in the previous section, the value function of the Asset-or-Nothing put is equal to the one of an Asset-or-Nothing Barrier option (Down-and-In) for which we know that the regularity conditions of Theorem 4.1.2 are satisfied. Moreover, $(r - \bar{\mathcal{G}}) y \mathbb{1}\{y < K\} = (r - \bar{\mathcal{G}}) y = (r - r) y = 0$ for $y \in \mathrm{int}(S)$. Thus Theorem 4.1.2 yields

$$V(s,x) = \hat{\mathbb{E}}_{(s,x)}(e^{-r(T-s)} X_T \mathbb{1}\{X_T < K\}).$$

So, the boundary is characterized by the following equation: For all $s > 0$

$$b(s) = \hat{\mathbb{E}}_{(s,b(s))}(e^{-r(T-s)} X_T \mathbb{1}\{X_T < K\}). \tag{4.7}$$

A closer look at this equation shows that the right hand side is the price of a European Asset-or-Nothing put option with time to maturity $T - s$, strike price K and underlying stock started in $b(s)$, which price is $b(s)\Phi(-d_1(T, s))$ (see [3, p. 553]). So the characterization reduces to

$$\begin{aligned} b(s) &= b(s)\Phi(-d_1(T,s)) \\ \Leftrightarrow \qquad 1 &= \Phi(-d_1(T,s)) \\ \Leftrightarrow \quad d_1(T,s) &= -\infty \\ \Leftrightarrow \log(b(s)/K) &= -\infty \\ \Leftrightarrow \qquad b(s) &= 0, \end{aligned}$$

which tells as to stop as soon us the underlying hits 0, which it does with probability 0. So this stopping boundary does not make sense as well and we must have an error in the argument.

4.5 Discussion on Sections 4.3 and 4.4

In this section we want to briefly discuss why the examples with the Cash-or-Nothing put option and the Asset-or-Nothing put option might not work. Both of the stopping sets are $\{(t, x) : x < K, 0 \le t \le T\}$ which is not a closed set and furthermore not containing the stopping boundary. So the characterizing equations for the boundary (4.6) and (4.7) might not be correct.

Another thought was to use $\mathbb{1}\{X_t \le K\}$ as reward function for the Cash-or-Nothing put options and respectively $X_t\mathbb{1}\{X_t \le K\}$ for the Asset-or-Nothing put option. This leads in both cases to a closed stopping set which therefore includes the stopping boundary. However, we face the problem that the reward functions are not lower semicontinuous and therefore Shiryaev's theorem for the r-excessive property of the value function might not apply.

Chapter 5

Summary

Based on the research of Christensen and Salminen, we reviewed theoretical results about the Riesz representation theorem. We specified on the multi-dimensional geometric Brownian motion and the one-dimensional Space-Time geometric Brownian motion with limited time. We found a more general formula for the optimal stopping boundary for the optimal investment problem and verified the formula for the special case $\mu = r$. Christensen and Salminen mentioned in their work that they did not find a numerical algorithm for the approximation of the stopping boundary for the two-dimensional optimal investment problem. We suggested an algorithm and found out that an ellipsoid is not a good approximation for the optimal stopping boundary. Although we did not get a convergence for the algorithm, we got a better understanding of the shape of the stopping set. For the Space-Time process we reviewed the theoretical results and furthermore considered three different examples of reward functions with limited payoff. Moreover, we provided an algorithm for the approximation of the stopping boundary of an American put option.

An interesting starting point for further research might be to analyze why the error of the objective function in Algorithm 1 is that big on the left boundary and whether we manged to eliminate that phenomena with Algorithm 2. Furthermore it would be interesting to prove convergence theorems for the algorithms. Considering why the approach did not work in the cases of the American Cash-or-Nothing put option and American Asset-or-Nothing put option might be of interest as well.

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APPENDIX

Matlab code

Listing 1: Test script for the 2-dim optimal investment problem

```
clear all
close all
n = 512
a1 = 0.2; a2 = 0.3;
mu1 = 0.05; mu2 = 0.04;
r = 0.06;
m1 = (mu1-0.5*a1^2)/a1;
m2 = (mu2-0.5*a2^2)/a2;
rho = 0.2;
K = 1;

[weights,nodes] = gauss(n);
nodes = (nodes + 1)/2;

x1_star = star(K,r,mu1,a1);
x2_star = star(K,r,mu2,a2);

numC = double(feature('numCores'));
fprintf('Number of cores available: %d\n',numC)
matlabpool close force local
matlabpool 8

poolSize=matlabpool('size')
if poolSize == 0
  fprintf('hello world1\n');
  error('parallel:demo:poolClosed', ...
        'This demo needs an open MATLAB pool to run.');
end

F_a = @(y) Func_a_par(y,a1,a2,mu1,mu2,rho,m1,m2,r,K,weights,nodes,x1_star);

y_circ = x2_star* sqrt(1-(nodes).^2);
y_circ_a = y_circ;
for i=n:-1:2
```

```
    y_circ_a(i) = y_circ_a(i)/y_circ_a(i-1);
end
tic
  fprintf('Start error is %d\n',norm(F_a(y_circ_a)));
toc

options=optimset('Display','iter','MaxFunEvals', 3700);    % Option to display output
lb = zeros(n,1);
ub = [x2_star;ones(n-1,1)];
y_a = lsqnonlin(F_a,y_circ_a,lb,ub,options);

matlabpool close

y = y_a;
for i=2:n
    y(i) = y(i-1)*y(i);
end

fid = fopen(['y',num2str(n),'_a.txt'],'w');
fprintf(fid,'%.9f \n',y);
fclose(fid);

fig = figure;
plot(nodes*x1_star,y);
hold on;
plot(nodes*x1_star,y_circ,'k')
xlabel('x1'); ylabel('x2');
print(fig,['y',num2str(n),'_a'],'-dpng')
print(fig,['y',num2str(n),'_a'],'-depsc','-tiff')
exit;
```

Listing 2: Objective function for the 2-dim optimal investment problem

```
function F = Func_a_par(y,a1,a2,mu1,mu2,rho,m1,m2,r,K,weights,nodes,x1_star)
%y = [y1,al2,...,aln]
n = length(weights);
for i=2:n
    y(i) = y(i-1)*y(i);
end
% y = [y1,y2,...,yn]
F = zeros(n,1);
```

```
Bp = @(u,v) u.^2 -2*rho*u.*v + v.^2;
Ap = @(u,v,m1_,m2_) 2*rho*(m2_*u+m1_*v)-2*(m1_*u+m2_*v);
r_hat = r+Bp(m1,m2)/2/(1-rho^2);

Gr = @(x1,x2,u,v) 1./(pi*sqrt(1-rho^2)*a1*a2*(u.*v)).*...
      exp(-1/(2*(1-rho^2))*Ap(1/a1*log(abs(u/x1)),1/a2*log(abs(v/x2)),m1,m2)).*...
      besselk(0,sqrt(r_hat)*sqrt(2*Bp(1/a1*log(abs(u/x1)),1/a2*log(abs(v/x2)))/(1-rho^2)));

sigma = @(t1,t2) r*K + (mu1-r)*t1 + (mu2-r)*t2;

i1nodes = (nodes*x1_star)*ones(1,n);
i2nodes = y*nodes';

allin1 = diag(1/4*x1_star*y)*(weights * weights') .* sigma(i1nodes,i2nodes);

parfor j=1:n
    F(j) = -(K - i1nodes(j,1)-y(j)) + sum(sum(allin1 .* Gr(i1nodes(j,1),y(j),i1nodes,i2nodes)));
end
```

Listing 3: Optimal stopping boundary for the 1-dim optimal investment problem

```
function x_star = star(K,r,mu,a)
%Computes the optimal stopping boundary for the optimal investment problem in 1d
if mu==r
  gamma = 2*r/a^2;
  x_star = gamma*K/(1+gamma);
else
  m = (mu-0.5*a^2)/a;
  r_hat = r + m^2/2;
  nom = K - (2*r_hat)^(-0.5)*r*K/(m+sqrt(2*r_hat));
  denom = (2*r_hat)^(-0.5)*(mu-r)/(m+sqrt(2*r_hat)+a)+1;
  x_star = nom/denom;
end
```

Listing 4: Sourcecode for the American put option

```
clear all; close all
n = 1000;
```

```
T = 10;
r = 0.03;
K = 2;
sigma = 0.09;

b = am_opt(n,T,r,K,sigma);
plot(linspace(0,T,n+1),b)
xlabel('Time')
ylabel('Space')
```

Listing 5: Function for the American put options

```
function b = am_opt(n,T,r,K,sigma)

  b = zeros(n+1,1);
  b(n+1) = K;
    x_star = star(K,r,r,sigma);
  for i=n:-1:1
    bi_fun = @(bi) eqn([zeros(i-1,1);bi;b(i+1:end)],i,n,T,r,K,sigma);
    b(i) = bisection(x_star,K,bi_fun);
  end

end

function res = eqn(b,i,n,T,r,K,sigma)
  h = T/n;
  d1i = d1(b,i,n,T,r,K,sigma);
  d2i = d1i - sigma*sqrt(h*(n+1-i));
  d3ijs = d3(b,i,n,T,r,sigma);

  res = r*K*h/2*(0.5 + exp(-r*h*(n+1-i))*normcdf(d3ijs(end))) + K*exp(-r*h*(n+1-i))*normcdf(-d2i)-b(i)*normcdf(-d1i) - (K-b(i));
  res = res + r*K*h*sum(exp(-r*h*(1:n-i))'.*normcdf(d3ijs(2:end-1)));

end

function val = d1(b,i,n,T,r,K,sigma)
  h = T/n;
  val = 1/sigma/sqrt(T-h*(i-1))*(log(b(i)/K)+(r+sigma^2/2)*(T-h*(i-1)));
```

```
end

function val = d3(b,i,n,T,r,sigma)
  val = zeros(n-i+2,1);
  h = T/n;
  val(2:end) = 1/sigma./sqrt(h*(1:n-i+1)').*(log(b(i+1:n+1)/b(i))-(r-sigma^2/2)*(h*(1:n-i+1)'));
end
```

Listing 6: Sourcecode for the Cash-or-Nothing put option

```
clear all;close all
n = 1000;
T = 10;
r = 0.03;
K = 2;
sigma = 0.09;

b = CoN_opt(n,T,r,K,sigma);
plot(linspace(0,T,n+1),b)
xlabel('Time')
ylabel('Space')
```

Listing 7: Function for the Cash-or-Nothing put options

```
function b = CoN_opt(n,T,r,K,sigma)

  b = zeros(n+1,1);
  b(n+1) = K;
  for i=n:-1:1
    bi_fun = @(bi) eqn([zeros(i-1,1);bi;b(i+1:end)],i,n,T,r,K,sigma);
        b(i) = fsolve(bi_fun,K);
  end

end

function res = eqn(b,i,n,T,r,K,sigma)
  h = T/n;
  d1i = d1(b,i,n,T,r,K,sigma);
  d2i = d1i - sigma*sqrt(h*(n+1-i));
  d3ijs = d3(b,i,n,T,r,sigma);

```

```
    res = K*h/2*(0.5 + exp(-r*h*(n+1-i))*normcdf(d3ijs(end))) + exp(-r*h*(n+1-i))*normcdf(-d2i) - 1;
    res = res + r*K*h*sum(exp(-r*h*(1:n-i))'.*normcdf(d3ijs(2:end-1)));

end

function val = d1(b,i,n,T,r,K,sigma)
   h = T/n;
   val = 1/sigma/sqrt(T-h*(i-1))*(log(b(i)/K)+(r+sigma^2/2)*(T-h*(i-1)));
end

function val = d3(b,i,n,T,r,sigma)
   val = zeros(n-i+2,1);
   h = T/n;
   val(2:end) = 1/sigma./sqrt(h*(1:n-i+1)').*(log(b(i+1:n+1)/b(i))-(r-sigma^2/2)*(h*(1:n-i+1)'));
end
```