May 2016

The Root Finite Condition on Groups and Its Application to Group Rings

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by

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A Dissertation Submitted in
Partial Fulfillment of the
Requirements for the Degree of

Doctor of Philosophy
in Mathematics

at

The University of Wisconsin–Milwaukee

May 2016

ABSTRACT

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The University of Wisconsin-Milwaukee, 2016
Under the Supervision of Allen D. Bell

A group G is said to satisfy the root-finite condition if for every $g \in G$, there are only finitely many $x \in G$ such that there exists a positive integer n such that $x^n = g$. It is shown that groups satisfy the root-finite condition iff they satisfy three subconditions, which are shown to be independent. Free groups are root-finite. Ordered groups are shown to satisfy one of the subconditions for the root-finite condition. Finitely generated abelian groups satisfy the root-finite condition. If, in a torsion-free abelian group G, there exists a positive integer r such that the subgroup A_r of elements of G taken to the r^{th} power has index less than r in G, then G does not satisfy the root-finite condition. Finitely generated finite conjugate groups satisfy the root-finite condition. Infinite groups with finitely many conjugacy classes fail to satisfy the root-finite condition. Torsion-free polycyclic-by-finite groups satisfy two of the subconditions for the root-finite condition. Finitely generated nilpotent groups satisfy the root-finite condition. If KG is a group ring, for every nonidentity element x of G, the following left module is defined $\mathcal{M}_x = KG/KG(x-1)$. This module is shown to be faithful if G satisfies the root-finite condition and x has an infinite conjugacy class. If KG is a prime group ring, then $\mathcal{M}_x$ is not faithful if the conjugacy class of x is finite. An analogous problem concerning skew polynomial and skew-Laurent polynomial rings is discussed.

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ACKNOWLEDGMENTS

I would like to express my appreciation for my adviser, Allen Bell, for guiding me through this process. I am grateful to him for his patience, for letting me follow my inquiries wherever they happened to lead me, and for setting me straight when I drifted off course. It has been a pleasure working with him the last two years. I benefited as well from working with Ian Musson in an earlier phase of my research, as it was he who first directed my attention to the field of group rings. I would also like to express appreciation for the Department of Mathematics at UWM in general. I have learned a great deal from the courses that I have taken here, and have benefited from the many excellent teachers whom I have been fortunate enough to study with. I am grateful, as well, to the other graduate students with whom I have studied and exchanged ideas over the past few years. Finally, I would like to thank my family and friends for their support and encouragement.

Chapter 1

Overview

There are two principal topics with which this work will be concerned: groups rings and finiteness conditions on groups. This research stems from a problem dealing with certain modules over group rings and determining under what conditions such modules are faithful. In investigating this problem, it was discovered that a key factor in determining whether the module was faithful hinged on the question of whether the underlying group of the group ring possessed a certain finiteness condition, which in this work is called the root-finite condition. It appears that this finiteness condition has not been previously studied.

The structure of the dissertation is as follows:

Chapter 2 introduces definitions and background concerning group rings. Some ring-theoretic concepts are also introduced in this chapter that will recur in various places throughout this work.

Chapter 3 introduces the root-finite condition. The root-finite condition is defined, and a group is shown to satisfy the root-finite condition if it simultaneously satisfies three sub-conditions. This chapter also looks at closure operations on the class of root-finite groups.

The next several chapters look at certain important classes of groups with respect to the two general topics that are the focus of this work: We attempt to delineate criteria for when groups from these classes satisfy the root-finite condition, and we look at some of the major theorems regarding group rings constructed from groups in these classes.

Chapter 4 focuses on free groups, which are shown to satisfy the root-finite condition.

Chapter 5 is concerned with ordered groups, which are shown to satisfy one of the sub-conditions for the root-finite condition.

Chapter 6 is concerned with abelian groups. Finitely generated abelian groups are shown to satisfy the root-finite condition. The question of whether abelian groups that are not finitely generated satisfy the root-finite condition is shown to be connected to the density of roots in the group.

Chapter 7 presents findings regarding finite conjugate groups, a class of groups which plays a prominent role in the theory of group rings. It is shown that finitely generated finite conjugate groups satisfy the root-finite condition. Groups that are not finite conjugate groups, but whose delta subgroup has finite index are shown to fail to satisfy the root-finite condition.

Chapter 8 looks at groups that consist of finitely many conjugacy classes. These groups are shown to violate one or more of the subconditions for the root-finite condition.

Chapter 9 looks at polycyclic and polycyclic-by-finite groups. The question of whether these groups satisfy the root-finite condition is complicated. In the case of torsion-free polycyclic-by-finite groups, it is shown that two of the subconditions for the root-finite condition are satisfied.

Chapter 10 is concerned with nilpotent groups. It is shown that finitely generated nilpotent groups satisfy the root-finite condition.

Chapter 11 discusses the question of the faithfulness of certain modules over group rings, with special emphasis on prime group rings and on the role played by the root-finite condition on the underlying group. This was chronologically the first question to be addressed in this research, and it provided the motivation for studying the theory of root-finite groups in greater depth. For a group ring KG, and for any nonidentity element x of G, we consider the left module $\mathcal{M}_x = KG/KG(x-1)$. The problem is to determine under what conditions this module is faithful. The main results of the chapter are as follows: If G satisfies the root-finite condition and $x \in G$ has an infinite conjugacy class, then $\mathcal{M}_x$ is faithful. If $[x]$ represents the conjugacy class of x, then the annihilator of $\mathcal{M}_x$ is equal to $\bigcap_{y\in[x]} KG(y-1)$. Finally, if KG is a prime group ring and x has a finite conjugacy class, then $\mathcal{M}_x$ is not faithful.

Chapter 12 discusses the implications of the preceding chapter with respect to group rings over the infinite dihedral group.

Chapter 13 extends the methods on group rings to similar questions concerning skew polynomials and skew-Laurent polynomials.

Chapter 14 proposes some topics for further research.

Chapter 2

Preliminaries

The purpose of this chapter is to present the necessary background for the later investigation of modules over group rings and their annihilators. We define group rings and several key concepts that are of central importance in discussing group rings. We also define some basic concepts of ring theory that will be useful in our discussion, and present some theorems from the literature.

2.1 Group Rings: Definition and Background

Given a field K and a group G, the group ring KG is defined as consisting of all formal finite sums of the form

$$\alpha = \sum_{x \in G} a_x \cdot x$$

with $a_x \in K$. (This definition, and those that immediately follow, are from [19].) For $\beta = \sum b_x \cdot x$, the operations of addition and multiplication in KG are defined naturally as follows:

$$\alpha + \beta = \sum_{x \in G} (a_x + b_x) \cdot x$$

and

$$\alpha\beta = \sum_{z \in G} c_z \cdot z$$

where

$$c_z = \sum_{xy=z} a_x b_y$$

Given an element of a group ring KG, we are interested in the set of group elements that actually occur in the finite sum that constitutes the element of the group ring. So, for any

$$\alpha = \sum_{x \in G} a_x x \in KG$$

we define the *support* of α, denoted $\operatorname{Supp} \alpha$, to be

$$\operatorname{Supp} \alpha = \{x \in G | a_x \neq 0\}$$

The *trivial units* of the group ring KG are those elements of the form λg, where λ is a nonzero element of the field K and $g \in G$. With slight abuse of notation, when $\lambda = 1$ or when $g = e$, we generally drop the identity elements when writing out these trivial units of the group ring, so long as there is no confusion about whether we are referring to an element of KG or to an element of the constituent field or group.

An important ideal of KG is the *augmentation ideal*, denoted $\omega(KG)$, defined as

$$\omega(KG) = \{\sum a_x x | \sum a_x = 0\}$$

2.2 Some Ring Theoretic Concepts

We now review the definitions of several concepts from ring theory that will be referred to in this dissertation.

Following [5], a *polynomial identity (PI)* on a ring R is defined as a polynomial $p(x_1, \cdots, x_n)$ in noncommuting variables $x_1, \cdots, x_n$ with coefficients from $\mathbb{Z}$ such that $p(r_1, \cdots, r_n) = 0$ for all $r_1, \cdots, r_n \in R$. A *polynomial identity ring (PI ring)*, is a ring R that satisfies some monic polynomial identity $p(x_1, \cdots, x_n)$ (that is, among the monomials of highest total degree which appear in p, at least one has coefficient 1).

A (left or right) module M of a ring is said to be *Noetherian* if it satisfies the following three equivalent properties:

- Every submodule of M is finitely generated.
- Every ascending chain of submodules $M_1 \subseteq M_2 \subseteq \cdots \subseteq M_i \subseteq \cdots$ has only finitely many distinct submodules.
- Every nonempty set S of submodules of M has a maximal member.

A ring R is said to be left Noetherian if it is a Noetherian left R-module, and right Noetherian if it is a Noetherian right R-module. If a ring is both left and right Noetherian, it is said to be Noetherian.

An *essential left ideal* is a left ideal that has a nonzero intersection with all other nonzero left ideals.

A *prime ideal* in a ring R is any proper ideal P of R such that, whenever I and J are ideals of R with $IJ \subseteq P$, either $I \subseteq P$ or $J \subseteq P$. A *prime ring* is a ring in which 0 is a prime ideal.

When studying a group ring KG, we are often interested in whether the group ring satisfies a polynomial identity over the field K, and not merely over $\mathbb{Z}$. An algebra E over a field K is said to be a *PI algebra* or to satisfy a polynomial identity if there exists $f(x_1, \cdots, x_n) \in K\langle x_1, \cdots, x_n\rangle \neq 0$ with $f(\alpha_1, \cdots, \alpha_n) = 0$ for all $\alpha_1, \cdots, \alpha_n \in E$.

The following lemma will prove useful:

Lemma 2.2.1. *If R is a prime Noetherian ring, then for $c \in R, Rc$ is essential iff c is a regular element of R.*

Proof. This lemma was proven in [4], where it is Lemma 3.8. □

A ring R is said to be *bounded* if every essential one-sided ideal of R contains an essential (two-sided) ideal. R is said to be *fully bounded Noetherian (FBN)* if R is Noetherian and if every prime image of R is bounded. This has implications for the question of whether KG satisifes a polynomial identity.

We introduce some additional definitions from [14]. Q is said to be a *central simple algebra* over a field Z if Z is the center of Q and Q is a simple Artinian ring, finite dimensional over Z. A ring Q is said to be a *quotient ring* if every regular element of Q is a unit. Given a quotient ring Q, a subring R, not necessarily containing 1, is said to be a *right order* in Q if each $q \in Q$ has the form rs^{-1} for some $r, s \in R$. A *left order* is defined analogously. A left and right order is said to be an *order*.

Theorem 2.2.2. *(Posner's theorem). Let R be a prime PI ring with center C. Let $S = C\backslash\{0\}, Q = R_S$, and $Z = C_S$, the quotient field of C. Then Q is a central simple algebra with center Z, R is an order in Q and $Q = RZ$.*

Proof. For a proof, see [14]. □

Posner's theorem enables us to prove the following theorem, which establishes the relationship between PI rings and FBN rings.

Theorem 2.2.3. *If R is a Noetherian ring with a polynomial identity, then R is FBN.*

Proof. With no loss of generality, we can take R to be a prime ring. Let I be an essential left ideal. By Lemma 1.4, I contains a regular element c and $c^{-1} \in Q = R_S$. Then $c^{-1} = rz^{-1}$ by Posner's Theorem. Therefore $c^{-1}z = r$, and, since z is central, $zc^{-1} = r$, so $z = rc \in I$. Thus I contains the two-sided ideal $zR = Rz$. $\square$

2.3 Prime Group Rings

In this section, we cite a key theorem relating to prime group rings, which plays an important role in several of the results of this dissertation.

A theorem of [2] sets forth equivalent conditions for a group ring to be a prime ring.

Theorem 2.3.1. *Let KG be a group ring. Then the following are equivalent:*

1. *KG is prime.*
2. *The center of KG is prime.*
3. *G has no nonidentity finite normal subgroup.*
4. *$\Delta(G)$ is torsion-free abelian, where $\Delta(G)$ is the subgroup of G consisting of elements of G with a finite conjugacy class.*

Proof. See [19], Theorem 4.2.10. $\square$

For a further discussion of $\Delta(G)$, including an explanation of why this subset is a subgroup, see Section 7.1.

Chapter 3

The Root-Finite Condition on Groups

The purpose of this chapter is to introduce the root-finite condition on groups. This chapter begins with a brief discussion of the general topic of finiteness conditions on groups. Then the root-finite condition is defined, and it will be shown that the root-finite condition is satisfied if and only if three subconditions are satisfied. These subconditions are shown to be independent. The topic of closure operations on the class of root-finite groups is explored. The chapter concludes with an investigation of two topics relating to the root-finite condition. The first of these topics is to look at some groups defined by relations on two generators and to see what sorts of relations can give rise to groups that fail to satisfy the three subconditions of the root-finite condition. Finally, we present some theorems relating to subgroups that satisfy the root-finite condition even though the larger group may not.

3.1 Finiteness Conditions on Groups

A finiteness condition on groups is any property of a group that holds for all finite groups and for some, but not all, infinite groups. Associated with each finiteness condition is a class of groups, consisting of those groups that satisfy the particular finiteness condition, and the finiteness condition itself is sometimes identified with this class of groups.

Several finite conditions on groups play a role in this work. Among them are the following:

- The finiteness condition which is generally first encountered in studying groups is the condition that a group be finitely generated, that is, for a group G there exists a finite set A of elements of G, the *generating set*, such that every element of G can be expressed as a product of positive and negative powers of elements of A. All finite groups can be viewed as finitely generated by identifying the generating set A with the group G.

- Finiteness conditions can arise from considering some sort of ascending chain within the group G and requiring that the chain stabilize in a finite number of steps. Thus, we have the max condition on subgroups of a group G, which is the condition that any ascending chain of subgroups of G

$$A_1 \subseteq A_2 \subseteq \cdots \subseteq A_i \subseteq$$

can have only finitely many distinct subgroups. This can be shown to be equivalent to the finiteness condition that all subgroups of a group G are finitely generated (the Axiom of Choice being welcome in this work). There are other finiteness conditions that are defined in terms of finiteness of ascending chains, such as the max-n condition, under which an ascending chain of normal subgroups can consist of only finitely many distinct subgroups.

- A finiteness conditions that plays a significant role in this work is the finite-conjugate condition. A group G satisfies the finite conjugate condition if all of its elements have finite conjugacy classes.

- Another finiteness condition involving conjugacy classes which is considered in this work is the condition that a group G have only finitely many conjugacy classes. Only finite groups can satisfy both this condition and the finite conjugate condition.

- If X is any class of groups which contains all trivial groups (groups consisting of only an identity element), then a group G satisfies the X-by-finite finiteness condition if G has a normal subgroup of finite index belonging to X. (All finite groups are X-by-finite, since the trivial group is a normal subgroup of finite index.) Examples of this sort of finiteness condition that are discussed in this work include abelian-by-finite groups, polycyclic-by-finite groups, and finite-conjugate-by-finite groups.

- Two finiteness conditions, the condition of being locally finite and the condition of being residually finite, are noted here and are defined in the discussion of closure operations on classes of groups (Section 3.4).

3.2 Definition and Preliminary Discussion of the Root-Finite Condition

An element x of a group G is said to be a *root* of $g \in G$ if for some positive integer r, $x^r = g$. Any such x is called an r^{th} root of g. A group G is said to be *root finite* or is said to satisfy the root-finite condition if all elements g of G have only finitely many roots. If an element g of G has no r^{th} roots for any integer $r \geq 2$, then g is said to be *rootless*. If an element g of G has only finitely many roots, that element is said to be a *root-finite element*, irrespective of whether the group G satisfies the root-finite property. The root-finite condition on G is equivalent to the condition that no cyclic subgroup of G is contained in infinitely many cyclic subgroups.

It should be noted that in any group, the relation "g is a root of h" is a preorder. The relation is reflexive, since any element g of G satisfies the equation $g^1 = g$. The relation is transitive, since if g is an r^{th} root of h and h is an s^{th} root of k, then g is an rs^{th} root of k, since $g^r = h$ and $h^s = k$ gives us $(g^r)^s = g^{rs} = k$. The relation is not necessarily a partial order, since antisymmetry will not always hold. For example, if an element g of a group G has order 5, and if $g^2 = h$, then $h^3 = (g^2)^3 = g^6 = g^5 g = eg = g$, so g is a root of h, and h is also a root of g, but $g \neq h$. We can denote this preorder "g is a root of h" by $g \leq_r h$ (the r subscript denoting "root"). If G is a torsion-free group, $\leq_r$ will be a partial order, since antisymmetry fails only when there is an element of finite order in G.

As a familiar example, one in which the order relation $\leq_r$ is well known, albeit by a different name, consider the integers as a group with the operation of addition. Then integers g and h satisfy the relation $g^r = h$ (which, to avoid confusing notation, should better be written in additive notation as $rg = h$) for some positive integer r precisely when h is a multiple of g. The group $(\mathbb{Z}, +)$ is ordered by the relation $\leq_r$ (since $\mathbb{Z}$ is torsion-free), which is to say that the integers are partially ordered by the factor relation $a|b$.

3.3 The Three-Condition Theorem

3.3.1 The Root Chain Condition

There is a chain condition that comes into play when discussing the root-finite condition. If there is some chain of group elements g_i satisfying the relations

$$g_1 \geq_r g_2 \geq_r \cdots \geq_r g_i \geq_r \cdots$$

that is, if g_{i+1} is a root of g_i for all i, this is said to be a *root chain* or a chain of roots originating at g_1. If a group G is root finite, then for any element g of G, any root chain originating at g can consist of only finitely many distinct elements. If g_{i+1} is a root of g_i, then there is a positive integer r such that $g_{i+1}^r = g_i$ and so g_i is an element of $\langle g_{i+1} \rangle$, the cyclic subgroup of G generated by g_{i+1}. Therefore $\langle g_i \rangle \subseteq \langle g_{i+1} \rangle$, so that the root chain

$$g_1 \geq_r g_2 \geq_r \cdots \geq_r g_i \geq_r \cdots$$

is seen to be equivalent to

$$\langle g_1 \rangle \subseteq \langle g_2 \rangle \subseteq \cdots \subseteq \langle g_i \rangle \subseteq \cdots$$

so that groups satisfying the root-finiteness condition are also seen to satisfy the ascending chain condition on cyclic subgroups. We thus can see that the root chain condition is equivalent to the property that there does not exist an infinite strict ascending chain of cyclic subgroups, that is, that the group satisfies the max condition on cyclic subgroups.

If C_1 and C_2 denote two root chains originating at the same element g in a group G, we say that C_2 is an *extension* of C_1 if all of the roots in the root chain C_1 also appear in C_2 and there is at least one root in the root chain C_2 that does not appear in C_1. Note that this definition does not require that the additional root or roots in C_2 follow the roots that appear in both C_1 and C_2.

3.3.2 The Prime Root Condition

We introduce another term which will be used to identify another condition which is satisfied whenever a group satisfies the root-finite condition. We say that some element g of a group G is a *prime root* of h if there is some prime p such that $g^p = h$. The following lemma presents an observation concerning root chains and prime roots.

Lemma 3.3.1. *Given any group G:*

(a) *a root chain originating at some element g of G contains at most one distinct prime root of g if g has infinite order.*

(b) *any nonconstant root chain originating at some element g of G (not necessarily an element g having infinite order) can be extended to include a prime root of g.*

Proof. (a) Let

$$g = x_0 \geq_r x_1 \geq_r \cdots \geq_r x_i \geq_r \cdots$$

be a root chain in G originating at g. If the root chain is constant, there is nothing to prove. If the root chain is not constant let i be minimal such that $x_i \neq x_0$. Then we have $g = x_i^r$ for some positive integer $r \geq 2$. If $x_j = x_i$ for all $j > i$, then there is only one distinct root of g in the chain, and therefore at most one distinct prime root. If otherwise, there is some j minimal such that $x_j \neq x_i$. Since, in a root chain, if $m < n$ then g_n is a root of g_m, we have, in particular that for $k \geq j$, x_k is a root of x_i. Therefore, for some integer $s \geq 2$, $x_i = x_k^s$, and therefore $g = (x_k^s)^r = x_k^{rs}$. Since r and s are greater than or equal to 2, the integer rs is obviously not prime. If it were the case that x_k^p were also equal to g for some prime p, then $x_k^p = x_k^{rs}$, so $x_k^{p-rs} = e$, and thus x_k has finite order in G. However, since $x_k \leq_r g$, the cyclic subgroup generated by g is contained in the cyclic subgroup generated by x_k. Since $\langle g \rangle$ is assumed to be infinite and $\langle x_k \rangle$ is finite under the assumption that x_k is a prime root of g, the inclusion $\langle g \rangle \subseteq \langle x_k \rangle$ is impossible. Thus, for $k \geq j$, x_k cannot be a prime root of g, and thus the root chain can have at most one prime root.

(b) Now suppose that there is a root chain originating at g

$$g = x_0 \geq_r x_1 \geq_r \cdots \geq_r x_i \geq_r \cdots$$

that does not contain any prime roots of g. We take the first element of the root chain that is not equal to g. Let us suppose, for ease of notation, that it is x_1. Then $g = x_1^r$ for some positive composite integer r. We then let p be one of the prime factors of r, so that $r = pk$ for some positive integer k. Now we create a new root chain

$$g = y_0 \geq_r y_1 \geq_r \cdots \geq_r y_i \geq_r \cdots$$

where $y_1 = x_1^k$ and $y_i = x_{i-1}$ for $i \geq 2$. To show that this is a root chain, we need to have that the two additional relations $g \geq_r y_1$ and $y_1 \geq_r y_2$ hold, as all the other relations hold given the prior root chain. We see that $y_1 \geq_r y_2$ since $y_2 = x_1$ and $y_1 = x_1^k$, so that $y_1 = y_2^k$, so y_2 is a root of y_1. Also, $y_1^p = (x_1^k)^p = x_1^{kp} = x_1^r = g$, so that y_1 is a root of g, and, in fact, it is a prime root, as required.

$\square$

In some contexts, it may only be necessary to concern ourselves with prime roots. Whenever $g \leq_r h$, if g is not a prime root of h, we can use the method of the proof of the second part Lemma 3.3.1 to insert additional roots between g and h and arrive at a chain where every element is a prime root of the next element.

Intuitively, the prime root condition would be stated something like this: In a group satisfying the root-finite condition, for every element g there are at most finitely many primes p_i such that g has a p_i^{th} root. Unfortunately, we are not going to be able to get away with a formulation that simple. One problem that we encounter is that the identity element of the group is its own r^{th} root for all positive integers r. Also, if a group G has finite cyclic subgroups, the above formulation fails. Suppose an element g of G has order t. Then if we consider a positive integer s less than t, $g^s = g^{s+kt}$ for all positive integers k. So, provided that there are infinitely many primes equal to s modulo t (which, according to a well-known theorem of Dirichlet, will hold if and only if s and t are coprime), the element g^s of G has prime roots for infinitely many primes p_i, and all of those prime roots are the same element g.

In order to avoid these complications, we use the following formulation of the condition that we continue to refer to as the "prime root condition", even though in this formulation, prime roots are not explicitly mentioned. If a group G satisfies the root-finite condition, it necessarily satisfies the finiteness condition that for all g in G there are only finitely many primes p_i for which there exists an element x_i of G such that $g \in \langle x_i \rangle$ and $[\langle x_i \rangle : \langle g \rangle] = p_i$. This avoids the problem of finding prime roots of g in the cyclic subgroup generated by g whenever g has finite order, since in those circumstances the subgroups $\langle x_i \rangle$ and $\langle g \rangle$ coincide, so the index $[\langle x_i \rangle : \langle g \rangle]$ will always be 1.

3.3.3 The r^{th}-Root Condition

We now introduce a third finiteness condition which is satisfied by all groups G that satisfy the root-finite condition: For every positive integer r, every element g of G has at most finitely many r^{th} roots. We will have occasion to look at groups in which, for every positive integer r and every element g of G, r^{th} roots are unique, that is, $x^r = y^r$ implies that $x = y$. Groups with this property are called R-groups. We might have referred to the r^{th}-root condition as the "weak R-group condition", but since R-groups will play only a peripheral role in this discussion, and especially since the property of being an R-group is not a finiteness condition (finite groups other than trivial groups are not R-groups since if a group has order r every element of the group is an r^{th} root of the identity), that terminology will not be used.

It should also be noted that since the existence of an r^{th} root of a group element g implies the existence of a p^{th} root of g for all prime factors p of r, we could limit the r^{th} root condition to prime roots with no loss of generality. We can thus alternatively define the r^{th}-root condition in the following manner: If G is a group, C any cyclic subgroup of G, and p any prime, then there are only finitely many cyclic subgroups D of G with $D \supseteq C$ and $[D : C] = p$.

3.3.4 Proof of the Three-Condition Theorem

In each of the previous three subsections, we encountered a finiteness condition that is satisfied by all groups that satisfy the root-finite condition. In this subsection, we will prove that if a group satisfies all three of these finiteness conditions, it satisfies the root-finite condition. We also show that the three conditions are independent, by exhibiting for each condition an example of a group that fails to satisfy that condition while satisfying the remaining two conditions.

Theorem 3.3.2. *A group G satisfies the root-finite condition if and only if it satisfies all of the following conditions:*

(a) *For all $g \in G$, all root chains originating at g have only finitely many distinct elements*

(b) *For all $g \in G$ there are only finitely many primes p_i for which there exists an element x_i of G such that $g \in \langle x_i \rangle$ and $[\langle x_i \rangle : \langle g \rangle] = p_i$.*

(c) *For all $g \in G$ and for all positive integers r, there are only finitely many (possibly zero) elements x of G such that $x^r = g$.*

Proof. It is clear that if a group G satisfies the root-finite condition, then all three of these conditions hold.

We now show that if a group G is does not satisfy the root-finite condition, then one of the conditions fails to hold. Suppose that a group G does not satisfy the root-finite condition. The proof will proceed on the following line of reasoning. Having assumed that G does not satisfy the root-finite condition, we are trying to show that it fails to satisfy at least one of the three conditions of the theorem. So our goal is to prove

$$\neg(a) \vee \neg(b) \vee \neg(c)$$

This is logically equivalent to

$$\neg[(a) \wedge (b)] \vee \neg(c)$$

which is, in turn, logically equivalent to

$$[(a) \wedge (b)] \Rightarrow \neg(c)$$

Thus it suffices to show that if G satisfies conditions (a) and (b), then G does not satisfy condition (c).

Let us then proceed by assuming that G is a group that does not satisfy the root-finite condition and that G satisfies conditions (a) and (b) of the theorem. Since G does not satisfy the root-finite condition, we can fix some element g of G that has infinitely many roots. Every root of g is part of a root chain originating at g, even if that chain consists of only two elements. Let $\mathscr{C}$ be a collection of root chains such that every root of g is contained in at least one chain C in $\mathscr{C}$. By Lemma 3.3.1, we can extend all of the chains C in $\mathscr{C}$ to contain a prime root of g, and that prime root will be the unique prime root of g in C. Since G satisfies condition (a) of the theorem, all of the chains C contain only finitely many distinct elements. If there were only finitely many chains C in $\mathscr{C}$, then only finitely many of the roots of g would be covered by the chains of $\mathscr{C}$, but since $\mathscr{C}$ was constructed to contain chains covering all the roots of g, there must be infinitely many root chains in $\mathscr{C}$, each with its unique prime root.

Since G satisfies condition (b) of the theorem, g has prime roots for only finitely many primes p_i. This means that that there is some prime p that has the property that infinitely many of the root chains in $\mathscr{C}$ contain a p^{th} root of g. If infinitely many of these p^{th} roots are distinct, then g has infinitely many p^{th} roots, and we have succeeded in showing that condition (c) of the theorem does not hold.

If there are only finitely many distinct p^{th} roots among the infinitely many root chains with p^{th} roots, then there must be a p^{th} root that occurs in infinitely many root chains. Fix one such p^{th} root and call it g_1. Since g_1 occurs in infinitely many root chains, it has infinitely many roots, and we repeat the process, using g_1 in place of g. If we continue this process, obtaining elements $g_2, g_3, \cdots$, we note that the g_i form a root chain of prime roots originating at g, which, since condition (a) of the theorem holds, cannot be infinite. So at some point we will arrive at an element g_i that has infinitely many p^{th} roots for some prime p, so condition (c) does not hold. $\square$

Example 3.3.3. The following examples illustrate the independence of the three conditions of Theorem 3.3.2. These are examples of groups that satisfy all but one of the conditions of Theorem 3.3.2 for each of the three conditions, so that no two conditions imply the third.

- We first look at a case where only the first condition, the root chain condition, fails. We consider, first, the rational numbers as a group with the operation of addition. This group does not satisfy the root chain condition because, starting with any rational number, we can take an infinite chain of square roots by successively multiplying by $1/2$. However, this does not provide us with the example that we need, as every nonzero rational number has a unique p^{th} root for all primes p, which is obtained by multiplying by $1/p$, so the prime root condition is not satisfied. However, the r^{th} root condition is satisfied, as all r^{th} roots are unique and are equal to $1/r$ times a given rational number. (In other words, $\mathbb{Q}$ is an R-group.)

We can, however, find a subgroup of the rational numbers that satisfies the prime root condition. Consider the group consisting of all rational numbers of the form $m/2^n$, with m an integer and n a positive integer, with the operation of addition. This group does not satisfy the root chain condition, as we can form an infinite chain of square roots originating at any nonzero group element by multiplying successively by $1/2$. In this subgroup of $\mathbb{Q}$, however, the prime root condition is satisfied, since if p is a prime other than 2, the subgroup element $m/2^n$ has a p^{th} root in the subgroup only if p is a factor of m, which necessarily occurs for only finitely many primes. Of course, the subgroup maintains the property of being an R-group, so the r^{th} root condition is still satisfied.

- As an example of a group that fails to satisfy only the prime root condition, consider the abelian group with a generator x of infinite order and countably infinitely many generators $x_1, x_2, \cdots$ and relations $x_i^{p_i} = x$, where p_i denotes the i^{th} prime. In this group, the element x would have a p^{th} root for every prime p, but it does not appear that the relations would give rise to an infinite root chain or to any element of the group having infinitely many r^{th} roots for some positive integer r.

- As an example of a group that fails to satisfy only the r^{th} root condition, consider the group formed by taking the direct product of infinitely many copies of $\mathbb{Z}/2\mathbb{Z}$. In this group, every element squared equals the identity, so the identity has infinitely many square roots, but no other element has any root other than itself.

3.4 Closure Operations on Root-Finite Groups

In discussions of finiteness conditions on groups, a topic that is often explored involves closure operations on classes of groups. In providing some background discussion of this topic, it will be useful to summarize the following relevant material that can be found in [20]. An *operation* A on classes of groups is a function mapping any class of groups X to a class of groups AX, fulfilling the following conditions:

- All trivial groups are members of the class X;

- If X and Y are classes of groups such that $X \subseteq Y$, then $AX \subseteq AY$;
- For all classes of groups X, $X \subseteq AX$;
- A maps the class of trivial groups to itself.

A class of groups X is said to be A-*closed* if it turns out that $AX = X$. When studying a particular finiteness condition, the class of groups of interest is the class consisting of all groups that possess that finiteness condition. The question that often arises is whether the class of groups possessing the finiteness condition is closed under various operations.

Operations can be multiplied in the natural way. For two operations A and B and a class of groups X, $(AB)X = A(BX)$. This makes sense since BX is a class of groups and the operation A acts on classes of groups. An operation is said to be a *closure operation* if $A^2 = A$. Some of the well-known closure operations are as follows:

- The subgroup operation, denoted by S, which maps a class of groups X to the class SX of all subgroups of members of X.
- The direct product operation, denoted by D, which maps a class of groups X to the class DX of all direct products of members of X. (There is also a closure operation D_0, which maps a class of groups X to the class D_0X of direct products of two members of X.)
- The operation H which maps a class of groups X to the class HX of all homomorphic images of members of X.
- The local operation L which maps a class of groups X to the class LX of all groups G with the following property: For every finite subset F of G there is a subgroup H of G containing F that belongs to X. For example, a group G is locally finite if every finite subset of G is contained in a finite subgroup.
- The residual operation R which maps a class of groups X to the RX of all groups G with the following property: For every element g of G there is a homomorphism from G to some group H in X such that g is not an element of the kernel of the homomorphism.

We can now proceed to analyze the class of all root-finite groups with respect to these closure operations. In deciding how to denote this class, the seemingly natural choice, RF, is to be avoided since it is too likely to be mistaken for the class of residually finite groups. Instead, in this discussion, the class of root finite groups will be denoted by T (the only letter to appear in both "root" and "finite").

We now consider which of the common closure operations defined above are T-closed:

- $ST = T$, that is, T is S-closed. If a group G is root-finite and H is a subgroup of G, then for any element h of H, all the roots of h in H are also going to be roots of h in G, so there cannot be infinitely many such roots.

- $D_0T = T$, that is, T is D_0-closed. If we let G and H be root-finite groups and consider an element (g, h) of $G \times H$, then an element (g', h') of $G \times H$ is an n^{th} root of (g, h) if and only if g' is an n^{th} root of g in G and h' is an n^{th} root of h in H. The elements g and h have finitely many roots in their respective groups, say g has n_g roots and h has n_h roots. Then the number of roots that (g, h) has in $G \times H$ is bounded by $n_g n_h$, and thus $G \times H$ is root finite.

- $DT \neq T$, that is T is not D-closed. Consider the direct product of infinitely many copies of $\mathbb{Z}/2\mathbb{Z}$. The constituent groups $\mathbb{Z}/2\mathbb{Z}$ are finite, and hence root-finite, groups, and yet their direct product is not root finite, as every element is a square root of the identity.

- $HT \neq T$, that is T is not H-closed. As will be shown in this work, the free groups are root finite. Since every group is the homomorphic image of a free group, HT thus includes all groups, so the root-finite condition is not closed under taking of homomorphic images. T actually fails this closure condition in a spectacular fashion; it will be seen that there is an injective homomorphism from every torsion-free root-finite group to a group that is not root finite.

- $LT \neq T$, that is, T is not L-closed. Locally root-finite groups need not be root finite. Take the example of $\mathbb{Q}$ as an additive group. Since any finite subset of rational numbers has a common denominator, every subset of $\mathbb{Q}$ is contained in an infinite cyclic subgroup of $\mathbb{Q}$. Since this subgroup is isomorphic to $\mathbb{Z}$, it must be root finite. Thus $\mathbb{Q}$ is locally root finite, but it is not root finite.

- $RT \neq T$, that is, T is not R-closed. The condition of being residually finite is a stronger condition than that of being residually root finite. Whatever homomorphisms you would use to establish that a group is residually finite work for establishing that the group is residually root finite as well. However, it is not the case that residually finite groups are necessarily root finite. An example, which will be discussed at greater length later in this work, is the infinite dihedral group. This group is polycyclic-by-finite, and thus residually finite, but it is not a root-finite group. See Chapter 12 for a further discussion.

3.5 The Root-Finite Condition and Relations on Two Generators

If a group is defined by its generators and relations, it may not be immediately apparent whether the group is root finite. We now consider some examples of relations that give rise to groups that lack the property of root finiteness.

Suppose, for example, that in some group G there are generators a and b satisfying the relation $a^2 = b^2$ (the infinite dihedral group is an example of a group which has such a relation). Then, if there is no further relation that restricts the order of ab^{-1}, the group is not root finite, as is shown by the following theorem. (If a and b commute, then $(ab^{-1})^2 = a^2b^{-2}$, which is equal to the identity e by the relation $a^2 = b^2$, so ab^{-1} would have order 2.)

Theorem 3.5.1. *If in a group G there are noncommuting elements a and b such that $a^2 = b^2$ and ab^{-1} has infinite order, then there is an element of G having infinitely many square roots.*

Proof. Denote by c the group element that is equal to a^2. Since $a^2 = b^2$, by multiplying on the left by a^{-1} and on the right by b^{-1}, we obtain the identity $ab^{-1} = a^{-1}b$. The claim is that $[a(ab^{-1})^i]^2 = c$ for all positive integers i. For $i = 1$,

$$[a(ab^{-1})]^2 = [a(a^{-1}b)]^2 = b^2 = c$$

Now, suppose that for some i, $[a(ab^{-1})^i]^2 = c$. The following computation shows that the equation holds for $i+1$: We first write $[a(ab^{-1})^{i+1}]^2$ in the form $a(ab^{-1})^i[(ab^{-1})a(ab^{-1})](ab^{-1})^i$. Using the identity $ab^{-1} = a^{-1}b$, the sequence of elements in brackets becomes $[ab^{-1}a(a^{-1}b)]$, which simplifies to a. We thus obtain that $[a(ab^{-1})^{i+1}]^2$ equals $[a(ab^{-1})^i]^2$, which is equal to c by the inductive assumption. Since ab^{-1} is assumed to have infinite order, the element c is thus shown to have infinitely many square roots.

□

We now consider a relation that gives rise to a group with an infinite chain of roots.

Theorem 3.5.2. *If in a group G there are two elements a and b of infinite order such that $a^{-1}ba = b^n$, where n is an integer with absolute value greater than or equal to 2. Then there is an infinite chain of roots in G, and thus G is not root finite.*

Proof. We adopt the following notation: Set $g_0 = b$, and let $g_i = a^i b a^{-i}$ for $i = 1, 2, \cdots$. The claim is that for $i = 1, 2, \cdots$, g_i is an n^{th} root of g_{i-1}. For the case $i = 1$, the given relation $a^{-1}ba = b^n$ gives us $b = ab^n a^{-1}$. Since $ab^n a^{-1}$ is equal to $(aba^{-1})^n$, we have that $g_1 (= aba^{-1})$ is an n^{th} root of $g_0 (= b)$. For $i \geq 2$, we have

$$g_i^n = (a^i b a^{-i})^n = a^i b^n a^{-i} = a^i (a^{-1} b a) a^{-i} = a^{i-1} b a^{-(i-1)} = g_{i-1}$$

It remains to be shown that the g_i are distinct. Suppose that for two distinct positive integers p and q, $g_p = g_q$ (with no loss of generality, we assume $p < q$). Then $a^p b a^{-p} = a^q b a^{-q}$. Multiplying on the left by a^{-p} and on the right by a^p, we get $b = a^{q-p} b a^{-(q-p)} = g_{q-p}$. However, since g_{q-p} is one of the g_i and hence a root of b, this would imply that b is equal to one of its roots. Since the conditions of the theorem give us that b is of infinite order, b cannot equal any of its roots, and thus we have a contradiction. So we can conclude that all of the g_i are distinct, and hence there is an infinite root chain beginning at $b = g_0$. $\square$

Corollary 3.5.3. *There are finitely presented groups that do not satisfy the root-finite condition.*

Proof. Theorems 3.5.2 and 3.5.1 provide examples of such groups. $\square$

We have given examples of relations in two generators that give rise to groups that violate two of the three conditions for groups to be root finite. It would be fitting to round out this discussion by showing a relation that gives rise to a group in which some element has p^{th} roots for infinitely many prime numbers p. It may well be the case, however, that there is no such group that can be finitely presented.

3.6 Subgroups Consisting of Root-Finite Elements

We now look at some theorems dealing with subsets of torsion-free groups that consist entirely of root-finite elements. We begin with the following theorem establishing that there is a maximal subgroup of this type.

Theorem 3.6.1. *If G is a torsion-free group and S denotes the set of all root-finite elements of G, then there is a subgroup of G maximal with respect to the property of consisting of elements of S.*

Proof. Since G is assumed to be torsion free, the identity element e of G is an element of S, and $\langle e \rangle$ is a subgroup of G consisting of elements of S. All such subgroups can be partially ordered by inclusion. If

$$A_1 \subseteq A_2 \subseteq \cdots \subseteq A_i \subseteq \cdots$$

is a chain of such subgroups, then $A = \bigcup_i A_i$ is a subgroup of G consisting of elements of S. For, if g and h are any elements of A, $g \in A_m$ and $h \in A_n$ for some m and n. If $k = \max(m, n)$, then $gh^{-1} \in A_k$, so gh^{-1} is in A, establishing that A is a subgroup of G. Since A is the union of the A_i, for any given subgroup A_n in the chain, we have that $A_n \subseteq A$. So every chain of subgroups of G that are subsets of S has an upper bound that is a subgroup of G contained in S. Thus, by Zorn's lemma, we conclude that there is a subgroup of G maximal with respect to the property of consisting of elements of S. □

We should not, however, form the impression that this maximal subgroup is all that large if the group itself does not satisfy the root-finite condition. In fact, as the next theorem tells us, in a torsion-free group that does not satisfy the root finite condition, any subgroup of consisting entirely of root-finite elements will have infinite index.

Theorem 3.6.2. *Let G be a torsion-free group that is not root-finite, and let H be a subgroup of G consisting of root-finite elements of G. Then $[G : H] = \infty$.*

Proof. Let S denote the set of all root-finite elements of G. Suppose that G and S do not coincide, so that there is some element g of G which is not in S (and hence also not in H). Consider the cosets $H, gH, g^2H, g^3H, \cdots$. If $[G : H] < \infty$, then it must be the case that $g^iH = g^jH$ for some $i \neq j$, and we may take $j > i$. Then $g^{j-i} \in H$. Since H is assumed to consist entirely of root-finite elements of G, this implies that g^{j-i} is a root-finite element of G. However, g has infinitely many roots by assumption, and all of these roots are also roots of g^{j-i}. This is a contradiction, and so $[G : H] = \infty$. □

A word of caution may be in order at this point. At first blush, this theorem might appear to tell us that a torsion-free group with a normal root-finite subgroup of finite index satisfies the root-finite condition. A closer look, however, reveals that this is not the case. Let G be a torsion-free group and H be a subgroup of finite index. If H consists entirely of root-finite elements of G, then H, viewed as a group on its own, necessarily satisfies the root-finite condition. However, if H itself satisfies the root-finite condition, it is not necessarily the case that H consists entirely of root-finite elements of G. For example, an element h of H

that has only finitely many roots in G that are also in H may have infinitely many roots in $G \backslash H$. So, if we have an exact sequence of groups,

$$1 \to H \to G \to Q \to 1$$

with H and G torsion-free, H satisfying the root-finite condition and Q finite, Theorem 3.6.2 does not permit us to conclude that G satisfies the root-finite condition.

There is, however, one of the subconditions in Theorem 3.3.2 where we can make such an inference. If, in the above exact sequence of groups, H and G are torsion-free, H is normal in G, Q is finite, and H satisfies condition (a) of Theorem 3.3.2 (the root chain condition), then we are able to conclude that G also satisfies the root chain condition, as the following theorem demonstrates.

Theorem 3.6.3. *If G is a torsion-free group and H is a normal subgroup of finite index satisfying condition (a) of Theorem 3.3.2, then G satisfies condition (a) of 3.3.2.*

Proof. Let G and H be as given in the theorem. First, note that no element of $G \backslash H$ can have a root that lies in H, for if $g \in G \backslash H$ had a root h in H, then $g = h^r$ for some positive integer r, and since H is a group and thus closed under multiplication, that would force g to lie in H.

Since H has no root chains that contain infinitely many distinct elements, and since no root chain originating in an element of $G \backslash H$ can contain any elements of the subgroup H, we have that if G contains a root chain with infinitely many distinct elements, all of those elements must lie in $G \backslash H$. Suppose that there exists a root chain of distinct elements:

$$g_1 \geq_r g_2 \geq_r \cdots \geq_r g_i \geq \cdots \tag{3.1}$$

with each of the g_i in $G \backslash H$. Since the index of H in G is assumed to be finite, we can refer to $[G : H]$ as q, and we have that for all elements $\bar{g}$ of G/H, $\bar{g}^q = \bar{e}$, the identity element of the quotient group G/H. Thus, for all g in $G \backslash H$, we have that g^q is in the subgroup H. Also, note that if $g_i \geq_r g_j$, then there is some positive integer r such that $g_j^r = g_i$ and thus $(g_j^r)^q = (g_j)^{rq} = (g_j^q)^r = g_i^q$, so that we have $g_i^q \geq_r g_j^q$. So from (3.1) we have the following:

$$g_1^q \geq_r g_2^q \geq_r \cdots \geq_r g_i^q \cdots$$

Since each of the g_i^q lies in H, this is an infinite root chain in H. Since by assumption there is no infinite root chain of distinct elements of H, it must be the case that not all elements of the chain are distinct. Suppose that g_i^q and g_j^q are two elements of the root chain with $j > i$ and that $g_i^q = g_j^q$. Since $g_j^q \geq_r g_i^q$, we have that $(g_i^q)^r = g_j^q$ for some positive integer $r \geq 2$. However, this gives us that $(g_i^q)^r = g_i^q$, and this cannot occur since it was assumed that G is a torsion-free group. In light of this contradiction, we conclude that the root chain of (3.1) cannot exist in G and thus G satisfies condition (a) of Theorem 3.3.2. $\square$

We might also consider the question of whether anything can be said about the case where the conditions of Theorem 3.6.3 are weakened so that the subgroup H is required to be torsion free, but not necessarily the group G. We can then demonstrate the following:

Theorem 3.6.4. *If a group G has a torsion-free subgroup H of finite index satisfying condition (a) of Theorem 3.3.2, then G satisfies condition (a) of 3.3.2 or there is an infinite root chain of distinct elements of $G \backslash H$ such that each element of the root chain taken to the power $[G : H]$ equals the identity element of G.*

Proof. Let G and H be as given in the theorem. As argued in the proof of Theorem 3.6.3, no element of $G \backslash H$ can have a root that lies in the subgroup H, and H satisfies the root chain condition of Theorem 3.3.2, so if G fails to satisfy the root chain condition, it must have an infinite chain of distinct elements lying entirely in $G \backslash H$. If we take each element of this root chain to the power $[G : H]$, again as in the proof of Theorem 3.6.3, we arrive at a root chain that lies entirely in the subgroup H. Since it is given that this subgroup satisfies the root chain condition, the elements of this root chain cannot all be distinct. Since H is torsion free, the root chain cannot wrap around itself to produce recurring elements. The only remaining possibility is that all of the elements in the root chain when taken to the power $[G : H]$ are equal to the identity element of G. $\square$

Chapter 4

Free Groups

4.1 Definition and Preliminary Remarks

The free group on a generating set A is a group with elements ("words") of the form $w = x_1^{m_1} \cdots x_n^{m_n}$ where the x_i ("letters") are elements of A and $m_i = \pm 1$, as well as an additional element, the word consisting of zero letters, or the empty word. The group operation is concatenation, with the empty word being the identity element. Adjacent letters that are inverses of each other cancel each other out. A *reduced word* is a word in which there are no consecutive letters that are inverses of each other. Every word is equal to a unique reduced word.

4.2 Free Groups and the Root-Finite Condition

Let $w = x_1^{m_1} \cdots x_n^{m_n}, m_i = \pm 1$ be a reduced word. We denote by $\ell(w)$ the length of w, that is, the number of letters in the word. We denote by $c(w)$ the cancellation length of w, that is, the number of letters of w that cancel when w is concatenated with itself. So, for example, if $w = x_1^{m_1} \cdots x_n^{m_n}$ with $m_i = \pm 1$ is a reduced word, and if $x_1 = x_n, m_1 = -m_n$, and $x_2 \neq x_{n-1}$, then $c(w) = 1$.

It is not difficult to see that for all nonempty reduced words w, $c(w) < \ell(w)/2$. If $\ell(w)$ is even, then $c(w) \geq \ell(w)/2$ would imply that for $i = 1, \cdots, \ell(w)/2$, it must be the case that $x_i = x_{\ell(w)-i+1}$ and $m_i = -m_{\ell(w)-i+1}$, which would cause the entire word to cancel and become the empty word. If, on the other hand, we suppose that $\ell(w)$ is odd, then $c(w) > \ell(w)/2$ means that at least $(\ell(w)+1)/2$ letters will cancel. Thus for $i = 1, \cdots, (\ell(w)+1)/2, x_i = x_{\ell(w)-i+1}$ and $m_i = -m_{\ell(w)-i+1}$. However, for $i = (\ell(w)+1)/2$, $i = \ell(w) - i + 1$, so the condition $m_i = -m_{\ell(w)-i+1}$ cannot hold. Thus for all nonempty words, $c(w)$ is strictly less than $\ell(w)/2$.

Suppose now that one has a word that consists of k iterations of the nonempty reduced word w. If we concatenate this word with a $(k+1)^{\text{th}}$ copy of w, we add $\ell(w)$ letters, and then must subtract the $2c(w)$ letters that cancel ($c(w)$ letters from the end of the k^{th} iteration of w and $c(w)$ letters from the beginning of the $(k+1)^{\text{th}}$ iteration) to arrive at a reduced word. That is, $\ell(w^{k+1}) = \ell(w^k)+\ell(w)-2c(w)$. Since $\ell(w) > 2c(w)$, $\ell(w^{k+1})$ must be strictly greater than $\ell(w^k)$. This observation enables us to prove the following theorem, which, though it seems intuitively obvious, requires some work.

Theorem 4.2.1. *Free groups are root finite.*

Proof. Suppose that there exists a word w and a string of words $w = w_0, w_1, w_2, \cdots$ such that w_{i+1} is a root of w_i. Assume that all words are written in reduced form. From the observation that $\ell(w^{k+1})$ is strictly greater than $\ell(w^k)$ for all positive integers k, we see that $\ell(w^m) > \ell(w^n)$ for all positive integers m, n with $m > n$, and so we conclude that $\ell(w_0) > \ell(w_1) > \ell(w_2) > \cdots$, which implies that the string of roots cannot be longer than $\ell(w_0)$. Thus there cannot be an infinite string of roots in a free group.

Suppose that there exists a reduced word w and reduced words w_i such that $w_i^{p_i} = w$ for p_i prime. Since each successive concatenation of w_i produces a word that is strictly longer than its predecessor, $\ell(w_i^{p_i}) = \ell(w) \geq p_i$. Since there are only finitely many primes less than or equal to $\ell(w)$, there can be only finitely many distinct primes p_i such that w has a p_i^{th} root.

It remains to show that a word w cannot have infinitely many n^{th} roots for any positive integer n. In fact, it will be shown that in free groups, n^{th} roots are unique.

First, it will be shown that if $v^n = w^n$ and $v \neq w$, then $c(v) \neq 0$ and $c(w) \neq 0$. Suppose that we have two reduced words, $v = x_1^{j_1} \cdots x_r^{j_r}$ with $j_i = \pm 1$ for $i = 1, \cdots, r$ and $w = y_1^{k_1} \cdots y_s^{k_s}$ with $k_i = \pm 1$ for $i = 1, \cdots s$, and that $v^n = w^n$. Now if $c(v) = 0$ and $c(w) = 0$, then no cancellation would occur in concatenating either v or w, then since the n iterations of v and w would have to match letter by letter, we would immediately have that $v = w$. So, cancellation must occur in concatenating one of the words, say v. Thus $x_1 = x_r$ and $j_1 = -j_r$. Since the first letter of v^n must match the first letter of w^n, $x_1 = y_1$ and $j_1 = k_1$. Since the last letter of v^n must match the last letter of w^n, we must have $x_r = y_s$ and $j_r = k_s$. But since $x_1 = x_r$, this gives us that $y_1 = y_s$, and since $j_1 = -j_r$, we have $k_1 = -k_s$. Thus cancellation occurs when w is concatenated with itself as well, so $c(v) \geq 1$ and $c(w) \geq 1$.

Now, denote by v_i the word $x_{i+1}^{j_{i+1}} \cdots x_{r-i}^{j_{r-i}}$, in other words, the word v with the first i letters and the last i letters removed, and similarly denote by w_i the word w with the first i

letters and the last i letters removed. Given that both $c(v)$ and $c(w)$ are at least 1, we can conclude that $v^n = x_1^{j_1} v_1^n x_r^{j_r}$ and $w^n = y_1^{k_1} w_1^n y_s^{k_s}$. Since $v^n = w^n$, $x_1^{j_1} = y_1^{k_1}$, and $x_r^{j_r} = y_s^{k_s}$, we can conclude that $v_1^n = w_1^n$.

The same argument can be made again, with v_1 and w_1 in place of v and w, and continuing in this fashion, we get shorter words whose n^{th} powers are equal. We can continue until one of the $\ell(v_i)$ or $\ell(w_i)$ is equal to 1 or 2. Since it was shown that cancellation must occur with both of the roots, we arrive at a contradiction. With a word length of 1, there is nothing to cancel, and with a word length of 2, cancellation produces the empty word. Thus we conclude that n^{th} roots are unique in free groups. $\square$

We will see that the fact that n^{th} roots are unique in free groups can be established independently using Lemma 5.2.1.

Chapter 5

Ordered Groups

5.1 Definition and Preliminary Remarks

An *ordered group* consists of a group G and a transitive relation $<$ such that for all elements g, h of G, $g < h, h < g$, or $g = h$, and only one of those three statements holds, and such that the order respects the group operation, that is, $g < h$ implies that for any $a \in G, ag < ah$ and $ga < ha$. Thus the integers as an additive group are an ordered group with the usual ordering, but the nonzero rational numbers with the operation of multiplication are not an ordered group with the usual ordering, because multiplying by a negative number changes the order relation.

In ordered groups, if there are elements a, b, c and d, satisfying the relations $a < b$ and $c < d$, it is easy to see that $ac < bd$. Multiplying the relation $a < b$ on the right by c and multiplying the relation $c < d$ on the left by b gives the relations $ac < bc$ and $bc < bd$, and the transitivity of the order relation gives $ac < bd$.

5.2 Ordered Groups and the Root-Finite Condition

When we consider the question of whether ordered groups are root finite, some familiar counterexamples leap to mind, such as the rational numbers with the operation of addition, which is an ordered group with the usual order relation, but which is not root finite. The rational numbers as an additive group do not possess two of the three conditions that we have seen are associated with root-finite groups, as there are infinite chains of roots, and a nonzero rational a/b has a p^{th} root a/bp for all primes p. However, the condition that no group element has infinitely many n^{th} roots is fulfilled by the rational numbers. This is, in fact, generally true for ordered groups, as the following theorem demonstrates.

Lemma 5.2.1. *If G is an ordered group with the order relation $<$ and $g \in G$, then g has at most one n^{th} root for all positive integers n.*

Proof. We show by induction that if g and h are group elements such that $g < h$, then $g^n < h^n$ for all positive integers n. For the case $n = 1$, there is nothing to prove. Assuming that the statement holds for all positive integers less than or equal to k, we show that it holds for $n = k + 1$. From the relations $g < h$ and $g^k < h^k$, we multiply the left sides and right sides together to get $g^{k+1} < h^{k+1}$. If g and h are distinct group elements, then either $g < h$ or $h < g$, so their n^{th} powers can never be equal, and thus n^{th} roots are unique. $\square$

Groups with the property that n^{th} roots are unique are referred to as *R groups*. A theorem of Vinogradov [21] establishes that every free group is an ordered group, so this lemma provides another way of showing that free groups satisfy the R group property.

It also follows from this lemma that ordered groups are torsion free. If an element g of an ordered group G has finite order n, then $g^n = e = e^n$, so that the identity element would have multiple n^{th} roots, but that is impossible according to the lemma.

5.3 Group Rings of Ordered Groups

Ordered groups play an important role in the study of group rings, particularly with respect to the zero-divisor problem (see [19]). If a group G has any elements of finite order, then the group ring KG will have zero divisors. In particular, if a group element g has order n then in the group ring KG, the product of $(1 - g)$ and $(1 + g + g^2 + \cdots + g^{n-1})$ is zero. If G is an ordered group, then KG cannot have zero divisors. Consider any two elements α and β of KG with Supp$\alpha = \{a_1, a_2, \cdots, a_m\}$ and Supp$\beta = \{b_1, b_2, \cdots, b_n\}$, and let the elements of the support be ordered from least to greatest. Consider the product $\alpha\beta = (j_1a_1 + j_2a_2 + \cdots + j_ma_m)(k_1b_1 + k_2b_2 + \cdots + k_nb_n)$, where the j_i and k_i are elements of the field K. This product will have a term with some nonzero coefficient and group element a_mb_n. Because the group operation respects the order relation and because a_m and b_n are the greatest elements in their respective supports, it is not possible for the product of any other element of Suppα with another element of Suppβ to be equal to a_mb_n. This shows that KG can have no zero divisors.

Chapter 6

Abelian Groups

6.1 Finitely Generated Abelian Groups and the Root-Finite Condition

We now turn our attention to abelian groups, and consider the question of whether there is a criterion or set of criteria that will allow us to determine whether a given abelian group is root finite. Indeed, the following straightforward result identifies a large class of abelian groups that are root finite.

Theorem 6.1.1. *Finitely generated abelian groups are root finite.*

Proof. The class T of root-finite groups has been shown to be D_0 closed, that is, every direct product of two root-finite groups is root finite. An inductive argument shows that every finite direct product of root-finite groups is root finite, as follows. Suppose that direct products of n root-finite groups are root finite. Consider the direct product $G_1 \times G_2 \times \cdots \times G_{n+1}$, where the G_i are root-finite groups. This can be viewed as the direct product of two groups $G_1 \times G_2 \times \cdots \times G_n$ and G_{n+1}. From the inductive assumption, we are assured that both of these groups are root finite, and thus their direct product is root finite by the D_0 closure of T.

Suppose, now, that G is a finitely generated abelian group. The fundamental theorem of finitely generated abelian groups enables us to write G as a finite direct sum of copies of $\mathbb{Z}$ and cyclic groups of prime power order. Since these groups are all root finite, G must be root finite as well. $\square$

6.2 Non-Finitely Generated Abelian Groups and the Root-Finite Condition

The situation with respect to the root-finite condition for abelian groups that are not finitely generated is considerably more complicated. We have only to consider the rational numbers under addition, a group which is not root finite, and the nonzero rational numbers under multiplication, a group which is root finite, to see that it can be a tricky matter to distinguish root-finite and non-root-finite groups from each other in the case of abelian groups that are not finitely generated. The approach that we will use to attempt to shed some light on this problem is to consider what can intuitively be thought of as the density of elements having n^{th} roots in a group. In the rational numbers under addition, all elements of the group have n^{th} roots for all n. That is the greatest possible density of roots. In the rational numbers under multiplication, group elements having n^{th} roots are quite sparse. This concept will be made more precise, and a key result concerning the density of roots will be presented.

We consider the set A_r of all group elements having r^{th} roots for a positive integer r, or equivalently the set of r^{th} powers of elements of G. This set has a nice property in abelian groups, as follows:

Lemma 6.2.1. *If G is an abelian group, then for all positive integers r, A_r is a subgroup of G.*

Proof. Suppose that g and h are elements of A_r for some positive integer r. Then there exist elements x and y of G such that $g = x^r$ and $h = y^r$. Then

$$gh^{-1} = x^r(y^r)^{-1} = x^r(y^{-1})^r = (xy^{-1})^r$$

So gh^{-1} is an element of A_r, from which we can conclude, by the one-element subgroup test, that A_r is a subgroup of G. □

Since the A_r are subgroups we can look at the number of cosets of A_r in any abelian group G. Let's first consider $\mathbb{Z}$ under addition. For any positive integer r, the elements of $\mathbb{Z}$ that have r^{th} roots are the multiples of r, and thus $[G : A_r] = r$ for all r. We might wish to consider the conjecture that this property constitutes a dividing line between abelian groups that were root finite and those that were not. It might turn out to be the case that if roots were denser in a group than they are in the integers, the group would not be root finite, and if roots were sparser in a group than they are in the integers, then the group is root finite. Indeed, the conjecture holds for torsion-free abelian groups in at least one direction, as the following theorem demonstrates.

Theorem 6.2.2. *If G is a torsion-free abelian group, and if $[G : A_r] < r$ for some positive integer r, then G is not root finite.*

Proof. Suppose $[G : A_r] = k < r$, and that h is some element of G that is not an element of A_r. Since $|G/A_r| = k$ the coset hA_r taken to the k^{th} power is the identity in G/A_r, which implies that h^k is an element of A_r. Of course, h^r is also an element of A_r, by construction of A_r.

If k and r are coprime, then $mk + nr = 1$ for some integers m and n. Since h^k and h^r are in A_r, so is $h^{mk+nr} = h^1 = h$. Since h was assumed not to be an element of A_r, we have a contradiction, and thus $G = A_r$. Since all elements of G have r^{th} roots, beginning with any nonidentity element of G, we can thus take an r^{th} root, and then an r^{th} root of the r^{th} root, and continue the process infinitely. Since G is assumed to be torsion free, all of these roots must be distinct. Since there is an infinite chain of distinct r^{th} roots, we conclude that G is not root finite. (The reason that we have to select a nonidentity element of G is that in a torsion-free group the identity is an element of A_r by virtue of the fact that $e^r = e$, in other words, the identity is its own r^{th} root. Thus the infinite chain of r^{th} roots constructed in this proof would simply be an infinite repetition of the identity element.)

If k and r are not coprime, let s be their greatest common divisor. Then for some integers m and n, $mk + nr = s$. Thus for the element h of G that is assumed not to be in A_r, since h^k and h^r are both in the group A_r, so is $h^{mk+nr} = h^s$. From this, we can see that $A_s \subseteq A_r$. The argument proceeds as follows: Suppose a is an element of A_s. Then $a = x^s$ for some $x \in G$. If x is an element of A_r, then x^s must also be an element of A_r, since A_r is a group. If x is not an element of A_r, then x^s is still an element of A_r, since it has been shown that the s^{th} power of an arbitrary element h of $G \backslash A_r$ must be in A_r. Thus $a \in A_r$, and thus $A_s \subseteq A_r$.

Since $s|r$, $r = js$ for some integer j. Suppose that b is an arbitrary element of A_r. Then $b = y^r$ for some $y \in G$. So $b = y^{js} = (y^j)^s$, and thus b is in A_s. Thus $A_r \subseteq A_s$, and since A_r and A_s are subsets of each other, they must coincide.

We now construct an infinite chain of roots beginning from any nonidentity element g_0 of A_r, and, recalling that s is a proper divisor of r, let $q = r/s$. we proceed to construct an infinite chain of q^{th} roots. Since g_0 has an r^{th} root by construction of A_r, it must have an q^{th} root (the r^{th} root taken to the power s), which we call g_1. Then g_1 has an s^{th} root (the r^{th} root of g_0), so g_1 is an element of A_s. But then g_1 is also in A_r, and since g_1 has an r^{th} root, it must also have a q^{th} root, which we call g_2. Continuing in this fashion, we can construct an infinite chain of roots, so G is not root finite.

$\square$

Though Theorem 6.2.2 has been presented in the context of providing a way of determining whether abelian groups that are not finitely generated are root finite, there is no assumption in the theorem that the group is not finitely generated. This leads to the following corollary.

Corollary 6.2.3. *If G is a torsion-free, finitely generated abelian group, then $[G : A_r] \geq r$ for all positive integers r.*

Proof. Let G be a torsion-free, finitely generated abelian group. If for some r, $[G : A_r] < r$, then we would conclude from the theorem that G is not root finite. However, since G is a finitely generated abelian group, we know that it must be root finite. Thus it must be the case that $[G : A_r] \geq r$ for all positive integers r. $\square$

Having examined the A_r subgroups in abelian groups, let us now consider a subset of A_r, the set I_r of group elements that do not just have r^{th} roots, but have infinitely many of them. In root-finite groups, of course, the I_r are all empty sets. For abelian groups having some nonempty I_r, we might wish to know whether the I_r were subgroups of the A_r, and, if so, what was the relationship between the two. The relationship, in fact, is quite striking. It turns out that I_r will either be empty or it will be all of A_r, as the following theorem shows.

Theorem 6.2.4. *If G is abelian and I_r is nonempty for some $r \geq 2$, then I_r is a subgroup of G, and, in fact, $I_r = A_r$.*

Proof. If g and h are any elements of I_r, fix an r^{th} root of g, say k, and consider $h_i(i = 1, 2, \cdots)$ such that $(h_i)^r = h$. Then for each i, $(h_i^{-1})^r = h^{-1}$. Then $(kh_i^{-1})^r = k^r(h_i^{-1})^r = gh^{-1}$. Since the elements kh_i^{-1} are all distinct, gh^{-1} has infinitely many r^{th} roots, and so is in I_r. Thus, by the one-step subgroup test, I_r is a subgroup of G.

Suppose $b \in A_r$ and $g \in I_r$. Fix one of the r^{th} roots of b, and call it h. Now let $(g_i)^r = g(i = 1, 2, \cdots)$, so that the g_i are infinitely many distinct r^{th} roots of g. Then for each i, $(hg_i)^r = h^r(g_i)^r = bg$. Sinch the hg_i are distinct elements of G, $bg \in I_r$. Since I_r is a group and $g \in I_r$, this implies that $(bg)g^{-1} = b$ is an element of I_r. Thus $A_r \subseteq I_r$. We have that $I_r \subseteq A_r$ by the definition of the two sets, so $A_r = I_r$. $\square$

6.3 Group Rings of Abelian and Abelian-by-Finite Groups

In the study of group rings, abelian groups provide the most basic instance of group rings that satisfy a polynomial identity. A group ring KG is said to satisfy a polynomial identity if there is some polynomial in n variables $f(x_1, x_2, \cdots, x_n)$ such that for all $a_1, a_2, \cdots, a_n \in KG$, $f(a_1, a_2, \cdots, a_n) = 0$. If the group G is abelian, then the group ring KG can be seen to satisfy the polynomial identity in two variables $f(x_1, x_2) = x_1x_2 - x_2x_1$. If G is abelian by finite, that is, if G has an abelian subgroup of finite index, then the group ring KG satisfies a polynomial identity. This was first shown by Kaplansky [11] for the case where K is a field of characteristic 0. Fifteen years later, Isaacs and Passman [10] showed that these were the only group rings over fields of characteristic 0 that satisfied a polynomial identity.

For the case of group rings KG satisfying a polynomial identity with K having characteristic $p > 0$, it is necessary to introduce the concept of a p-abelian group. A group G is said to be *p-abelian* if the commutator group G' is a finite p-group. (Recall that a *p-group*, for some prime number p, is a group in which the order of every element is a power of p.) Passman [18] proved that a group ring KG for K a field of characteristic $p > 0$ satisfies a polynomial identity if and only if the group G has a p-abelian subgroup of finite index.

Chapter 7

Finite Conjugate Groups

7.1 Definitions and Preliminary Remarks

In this chapter, we consider a more general class of groups than abelian groups, namely, groups having the property that all elements have a finite conjugacy class. We examine the conditions under which such groups may possess the root-finite condition, and look at some results concerning root-finite finite conjugate groups. This is a class of groups that is of great importance in the theory of group rings, and we review some of the major results on group rings of finite conjugate groups. We also look at groups that have a finite conjugate group of finite index, and discuss such groups in the context of the topics of root finiteness and the properties of the group rings of such groups.

For any group G, we define the delta subgroup $\Delta(G)$ to be the set of all elements g of G such that the conjugacy class of g is finite. To see that this is a subgroup of G, suppose that g and h are elements of $\Delta(G)$. If it can be shown that gh^{-1} has only finitely many conjugates, then $\Delta(G)$ will be seen to be a subgroup of G by the one-element subgroup test. Let k be an arbitrary element of G, so that $k^{-1}gh^{-1}k$ is an arbitrary conjugate of gh^{-1}. Then, since $k^{-1}gh^{-1}k = (k^{-1}gk)(k^{-1}h^{-1}k)$, we see that all conjugates of gh^{-1} are the product of a conjugate of g and a conjugate of h^{-1}. Since h is an element of $\Delta(G)$, it has only finitely many conjugates, and since every conjugate of h^{-1} is an inverse of a conjugate of h, there can be only finitely many conjugates of h^{-1} as well. So, since g and h^{-1} both have finitely many conjugates, there can only be finitely many products formed from a conjugate of g and a conjugate of h^{-1}. Thus gh^{-1} can have only finitely many conjugates and so we are justified in referring to $\Delta(G)$ as a subgroup.

An equivalent formulation is as follows: We denote by $C_G(g)$ the centralizer of g in G, that is, the set of all elements of G that commute with g. The centralizer of an element is a subgroup of G. The delta subgroup is defined to be the set of all elements g of G such that the centralizer $C_G(g)$ has finite index in G. These two definitions are seen to be equivalent when we fix an element g of G and consider the mapping $h \mapsto h^{-1}gh$. There is a one-to-one correspondence between the conjugates of h and the cosets of $C_G(g)$. To see this, suppose that h and k are elements of G such that $h^{-1}gh = k^{-1}gk$. By multiplying this equation on the left by h and on the right by k^{-1}, we see that this equation is equivalent to $g(hk^{-1}) = (hk^{-1})g$, in other words, that hk^{-1} is an element of $C_G(g)$. The fact that hk^{-1} is an element of $C_G(g)$ is, in turn, equivalent to h and k lying in the same coset of $C_G(g)$, thus establishing the one-to-one correspondence, and, hence, the equivalence of the two definitions of the delta subgroup.

A group G is said to be a *finite conjugate group* if $\Delta(G) = G$.

7.2 Finite Conjugate Groups and the Root-Finite Condition

We now return to the topic of root-finite groups and consider whether the delta subgroup of a root-finite group has any special properties. We begin with the following theorem, which establishes a relationship between the set of elements of finite order and the delta subgroup of a root-finite group.

Theorem 7.2.1. *Let G be a root-finite group, and let T be the set of elements of finite order in G. Then T is finite, and $T \subseteq \Delta(G)$.*

Proof. Since G is assumed to be root finite, in particular, the identity element of the group can have only finitely many roots, which is the same as saying that the group has only finitely many elements of finite order. So T must be finite.

Suppose that there is an element t of T that has infinitely many conjugates. For each conjugate element $g^{-1}tg$ and each positive integer n, consider the element $(g^{-1}tg)^n = g^{-1}t^n g$. Since t is assumed to be an element of T, there is some power of t that equals the identity. But if $t^n = e$, it follows that $g^{-1}t^n g = e$ as well, so that each of the infinitely many conjugates of t has finite order. However, this is not possible, since G is assumed to be root finite, so the identity can only have finitely many roots. Therefore t must be in $\Delta(G)$, and thus $T \subseteq \Delta(G)$. □

This theorem, along with a classical result concerning elements of finite order in finite conjugate groups, enables us to draw the following conclusion about the set of torsion elements in root-finite groups.

Corollary 7.2.2. *The set of elements of finite order in a root-finite group G is a normal subgroup of G.*

Proof. A theorem of B. H. Neumann [16] states that the set of elements of finite order in a finite conjugate group constitutes a subgroup. Given a root-finite group G and the set of its torsion elements T, we saw in Theorem 7.2.1 that $T \subseteq \Delta G$. Since $\Delta(\Delta(G)) = \Delta(G)$ ($\Delta(G) \subseteq G$, so an element of $\Delta(G)$ cannot have infinitely many conjugates in $\Delta(G)$ if it has only finitely many conjugates in G), $\Delta(G)$ is a finite conjugate group, and thus applying Neumann's theorem, we have that the set of elements of finite order in $\Delta(G)$ is a subgroup of $\Delta(G)$ and hence of G. We must still show that the set of elements of finite order in $\Delta(G)$, which we will refer to as T_Δ, is equal to T, the set of elements of finite order in G. Let t be an element of T. Since $T \subseteq \Delta(G)$, t is also an element of $\Delta(G)$, and since G and $\Delta(G)$ have the same identity, t is of finite order in $\Delta(G)$ and thus is in T_Δ. If t is in T_Δ, then it is of finite order in $\Delta(G)$ and thus also in G, and so $t \in T$. To see that T is a normal subgroup of G, suppose that t is an element of T with order n. Then any conjugate $g^{-1}tg$ of t will also have finite order, since $(g^{-1}tg)^n = g^{-1}t^n g = g^{-1}eg = e$. Thus T is a normal subgroup of G. $\square$

In general, it is possible for an element of a group G that is not in $\Delta(G)$ to have finite order. For example, in the group of 2×2 upper-triangular matrices over $\mathbb{Z}$ with determinants equal to 1 or -1, the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

has order 2, and yet it has an infinite conjugacy class, as for any $a \in \mathbb{Z}$

$$\begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2a \\ 0 & -1 \end{pmatrix}$$

This is also illustrative of a phenomenon that cannot occur in a root-finite group, taking a power of an element that is not in the delta subgroup and ending up inside the delta subgroup, as the following theorem demonstrates.

Theorem 7.2.3. *Let G be a root-finite group. Then $\Delta(G)$ and $(\Delta(G))^C = G\backslash\Delta(G)$ are closed under the operations of taking powers and taking roots.*

Proof. $\Delta(G)$ is a group, so it is closed under taking powers, regardless of whether G is root finite.

Suppose g is some element of G that is not in $\Delta(G)$ and g^n is some power of this element that is in $\Delta(G)$. Then since g is not in $\Delta(G)$, there are infinitely many distinct elements of G of the form $h^{-1}gh$, each of which has an n^{th} power $(h^{-1}gh)^n = h^{-1}g^nh$. Since g^n is assumed to be in $\Delta(G)$, it has only finitely many conjugates, at least one of these finitely many conjugates of g^n must be the n^{th} power of infinitely many of the conjugates of g. So there is some element of G with infinitely many distinct n^{th} roots, contradicting the assumption that G is root finite. It can then be concluded that $(\Delta(G))^C$ is closed under taking powers.

It now follows that $\Delta(G)$ and $(\Delta(G))^C$ are closed under taking roots. The assertion that $\Delta(G)$ is closed under taking powers is equivalent to the assertion that $(\Delta(G))^C$ is closed under taking roots, and likewise $(\Delta(G))^C$ being closed under taking powers is equivalent to $\Delta(G)$ being closed under taking roots. $\square$

The following corollary makes an observation concerning the delta subgroup of torsion-free, root-finite groups. As previously noted, such groups must contain elements with no n^{th} roots for any $n \geq 2$, as otherwise the group would contain infinite root chains. We will denote the set of rootless elements in a group by C, that is,

$$C = \bigcap_{n=2}^{\infty} A_n^C$$

We have the following result concerning the relationship of C and the delta subgroup.

Corollary 7.2.4. *If G is a torsion-free, root-finite group, then the intersection of $\Delta(G)$ and the set C of rootless elements of G is nonempty.*

Proof. Take G to be a torsion-free, root-finite group. Then G contains at least one rootless element, and since every root chain can be extended until it eventually terminates in a rootless element, every element of G is either rootless or the power of a rootless element, that is,

$$G = \bigcup_{c \in C} \langle c \rangle$$

where $\langle c \rangle$ is the cyclic subgroup of G generated by c. Let g be an element of $\Delta(G)$. If g is rootless, then we are done. If g is not rootless, it is a power of some rootless element c, and according to Theorem 7.2.3, c must be in $\Delta(G)$, so the intersection of C and $\Delta(G)$ is nonempty. $\square$

We now consider what conditions might lead to the circumstance that an element g of G lying outside of the delta subgroup has a power in the delta subgroup. The following theorem identifies a large class of groups with this property.

Theorem 7.2.5. *Suppose G is a group that is not a finite conjugate group, and that the index of the subgroup $\Delta(G)$ in G is finite. Then $(\Delta(G))^C$ is not closed under taking powers, and, in fact, every $g \in (\Delta(G))^C$ has a power in $\Delta(G)$.*

Proof. Suppose that g is an element of G that is not in the delta subgroup of G. If g has finite order, then $g^n = e$ for some positive integer n, and since the identity element is in the delta subgroup, we are done. If g has infinite order, consider the cosets of the delta subgroup $g\Delta(G), g^2\Delta(G), g^3\Delta(G), \cdots$. Since $[G : \Delta(G)]$ is assumed to be finite, these cosets are not all distinct, and so for some positive integers k and j with $k > j$, it must be the case that $g^k\Delta(G) = g^j\Delta(G)$, which implies that g^{k-j} is an element of the delta subgroup. Thus every element of G that is not in the delta subgroup has some power that lies in the delta subgroup. $\square$

The *division hull* of a subset S of a group G is defined to be the set of all elements g of G such that g^n is an element of S for some positive integer n. It should be remarked that the concept of a division hull occurs often in the literature, but different terminology is used to refer to the concept. Some authors (for example, [22]) refer to the division hull of a set as the *isolator* of the set. Another term that is used for this set is the *root set* (for example, [19]). Thus Theorem 7.2.5 tells us that if G is a group such that $1 < [G : \Delta(G)] < \infty$, the division hull of $\Delta(G)$ equals G itself.

We are now able to identify another class of groups that are not root finite, as demonstrated in the following corollary.

Corollary 7.2.6. *If G is a group such that $1 < [G : \Delta(G)] < \infty$, then G is not root finite.*

Proof. Since $1 < [G : \Delta(G)]$, the delta subgroup does not encompass all of G, so there is some element g of G that lies outside of $\Delta(G)$. According to Theorem 7.2.5, there is some power g^n of g that lies in $\Delta(G)$. This tells us that $\Delta(G)$ is not closed under the operation of taking roots. Since the delta subgroup of a root-finite group is closed under the operation of taking roots (Theorem 7.2.3), it therefore follows that a group G satisfying the conditions of this corollary is not root finite. $\square$

This theorem tells us that if G is a root-finite group, then the quotient group $G/\Delta(G)$ cannot be finite, unless it is trivial. We can actually use a similar argument to say a bit more, as demonstrated in the following theorem.

Theorem 7.2.7. *If G is a root-finite group and $G \neq \Delta(G)$, then $G/\Delta(G)$ contains no elements of finite order.*

Proof. Let g be an element of the root-finite group G which is not an element of the delta subgroup of G. Then $\bar{g}$ is not the identity in $G/\Delta(G)$. Suppose that $\bar{g}$ has finite order in $G/\Delta(G)$, so that for some integer n, $\bar{g}^n = e$, the identity element of $G/\Delta(G)$. But that implies that g^n is in $\Delta(G)$. This cannot occur in the root-finite group G, because according to Theorem 7.2.3, the delta subgroup is closed under the operation of taking roots. So $G/\Delta(G)$ contains no elements of finite order. □

We now consider finite conjugate groups that are not root finite, particularly those having the property that there is some element of the group with infinitely many r^{th} roots for some positive integer r. We first consider the case $r = 2$, which is the subject of the following lemma, which deals with the commutator elements of a group, that is, elements of the form $g^{-1}h^{-1}gh$.

Lemma 7.2.8. *If G is a finite conjugate group and there exists an element g of G that has infinitely many square roots, then there is a commutator element of G with infinitely many square roots.*

Proof. Let G be a finite conjugate group and let g be an element of G such that for infinitely many elements x of G, $x^2 = g$. Fix one of these elements, a, with $a^2 = g$. Then for every element b_i such that $b_i^2 = a^2$, we can multiply on the left by a^{-1} and on the right by b_i^{-1} to obtain $a^{-1}b_i = ab_i^{-1}$. From this we get that $(a^{-1}b_i)^2 = a^{-1}b_iab_i^{-1}$, so $(a^{-1}b_i)^2$ is a commutator element of G. The expression $a^{-1}b_iab_i^{-1}$ is also the product of a^{-1} and a conjugate of a, so since G is a finite conjugate group, there are only finitely many elements of G that can be obtained as b_i varies among all the infinitely many elements of G such that $b_i^2 = a$. Thus there is at least one commutator element that is equal to $a^{-1}b_iab_i^{-1}$ for infinitely many of the b_i, and this element has infinitely many distinct square roots $a^{-1}b_i$. □

The next theorem looks further at finite conjugate groups that fail to have the root-finite condition, in particular, groups that have an element with infinitely many r^{th} roots for some positive integer r.

Theorem 7.2.9. *If G is a finite conjugate group such that there is some element g of G having infinitely many r^{th} roots for some positive integer r, then there are infinitely many elements of G having finite order.*

Proof. Let G be a finite conjugate group and let T be the set of elements of finite order in G. It is known that the elements of finite order in a finite conjugate group form a normal subgroup and that G/T is torsion-free abelian (this was proven in [16]; see also Lemma 4.1.6 of [19]).

Suppose g is an element of G with infinitely many r^{th} roots. If g itself is an element of T, we are done, since if g has order n, then for each of the infinitely many elements h of G such that $h^r = g$, we have that $h^{nr} = e$, so they have finite order.

If, on the other hand, g has infinite order, the argument proceeds as follows. Let h_1 and h_2 be two r^{th} roots of g, and denote by $\bar{h_1}$ and $\bar{h_2}$ the respective cosets of these elements in G/T. Then $\bar{h_1}^r \bar{h_2}^{-r} = (\bar{h_1}\bar{h_2}^{-1})^r$ since G/T is abelian. But since $h_1^r h_2^{-r} = gg^{-1} = e$, we have that $(\bar{h_1}\bar{h_2}^{-1})^r = \bar{e}$, and since G/T is torsion free, this implies that $\bar{h_1}\bar{h_2}^{-1} = \bar{e}$, so $\bar{h_1} = \bar{h_2}$, in other words, all the r^{th} roots of g are in the same coset of T.

Now we fix an element h of G such that $h^r = g$, and consider the elements hj_i^{-1}, where the j_i are infinitely many distinct elements of G such that $j_i^r = g$. Since all of the r^{th} roots of g lie in the same coset of T, this implies that for each j_i, $\bar{h}\bar{j_i}^{-1} = \bar{e}$ in the group G/T. Therefore, hj_i^{-1} is in T for all i. Since T is the set of elements of G of finite order and the elements hj_i^{-1} are all distinct, this gives us what we set out to prove. $\square$

An immediate consequence of this theorem is that in any torsion-free, finite conjugate group, there can be no elements with infinitely many r^{th} roots for any integer r. It should also be emphasized that this theorem holds for a fixed integer r. The rational numbers as an additive group, for example, are a torsion-free, finate conjugate group in which all of the nonidentity elements have infinitely many roots, but not infinitely many r^{th} roots for any particular r.

We also can deduce the following corollary regarding finitely generated finite conjugate groups.

Corollary 7.2.10. *No element of a finitely generated finite conjugate group has infinitely many r^{th} roots for any positive integer r.*

Proof. Suppose that G is a finitely generated finite conjugate group and g is an element of G with infinitely many r^{th} roots for some positive integer r. Then by Theorem 7.2.9, there are infinitely many elements of G with finite order. However, this contradicts a theorem of [16] that finitely generated finite conjugate groups have a finite torsion subgroup. $\square$

A more general theorem regarding finitely generated finite conjugate groups will be presented later in this chapter. First, though, we note the following property of the delta subgroup which may on occasion be useful.

Lemma 7.2.11. *In a group G, if $g \in \Delta(G)$ and h is an element of G with infinite order, then g commutes with some positive power of h.*

Proof. We take conjugates of g by powers of h, that is, elements of the form $h^{-n}gh^n$ for $n = 1, 2, \cdots$. Since $g \in \Delta(G)$, these elements are not all distinct, so we can choose positive integers j and k with $j < k$ such that $h^{-j}gh^j = h^{-k}gh^k$. Then, multiplying on the left by h^j and on the right by h^{-j} we arrive at $g = h^j h^{-k} g h^k h^{-j} = h^{-(k-j)} g h^{k-j}$, so we see that g commutes with h^{k-j}. $\square$

This enables us to make the following observation about torsion-free, finite conjugate groups.

Corollary 7.2.12. *In torsion-free, finite conjugate groups, every element commutes with a power of every other element.*

Proof. Let G be a torsion-free, finite conjugate group, and let g and h be arbitrary elements of G. Since G is a finite conjugate group, G coincides with its delta subgroup so $g \in \Delta(G)$, and since G is torsion-free, h has infinite order. Applying Lemma 7.2.11 to g and h, we conclude that g commutes with some power of h, and since g and h are arbitrary, the assertion of the corollary is proven. $\square$

Another immediate consequence of Lemma 7.2.11 is the following.

Corollary 7.2.13. *In a finite conjugate group G, with T the torsion subgroup of G,*

$$T^C \subseteq \bigcap_{g \in G} \mathrm{dh}(C_G(g))$$

where dh denotes the division hull.

Proof. Since G is a finite conjugate group, G coincides with its delta subgroup. So if $x \in T^C$, that is, if x has infinite order, then Lemma 7.2.11 tells us that some power of x commutes with all g in G. So for all g, x is in the division hull of $C_G(g)$. $\square$

The following theorem identifies another large class of root-finite groups. It turns out to be a generalization of both Theorem 6.1.1 and Corollary 7.2.10.

Theorem 7.2.14. *Finitely generated finite conjugate groups are root finite.*

Proof. A theorem of Nishigori [17] states that finitely generated finite conjugate groups are direct products of a finite number of copies of $\mathbb{Z}$ and a finite group. Since the class of root-finite groups is D_0-closed, any finite direct product of root-finite groups is root finite. Since $\mathbb{Z}$ is root finite, and any finite group is, of course, root finite, we can conclude that finitely generated finite conjugate groups are root finite. $\square$

7.3 Group Rings of Finite Conjugate Groups

Turning our attention now to group rings, we make an observation concerning the relationship between root-finite groups and group rings that satisfy a polynomial identity.

Theorem 7.3.1. *If G is a root-finite group, K is a field, and the group ring KG satisfies a polynomial identity, then G is a finite conjugate group.*

Proof. According to Theorem 5.2.14 of [19], if a group ring KG satisfies a polynomial identity of degree n, then $[G : \Delta(G)] \leq n/2$. So since the index of the delta subgroup of G is finite, Corollary 7.2.6 tells us that G can only be root finite if the index is equal to 1. Since G is assumed to be root finite, we conclude that G and $\Delta(G)$ coincide, that is, G is a finite conjugate group. $\square$

Since we know that group rings of abelian-by-finite groups satisfy polynomial identities, this theorem tells us that an abelian-by-finite group that is not a finite conjugate group is not root finite.

Chapter 8

Groups With Finitely Many Conjugacy Classes

8.1 Preliminary Remarks

Having looked at groups with the finiteness property that every element has a finite number of conjugate classes, we can also look at the other extreme, groups with the finiteness property that there are only a finite number of conjugacy classes in the group.

Recall that we have introduced the notation A_r to indicate the set of elements of G that have r^{th} roots. The following simple theorem will be useful throughout the discussion of this topic.

Theorem 8.1.1. *In any group G, for $n = 2, 3, \cdots, A_n$ is the union of conjugacy classes of G.*

Proof. Let g be an element of A_n for some integer n greater than or equal to 2. Then for some x in G, $x^n = g$. Now suppose that h is an element of G that is conjugate to g, so that for some element k of G, $k^{-1}gk = h$. But then we have $h = k^{-1}gk = k^{-1}x^nk = (k^{-1}xk)^n$, so that h has an n^{th} root, namely $k^{-1}xk$, and thus h is in A_n. Since h was an arbitrary element of the conjugacy class of g, we can conclude that the conjugacy class of g is a subset of A_n, and thus A_n is seen to be the union of conjugacy classes of G. □

8.2 Groups With Finitely Many Conjugacy Classes and the Root-Finite Condition

In exploring whether groups with finitely many conjugacy classes may possess the root-finite condition, we begin by considering the most extreme case, groups that have only two conjugacy classes, the identity belonging to one conjugacy class and everything else belonging to the other conjugacy class. For such groups, we can make the following observation.

Theorem 8.2.1. *If G is a group in which all the nonidentity elements are conjugate, then for all but at most one prime p, $A_p = G$.*

Proof. For a group G in which all the nonidentity elements are conjugate, and for any prime p, Theorem 8.1.1 tells us that there are technically four conceivable possibilities for what A_p can be: the empty set, the identity, everything that is not the identity, or the entire group. The empty set is not a possibility, however, since every group element has a p^{th} power and thus A_p can never be empty. It is also clear that A_p contains the identity element, since $e^p = e$. So the only possibilities that remain are $A_p = \{e\}$ or $A_p = G$. If $A_p = \{e\}$, then every nonidentity element of G must have order p. That situation can arise for at most one prime p, and in all other cases A_p must be the entire group. □

It was previously noted that homomorphic images of root-finite groups need not be root finite. We see in the next corollary just how far the class of root-finite groups is from being closed under the operation of homomorphic images.

Corollary 8.2.2. *If G is a torsion-free group, then there is a group H and an injective homomorphism $\iota : G \to H$ such that every nonidentity element of H has an n^{th} root for $n = 2, 3, \cdots$.*

Proof. If G is a torsion-free group, then it is known that there is an injective homomorphism from G to a group H in which all of the nonidentity elements are conjugates, and thus satisfying the conditions of Theorem 8.2.1 [12]. We can eliminate the case where in the group H, there is a prime p such that $A_p = \{e\}$, since in an injective homomorphism an element g of G having infinite order cannot be mapped to an element h of H having order p. If that were the case, then g^p would map to the identity in H, but injective homomorphisms have trivial kernels. So it must be the case that $A_p = H$ for all primes p, and thus every nonidentity element of H has a p^{th} root for all primes p. If n is not a prime, then we obtain the n^{th} root of a nonidentity element of H by writing n in its prime factorization form and successively taking prime roots, which will always exist given Theorem 8.2.1. □

We now consider the more general case of groups having a finite number of conjugacy classes. Although the results are somewhat weaker than the case for groups with exactly two conjugacy classes, we will see that if one is looking for root-finite groups, this is not the class of groups in which to find them.

Theorem 8.2.3. *If a group G has only finitely many conjugacy classes and at least one element of infinite order, then there exists a conjugacy class of G and an infinite set of primes P such that every element of that conjugacy class has a p^{th} root for every $p \in P$.*

Proof. Let $Q = \{p_1, p_2, \cdots, p_i, \cdots\}$ be an infinite set of primes and let g be an element of G with infinite order. Then consider the elements $g^{p_1}, g^{p_2}, \cdots, g^{p_i}, \cdots$ in G. Since G has only finitely many conjugacy classes, at least one of the conjugacy classes has to contain infinitely many of the g^{p_i}. We choose one such conjugacy class, and take as the set P the subset of Q consisting of those primes p_i such that g^{p_i} are in the conjugacy class. So the conjugacy class contains at least one element that has a p_i^{th} root for infinitely many primes. But by Theorem 8.1.1, this means that the entire conjugacy class must be a subset of A_{p_i} for all of those primes. Thus every member of the conjugacy class has a p_i^{th} root for each p_i in P. $\square$

We continue our investigation of groups with finitely many conjugacy classes and at least one element of infinite order. As the next theorem demonstrates, these groups, which have already been shown not to satisfy one of the criteria for a group to be root finite, fail to satsify a second criterion as well, that of having an infinite root chain. To demonstrate that this criterion fails, we can actually relax the condition that the group have an element of infinite order, provided that it has sufficiently large order, as seen in the following theorem.

Theorem 8.2.4. *If a group G has only finitely many conjugacy classes and there is an element of G with order greater than 2^n, where n is the number of conjugacy classes, then G contains an infinite root chain.*

Proof. Let G be a group that has only finitely many conjugacy classes, and let n represent the number of conjugacy classes. We suppose that there exists an element g of G with order greater than 2^n. We consider the elements $g_0, g_1, \cdots, g_n$, where g_i is defined to be g^{2^i}. Since the order of g is assumed to be greater than 2^n these $n+1$ elements are distinct. We also notice that if j and k are integers and $0 \le j < k \le n$, $g_k = g_j^{2^{k-j}}$. Since there are $n+1$ distinct group elements g_i and n conjugacy classes, by the pigeonhole principle, there is a conjugacy class having at least two of the g_i. Now, it has previously been shown (Theorem 3.5.2) that whenever an element of a group is conjugate to one of its r^{th} powers, for $r \ge 2$, then the group contains an infinite root chain. Thus G contains an infinite root chain. $\square$

Chapter 9

Polycyclic Groups and Polycyclic-by-Finite Groups

9.1 Definitions and Preliminary Remarks

In this chapter, we consider polycyclic and polycyclic-by-finite groups. First, this class of groups is defined, and certain important and well-known properties of these groups are mentioned. The question that is foremost in this chapter is under what conditions are these groups root finite. While a complete resolution of this question remains elusive, several results are presented that provide some insight into the problem. Finally, material is presented regarding group rings of polycyclic and polycyclic-by-finite groups.

For any class of groups X, we can define a class of groups that are referred to as *poly-X* and have the following property: A group G is poly-X if there is a finite sequence of subgroups

$$G_0 = \langle e \rangle \subseteq G_1 \subseteq G_2 \subseteq \cdots \subseteq G_n = G$$

such that each G_i is normal in G_{i+1} (though not necessarily in G) for $i = 0, 1, \cdots, n-1$ and each of the quotient groups G_{i+1}/G_i is in the class X. So if each of the G_{i+1}/G_i is cyclic, G is *polycyclic*, and if G has a normal polycyclic subgroup of finite index, then G is referred to as *polycyclic-by-finite*. There is also a class of groups where each of the G_{i+1}/G is infinite cyclic, and these groups are referred to as *poly-(infinite cyclic)* or *poly-$\mathbb{Z}$* groups. We could also discuss groups where each of the G_{i+1}/G_i is either finite or cyclic and call these groups poly-(finite or cyclic). However, it turns out that we need not resort to such clumsy nomenclature as it has been proven that poly-(finite or cyclic) groups are, in fact, poly-$\mathbb{Z}$-by-finite (see, for example, [22]), and are generally referred to as polycyclic-by-finite groups.

Consider a polycylic-by-finite group G and the sequence of subsets:

$$G_0 = \langle e \rangle \subseteq G_1 \subseteq G_2 \subseteq \cdots \subseteq G_n \subseteq G_{n+1} = G$$

with G_i normal in G_{i+1} for $i = 0, 1, \cdots n$, G_{i+1}/G_i infinite cyclic for $i = 0, 1, \cdots n-1$, and G_{n+1}/G_n finite. In general, there may be more than one such sequence of subsets from $\langle e \rangle$ to G, but over all such sequences the number of infinite cyclic quotients is invariable and is referred to as the *Hirsch number* of the group.

Some of the key properties of polycyclic-by-finite groups are as follows:

- Polycyclic-by-finite groups satisfy the finiteness condition known as the maximal condition on subgroups, that is, the property that any strictly ascending chain of subgroups is finite (see, for example, [22]).
- Polycyclic-by-finite groups are residually finite [9].
- Every soluble subgroup of $GL(n, \mathbb{Z})$ is polycyclic [13].
- Any polycyclic-by-finite groups is isomorphic to a subgroup of $GL(n, \mathbb{Z})$ for some n [1].

9.2 Polycyclic, Polycyclic-by-Finite Groups and the Root-Finite Condition

We now turn our attention to the question of when polycyclic groups (or poly-$\mathbb{Z}$ or polycyclic-by-finite groups) are root finite or fail to meet the root-finite condition. We recall that there are three subconditions which must be fulfilled in order for the root-finite condition to apply to any given group. The following theorems demonstrate the relationships between these subconditions and polycyclic groups.

Theorem 9.2.1. *Polycyclic-by-finite groups satisfy condition (a) of Theorem 3.3.2, that is, in polycyclic-by-finite groups, there are no infinite root chains.*

Proof. Suppose that G is a polycyclic-by-finite group and that G fails to satisfy condition (a) of Theorem 3.3.2. Then for some element g of G, there is an infinite chain of elements $(g = x_0, x_1, x_2, \cdots)$ such that for $i = 1, 2, \cdots$, there is some positive integer n_i such that $x_{i-1} = x_i^{n_i}$. Then looking at the cyclic subgroups generated by each successive element in the chain, we have an infinite chain of subgroups

$$\langle x_0 \rangle \subset \langle x_1 \rangle \subset \langle x_2 \rangle \subset \cdots$$

in which each of the inclusions is strict. However, this is not possible if G is polycylic-by-finite, since polycyclic-by-finite groups have the max condition on subgroups. Thus we conclude that if G is polycyclic-by-finite, it satisfies condition (a) of Theorem 3.3.2. $\square$

We now consider subcondition (b) of Theorem 3.3.2 and show that it holds in the case of poly-(infinite cyclic) or poly-$\mathbb{Z}$ groups.

Theorem 9.2.2. *If G is a poly-$\mathbb{Z}$ group, then it satisfies subcondition (b) of Theorem 3.3.2, that is, for all g in G, there are only finitely many distinct primes p_i such that there exists an element x of G such that g is a power of x and $[\langle x\rangle : \langle g\rangle] = p_i$.*

Proof. Now we shall assume that G is a poly-$\mathbb{Z}$ and that G fails to satisfy condition (b) of Theorem 3.3.2. Then there is some element g of G and some infinite root set R consisting of elements of G, $R = \{x_1, x_2, \cdots\}$, such that for each $i = 1, 2, \cdots$, there exists some prime integer p_i such that $g = x_i^{p_i}$ and such that all of the x_i and all of the p_i are distinct. Since G is poly-$\mathbb{Z}$, there is a finite subnormal chain of subgroups of G,

$$\langle e\rangle = H_0 \subset H_1 \subset H_2 \subset \cdots \subset H_n = G$$

such that H_i/H_{i-1} is infinite cyclic for $i = 1, 2, \cdots, n$.

There must be some subgroup H_i in which the element g first appears, that is $g \in H_i$ and $g \notin H_k$ for all $k < i$. We can then infer that no element of the root set R is in any of the H_k for $k < i$, since the H_k are subgroups and if some element of R were in H_k, all of its powers, including g, would be in H_k.

We can also infer that there cannot be an element of R that first appears in H_k for $k > i$. To see this, suppose that there is an element x_r of R that is an element of $H_k \backslash H_{k-1}$ for some $k > i$. Since $k > i$ and the subgroup H_i is the subgroup in which g first appears, we have that $g \in H_{k-1}$. Now, for x_r in $H_r \backslash H_{r-1}$, some prime power p_r of x_r is equal to g by the construction of the root set R, and since g is in H_{r-1}, $\bar{x_r}$ has order p_r in the quotient group H_r/H_{r-1}. However, H_r/H_{r-1} is assumed to be infinite cyclic, so there are no elements of finite order. However, each of the p_j is assumed to be distinct, and since H_r/H_{r-1} is infinite cyclic, it cannot have elements of prime order. Thus we can conclude that there cannot be an element of R making its first appearance in the subgroup chain in some subgroup H_k, where $k > i$.

Thus we have seen that for all the elements of R make their first appearance in the subgroup chain in the same subgroup H_i in which g first appears. Since H_i/H_{i-1} is cyclic, we can denote its generator by $\bar{a}$ and for some positive integer t, $\bar{g} = \bar{a}^t$. Furthermore, for each of the elements x_j of R in H_i, there is some positive integer m_j such that $\bar{x}_j = \bar{a}^{m_j}$. Then for each x_j in R, $\bar{g} = \bar{a}^{m_j p_j}$ since $g = x_j^{p_j}$. This implies that t has infinitely many prime factors p_j. As this cannot be the case, we conclude that poly-$\mathbb{Z}$ groups satisfy condition (b) of Theorem 3.3.2. $\square$

We can extend this result to torsion-free polycyclic and polycyclic-by-finite groups.

Corollary 9.2.3. *Torsion-free polycyclic and polycyclic-by-finite groups satisfy condition (b) of Theorem 3.3.2.*

Proof. Suppose that the group G is polycyclic or polycyclic-by-finite and that G is torsion-free. If G is poly-infinite cyclic, then Theorem 9.2.2 tells us that we are done. So we say that G is either polycyclic group with at least one of the quotient groups in the subnormal series a finite cyclic group or a polycyclic-by-finite group, and in either case, we have a series of subgroups

$$\langle e \rangle = G_0 \subseteq G_1 \subseteq G_2 \subseteq \cdots \subseteq G_{n-1} \subseteq G_n = G$$

with G_{i-1} a normal subgroup of G_i for $i = 1, 2, \cdots n$, G_i/G_{i-1} normal for $i = 1, 2, \cdots n-1$, and G_n/G_{n-1} finite. For convenience of notation, we assign the letter q to the index of G_{n-1} in G_n, so q will be some integer greater than or equal to 2.

Now we suppose that there is some element g of G that has a p_i^{th} root for infinitely many primes p_i. Necessarily, these are distinct roots, for if r was a p_j^{th} root and a p_k^{th} root of g for $j \neq k$ and we take k to be greater than j, then $r^{p_k} = r^{p_j} = g$, and so $r^{p_k - p_j} = e$, which violates the assumption that G is a torsion-free group. Theorem 9.2.2 tells us that it is not the case that g and infinitely many of the p_i^{th} roots lie in G_{n-1}. So, all but finitely many of the p_i^{th} roots lie in $G \backslash G_{n-1}$. Since the index of G_{n-1} in G_n is q, then the q^{th} power of all the elements of $G \backslash G_{n-1}$ lies in G_{n-1}. Moreover, if an element h of G_n is a p_i^{th} root of g, then the element h^q in G_{n-1} is a p_i^{th} root of g^q (also in G_{n-1}) and since G is torsion-free, the p_i^{th} roots are distinct. However, G_{n-1} is a poly-$\mathbb{Z}$ group, so according to Theorem 9.2.2, no element of G_{n-1} can have a p_i^{th} root for infinitely many distinct primes p_i. This contradiction tells us that there is no element g in G with infinitely many p_i^{th} roots for distinct primes p_i, and thus that torsion-free polycyclic or polycyclic-by-finite groups satisfy condition (b) of Theorem 3.3.2. $\square$

We can now combine some results to obtain the following useful result.

Corollary 9.2.4. *If a torsion-free polycyclic-by-finite group does not satisfy the root-finite condition, then there is some element of the group with infinitely many r^{th} roots for some positive integer r.*

Proof. Suppose that G is a torsion-free polycyclic-by-finite group and that G does not satisfy the root-finite condition. Failure to satisfy the root-finite condition implies failure to satisfy one of the three conditions of Theorem 3.3.2. However, by Theorem 9.2.1 and Corollary 9.2.3, G satisfies conditions (a) and (b) of Theorem 3.3.2. Therefore G must fail to satisfy condition (c), and so there is some element g of G with infinitely many r^{th} roots for some positive integer r. $\square$

We now look at some additional theorems regarding poly-$\mathbb{Z}$ groups. The set C of rootless elements, that is, elements that have no r^{th} roots for any positive integer $r \geq 2$, cannot be empty in poly-$\mathbb{Z}$ (and more generally in polycyclic-by-finite groups), since if there are no rootless elements, root chains could be extended indefinitely, and Theorem 9.2.1 insures that this is not the case. The next theorem identifies some of these rootless elements in poly-$\mathbb{Z}$ groups.

Theorem 9.2.5. *Suppose that G is a poly-$\mathbb{Z}$ group, such that there is a series of subgroups*

$$\langle e\rangle = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_n = G$$

with $G_i/G_{i-1} = \langle \bar{a}_i \rangle$ infinite cyclic, $i = 1, 2, \cdots, n$. Then the a_i are elements of G that do not have r^{th} roots for any positive integer $r \geq 2$.

Proof. Suppose that G is a poly-$\mathbb{Z}$ group, g is an element of G, and g is an r^{th} root of one of the a_i. We consider first where g would have to occur in the series of subgroups G_j. If g were an element of G_{i-1}, then since $a_i = g^r$, a_i would lie in G_{i-1}, so $\bar{a}_i$ could not be a generating element of the quotient group G_i/G_{i-1}. Thus g is not in G_{i-1}.

If g were an element of G_i, then g would lie in some coset $a_i^k G_{i-1}$ for some $k \neq 0$. Then, since $g^r = a_i$, we can conclude that $a_i \in a_i^{kr} G_{i-1}$, which implies that $\bar{a}_i = \bar{a}_i^{kr}$, contradicting the assumption that $G_i/G_{i-1} = \langle \bar{a}_i \rangle$ is infinite cyclic. Thus g cannot lie in the subgroup G_i.

Therefore $g \in G_j$ for some $j > i$ and we can choose j to be minimal, so that $g \in G_j \backslash G_{j-1}$. Then g is in some coset $a_j^m G_{j-1}$ of G_{j-1} for some $m \neq 0$. Then, since $a_i = g^r$, we conclude that a_i is in the coset $a_j^{mr} G_{j-1}$. But a_i is in G_{j-1}, since $a_i \in G_i$ and $i < j$. This gives us that ${\bar{a}_j}^{mr}$ is equal to the identity in G_j/G_{j-1}, which contradicts that the quotient groups are infinite cyclic.

Since it has been shown that g cannot lie in any of the subgroups G_j for $j < i, j = i$ or $j > i$, we conclude that no such element exists, and that the a_i are rootless elements of G.

$\square$

We now explore under what conditions poly-$\mathbb{Z}$ groups are root finite. First, we will need the following lemma.

Lemma 9.2.6. *Suppose that G is a poly-$\mathbb{Z}$ group, such that there is a series of subgroups*

$$\langle e \rangle = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_n = G$$

with $G_i/G_{i-1} = \langle \bar{a}_i \rangle$ infinite cyclic, $i = 1, 2, \cdots, n$. Then if $g \in G_j \backslash G_{j-1}$ for some $j = 1, 2, \cdots, n$, then $g^k \in G_j \backslash G_{j-1}$ for all integers $k \geq 1$.

Proof. Let G be a poly-$\mathbb{Z}$ group with notation as in the statement of the theorem, and let g be some element of $G_j \backslash G_{j-1}$. Then g is in some coset $a_j^m G_{j-1}$ with $m \neq 0$, so that for any positive integer k, $\bar{g}^k = \bar{a_j}^{km}$ in the quotient group G_j/G_{j-1}. If it were the case that g^k were in the subgroup G_{j-1}, then $\bar{a_j}^{km}$ would be equal to the identity in G_j/G_{j-1}, contradicting that the quotient groups are infinite cyclic. Therefore all of the powers of g must lie in the subgroup G_j but not in the subgroup G_{j-1}. □

In the next theorem, we identify a sufficient condition for r^{th} roots to be unique in poly-(infinite cyclic) groups. Since those groups have been shown to satisfy two of the three conditions for groups to be root finite (Theorems 9.2.1 and 9.2.2, the additional property of uniqueness of r^{th} roots gives us that the groups meeting the conditions of the following theorem are root finite.

Theorem 9.2.7. *Suppose that G is a poly-$\mathbb{Z}$ group, such that there is a series of subgroups*

$$\langle e \rangle = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_n = G$$

with $G_i/G_{i-1} = \langle \bar{a}_i \rangle$ infinite cyclic, $i = 1, 2, \cdots, n$. If the elements a_i are central in G_i for all i, then r^{th} roots are unique in G for all positive integers r.

Proof. We use induction on the Hirsch number h. If $h = 1$, G is isomorphic to $\mathbb{Z}$, so r^{th} roots are unique.

Suppose that the theorem holds for Hirsch numbers $h \leq k$, and G is a poly-$\mathbb{Z}$ group with Hirsch number $h = k + 1$. We may assume that $g \in G \backslash G_k$, since if g were in G_k, the r^{th} roots of g would also be in G_k by Lemma 9.2.6, and that would contradict the induction hypothesis.

Let $g = x^r = y^r \in G \backslash G_k$. By Lemma 9.2.6, x and y are also in $G \backslash G_k$, so they are equal to $a_{k+1}^s h_1$ and $a_{k+1}^t h_2$ for some nonzero integers s and t and some elements h_1 and h_2 of G_k.

Since $x^r = y^r$ and since $\bar{x} = \bar{a}_{k+1}^s$ and $\bar{y} = \bar{a}_{k+1}^t$ in the quotient group G_{k+1}/G_k, and so $s = t$ since $\langle \bar{a}_{k+1} \rangle$ is infinite cyclic. So $(a_{k+1}^s h_1)^r = (a_{k+1}^s h_2)^r$, and, since a_{k+1} is assumed central in G_k, we have that $a_{k+1}^{sr} h_1^r = a_{k+1}^{sr} h_2^r$, and thus $h_1^r = h_2^r$. Since h_1 and h_2 are in G_k and since r^{th} roots are unique in G_k by the induction hypothesis, we conclude that $x = y$ and thus that r^{th} roots are unique in G. □

9.3 Groups Rings of Polycyclic and Polycyclic-by-Finite Groups

In this section we make note of some of the important theorems relating to group rings of polycyclic and polycyclic-by-finite groups.

With respect to the zero-divisor problem for groups rings, if G is a torsion-free polycyclic-by-finite group and if K is a field of characteristic 0, then the group ring KG has no proper zero divisors [3].

A well-known property of group rings of polycyclic-by-finite groups is that such rings are Noetherian. The only known examples of Noetherian group rings are group rings of polycyclic-by-finite groups. It remains an open question whether any other Noetherian group rings exist [22].

Chapter 10

Nilpotent Groups

10.1 Definition and Preliminary Remarks

Another important class of groups consists of nilpotent groups. In this chapter, a definition of nilpotent groups is given, and several important properties of these groups are discussed. Then we examine the circumstances under which nilpotent groups can be determined to possess the root-finite condition. Finally, we review some theorems relating to group rings of nilpotent groups.

If x and y are two elements of a group G, the *commutator* of x and y, denoted $[x, y]$, is the group element $x^{-1}y^{-1}xy$. Of course, if x and y commute, $[x, y]$ is the identity element of the group.

We can generalize this concept, so that for any two subgroups H and K of G, we define the commutator subgroup of H and K, denoted as $[H, K]$, the subgroup of G that is generated by the set of all the commutators of pairs of elements from H and K, that is, $[H, K] = \langle [h, k] | h \in H, k \in K \rangle$.

We now construct a descending series of subgroups of G, known as the *lower central series*, in the following recursive manner:

$$\gamma_1(G) = G$$

and

$$\gamma_{i+1}(G) = [\gamma_i(G), G] \qquad i \geq 1$$

If the lower central series stabilizes at the identity, that is, if there is some c such that $\gamma_{c+1} = \langle e \rangle$, then the group G is said to be *nilpotent*, and the least c for which $\gamma_{c+1} = \langle e \rangle$ is called the *nilpotency class* of G.

There is another series of subgroups, known as the *upper central series*, which is constructed recursively as follows. $Z_0(G)$ is the subgroup of G consisting of only the identity element. Then Z_{n+1} is the unique subgroup of G that satisfies the relation $Z_{n+1}(G)/Z_n(G) = Z(G/Z_n(G))$. If the upper central series stabilizes at G after a finite number of steps, then G is nilpotent. The definitions based on the construction of lower and upper central series can be shown to be equivalent. Moreover, the nilpotency class of G as defined in terms of lower central series is equal to the lowest n for which $Z_n(G) = G$ in terms of the upper central series.

There are two theorems regarding nilpotent groups that will be needed when developing criteria for determining if a nilpotent group satisfies the root-finite condition. It is not in general true that subgroups of finitely generated groups are finitely generated. It is known, for example, that every group with countably many elements can be embedded in a group generated by two elements [6]. Since groups with countably many elements need not be finitely generated (the rationals as a group with the operation of addition is an example), it is possible to have finitely generated groups with subgroups that are not finitely generated.

There are, however, classes of groups for which it is the case that subgroups of finitely generated groups are finitely generated. One such example is the class of abelian groups. Since nilpotent groups are defined in terms a finite series involving commutators, it is plausible that nilpotent groups are sufficiently like abelian groups that they will share some of the properties of abelian groups. In this instance, that indeed is the case, according to the following theorem.

Theorem 10.1.1. *Subgroups of finitely generated nilpotent groups are finitely generated.*

Proof. See Lemma 3.4.2 of [19]. □

We will have occasion to use the following property of finitely generated nilpotent groups, proven in [7] and [8].

Theorem 10.1.2. *Finitely generated nilpotent groups are polycyclic.*

Proof. See Theorem 2.13 of [22]. (Wehrfritz offers three proofs for this theorem.) □

There is an interesting result concerning the division hull of subgroups of finitely generated nilpotent groups. In general, the division hull of a subgroup need not be a subgroup. For example, the division hull of the trivial subgroup $\langle e \rangle$ is the set of elements of finite order, and that set is not generally a subgroup. However, in finitely generated nilpotent groups, the division hull of a subgroup is a subgroup, as stated in the following theorem.

Theorem 10.1.3. *If G is a finitely generated nilpotent group and H is a subgroup of G, then the division hull of H,* $\mathrm{dh}(H)$*, is a subgroup of G and* $[\mathrm{dh}(H) : H] < \infty$.

Proof. See [22], Theorem 5.11. $\square$

An immediate consequence of this, taking the subgroup H to be $\langle e \rangle$, is that in finitely generated nilpotent groups, there are only finitely many elements of finite order.

The following technical lemma will turn out to be useful for deriving several results concerning nilpotent groups and the root-finite condition.

Lemma 10.1.4. *Let G be a nilpotent group, and let $x, y \in G$ with $(x^r, y^s) = e$ for some integers $r, s \geq 1$. Then the commutator (x, y) has finite order.*

Proof. This is Lemma 11.1.4 of [19]. $\square$

10.2 Nilpotent Groups and the Root-Finite Condition

In this section, we explore the question of which nilpotent groups satisfy the root-finite condition. We first present a theorem that follows from Lemma 10.1.4 and which establishes the relationship between the delta subgroup of a torsion-free nilpotent group and its center.

Theorem 10.2.1. *If G is a torsion-free nilpotent group, then $\Delta(G) = Z(G)$.*

Proof. The inclusion $Z(G) \subseteq \Delta(G)$ is immediate.
Suppose $g \in \Delta(G)$, and let $x \neq e$ be an element of G. By Lemma 7.2.11, it is known that there exists an integer $r \geq 1$ such that x^r commutes with g. Then by Lemma 10.1.4, the commutator (g, x) has finite order. Since G is assumed to be torsion-free, it follows that $(g, x) = e$, so we conclude that g is central, thus giving the inclusion $\Delta(G) \subseteq Z(G)$. Therefore $Z(G) = \Delta(G)$. $\square$

It is always the case in groups that all the powers of a fixed element commute with each other. The fixed element generates a cyclic subgroup, and all cyclic groups are abelian. It is, however, not generally the case that all the roots of a fixed element commute with one another. A familiar example is the quaternion group, in which i, j, and k are all square roots of -1, but they do not commute. Continuing our investigation of torsion-free nilpotent groups, we see in the following theorem that all the roots of a fixed element do commute with one another.

Theorem 10.2.2. *Let G be a torsion-free nilpotent group and let g be an element of G. Then all the roots of g commute with each other.*

Proof. Let x and y be roots of g, i.e, there exist integers $r, s \geq 1$ such that $x^r = y^s = g$. Since x^r and y^s represent the same group element, they certainly commute with other, and so their commutator (x^r, y^s) equals the identity. Since $(x^r, y^s) = e$, it follows from Lemma 10.1.4 that (x, y) has finite order. Since G is assumed to be torsion-free, this implies that $(x, y) = e$, and thus x and y commute.

□

We can say more about the roots of finitely generated torsion-free nilpotent groups. It turns out that such groups are R-groups, as the following theorem demonstrates.

Theorem 10.2.3. *In finitely generated torsion-free nilpotent groups, n^{th} roots are unique.*

Proof. Let G be a finitely generated torsion-free nilpotent group, and let $g \in G$. Denote by R_g the set of all roots of g, $R_g = \{x \in G | x^r = g \text{ for some } r \geq 1\}$. Let $H_g = \langle R_g \rangle$ be the subgroup of G generated by the elements of R_g. By construction H_g contains all the roots of g in G. Since, by Theorem 10.2.2, the generators of H_g commute with each other, H_g is abelian. Furthermore, H_g is finitely generated, since by Theorem 10.1.1 subgroups of finitely generated nilpotent groups are finitely generated. H_g is torsion-free, since G is. Thus H_g is isomorphic to a finite direct sum of copies of $\mathbb{Z}$, and in such groups n^{th} roots are unique. □

We are now ready to prove that another class of groups satisfies the root-finite condition.

Corollary 10.2.4. *Finitely generated torsion-free nilpotent groups are root-finite.*

Proof. This follows from the fact, given in Theorem 10.1.2, that finitely generated nilpotent groups are polycyclic. It was previously shown that in polycyclic groups that are not root-finite, there exists an element with infinitely many n^{th} roots for some $n \geq 2$. Theorem 10.2.3 shows that such an element does not exist in finitely generated torsion-free nilpotent groups, so these groups are root-finite. □

10.3 Group Rings of Nilpotent Groups

We now make some observations about group rings of nilpotent groups. We make use of the fact, a direct consequence of Theorem 10.1.3, that in finitely generated nilpotent groups, the set of elements of finite order forms a finite subgroup. This subgroup is normal, since group elements that are conjugate have the same order. Furthermore, for a group G and a field K, the group ring KG is prime if and only if G has no nonidentity finite normal subgroup. This leads us to observe that if G is a finitely generated nilpotent group and if G is not torsion-free, then the group ring KG is not prime.

Chapter 11

The Module Problem for Group Rings

11.1 Preliminary Lemmas

The question to be addressed in this chapter and which is referred to herein as "the module problem for group rings" was raised by [15]: The question deals with a class of modules of a group ring that are indexed by the elements of the group and are constructed in the following manner. In the group ring KG, for any $x \in G \setminus \langle e \rangle$, consider the left module

$$\mathcal{M}_x = KG/KG(x-1).$$

Under what circumstances is $\mathcal{M}_x$ a faithful module?

The significance of this condition is that if KG is prime and $KG(x-1)$ is essential and $\mathcal{M}_x$ is faithful, then KG is not bounded. The following lemma will prove useful:

Lemma 11.1.1. *Let $x \in G$. If $\alpha \in KG$ is a nontrivial element of $\operatorname{ann} \mathcal{M}_x$, then $\alpha \in \operatorname{ann} \mathcal{M}_y$ for all $y \in G$ that are conjugate to x.*

Proof. Let β be an arbitrary element of KG. Since $\alpha \in \operatorname{ann} \mathcal{M}_x$, there exists some $\gamma_\beta \in KG$ such that

$$\alpha\beta = \gamma_\beta(x-1).$$

Let $y = h^{-1}xh$ for some $h \in G$. Applying the previous observation to the element βh^{-1} of KG, there exists some $\gamma_{\beta h^{-1}} \in KG$ such that

$$\alpha\beta h^{-1} = \gamma_{\beta h^{-1}}(x-1)$$

Then

$$\begin{aligned}
\alpha\beta &= \gamma_{\beta h^{-1}}(x-1)h \\
\alpha\beta &= \gamma_{\beta h^{-1}}h(h^{-1}xh-1) \\
\alpha\beta &= \gamma_{\beta h^{-1}}h(y-1)
\end{aligned}$$

Thus

$$\alpha\beta = \delta_\beta(y-1)$$

where

$$\delta_\beta = \gamma_{\beta h^{-1}}h$$

Since β is arbitrary, this shows that $\alpha \in \mathcal{M}_y$. □

For every $x \in G$, we define an equivalence relation $\sim_x$ on $\operatorname{Supp}\alpha$ as follows: $g \sim_x h$ if $g = hx^n$ for some integer n. For any $g \in \operatorname{Supp}\alpha$, the equivalence class containing g under such a partition is denoted by $[g]_x$.

We have the following lemma.

Lemma 11.1.2. *Let KG be a group ring and let α be an element of KG. If x is an element of G such that $\alpha \in \operatorname{ann}\mathcal{M}_x$, then for all $y \in \operatorname{Supp}\alpha$, $[y]_x$ has at least two elements.*

Proof. Since $\alpha \in \operatorname{ann}\mathcal{M}_x$, there is some $\beta \in KG$ such that $\alpha = \beta(x-1)$. We write α as

$$\sum_{i=1}^{n} a_i g_i$$

for some finite n, with $a_i \in K$ and $g_i \in G$. Similarly,

$$\beta = \sum_{j=1}^{m} b_j h_j$$

for some finite m, $b_j \in K$, $h_j \in G$. Now we break these expressions down into a component with support in $[y]_x$ and a component with support disjoint from $[y]_x$:

$$\alpha^{(1)} = \sum_{g_i \in [y]_x} a_i g_i$$

$$\alpha^{(2)} = \sum_{g_i \notin [y]_x} a_i g_i$$

$$\beta^{(1)} = \sum_{h_j \in [y]_x} b_j h_j$$

$$\beta^{(2)} = \sum_{h_j \notin [y]_x} b_j h_j$$

We then have

$$\alpha^{(1)} + \alpha^{(2)} = \beta^{(1)}(x-1) + \beta^{(2)}(x-1)$$

Note that $\operatorname{Supp} \alpha^{(1)}$ is precisely the equivalence class $[y]_x$. The claim now is that $\alpha^{(1)} = \beta^{(1)}(x-1)$. Since by construction the equivalence class $[y]_x$ is closed under multiplication by x, all elements of $\operatorname{Supp} \beta^{(1)}(x-1)$ are in the equivalence class $[y]_x$, and no elements of $\operatorname{Supp} \beta^{(2)}(x-1)$ are in that equivalence class. Thus it must be the case that $\alpha^{(1)} = \beta^{(1)}(x-1)$. Now because $\alpha^{(1)}$ is a multiple of $x-1$, it is in the augmentation ideal of KG. But this implies that $\operatorname{Supp} \alpha^{(1)}$ is not a singleton, and since $\operatorname{Supp} \alpha^{(1)}$ coincides with $[y]_x$, the lemma is proved. □

11.2 Main Theorems

We are now ready to prove the main results of this chapter.

Theorem 11.2.1. *Let G be a root-finite group. If $x \in G$ has an infinite conjugacy class, then $\mathcal{M}_x$ is faithful.*

Proof. Suppose $\alpha \in \operatorname{ann} \mathcal{M}_x$. Denote the elements of $\operatorname{Supp} \alpha$ by $y_1, \cdots, y_n$, and let $[x]$ denote the conjugacy class of $x \in G$. For each $y_i, 2 < i \leq n$, define the set $\mathcal{A}_i$ by $\mathcal{A}_i = \{z \in [x]$ such that $y_1 \sim_z y_i\}$. The claim now is that $\mathcal{A}_i$ must be infinite for some i. Suppose that this is not the case. Then $\sum_{i=2}^{n} |\mathcal{A}_i| < \infty$, and since $[x]$ is assumed to be infinite, there must be some $z^* \in [x]$ such that $z^* \notin \mathcal{A}_i$ for all i. However, this means that there is no y_i for $2 \leq i \leq n$ such that $y_1 \sim_z y_i$. This cannot be the case because of Lemma 11.1.2. So at least one of the $\mathcal{A}_i$ must be infinite. With no loss of generality, we can let $\mathcal{A}_2$ be one such infinite set. This implies that $y_1 = y_2 z^{n_z}$ for infinitely many $z \in [x]$ (and hence in G), and with no loss of generality we can take $n_z > 0$ for infinitely many z. Thus $y_2^{-1} y_1 = z^{n_z}$ for infinitely many distinct z, which is impossible if G is root finite. Thus there can be no $y_i \in \operatorname{Supp} \alpha$, meaning that the only element of $\operatorname{ann} \mathcal{M}_x$ is zero, and thus $\mathcal{M}_x$ is faithful.

□

Theorem 11.2.2. *Let $\mathcal{M}_x = \frac{KG}{KG(x-1)}$ and $[x]$ be the conjugacy class of $x \in G$. Then $\operatorname{ann} \mathcal{M}_x = \bigcap_{y \in [x]} KG(y-1)$.*

Proof. I. $\operatorname{ann}\mathcal{M}_x \subseteq \bigcap_{y\in[x]} KG(y-1)$.

Let $\alpha \in \operatorname{ann}\mathcal{M}_x$. Then for all $\beta \in KG$ there exists $\gamma_\beta \in KG$ such that $\alpha\beta = \gamma_\beta(x-1)$. In particular, $\alpha \cdot 1 = \gamma_1(x-1)$, so $\alpha \in KG(x-1)$. It was shown in Lemma 11.1.1 that if $\alpha \in \operatorname{ann}\mathcal{M}_x$, then $\alpha \in \operatorname{ann}\mathcal{M}_y$ for all $y \in [x]$. Thus $\alpha \in \bigcap_{y\in[x]} KG(y-1)$.

II. $\bigcap_{y\in[x]} KG(y-1) \subseteq \operatorname{ann}\mathcal{M}_x$.

Let $\alpha \in \bigcap_{y\in[x]} KG(y-1)$ and let $\gamma = \sum_{i=1}^n a_i g_i$ be an arbitrary element of KG. Then, for each $g_i \in \operatorname{Supp}\gamma$, we can write $\alpha = \beta_i(g_i x g_i^{-1} - 1)$ for some $\beta_i \in KG$, since $\alpha \in \bigcap_{y\in[x]} KG(y-1)$. We then compute

$$\begin{aligned}\alpha(\sum_{i=1}^n a_i g_i) &= \sum_{i=1}^n a_i \alpha g_i \\ &= \sum_{i=1}^n a_i \beta_i (g_i x g_i^{-1} - 1) g_i \\ &= \sum_{i=1}^n a_i \beta_i g_i (x-1)\end{aligned}$$

So $\alpha \in \operatorname{ann}\mathcal{M}_x$.

Therefore $\bigcap_{y\in[x]} KG(y-1) = \operatorname{ann}\mathcal{M}_x$.

□

Theorem 11.2.3. *Let KG be a prime group ring and let $x \in \Delta(G)$. Then $\mathcal{M}_x$ is not faithful.*

Proof. Since KG is prime, we have that $\Delta(G)$ is torsion-free and abelian (Theorem 2.3.1). Since $[x]$ is finite (by definition of $\Delta(G)$) and since all elements of $[x]$ are in $\Delta(G)$ and therefore commute with each other,

$$\alpha = \prod_{y\in[x]} (y-1)$$

is well defined. By construction, α is an element of the group ring $K\Delta(G)$. Since $\Delta(G) = \Delta(\Delta(G))$ and since $\Delta(G)$ is torsion-free abelian, it follows from Theorem 2.3.1 that $K\Delta(G)$ is a prime ring. But since $\Delta(G)$ is abelian, the ring $K\Delta(G)$ is commutative. Since commutative prime rings have no proper nonzero divisors, we can conclude that α is nonzero. So

$$\alpha \in \bigcap_{y\in[x]} KG(y-1)$$

and thus $\alpha \in \operatorname{ann}\mathcal{M}_x$, so $\mathcal{M}_x$ is not faithful. □

This leads to the following corollary:

Corollary 11.2.4. *If KG is a prime group ring and G is root-finite, then $\mathcal{M}_x$ is not faithful if and only if $x \in \Delta(G)$.*

Proof. This is an immediate consequence of Theorems 11.2.1 and 11.2.3. $\square$

Chapter 12

A Case Study: The Infinite Dihedral Group

As a case study of the results in the previous chapter, we now turn our attention to the infinite dihedral group. There are two presentations of this group that are commonly used:

$$G = \langle a, b | a^2 = b^2 = 1 \rangle$$

and

$$G = \langle x, y | x^2 = 1, x^{-1}yx = y^{-1} \rangle$$

where b in the first presentation corresponds to xy in the second. The second presentation will be used in this chapter.

Since the second relation gives us that

$$yx = xy^{-1}$$

it can be seen that all elements of G can be written uniquely in the form

$$x^i y^j, i \in \{0, 1\}, j \in \mathbb{Z}$$

It should also be noted that if $g \in G$ has the form xy^j, then

$$g^2 = (xy^j)(xy^j) = (xy^j x)y^j = y^{-j}y^j = 1$$

Thus G is not a root-finite group, since the identity has infinitely many square roots.

We now look at the conjugacy classes of G. Consider, first, an element of G of the form xy^j. If we conjugate by the element xy^k, we obtain

$$(xy^k)^{-1}xy^j(xy^k) = (xy^k)(xy^j x)(y^k) = (xy^k)(y^{-j})(y^k) = xy^{2k-j}$$

If we conjugate xy^j by y^k, we obtain

$$(y^{-k})(xy^j)(y^k) = (y^{-k}x)(y^{j+k}) = (xy^k)(y^{j+k}) = xy^{j+2k}$$

So this gives us two infinite conjugacy classes, one of elements of the form xy^{2k} and the other of the elements of the form xy^{2k+1}, with $k \in \mathbb{Z}$.
Now suppose we have an element of G of the form y^j. If we conjugate by xy^k, we obtain

$$(xy^k)^{-1}y^j(xy^k) = (xy^k)(y^jx)(y^k) = (xy^k)(xy^{-j})y^k = (xy^kx)y^{-j+k} = y^{-k}y^{-j+k} = y^{-j}$$

Since conjugation of y^j by another power of y has no effect, we see that the remaining conjugacy classes of G are $\{y^{\pm j}\}, j \in \mathbb{Z}$, and thus the Delta subgroup is $\Delta(G) = \{y^j, j \in \mathbb{Z}\}$. Since $\Delta(G)$ is torsion free abelian, we have by Theorem 2.3.1 that KG is a prime ring, and thus by Theorem 11.2.3, $\mathcal{M}_{y^j}$ is not faithful, and we obtain that

$$\operatorname{ann}(\mathcal{M}_{y^j}) = KG(y^j - 1)(y^{-j} - 1) = KG(2 - y^j - y^{-j})$$

Note that KG has an abelian subgroup of index 2, so it is PI by Corollary 5.3.8 of [19], and it is FBN by Theorem 2.2.3. So every essential left ideal contains a nonzero two-sided ideal.

Chapter 13

Skew Polynomial Rings and Skew-Laurent Polynomial Rings

13.1 Definitions

We now consider skew polynomial rings and skew-Laurent polynomial rings and raise the same sorts of questions that we have been exploring for group rings. Following [5], we adopt the following definitions for skew polynomial rings and skew-Laurent polynomial rings.

To define a skew polynomial ring T, we let R be a ring, σ an automorphism of R, and δ a derivation on R, that is, an additive map satisfying $\delta(rs) = r\delta(s) + \delta(r)s$. We write

$$T = R[x; \sigma, \delta]$$

to mean that

1. T is a ring containing R as a subring,

2. x is an invertible element of T,

3. T is a free left R-module with basis $\{x^n | n = 0, 1, 2, \cdots\}$,

4. for all $r \in R, xr = \sigma(r)x + \delta(r)$.

Similarly, to define a skew-Laurent polynomial ring T, we let R be a ring and σ an automorphism of R. We write

$$T = R[x^{\pm}; \sigma]$$

to mean that

1. T is a ring containing R as a subring,

2. x is an invertible element of T,

3. T is a free left R-module with basis $\{1, x, x^{-1}, x^2 . x^{-2}, \cdots\}$,

4. for all $r \in R, xr = \sigma(r)x$.

13.2 The Module Problem

We consider the left module $\mathcal{M}_r = T/T(x-r)$ for some $r \in R$, and ask the question, analogous to the question that examined for modules of group rings, for what r is $\mathcal{M}_r$ faithful.

We first prove the following lemma:

Lemma 13.2.1. *If* $\alpha \in \operatorname{ann}(\mathcal{M}_r)$, *then* $\alpha \in \operatorname{ann}(\mathcal{M}_{\sigma(r)})$.

Proof. If $\alpha \in \operatorname{ann}(\mathcal{M}_r)$, then for all $\beta \in T$ there exists some $\gamma \in T$ such that

$$\alpha\beta = \gamma_\beta(x-r)$$

In particular, there is some $\gamma_{\beta x} \in T$ such that

$$\alpha\beta x = \gamma_{\beta x}(x-r)$$

Then

$$\begin{aligned}
\alpha\beta &= \gamma_{\beta x}(x-r)x^{-1} \\
&= \gamma_{\beta x}(x^{-1})(x^2 - xr)(x^{-1}) \\
&= \gamma_{\beta x}(x^{-1})(x^2 - \sigma(r)x)(x^{-1}) \\
&= \gamma_{\beta x}(x^{-1})(x - \sigma(r))
\end{aligned}$$

So, $\alpha \in \operatorname{ann}(\mathcal{M}_{\sigma(r)})$.

□

Corollary 13.2.2. $\operatorname{ann}(\mathcal{M}_r) = \operatorname{ann}(\mathcal{M}_{\sigma^k(r)})$ *for all* $k \in \mathbb{Z}$*.*

Proof. The inclusion $\operatorname{ann}(\mathcal{M}_r) \subseteq \operatorname{ann}(\mathcal{M}_{\sigma^k(r)})$ for $k > 0$ follows from repeated applications of Lemma 13.2.1, and for $k < 0$, it follows from repeated applications of Lemma 13.2.1 substituting the automorphism σ^{-1} for σ.
The inclusion $\operatorname{ann}(\mathcal{M}_r) \supseteq \operatorname{ann}(\mathcal{M}_{\sigma^k(r)})$ for $k > 0$ follows from repeated applications of Lemma 13.2.1 substituting $\sigma^k(r)$ for r as the ring element and substituting σ^{-1} for σ as the automorphism. The inclusion $\operatorname{ann}(\mathcal{M}_r) \supseteq \operatorname{ann}(\mathcal{M}_{\sigma^k(r)})$ for $k < 0$ follows from repeated applications of Lemma 13.2.1 substituting $\sigma^k(r)$ for r as the ring element and retaining σ as the automorphism. □

Theorem 13.2.3. $\operatorname{ann}(\mathcal{M}_r) \subseteq \bigcap_{k\in\mathbb{Z}} T(x - \sigma^k(r))$

Proof. Suppose $\alpha \in \operatorname{ann}(\mathcal{M}_r)$. Then, by the Corollary 13.2.2, we have that $\alpha \in \operatorname{ann}(\mathcal{M}_{\sigma^k(r)})$ for $k \in \mathbb{Z}$. This implies that for all $\beta \in T$, there exists an element of T, say, γ_β such that

$$\alpha\beta = \gamma_\beta(x - \sigma^k(r))$$

In particular, there exists $\gamma_1 \in T$ such that

$$\alpha \cdot 1 = \gamma_1(x - \sigma^k(r))$$

so that $\alpha \in T(x - \sigma^k(r))$ for all $k \in \mathbb{Z}$, and the theorem is proved. □

In the following lemmas, we assume that R is a commutative integral domain and ϕ is an automorphism of R. We set $T = R[x;\phi]$. The same results should hold for $T = R[x^{\pm};\phi]$.

Lemma 13.2.4. *Let I be a nonzero ideal of T, and suppose ϕ has infinite order. If n is the minimal degree of a nonzero element of I, then I has an element of the form rx^n for $r \neq 0$. (In fact, every element of I of degree n has this form.)*

Proof. Let $f = \sum_{i=0}^{n} r_i x^i \in I$ be an element of degree n (so $r_n \neq 0$). Since ϕ has infinite order, R must contain elements with infinite ϕ-orbits or elements with arbitrarily large finite ϕ-orbits. In particular, R must contain an element s whose ϕ-orbit has greater than n elements. Consider the element $g = \phi^n(s)f - fs$ of T. Since $I \triangleleft T$, we have that $g \in I$. Moreover, the degree n term of g is $(\phi^n(s)r_n - r_n\phi^n(s))x^n$, which is zero. Thus $\deg g < n$ and so $g = 0$. This implies that $\phi^n(s)r_i = r_i\phi^i(s)$ for $i = 0, \cdots, n-1$, that is, $(\phi^n(s) - \phi^i(s))r_i = 0$. Since the orbit of s has more than n elements, no $\phi^n(s) - \phi^i(s)$ can be 0. Thus each r_i, $i = 0, \cdots, n-1$ must be 0, and the lemma is proven. □

Lemma 13.2.5. *Let $a \in R$ be nonzero. Then no nonzero element of the left ideal $T(x-a)$ has the form rx^n for $r \neq 0$ and $n \in \mathbb{N}$.*

Proof. Let $f = \sum_{i=k}^{m} r_i x^i \in T$ be nonzero where $k \leq m$, $r_k \neq 0$, and $r_m \neq 0$. Then $f(x-a)$ has leading term $r_m x^{m+1}$ and lowest term $-r_k \phi^k(a) x^k$; neither of these terms is 0 since R is an integral domain and ϕ is an automorphism. Thus $f(x-a)$ must have at least two terms $(k < m+1)$, and thus cannot equal any $r_n x^n$. $\square$

Corollary 13.2.6. *Let $a \in R$ be nonzero and let ϕ have infinite order. Then the left T-module $mathcal M_a$ is faithful.*

Proof. If I is the annihilator of $mathcal M_a$, then I is an ideal of T contained in $T(x-a)$. However, Lemmas 13.2.4 and 13.2.5, taken together, show that there is no nonzero ideal contained in $T(x-a)$. $\square$

Chapter 14

Questions for Further Research

There remain many unresolved questions related to the topics of this dissertation. Among these questions, the following are of particular interest:

- Having identified two relations that give rise to elements with infinitely many roots in groups with two generators, we might be interested in knowing what other such relations could be found. In particular, one of the relations gave rise to elements with infinitely many square roots. It may be possible to come up with some sort of analogous relation that would produce elements with infinitely many r^{th} roots for some r greater than 2.

- It was shown in Theorem 6.2.2 that torsion-free abelian groups do not satisfy the root-finite condition if the r^{th} roots in the group are denser than in $\mathbb{Z}$ for some positive integer r. It would be interesting to know whether the converse holds as well so that we could have a criterion for determining if torsion-free abelian groups satisfy the root-finite condition. A further exploration of root density could also be extended to nonabelian groups. The set of group elements having r^{th} roots, A_r, is not necessarily a group if A is not abelian, but we can still speak of the index of A_r in G as the minimum number of translations of A_r that can achieve a covering of G. (The concept of index of a subset of a group is discussed in [19] (see pages 180-190); in general, there may be different left and right indices, but since A_r is defined in such a way that it is a union of conjugacy classes, this complication would not occur.) It might be possible to come up with a more general criterion linking the indices of the A_r to the question of whether a group satisfies the root-finite condition.

- One of the lacunae in the theory of the root-finite condition as developed in this work is the lack of any criteria for determining whether groups whose delta subgroup is of infinite index satsify the root-finite condition. This class of groups includes many matrix groups where the delta subgroup and the center coincide. It would be interesting to consider under what circumstances such groups satisfy the root-finite condition.

- The conjecture that polycyclic-by-finite groups necessary satisfy conditions (a) and (b) of Theorem 3.3.2 remains under investigation. Corollary 9.2.4 is a weakened version of that assertion, applying only to the case of torsion-free polycyclic-by-finite groups. It would be interesting arrive at a proof of the more general assertion or to find a counterexample.

Bibliography

[1] Auslander, L., On a problem of Philip Hall, *Ann. Math. (2)*, **86**, 112-116, 1967.

[2] Connell, I. G., On the group ring, *Can. J. Math.*, **15**, 650-685, 1963.

[3] Farkas, D. R., and R. L. Snider, K_0 and Noetherian group rings, *J. Algebra*, **42**, 192-198, 1976.

[4] Goldie, A.W., The structure of prime rings under ascending chain conditions, *Proc. London Math. Soc. (3)*, **8**, 589-608, 1958.

[5] Goodearl, K. R., and R. B. Warfield, *An Introduction to Noncommutative Noetherian Rings*, 2nd ed., London Mathematical Society Student Texts, Vol. 61, Cambridge University Press, Cambridge, U.K., 1989.

[6] Higman, G., B. H. Neumann, and H. Neumann, Embedding theorems for groups, *J. London Math. Soc.*, **24**, 247-254, 1949.

[7] Hirsch, K. A., On infinite soluble groups I, *Proc. London Math. Soc. (2)*, **44**, 53-60, 1938.

[8] Hirsch, K. A., On infinite soluble groups II, *Proc. London Math. Soc. (2)*, **44**, 336-344, 1938.

[9] Hirsch, K. A., On infinite soluble groups IV, *J. London Math. Soc.*, **27**, 81-85, 1952.

[10] Isaacs, I. M., and D. S. Passman, Groups with representations of bounded degree, *Can. J. Math.*, **16**, 299-309, 1964.

[11] Kaplansky, Groups with representations of bounded degree, *Can. J. Math*, **1**, 105-112, 1949.

[12] Kurosh, A. G., *Theory of Groups*, 2nd English ed., Chelsea, New York, 1960.

[13] Mal'cev, A. I., On certain classes of infinite soluble groups (in Russian), *Mat. Sb.*, **28**, 567-588, 1951. (*Am. Math Soc. Transl. (2)*, **2**, 1-21, 1956.)

[14] McConnell, J. C., and J. C. Robson, *Noncommutative Noetherian Rings*, *Grad. Stud. Math.*, vol. 30, Am. Math. Soc., 1987.

[15] Musson, I. M., Faithful cyclic modules for enveloping algebras and Sklyanin algebras, in *Groups, Rings and Algebras, a Conference in Honor of Don Passman*, *Contemp. Math.*, vol. 420, pp. 269-276, Am. Math. Soc., 2006.

[16] Neumann, B. H., Groups with finite classes of conjugate elements, *Proc. Math Soc. London*, **3**, 178-187, 1951.

[17] Nishigori, On some properties of FC-groups *J. Sci. Hiroshima Univ. Ser. A*, **21**, 99-105, 1957-1958.

[18] Passman, D. S., Group rings satisfying a polynomial identity, *J. Algebra*, **20**, 103-117, 1972.

[19] Passman, D. S., *The Algebraic Structure of Group Rings*, Dover, Mineola, N.Y., 2011.

[20] Robinson, D. J. S., *Finiteness Conditions and Generalized Soluble Groups*, Part 1, Springer-Verlag, New York, 1972.

[21] Vinogradov, A. A., On the free product of ordered groups (in Russian), *Mat. Sb.*, **25**, 163-168, 1949.

[22] Wehrfritz, B.A.F., *Group and Ring Theoretic Properties of Polycyclic Groups*, Algebra and Applications, vol. 10, Springer-Verlag, New York, 2009.

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