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Heun Polynomials in the Construction of Vector Valued Slepian Functions on a Spherical Cap

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HEUN POLYNOMIALS IN THE CONSTRUCTION OF VECTOR VALUED SLEPIAN FUNCTIONS ON A SPHERICAL CAP

by

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ABSTRACT
HEUN POLYNOMIALS IN THE CONSTRUCTION OF VECTOR VALUED SLEPIAN FUNCTIONS ON A SPHERICAL CAP

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I summarize the existing work on the problem of finding vector valued Slepian functions on the unit sphere: separable vector fields whose energy is concentrated within a compact region; in this case, a spherical cap. The radial and tangential components are independent for an appropriate choice of basis, and for each component the problem is recast as that of finding real eigenfunctions of an integral operator. There exist Sturm-Liouville differential operators that commute with these integral operators and hence share their eigenfunctions. Therefore, the radial and tangential eigenfunctions are solutions to second order linear ODEs. After introducing the Heun differential equation and some of its basic properties, I show how our equations can be put into Heun form by a change of variables, at which point the Slepian functions can be expressed in terms of Heun polynomials: polynomial solutions to a Heun equation.

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1 Introduction

Our goal is to construct complex valued three dimensional vector fields on the unit sphere whose energy in the L^2 norm is concentrated within a spherical cap, hereafter referred to as **Slepian functions**. More precisely, if $R \subset S^2$ denotes the cap, and $d\sigma$ denotes surface measure on the unit sphere, then we wish to find maps $\mathbf{V} : S^2 \to \mathbb{C}^3$ for which the quantity $\int_R \mathbf{V}^* \cdot \mathbf{V} d\sigma / \int_{S^2} \mathbf{V}^* \cdot \mathbf{V} d\sigma$ is maximized. In considering the expansion of square-integrable functions as linear combinations of spherical harmonics, it is well known that **space-limited** functions, which are identically zero outside of some finite region, can not admit a finite expansion. Conversely, **band-limited** functions, which do admit a finite expansion, can not be space-limited. Space-limited functions maximize this quantity trivially, but do so at the loss of the relative simplicity afforded by a finite expansion. This suggests two complementary optimization problems: to find band-limited Slepian functions optimally concentrated within a spatial region, and to find space-limited Slepian functions whose expansion coefficients are optimally concentrated within a prescribed band. As it happens, these problems have identical solutions: the optimal space-limited function is the optimal band-limited function with all values outside the region of interest suppressed to zero; the optimal band-limited function is the optimal space-limited function with all expansion coefficients outside of the prescribed band suppressed to zero. [1] This paper will focus exclusively on the construction of band-limited Slepian functions where the region of interest is a given spherical cap, R. In particular, we will show how to construct an orthogonal basis for the subspace of band-limited square-integrable vector fields on the unit sphere ranked according to their concentration within R. In part two we will show how these Slepian functions arise from solutions to an ODE, in part three we introduce the Heun differential equation and its properties, and in part four we show that the Slepian functions can be constructed from solutions to a Heun equation.

We can assume that the spherical cap R is centered at the north pole without any loss of generality. Three dimensional vector fields on the unit sphere can be decomposed into one radial and two tangential components. The optimization problem for the radial component is identical to the problem for scalar fields and is independent from the tangential components. [2] We will tackle these problems separately, although the strategy will be the same. We start by expanding the component in some basis: for the radial component these will be the **surface spherical harmonics**, and for the tangential components we will use the so-called **mixed vector spherical harmonics** developed by Jahn and Bokor in [3]. For each component we obtain an integral operator whose eigenfunctions and eigenvalues determine the Slepian functions and the extent of their concentration in R respectively. Remarkably, there exist Sturm Liouville differential operators which commute with these integral operators. Since commuting operators share eigenfunctions, the problems become standard Sturm Liouville problems. The remainder of the paper concerns finding appropriate solutions to the associated ODEs.

The Heun differential equation is a second order linear ODE which generalizes the hyper-geometric equation. It is distinctive in the sense that any second order linear ODE with four regular singularities, or with three finite regular singularities and a regular singularity at infinity, can be made into Heun form by a simple change of variables. Further, under certain conditions, the equation admits polynomial solutions. [4] The equations for the radial and tangential components arrived at in part two satisfy these conditions, hence band-limited Slepian functions can be expressed in terms of polynomial solutions to Heun equations. This constitutes the main result of the paper.

In conclusion we find that the method of Heun polynomials gives a reasonable construction of Slepian functions. Construction of the Heun polynomials themselves only depends on diagonalizing a tridiagonal matrix with analytically prescribed values, so their computation can be implemented swiftly.

2 Slepian Functions on a Spherical Cap

2.1 Special Functions and Notation

Throughout we will be working on the unit 2-sphere S^2. Its points are identified with ordered pairs (θ, ϕ) where θ is its co-latitude measured from the north pole, and ϕ is its longitude.

$$S^2 = \{z \in \mathbb{R}^3 : |z| = 1\} \sim \{(\theta, \phi) : 0 \leq \theta \leq \pi, 0 \leq \phi < 2\pi\}$$

A spherical cap $R \subset S^2$ can be assumed without loss of generality to be centered at the north pole so that it is uniquely identified by the maximal co-latitude Θ of points within it.

$$R \sim \{(\theta, \phi) : 0 \leq \theta \leq \Theta, 0 \leq \phi < 2\pi\}$$

The following constructions depend critically on the expansion of vector fields on S^2 into vector spherical harmonics. One way of doing this is described in [2] but we will follow the example of Jahn and Bokor in [3]. Their expansion has the advantage of allowing us to reduce the optimization problem for tangential vector fields to an equivalent scalar problem. Indeed, in their basis, we can tackle the radial and tangential components with the same strategy. Let $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$ denote unit vectors on the sphere pointing in the outward radial direction, tangentially toward the south, and tangentially toward the east respectively. Introduce a new basis for tangential vector fields:

$$\hat{\boldsymbol{\tau}}_{\pm} := \frac{1}{\sqrt{2}}(\hat{\boldsymbol{\theta}} \pm i\hat{\boldsymbol{\phi}}) \tag{2.1}$$

Here, $i = \sqrt{-1}$ is the imaginary unit. These are orthonormal with respect to the complex dot product:

$$\hat{\boldsymbol{\tau}}_{\pm}^{*} \cdot \hat{\boldsymbol{\tau}}_{\mp} = 0, \quad \text{and} \quad \hat{\boldsymbol{\tau}}_{\pm}^{*} \cdot \hat{\boldsymbol{\tau}}_{\pm} = 1 \tag{2.2}$$

Where * denotes the complex conjugate. The standard tangential components can be written in this new basis as: $\hat{\boldsymbol{\theta}} = 2^{-1/2}(\hat{\boldsymbol{\tau}}_{+} + \hat{\boldsymbol{\tau}}_{-})$ and $\hat{\boldsymbol{\phi}} = -i\, 2^{-1/2}(\hat{\boldsymbol{\tau}}_{+} - \hat{\boldsymbol{\tau}}_{-})$. Define the **surface spherical harmonics**, $Y_{lm}(\theta, \phi)$ as: [5]

$$Y_{lm}(\theta, \phi) := X_{lm}(\cos\theta)e^{im\phi} \tag{2.3}$$

$$X_{lm}(x) := \left(\frac{(2l+1)(l-m)!}{4\pi(l+m)!}\right)^{1/2} P_{lm}(x) \tag{2.4}$$

where $P_{lm}(x)$ is the **associated Legendre function** of integer degree $l \geq 0$ and order $-l \leq m \leq l$.

$$P_{lm}(x) = \frac{(-1)^m}{2^l l!}(1-x^2)^{m/2}\left(\frac{d}{dx}\right)^{l+m}(x^2-1)^l \tag{2.5}$$

The normalized associated Legendre functions $X_{lm}(\cos\theta)$ are eigenfunctions of the fixed order Laplace-Beltrami operator $\nabla_m^2 := \partial_\theta^2 + \cot\theta\,\partial_\theta - m^2\csc^2\theta$. [1]

$$\nabla_m^2 X_{lm}(\cos\theta) = -l(l+1)X_{lm}(\cos\theta) \tag{2.6}$$

The constants have been chosen so that the spherical harmonics are orthonormal on the unit sphere.

$$\int_{S^2} Y_{lm}^* \cdot Y_{l'm'} d\sigma = \int_0^{2\pi} \int_0^{\pi} Y_{lm}^*(\theta, \phi) \cdot Y_{l'm'}(\theta, \phi) \sin\theta \; d\theta d\phi = \delta_{ll'}\delta_{mm'} \tag{2.7}$$

Here, $d\sigma$ is surface measure on S^2, and δ_{xy} is the Kronecker delta.

We wish to define a pair of orthogonal vector spherical harmonics by finding eigenfunctions of the fixed order vector Laplace-Beltrami operator on tangential vector fields. This can be written in the basis (2.1) as: [3]

$$\nabla_m^2 \mathbf{w} = (\Delta_m w_+)\hat{\boldsymbol{\tau}}_+ + (\Delta_{-m} w_-)\hat{\boldsymbol{\tau}}_- \tag{2.8}$$

where $\mathbf{w}(\theta, \phi) = w_+(\theta, \phi)\hat{\boldsymbol{\tau}}_+ + w_-(\theta, \phi)\hat{\boldsymbol{\tau}}_-$ and

$$\Delta_m := \nabla_m^2 - \frac{1 - 2m\cos\theta}{\sin^2\theta}. \tag{2.9}$$

Its eigenfunctions are the **Sheppard-Török functions** $F_{lm}(\cos\theta)$:

$$\Delta_m F_{lm}(\cos\theta) = -l(l+1)F_{lm}(\cos\theta), \quad l \geq 1, \; -l \leq m \leq l \tag{2.10}$$

where

$$F_{lm}(x) := \frac{-1}{\sqrt{l(l+1)}}[a_{lm}^+ X_{l,m+1}(x) + a_{lm}^- X_{l,m-1}(x) + b_{lm}^+ X_{l-1,m+1}(x) + b_{lm}^- X_{l-1,m-1}(x)] \tag{2.11}$$

with constants:

$$a_{lm}^{\pm} := \pm\frac{\sqrt{(l \mp m)(l \pm m + 1)}}{2} \tag{2.12}$$

and

$$b_{lm}^{\pm} := -\sqrt{\frac{2l+1}{2l-1}}\frac{\sqrt{(l \mp m)(l \mp m - 1)}}{2} \tag{2.13}$$

This differs from the definition used in [3] by a factor of $(2\pi)^{-1/2}$. From this it is evident that we can define the **mixed vector spherical harmonics** $Q^{\pm}_{lm}(\theta,\phi)$ as:

$$Q^{\pm}_{lm}(\theta,\phi) := F_{l,\pm m}(\cos\theta)e^{im\phi} \tag{2.14}$$

The Sheppard-Török functions have many identities and recurrence relations that can be derived from analogous properties of the associated Legendre functions. In particular, for fixed m, they are orthogonal on the interval $[-1,1]$: [6]

$$\int_{-1}^{1} F_{lm}(x)\cdot F_{l'm}(x)dx = \frac{1}{2\pi}\delta_{ll'} \tag{2.15}$$

This fact together with (2.2) can be used to check the following crucial relations:

$$\left[Q^{\pm}_{lm}(\theta,\phi)\hat{\boldsymbol{\tau}}_{\pm}\right]^{*}\cdot\left[Q^{\mp}_{l'm'}(\theta,\phi)\hat{\boldsymbol{\tau}}_{\mp}\right] = 0 \tag{2.16}$$

and

$$\int_{S^2} Q^{\pm *}_{lm}(\theta,\phi)\cdot Q^{\pm}_{l'm'}(\theta,\phi)d\sigma = \delta_{ll'}\delta_{mm'} \tag{2.17}$$

The surface spherical harmonics together with the mixed vector spherical harmonics form a basis for the Hilbert space of square-integrable vector fields on S^2. Hence, for any arbitrary vector field $\mathbf{V}(\theta,\phi)$ we have the expansion:

$$\mathbf{V}(\theta,\phi) = \sum_{lm}^{\infty} v^{r}_{lm}Y_{lm}(\theta,\phi)\hat{\mathbf{r}} + v^{+}_{lm}Q^{+}_{lm}(\theta,\phi)\hat{\boldsymbol{\tau}}_{+} + v^{-}_{lm}Q^{-}_{lm}(\theta,\phi)\hat{\boldsymbol{\tau}}_{-} \tag{2.18}$$

where, in the notation of [2], the double sum $\sum_{lm}^{L} := \sum_{l=0}^{L}\sum_{m=-l}^{l}$ wherever the $Y_{lm}(\theta,\phi)$ terms are involved, and $\sum_{lm}^{L} := \sum_{l=1}^{L}\sum_{m=-l}^{l}$ wherever the $Q^{\pm}_{lm}(\theta,\phi)$ terms are involved. In this context, a vector field is **band-limited** if the coefficients are zero for all $l > L$, for **maximal degree** L. Denote the subspace of band-limited vector fields with maximal degree L by S_L. Each member of this subspace is uniquely determined by the coefficients in (2.18), so its dimension is the total number of coefficients: $\dim S_L = (L+1)^2 + 2L(L+2) = 3L^2 + 6L + 1$.

2.2 Statement of the Problem

We seek to maximize the quantity:

$$\eta := \frac{\int_R \mathbf{V}^* \cdot \mathbf{V} d\sigma}{\int_{S^2} \mathbf{V}^* \cdot \mathbf{V} d\sigma} \tag{2.19}$$

when $\mathbf{V}(\theta, \phi)$ is band-limited. Substituting the expansion (2.18), the numerator becomes:

$$\begin{aligned}\sum_{lm}^{L} v_{lm}^{r\,*} \sum_{l'm'}^{L} \left[\int_R Y_{lm}^*(\theta,\phi) \cdot Y_{l'm'}(\theta,\phi) d\sigma \right] v_{l'm'}^{r} &+ \\ \sum_{lm}^{L} v_{lm}^{+\,*} \sum_{l'm'}^{L} \left[\int_R Q_{lm}^{+\,*}(\theta,\phi) \cdot Q_{l'm'}^{+}(\theta,\phi) d\sigma \right] v_{l'm'}^{+} &+ \\ \sum_{lm}^{L} v_{lm}^{-\,*} \sum_{l'm'}^{L} \left[\int_R Q_{lm}^{-\,*}(\theta,\phi) \cdot Q_{l'm'}^{-}(\theta,\phi) d\sigma \right] v_{l'm'}^{-}&\end{aligned} \tag{2.20}$$

while the orthonormality relations given in section 2.1 make the denominator into:

$$\sum_{lm}^{L} \left(v_{lm}^{r\,*} v_{lm}^{r} + v_{lm}^{+\,*} v_{lm}^{+} + v_{lm}^{-\,*} v_{lm}^{-} \right) \tag{2.21}$$

Since R is a spherical cap, the bracketed integrals in (2.20) can be simplified somewhat:

$$\begin{aligned}&\int_R Y_{lm}^*(\theta,\phi) \cdot Y_{l'm'}(\theta,\phi) d\sigma = \\ &\qquad \int_0^{2\pi} e^{i\phi(m'-m)} d\phi \cdot \int_0^{\Theta} X_{lm}(\cos\theta) \cdot X_{l'm'}(\cos\theta) \sin\theta \, d\theta = \delta_{mm'} D_{ll',m}\end{aligned} \tag{2.22}$$

$$\int_R Q^{\pm *}_{lm}(\theta,\phi)\cdot Q^{\pm}_{l'm'}(\theta,\phi)d\sigma =$$
$$\int_0^{2\pi} e^{i\phi(m'-m)}d\phi \cdot \int_0^{\Theta} F_{l,\pm m}(\cos\theta)\cdot F_{l',\pm m'}(\cos\theta)\sin\theta\, d\theta = \delta_{mm'}K_{ll',\pm m} \quad (2.23)$$

where

$$D_{ll',m} := 2\pi \int_{\cos\Theta}^{1} X_{lm}(x)\cdot X_{l'm}(x)dx \quad \text{and} \quad K_{ll',m} := 2\pi\int_{\cos\Theta}^{1} F_{lm}(x)\cdot F_{l'm}(x)dx \quad (2.24)$$

So, the terms in the sum (2.20) with $m \neq m'$ are zero.

These conditions can be expressed much more concisely in the language of matrices. Consolidate the coefficients v^r_{lm} into an $(L+1)^2$ element row vector $\mathbf{g}^r$. We enumerate its entries by fixing m, stepping through all l from $|m|$ to L, then moving on to the next m and repeating the process until all m between $-L$ and L have been exhausted.

$$\mathbf{g}^r := \left[v^r_{L,-L}; v^r_{L-1,-L+1}, v^r_{L,-L+1}; \ldots; v^r_{L,L}\right] \quad (2.25)$$

We can do this with the coefficients v^+_{lm} and v^-_{lm} as well, collecting them into two $L(L+2)$ element row vectors $\mathbf{g}^+$ and $\mathbf{g}^-$ enumerated with the same scheme. Since l cannot be zero, for fixed m we step through all l from $l_m := \max(|m|,1)$ to L instead. Finally we concatenate these to make a single column vector with all $3L^2+6L+1$ coefficients: $\mathbf{g} := [\mathbf{g}^r, \mathbf{g}^+, \mathbf{g}^-]^{\mathrm{T}}$. Construct a $(3L^2+6L+1)\times(3L^2+6L+1)$ block diagonal matrix $\mathbf{M}$:

$$\mathbf{M} := \begin{bmatrix} \mathbf{D} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}^+ & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{K}^- \end{bmatrix} \quad (2.26)$$

Here, $\mathbf{D}$ is a $(L+1)^2 \times (L+1)^2$ block diagonal matrix, and $\mathbf{K}^{\pm}$ are $L(L+2)\times L(L+2)$

block diagonal matrices with one block for each order m:

$$\mathbf{D} := \text{diag}\left[\mathbf{D}_{-L}; \mathbf{D}_{-L+1}; \ldots; \mathbf{D}_{L}\right];$$
$$\mathbf{K}^{+} := \text{diag}\left[\mathbf{K}_{-L}; \mathbf{K}_{-L+1}; \ldots; \mathbf{K}_{L}\right]; \quad \mathbf{K}^{-} := \text{diag}\left[\mathbf{K}_{L}; \mathbf{K}_{L-1}; \ldots; \mathbf{K}_{-L}\right] \tag{2.27}$$

Notice that $\mathbf{K}^{+}$ and $\mathbf{K}^{-}$ have the same blocks but in opposite order. These blocks carry the fixed order quantities (2.24). As such, $\mathbf{D}_m$ and $\mathbf{K}_m$ are real symmetric matrices of dimensions $(L-|m|+1)\times(L-|m|+1)$ and $(L-l_m+1)\times(L-l_m+1)$ respectively:

$$\mathbf{D}_m := \begin{bmatrix} D_{|m||m|,m} & \ldots & D_{|m|L,m} \\ \vdots & \ddots & \vdots \\ D_{L|m|,m} & \ldots & D_{LL,m} \end{bmatrix}; \quad \mathbf{K}_m := \begin{bmatrix} K_{l_m l_m,m} & \ldots & K_{l_m L,m} \\ \vdots & \ddots & \vdots \\ K_{L l_m,m} & \ldots & K_{LL,m} \end{bmatrix} \tag{2.28}$$

Finally, we change the order of summation in (2.20) and (2.21) from $\sum_{l=0}^{L}\sum_{m=-l}^{l}$ to $\sum_{m=-L}^{L}\sum_{l=|m|}^{L}$ for the radial component and to $\sum_{m=-L}^{L}\sum_{l=l_m}^{L}$ for the tangential components to agree with the scheme (2.25). Then, the quantity (2.19) is:

$$\eta = \frac{\mathbf{g}^{*}\mathbf{M}\mathbf{g}}{\mathbf{g}^{*}\mathbf{g}} \tag{2.29}$$

Since $\mathbf{M}$ is real, symmetric, and positive definite, its eigenvalues must be real and strictly positive; it follows that if $\mathbf{g}$ is an eigenvector of $\mathbf{M}$, then η is its associated eigenvalue. Since no band-limited function can be completely contained within a finite region, we have $0<\eta<1$. Hence, we are searching for the maximal eigenvalues of the matrix $\mathbf{M}$, and the optimally concentrated Slepian functions can be recovered from the elements of their associated eigenvectors by plugging them into the expansion (2.18).

Since $\mathbf{M}$ is block diagonal, its eigenvalues are simply those of the individual fixed order blocks, while its eigenvectors are those of the blocks with zeros filling in the extra space to make it the requisite dimension. This implies that eigenvectors of separate blocks are orthogonal, even if they have the same eigenvalue. Since $X_{l,-m}(x) = (-1)^m X_{lm}(x)$, we have $\mathbf{D}_m = \mathbf{D}_{-m}$. Because of this and the fact that $\mathbf{K}^+$ and $\mathbf{K}^-$ contain all of the same blocks, it suffices to find the eigenvalues and associated eigenvectors of the matrices $\mathbf{D}_m$ for $0 \leq m \leq L$, and $\mathbf{K}_m$ for $-L \leq m \leq L$. Since $\mathbf{M}$ is symmetric its eigenvectors are orthogonal; it follows that the Slepian functions are orthogonal with respect to the L^2 inner product, and since $\mathbf{M}$ has the same dimension as S_L, they can be normalized to form an orthonormal basis of S_L. That is,

$$\mathbf{g}_\alpha^* \mathbf{g}_\beta = \int_{S^2} \mathbf{V}_\alpha^* \cdot \mathbf{V}_\beta d\sigma = \delta_{\alpha\beta}, \quad \text{and} \quad \mathbf{g}_\alpha^* \mathbf{M}\mathbf{g}_\beta = \int_R \mathbf{V}_\alpha^* \cdot \mathbf{V}_\beta d\sigma = \eta_\alpha \delta_{\alpha\beta} \tag{2.30}$$

In the next section we will show how these problems are each equivalent to those of finding the eigenvalues and associated eigenfunctions of a Sturm-Liouville differential operator.

2.3 Commuting Differential Operators

2.3.1 The Radial Component

In this section we will examine the eigenvalue problem:

$$\mathbf{D}_m \mathbf{g}_m = \eta \mathbf{g}_m, \quad 0 \leq m \leq L \tag{2.31}$$

If $\mathbf{g}_m = [g_m, \ldots, g_L]^{\mathrm{T}}$ is an $(L-m+1)$ element eigenvector, then the associated Slepian function is:

$$\mathbf{G}^r_m(\theta,\phi) := \sum_{l=m}^{L} g_l Y_{lm}(\theta,\phi)\hat{\mathbf{r}} = \sum_{l=m}^{L} g_l X_{lm}(\cos\theta)e^{im\phi}\hat{\mathbf{r}} = G^r_m(\cos\theta)e^{im\phi}\hat{\mathbf{r}} \tag{2.32}$$

where

$$G^r_m(x) := \sum_{l=m}^{L} g_l X_{lm}(x). \tag{2.33}$$

Since the order is fixed, we can suppress the second subscript m in each of the entries of $\mathbf{D}_m$ for brevity. Written in full, (2.31) is the system of equations:

$$\sum_{l=m}^{L} D_{l'l}g_l = \eta g_{l'}, \quad m \le l' \le L \tag{2.34}$$

Multiplying both sides of each equation by $X_{l'm}(\cos\theta)$ and summing over all l' we get:

$$\int_0^{\Theta} \left[2\pi \sum_{l'=m}^{L} X_{l'm}(\cos\theta)X_{l'm}(\cos\theta')\right] G^r_m(\cos\theta')\sin\theta' d\theta' = \eta G^r_m(\cos\theta) \tag{2.35}$$

Upon letting $x = \cos\theta$, $x' = \cos\theta'$, and $\mathscr{D}(x,x') := 2\pi\sum_{l=m}^{L} X_{lm}(x)X_{lm}(x')$, this is:

$$\int_{\cos\Theta}^{1} \mathscr{D}(x,x')G^r_m(x')dx' = \eta G^r_m(x), \quad -1 \le x \le 1 \tag{2.36}$$

So, G^r_m are the eigenfunctions of the integral operator: $f(x) \mapsto \int_{\cos\Theta}^{1} \mathscr{D}(x,x')f(x')dx'$.

Define the **Grünbaum operator** on S_L as:

$$\mathscr{G} := (\cos\Theta - \cos\theta)\nabla^2_m + \sin\theta\frac{d}{d\theta} - L(L+2)\cos\theta \tag{2.37}$$

Rewritten with $x = \cos\theta$ this is:

$$\mathscr{G}_x := \frac{d}{dx}\left[(\cos\Theta - x)(1-x^2)\frac{d}{dx}\right] - L(L+2)x - \frac{m^2(\cos\Theta - x)}{1-x^2}. \tag{2.38}$$

This operator commutes with the integral operator above. That is,

$$\int_{\cos\Theta}^{1} \mathscr{D}(x,x')\mathscr{G}_{x'} f(x')dx' = \int_{\cos\Theta}^{1} \mathscr{G}_x \mathscr{D}(x,x') f(x')dx'. \tag{2.39}$$

The proof is given in [1]. It is a straight-forward but involved exercise in integration by parts and in the various relations involving Legendre functions. Eigenfunctions of $\mathscr{G}_x$ solve the Sturm-Liouville equation:

$$\frac{d}{dx}\left(p(x)\frac{dy(x)}{dx}\right) - q(x)y(x) + \chi y(x) = 0, \quad -1 \le x \le 1 \tag{2.40}$$

where $p(x) = (x - \cos\Theta)(1-x^2)$, $q(x) = m^2(1-x^2)^{-1}(x-\cos\Theta) - L(L+2)x$, and χ is an eigenvalue. A familiar property of Sturm-Liouville problems is that their eigenvalues are always real and distinct with an accumulation point at infinity. Another well-known fact about commuting operators is that if one of them has a simple spectrum, as the Grünbaum operator does, its eigenfunctions are shared with those of the other operator. In particular, solutions to (2.40) are eigenfunctions of the integral operator in (2.36). Hence, solutions to the ODE (2.40) are the co-latitudinal part of the radial Slepian functions $G^r_m(x)$. With the eigenvalues χ ordered according to size, only the first $L-m+1$ correspond to non-trivial Slepian functions; the rest lie in the null-space of the integral operator (2.36). The optimally concentrated Slepian function corresponds to the minimal eigenvalue χ. [1]

Remark: [1] states that $\mathscr{G}_x$ has a simple spectrum, i.e. its eigenvalues are distinct, so that if two eigenfunctions have the same eigenvalue then they must be constant multiples of each other. This fact is crucial to the construction of Slepian functions on the sphere since, together with the commutation relation (2.39), it ensures that eigenfunctions of $\mathscr{G}_x$ are also eigenfunctions of the integral operator in (2.36) albeit with different eigenvalues. This is a well known property of equations like (2.40) provided that a number of conditions hold: namely, $p(x)$ and $q(x)$ are continuous on a compact interval $[a, b]$, $p(x) > 0$ for all $x \in [a, b]$, and the solution $y(x)$ and its derivative $y'(x)$ are finite at the endpoints a and b. The boundary condition is easily satisfied by band-limited functions, but $q(x)$ is not continuous at 1 or -1, and $p(x)$ is only positive for $x \in (\cos\Theta, 1)$. If we restrict our attention to the sub-interval $[\cos\Theta + \epsilon, 1 - \epsilon]$ for ϵ very small, then the conditions are met and $\mathscr{G}_x$ has a simple spectrum of eigenvalues: that is, if $y(x)$ and $z(x)$ are eigenfunctions of $\mathscr{G}_x$ of the form (2.33) on $[-1, 1]$ with the same eigenvalue χ, then $y(x) = \xi z(x)$ for some constant ξ for all $x \in [\cos\Theta + \epsilon, 1 - \epsilon]$. Since ϵ is arbitrary, and y and z are continuous, it follows that $y(x) = \xi z(x)$ for all $x \in [\cos\Theta, 1]$

So, we can be confident that $\mathscr{G}_x$ has a simple spectrum when $x \in [\cos\Theta, 1]$. That is, for space-limited functions on R. The same argument holds for space-limited functions on the antipodal cap $S^2 - R$ with eigenvalues $-\chi$, simply multiply the equation by -1. As mentioned before, the band-limited Slepian functions are obtained from the space-limited functions by subtracting the residual terms outside of the band-limit. It follows that the band-limited function is the sum of the space-limited function on R with eigenvalue χ and the space-limited function on $S^2 - R$ with eigenvalue $-\chi$. Since this is continuous and the two space-limited functions agree at $x = \cos\Theta$, it follows that $y(x) = \xi z(x)$ on the whole interval $[-1, 1]$.

While the minimal eigenvalues χ optimize concentration in R, the maximal eigenvalues optimize concentration in $S^2 - R$. So, traversing the spectrum of eigenvalues from least to greatest we see eigenfunctions with energy moving from R into the antipodal cap $S^2 - R$.

2.3.2 The Tangential Components

For the eigenvalue problem:

$$\mathbf{K}_m^{\pm}\mathbf{g}_m = \eta\mathbf{g}_m, \quad -L \leq m \leq L \tag{2.41}$$

our analysis will parallel the previous section. If $\mathbf{g}_m = [g_{l_m}, \ldots, g_L]^{\mathrm{T}}$ is an $(L - l_m + 1)$ element eigenvector, then the associated Slepian function is:

$$\mathbf{G}_m^{\pm}(\theta, \phi) := \sum_{l=l_m}^{L} g_l Q_{lm}^{\pm}(\theta, \phi)\hat{\boldsymbol{\tau}}_{\pm} = \sum_{l=l_m}^{L} g_l F_{l,\pm m}(\cos\theta)e^{im\phi}\hat{\boldsymbol{\tau}}_{\pm} = G_{\pm m}^{t}(\cos\theta)e^{im\phi}\hat{\boldsymbol{\tau}}_{\pm} \tag{2.42}$$

where

$$G_m^t(x) := \sum_{l=l_m}^{L} g_l F_{lm}(x). \tag{2.43}$$

Since the order is fixed, we can suppress the second subscript m in each of the entries of $\mathbf{K}_m$ for brevity. Written in full, (2.41) is the system of equations:

$$\sum_{l=l_m}^{L} K_{l'l}g_l = \eta g_{l'}, \quad l_m \leq l' \leq L \tag{2.44}$$

Multiplying both sides of each equation by $F_{l'm}(\cos\theta)$ and summing over all l' we get:

$$\int_0^{\Theta} \left[2\pi \sum_{l'=l_m}^{L} F_{l'm}(\cos\theta)F_{l'm}(\cos\theta')\right] G_m^t(\cos\theta')\sin\theta' d\theta' = \eta G_m^t(\cos\theta) \tag{2.45}$$

Upon letting $x = \cos\theta$, $x' = \cos\theta'$, and $\mathscr{K}(x, x') := 2\pi\sum_{l=l_m}^{L} F_{lm}(x)F_{lm}(x')$, this is:

$$\int_{\cos\Theta}^{1} \mathscr{K}(x, x')G_m^t(x')dx' = \eta G_m^t(x), \quad -1 \leq x \leq 1 \tag{2.46}$$

So, G_m^t are the eigenfunctions of the integral operator: $f(x) \mapsto \int_{\cos\Theta}^{1} \mathscr{K}(x, x')f(x')dx'$.

Given the form of the Grünbaum operator, it seems reasonable to presume that a commuting differential operator will have the following form:

$$\mathscr{J} := (\cos\Theta - \cos\theta)\Delta_m + \sin\theta\frac{d}{d\theta} - L(L+2)\cos\theta \tag{2.47}$$

where Δ_m is defined as in (2.9). Rewritten with $x = \cos\theta$, this is:

$$\mathscr{J}_x := \frac{d}{dx}\left[(\cos\Theta - x)(1-x^2)\frac{d}{dx}\right] - L(L+2)x - \frac{(m^2 - 2mx + 1)(\cos\Theta - x)}{1-x^2} \tag{2.48}$$

Indeed, this operator commutes with the integral operator above. That is,

$$\int_{\cos\Theta}^{1} \mathscr{K}(x,x')\mathscr{J}_{x'}f(x')dx' = \int_{\cos\Theta}^{1} \mathscr{J}_x\mathscr{K}(x,x')f(x')dx' \tag{2.49}$$

The proof is given in [3]. Just as commutativity of the Grünbaum operator relied on relations among the associated Legendre functions, this proof is reliant on analogous relations among the Sheppard-Török functions. These relations are detailed in [3] and [6]. Eigenfunctions of $\mathscr{J}_x$ solve the Sturm-Liouville equation:

$$\frac{d}{dx}\left(p(x)\frac{dy(x)}{dx}\right) - q(x)y(x) + \chi y(x) = 0, \quad -1 \leq x \leq 1 \tag{2.50}$$

where $p(x) = (x - \cos\Theta)(1-x^2)$ and $q(x) = (m^2 - 2mx + 1)(1-x^2)^{-1}(x - \cos\Theta) - L(L+2)x$, and χ is an eigenvalue. It follows that co-latitudinal part of the tangential Slepian functions $G_m^t(x)$ are solutions to this ODE. The maximally concentrated functions correspond to minimal eigenvalues as before.

In the next chapter we will introduce the Heun differential equation with the intent of converting equations (2.40) and (2.50) into Heun form so that the Slepian functions can be expressed in terms of solutions to the Heun equation.

3 The Heun Equation

3.1 General Features of the Heun Equation

The citation for everything in this section is [4] unless otherwise stated. The **Heun Equation** is:

$$\frac{d^2y}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-a}\right)\frac{dy}{dz} + \frac{\alpha\beta z - q}{z(z-1)(z-a)}y = 0 \tag{3.1}$$

In general, the variables y and z and all the parameters are complex and arbitrary except that $a \neq 0, 1$. This equation has regular singularities at $z = 0, 1, a$, and ∞. The exponents at these singularities are respectively $\{0, 1-\gamma\}; \{0, 1-\delta\}; \{0, 1-\epsilon\}$; and $\{\alpha, \beta\}$. As such, $\gamma, \delta, \epsilon, \alpha$, and β are called the **exponent parameters** and are related via the equation:

$$\gamma + \delta + \epsilon = \alpha + \beta + 1 \tag{3.2}$$

a is called the **singularity parameter**. The remaining parameter q is called the **accessory parameter** and is vital in determining the existence of polynomial solutions, as we will see.

Since the singularities are regular, we can find two linearly independent solutions valid in a neighborhood about each singularity by the Frobenius method. These are called the **local solutions**. The term **Heun function** is reserved for solutions that are analytic at more than one singularity. In certain special cases a local (series) solution may have zeros for all coefficients past a certain number. These are called

Heun polynomials.

Here we find the fundamental solution of the Heun equation: the local solution about the singularity $z = 0$ with exponent zero. Substitute the Taylor series:

$$y = \sum_{r=0}^{\infty} c_r z^r \quad (c_0 \neq 0) \tag{3.3}$$

and equate coefficients of powers of z to obtain the following recurrence relation for c_r:

$$\begin{gathered} -qc_0 + a\gamma c_1 = 0 \\ P_r c_{r-1} - (Q_r + q)c_r + R_r c_{r+1} = 0 \quad (r \geq 1) \end{gathered} \tag{3.4}$$

where

$$\begin{aligned} P_r &:= (r - 1 + \alpha)(r - 1 + \beta) \\ Q_r &:= r\left[(r - 1 + \gamma)(1 + a) + a\delta + \epsilon\right] \\ R_r &:= (r + 1)(r + \gamma)a \end{aligned} \tag{3.5}$$

We adopt the convention $c_0 = 1$ then denote the series (3.3) with coefficients (3.4) as $Hl(a, q; \alpha, \beta, \gamma, \delta; z)$. Notice that ϵ is not included in this notation; that would be redundant since in light of the relation (3.2), we have $\epsilon = \alpha + \beta - \gamma - \delta + 1$. In general this function is only defined in the disc $|z| < \min(1, |a|)$ and when $\gamma \neq 0, -1, -2, \ldots$. [5]

3.2 Conversion to Heun Form

Any second order linear ODE with three finite regular singularities and a regular singularity at infinity can be converted into the form (3.1) by making a change of variables. The independent variable is changed so as to place the finite singularities at

$\{0, 1, a\}$, while the dependent variable is changed so as to reduce one of the exponents at each of these singularities to zero.

Say that the equation $y''(z) + A(z)y'(z) + B(z)y(z) = 0$ has regular singularities at $z = a_r$, $r = 1, 2, 3$ and at $z = \infty$ with exponents k_r, k'_r at a_r and k, k' at ∞. We map a_1, a_2 to $0, 1$ respectively by changing the independent variable to:

$$z \mapsto \zeta := \frac{z - a_1}{a_2 - a_1} \tag{3.6}$$

The remaining singularity a_3 has been mapped to $a := \frac{a_3 - a_1}{a_2 - a_1}$. Note that for equations of this type, (Fuchsian equations) the power transformation of the dependent variable:

$$y(z) = z^{\rho}(z-1)^{\sigma}(z-a)^{\tau}\tilde{y}(z) \tag{3.7}$$

converting an equation in $y(z)$ to an equation in $\tilde{y}(z)$ has the effect of reducing each of the exponents at the singularity $z = 0$ by ρ, those at $z = 1$ by σ, and those at $z = a$ by τ, while the exponents at ∞ are each increased by $\rho + \sigma + \tau$. With this in mind, we can reduce the exponents k_r to zero by changing the dependent variable to:

$$w(\zeta) = \zeta^{-k_1}(\zeta - 1)^{-k_2}(\zeta - a)^{-k_3}y(\zeta) \tag{3.8}$$

The resulting equation in $w(\zeta)$ will be in the form (3.1) with $\gamma = 1 - k'_1 + k_1$, $\delta = 1 - k'_2 + k_2$, $\epsilon = 1 - k'_3 + k_3$, $\alpha = k + k_1 + k_2 + k_3$, and $\beta = k' + k_1 + k_2 + k_3$. The value of q is dependent on the functions A and B so it must be computed by explicitly plugging in the change of variables then rearranging the resultant equation into the form (3.1) manually.

3.3 Heun Polynomials

We seek polynomial solutions to (3.1). Our strategy is to find a sequence $\{c_r\}$ with $r = 0, 1, \ldots$ satisfying the recurrence relation (3.4), (3.5) for which $c_r = 0$ for all $r > N$. If both c_{N+1} and c_{N+2} are zero, the recurrence relation implies all subsequent terms must be zero. In this case the local solution Hl is valid everywhere on $[0, 1]$. First:

$$c_{N+1} = \frac{(Q_N + q)c_N - P_N c_{N-1}}{R_N} = 0 \tag{3.9}$$

The condition that $\gamma \neq 0, -1, \ldots$ implies $R_r \neq 0$, so,

$$c_N = \frac{P_N}{Q_N + q} c_{N-1} \tag{3.10}$$

Next:

$$c_{N+2} = -\frac{P_{N+1}}{R_{N+1}} c_N = 0 \tag{3.11}$$

which implies

$$P_{N+1} = (N + \alpha)(N + \beta) = 0 \tag{3.12}$$

This will only ever happen if α or β are negative integers: this is a necessary condition for polynomial solutions to exist.

Say $\alpha = -N$ so that $P_{N+1} = 0$. If we wish for the local solution $Hl(a, q; -N, \beta, \gamma, \delta; z)$ to be a polynomial of degree N, then in addition to the recurrence relations (3.4) and (3.5) we also need (3.10) to hold. After some rearrangement we have the system of equations:

$$
\begin{aligned}
a\gamma c_1 &= qc_0 \\
P_1 c_0 - Q_1 c_1 + R_1 c_2 &= qc_1 \\
&\dots \\
P_{N-1} c_{N-2} - Q_{N-1} c_{N-1} + R_{N-1} c_N &= qc_{N-1} \\
P_N c_{N-1} - Q_N c_N &= qc_N
\end{aligned}
\tag{3.13}
$$

Upon introducing the $(N+1) \times (N+1)$ tridiagonal matrix:

$$
\mathbf{H} := \begin{bmatrix}
0 & a\gamma & 0 & \dots & 0 \\
P_1 & -Q_1 & R_1 & \dots & 0 \\
0 & P_2 & -Q_2 & \dots & \vdots \\
\vdots & \vdots & \vdots & \ddots & R_{N-1} \\
0 & 0 & \dots & P_N & -Q_N
\end{bmatrix}
\tag{3.14}
$$

and the $(N+1)$ element column vector $\mathbf{c} = [c_0, c_1, \dots, c_N]^{\mathrm{T}}$, we see that q must be an eigenvalue of $\mathbf{H}$ with eigenvector $\mathbf{c}$. We can then normalize the eigenvector so that $c_0 = 1$. Hence, there are at most $N+1$ values of q for which $Hl(a, q; -N, \beta, \gamma, \delta; z)$ is a polynomial of degree N whose coefficients are elements of the associated normalized eigenvectors of $\mathbf{H}$.

4 Solution by Heun Polynomials

4.1 The Radial Component

Here we seek solutions to (2.40) in the form (2.33). In light of definitions (2.4) and (2.5), our solutions should have the form:

$$G_m^r(x) = (1-x^2)^{m/2} P(x) \quad (0 \le m \le L, \quad -1 \le x \le 1) \tag{4.1}$$

where $P(x)$ is a polynomial of degree $L-m$.

We follow the procedure outlined in section 3.2 to convert (2.40) to Heun form. First, notice that (2.40) has four regular singularities: $x = \pm 1$, $\cos\Theta$, and ∞. The exponents are $\{\frac{m}{2}, -\frac{m}{2}\}$ for both $x = 1$ and $x = -1$, $\{0, 0\}$ for $x = \cos\Theta$, and $\{-L, 2+L\}$ for $x = \infty$. We map the singularity at $x = -1$ to $z = 0$ and $x = \cos\Theta$ to $z = \frac{\cos\Theta + 1}{2}$ by employing the transformation:

$$z := \frac{x+1}{2}, \quad \text{or,} \quad x = 2z - 1 \tag{4.2}$$

The singularity at $x = 1$ remains at $z = 1$. Next we reduce the exponents $\frac{m}{2}$ to zero for the singularities $z = 0, 1$ by changing the dependent variable as:

$$G_m^r(z) = z^{m/2}(1-z)^{m/2} y(z) \quad (0 \le z \le 1) \tag{4.3}$$

Notice that this differs from (3.7) by a factor of $(-1)^{m/2} = i^m$ though naturally that won't affect the properties we are interested in. The difference is to ensure that $G_m^r(z)$ is always real since $z \in [0,1]$. It follows that (2.40) is now in the form (3.1) with $\gamma = \delta = m+1$, $\epsilon = 1$, $a = \frac{\cos\Theta+1}{2}$, $\alpha = m-L$, and $\beta = 2+L+m$. Performing the change of variables manually reveals that $q = \frac{1}{2}[\chi - L(2+L)+m+m(1+\cos\Theta)(1+m)]$.

Since $\gamma \geq 1$ this equation has local solutions, and α is a negative integer provided $L \neq 0$ and $L \neq m$; so there exist $-\alpha+1 = L-m+1$ values of q, hence values of χ, for which $y(z)$ is a Heun polynomial of degree $L-m$. If $L=0$ or $L=m$ then $\alpha=0$. In this case the matrix $\mathbf{H} = [0]$, so the only eigenvalue is $q=0$, and the corresponding Heun polynomial is constant.

So, if $\chi = 2q + L(L+2) - m - m(1+\cos\Theta)(1+m)$ where q is an eigenvalue of the matrix $\mathbf{H}$ in (3.14) with $(r \geq 1)$:

$$\begin{aligned} P_r &= (r-1+m-L)(r+1+m+L) \\ Q_r &= \frac{r}{2}[(r+m)(3+\cos\Theta)+(1+\cos\Theta)(1+m)+2] \\ R_r &= \frac{1}{2}(1+\cos\Theta)(r+1)(r+1+m) \end{aligned} \tag{4.4}$$

then the equation (2.40) in z has the solution:

$$G_m^r(z) = z^{m/2}(1-z)^{m/2} Hl((1+\cos\Theta)/2, q; m-L, 2+L+m, m+1, m+1; z) \tag{4.5}$$

This agrees with our expectation (4.1) since given the definition (4.2), we have

$$G_m^r(x) = 2^{-m}(1-x^2)^{m/2} Hl((1+\cos\Theta)/2, q; m-L, 2+L+m, m+1, m+1; (x+1)/2) \tag{4.6}$$

and Hl with these parameters is a polynomial of degree $L-m$. The figures show the spectrum of eigenvalues χ and selected functions $G_m^r(\cos\theta)$ normalized so that their energy on the sphere is unity.

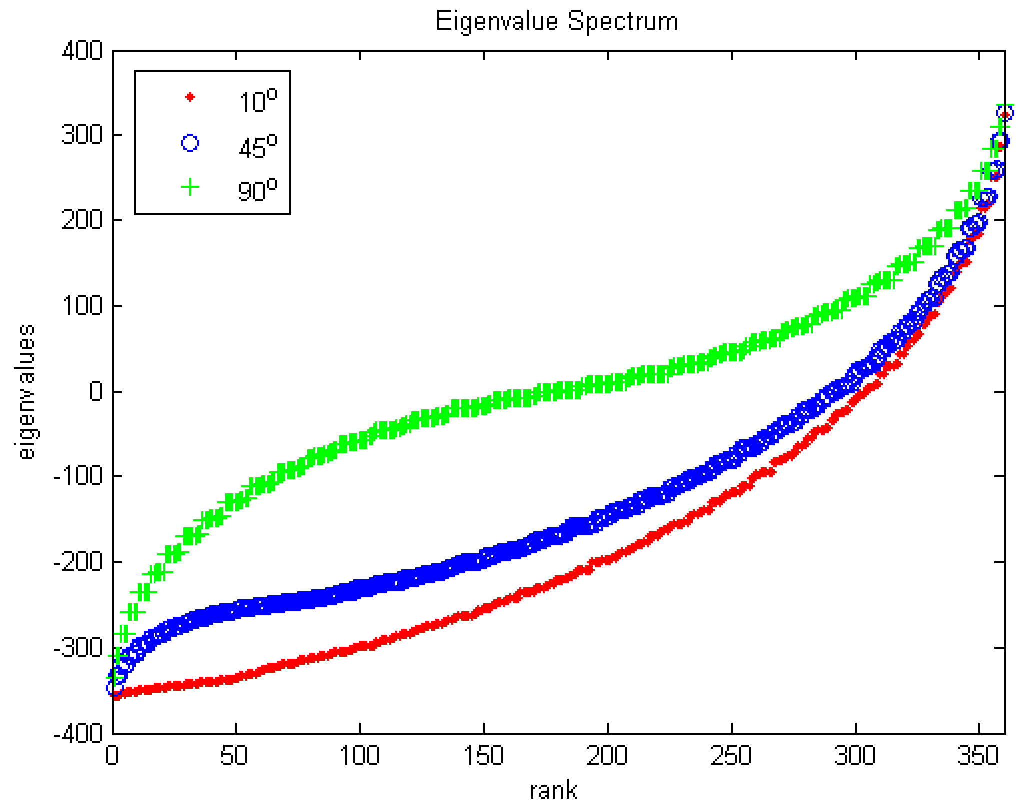

Figure 4.1: Eigenvalues χ for all $m = 0, \ldots, L$ for $L = 18$ and $\Theta = 10°, 45°$, and $90°$. Eigenvalues on the lower end of the spectrum correspond to Slepian functions with better concentration.

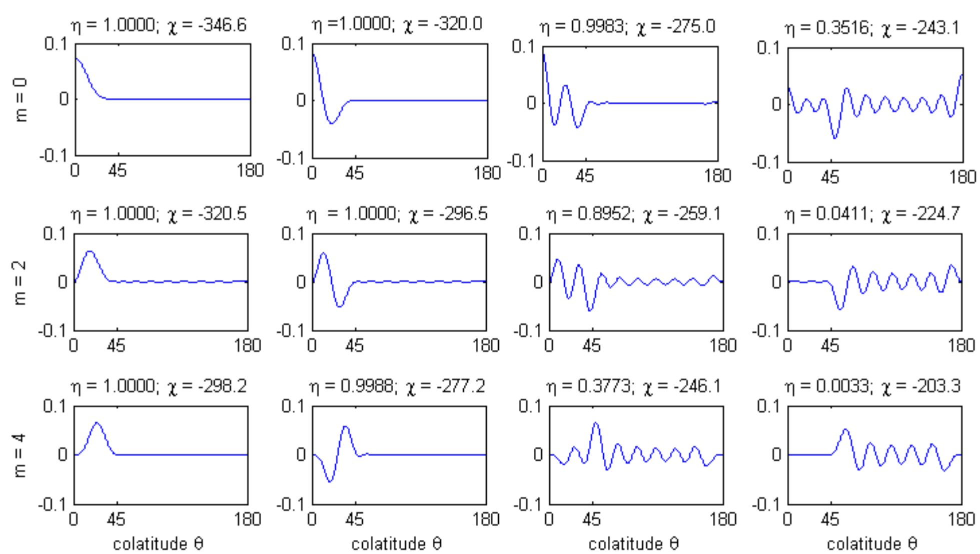

Figure 4.2: Selected normalized eigenfunctions $G_m^r(\cos\theta)$ for $L = 18$ and $\Theta = 45°$. χ is the eigenvalue and $\eta = \int_0^{\Theta}[G_m^r(\cos\theta)]^2 \sin\theta d\theta$ is the concentration in R.

4.2 The Tangential Component

This section will mostly parallel the previous section. Here we seek solutions to the equation (2.50) in the form (2.43). First, notice that (2.50) has regular singularities at $x = \pm 1$, $x = \cos\Theta$, and $x = \infty$. The exponents at these singularities are respectively: $\{\frac{m\mp 1}{1}, -\frac{m\mp 1}{2}\}$, $\{0, 0\}$, and $\{-L, 2+L\}$. Again we map $-1 \mapsto 0$, $\cos\Theta \mapsto a := (\cos\Theta + 1)/2$, and $1 \mapsto 1$ with the change of variable $z := (x+1)/2$. We must make a change of dependent variable so as to reduce one of the exponents at $z = 0, 1$ to zero, but in order for local solutions to exist, γ cannot be zero or a negative integer. We must also be sure that our changed variable has no singularities in $[0, 1]$. We look at three separate cases: $m = 0$, $m > 0$, and $m < 0$.

If $m = 0$, employ the transformation:

$$G_0^t(z) = z^{1/2}(1-z)^{1/2}y(z) \tag{4.7}$$

Then, (2.50) is in the form (3.1) with $\gamma = \delta = 2$, $\epsilon = 1$, $\alpha = -L+1$, $\beta = L+3$, and $q = \frac{1}{2}[\chi - L(2+L) + 1 + 2(1+\cos\Theta)]$.

If $m > 0$, let:

$$G_m^t(z) = z^{(m+1)/2}(1-z)^{(m-1)/2}y(z) \tag{4.8}$$

Our parameters are $\gamma = m+2$, $\delta = m$, $\epsilon = 1$, $\alpha = m - L$, $\beta = 2 + L + m$, and $q = \frac{1}{2}[\chi - L(2+L) + (m+1) + m(m+1)(1+\cos\Theta)]$.

If $m < 0$, let:

$$G_m^t(z) = z^{-(m+1)/2}(1-z)^{-(m-1)/2}y(z) \tag{4.9}$$

Our parameters are $\gamma = -m$, $\delta = 2 - m$, $\epsilon = 1$, $\alpha = -L - m$, $\beta = 2 + L - m$, and $q = \frac{1}{2}[\chi - L(2+L) - (m+1) + m(m-1)(1+\cos\Theta)]$.

In all of these transformations γ is a positive integer and α is zero or a negative integer, so polynomial solutions of degrees $L-1$, $L-m$, and $L+m$ respectively exist according to the theory of section 3.3 when q is an eigenvalue of the matrix $\mathbf{H}$ with the appropriate parameters plugged in. Going back to the original variables, we have the following solutions to (2.50) for the following eigenvalues χ:

If $m=0$ and $\chi = 2q + L(2+L) - 1 - 2(1+\cos\Theta)$, then (2.50) has the solution:

$$G_0^t(x) = 2^{-1}(1-x^2)^{1/2} Hl((1+\cos\Theta)/2, q; -L+1, L+3, 2, 2; (x+1)/2) \tag{4.10}$$

If $m>0$ and $\chi = 2q + L(2+L) - (m+1) - m(m+1)(1+\cos\Theta)$ then (2.50) has the solution:

$$\begin{aligned} G_m^t(x) = {} & 2^{-m}(1+x)^{(m+1)/2}(1-x)^{(m-1)/2} \times \\ & Hl((1+\cos\Theta)/2, q; m-L, 2+L+m, m+2, m; (x+1)/2) \end{aligned} \tag{4.11}$$

If $m<0$ and $\chi = 2q + L(2+L) + (m+1) - m(m-1)(1+\cos\Theta)$ then (2.50) has the solution:

$$\begin{aligned} G_m^t(x) = {} & 2^{m}(1+x)^{-(m+1)/2}(1-x)^{-(m-1)/2} \times \\ & Hl((1+\cos\Theta)/2, q; -L-m, 2+L-m, -m, 2-m; (x+1)/2) \end{aligned} \tag{4.12}$$

where again, q is an eigenvalue of $\mathbf{H}$ with appropriately chosen entries.

In light of definition (2.10), it is a simple exercise to show that these solutions agree with our expectation that $G_m^t(x)$ has the form (2.43). The figure shows the normalized functions $G_m^t(\cos\theta)$ The eigenvalue spectrum does not look significantly different from figure 4.1 so it is not displayed.

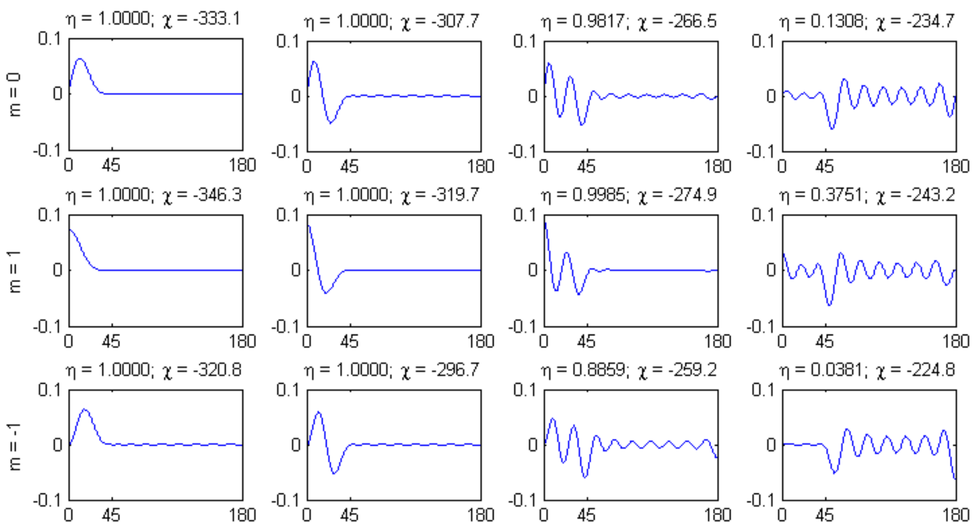

Figure 4.3: Selected normalized eigenfunctions $G_m^t(\cos\theta)$ for $L = 18$ and $\Theta = 45°$. χ is the eigenvalue and $\eta = \int_0^\Theta [G_m^t(\cos\theta)]^2 \sin\theta d\theta$ is the concentration in R.

5 Conclusions

In summary, we have found an orthogonal basis of the space of band-limited vector fields S_L ranked according to their concentration in the spherical cap R of maximum co-latitude Θ. These are the Slepian functions. If we choose to represent vector fields on the sphere in the basis $\{\hat{\mathbf{r}}, \hat{\boldsymbol{\tau}}_+, \hat{\boldsymbol{\tau}}_-\}$ defined in (2.1), then the Slepian functions have the form:

$$\mathbf{G}^r_m(\theta, \phi) = G^r_{|m|}(\cos\theta)e^{im\phi}\hat{\mathbf{r}} \tag{5.1}$$

and

$$\mathbf{G}^{\pm}_m(\theta, \phi) = G^t_{\pm m}(\cos\theta)e^{im\phi}\hat{\boldsymbol{\tau}}_{\pm} \tag{5.2}$$

for $-L \leq m \leq L$. The real-valued functions $G^r_m(x)$ for $0 \leq m \leq L$ and $G^t_m(x)$ for $-L \leq m \leq L$ solve equations (2.40) and (2.50) respectively on the interval $[-1, 1]$ for eigenvalues χ of the forms given in sections 4.1 and 4.2. This fact implies that they can be written in the form:

$$G^{\{r,t\}}_m(x) = (1+x)^{k_1}(1-x)^{k_2}Hl((1+\cos\Theta)/2, q; \alpha, \beta, \gamma, \delta; (x+1)/2) \tag{5.3}$$

with parameters specified as in sections 4.1 and 4.2. In particular, q must be an eigenvalue of the tridiagonal matrix with analytically prescribed entries $\mathbf{H}$ defined in (3.14). Then, Hl is a Heun polynomial whose coefficients form an eigenvector of $\mathbf{H}$ with eigenvalue q. Numerically, we normalize the $G^{\{r,t\}}_m$ so that $\int_{-1}^{1}[G^{\{r,t\}}_m(x)]^2 dx = 1$.

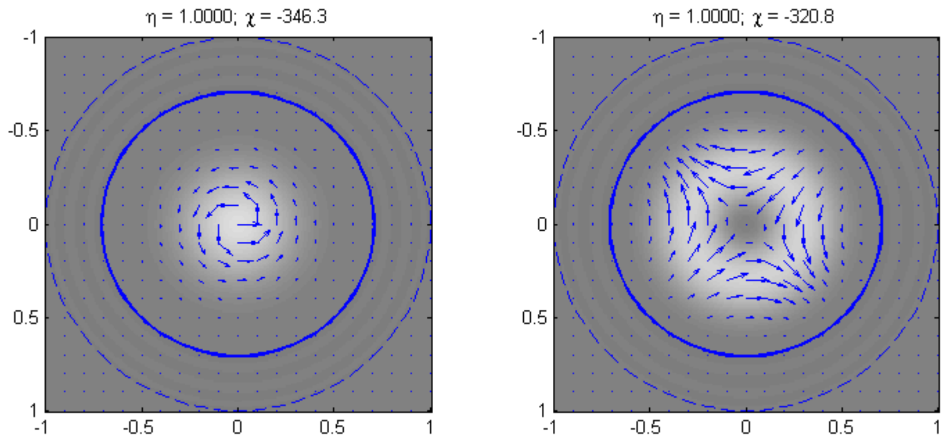

Figure 5.1: Real part of best concentrated tangential Slepian functions $\mathbf{G}_m^{\pm}(\theta,\phi)$ for $\Theta = 45°$ and $m = 1$. The left figure shows $\mathbf{G}_m^{+}$ while the right figure shows $\mathbf{G}_m^{-}$. The picture is of a sphere from the top down. The solid circle is the boundary of the cap R, the dashed circle is the boundary of the hemisphere. The gray-scale image is the magnitude of the overlaid vector field.

Then, the concentration in the spherical cap R is optimized when the quantity:

$$\eta = \int_{\cos\Theta}^{1} [G_m^{\{r,t\}}(x)]^2 dx \tag{5.4}$$

is maximized. The maximal values of η are attained for minimal values of χ of which there are $L - m + 1$ choices for each m. The figures show the real parts of selected radial and tangential Slepian functions. For the radial component this is:

$$\mathrm{Re}[\mathbf{G}_m^r(\theta,\phi)] = G_m^r(\cos\theta)\cos(m\phi)\hat{\mathbf{r}} \tag{5.5}$$

and the tangential components:

$$\mathrm{Re}[\mathbf{G}_m^{\pm}(\theta,\phi)] = 2^{-1/2} G_{\pm m}^t(\cos\theta)[\cos(m\phi)\hat{\boldsymbol{\theta}} \mp \sin(m\phi)\hat{\boldsymbol{\phi}}] \tag{5.6}$$

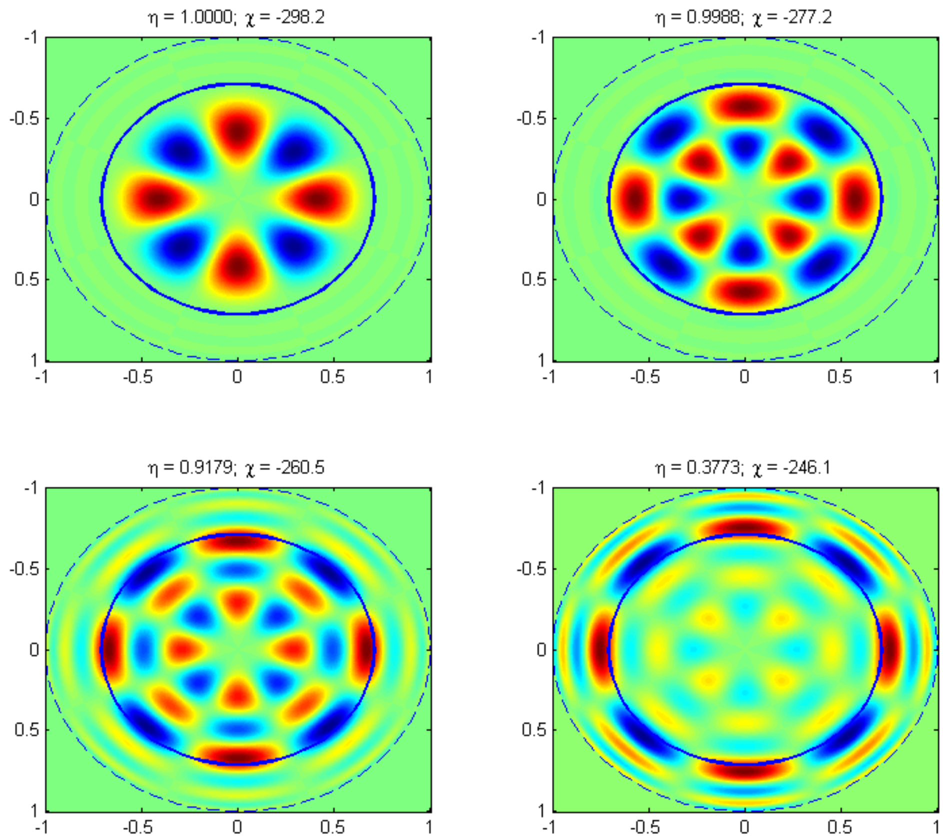

Figure 5.2: Real part of radial Slepian function $\mathbf{G}_m^r(\theta,\phi)$ with $\Theta = 45°$ and $m = 4$. χ is the eigenvalue and η is the concentration in R. The picture is of a sphere from the top down. The solid circle is the boundary of the cap R, the dashed circle is the boundary of the hemisphere.

REFERENCES

[1] F. J. Simons, F. A. Dahlen, and M. A. Wieczorek, "Spatiospectral concentration on a sphere," *SIAM Review*, vol. 48, no. 3, pp. 504–536, 2006. [Online]. Available: http://dx.doi.org/10.1137/S0036144504445765

[2] A. Plattner, F. J. Simons, and L. Wei, "Analysis of real vector fields on the sphere using slepian functions," in *Statistical Signal Processing Workshop (SSP), 2012 IEEE*. IEEE, 2012, pp. 1–4.

[3] K. Jahn and N. Bokor, "Revisiting the concentration problem of vector fields within a spherical cap: A commuting differential operator solution," *Journal of Fourier Analysis and Applications*, vol. 20, no. 2, pp. 421–451, 2014. [Online]. Available: http://dx.doi.org/10.1007/s00041-014-9324-7

[4] A. Ronveaux, Ed., *Heun's Differential Equations*. Oxford, England: Oxford University Press, 1995.

[5] "NIST Digital Library of Mathematical Functions," http://dlmf.nist.gov/, Release 1.0.9 of 2014-08-29. [Online]. Available: http://dlmf.nist.gov/

[6] C. J. R. Sheppard and P. Török, "Efficient calculation of electromagnetic diffraction in optical systems using a multipole expansion," *Journal of Modern Optics*, vol. 44, pp. 803–818, Apr. 1997.