August 2014

Option Pricing for a General Stock Model in Discrete Time

Cindy Lynn Nichols
University of Wisconsin-Milwaukee

Option Pricing for a General Stock Model in Discrete Time

by

Cindy Nichols

A Thesis Submitted in
Partial Fulfillment of the
Requirements for the Degree of

Master of Science
in
Mathematics

at

The University of Wisconsin-Milwaukee
August 2014

ABSTRACT

Option Pricing for a General Stock Model in Discrete Time

by

Cindy Nichols

The University of Wisconsin-Milwaukee, 2014
Under the Supervision of Professor Richard H. Stockbridge

As there are no arbitrage opportunities in an efficient market, the seller of an option must find a risk neutral price. This thesis examines different characterizations of this option price. In the first characterization, the seller forms a hedging portfolio of shares of the stock and units of the bond at the time of the option's sale so as to reduce his risk of losing money. Before the option matures, the present value stock price fluctuates in discrete time and, based on those changes, the seller alters the content of the portfolio at the end of each time period. The primal linear program captures the seller's hedging activities. We use linear programming to explore the pricing of options for both the Trinomial Asset Pricing Model and the General Asset Pricing Model, allowing us to consider the pricing of any style of option.

We first look at the Trinomial Asset Pricing Model. This model yields a finite-dimensional linear program and is included to motivate the results for the General Asset Pricing Model. We use the strong duality results for finite-dimensional linear programs to characterize the solution to the primal linear program through the solution of the dual linear program. The dual program can be interpreted as minimizing the expected present value of the contingent claim with respect to measures under which the present value stock price process is a martingale relative to its natural filtration. The dual program provides a second characterization of the option price.

We then present a general asset pricing model in which the present value stock price is a random process. The thesis examines the dual linear program corresponding to the primal linear program arising from the seller's hedging portfolio. The optimal values of the two linear programs are related by weak duality in the general case. In the interpretation of the dual linear program, this paper examines *expectations* and *conditional expectations* of stock prices over time. It is here that the use of *measure theory* in combination with the definition of *conditional expectation* reveals that, even for this general model, our dual optimization problem minimizes the expected present value of the contingent claim over measures under which the present value stock price process is a *martingale*. The validity of strong duality between the primal and dual linear programs is not addressed in this thesis.

At the end, we present possibilities for further work on this model.

Table of Contents

List of Figures

ACKNOWLEDGEMENTS

I would like to thank Prof. Stockbridge. Without his time and patience, this thesis would not have been possible.

Chapter 1

Introduction

We would like to find the market efficient price of a general stock option. The underlying assumption that the financial market consisting of stocks and bonds is efficient dictates that a seller can not profit without risk. If arbitrage were allowed, with the promise of a small risk free profit, a new seller would enter the market listing the asset for just under the current price. More sellers would continue to undercut each other until the asset was sold at the break even point.

A call option is a contract that gives the buyer of the option the right to purchase stock for a strike price (set price) at or before a specified date. If the buyer purchases an option that he can exercise any time up to the specified date, he would be able to watch the market evolve and choose when to exercise the option. If the buyer purchases an option that matures at the specified date, his decision would be dominated by the price of the stock on that date alone. If the current market price were higher than the strike price, the buyer could, and most likely would, choose to purchase the stock for the strike price. If not, the buyer has only lost the amount of money he has spent on the option.

Clearly, the seller benefits if the option is never exercised. The seller's task is to price the option to cover any shortfall between the strike price and the future stock value at the time of maturity. The contingent claim is the difference of stock value and the strike price, when the stock price exceeds the strike price, and zero otherwise. The seller must think about how to hedge his risk of paying the contingent claim. At the time of the option's sale, he will invest the proceeds from the option into a hedging portfolio of bonds and stocks. The seller will manage this portfolio in discrete time. As the bond price is fixed, the stock price will be a major factor in all decisions. As the price of the stock fluctuates, the seller may decide to re-balance by choosing a different mix of bonds and shares of stock. The goal is that the portfolio is self-financing. If the seller must add his own money to finance the portfolio, the option was not priced high enough. Pricing the option largely depends on the analysis of what the contingent claim will be.

Chapter 2

Background Theory and Definitions

This chapter provides the theorems and definitions that are used to explore the topic and write the paper. Although not every theorem or definition will be referenced, they support one another to further the reader's full understanding of the topic. It is also important to note that the process of computing the dual linear program from the primal linear program will not be fully discussed. The reader should acquaint himself with this process before examining the figures provided in the paper.

2.1 Dual Spaces

This section follows the development of both finite- and infinite-dimensional linear programming in Anderson and Nash [1]. We begin with the finite-dimensional case.

Definition 2.1. *For each $k \in \mathbb{N}$, and vectors $x, y \in \mathbb{R}^k$, $x \leq y$ means $x_i \leq y_i$ for all i.*

Definition 2.2. *(Finite Primal Linear Program)*
Let $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$. The finite primal linear program is defined for $x \in \mathbb{R}^n$ to be

$$\begin{array}{lll} \textit{FLP:} & \textit{minimize} & c^T x \\ & \textit{subject to} & Ax \geq b, \\ & & x \textit{ unrestricted.} \end{array}$$

Definition 2.3. *(Dual of the Finite Primal Linear Program) Let an FLP be as given in Definition 2.2. Then its finite dual linear program is defined by*

$$\begin{array}{lll} FLP^*: & \text{maximize} & b^T y \\ & \text{subject to} & A^T y = c, \\ & & y \geq \mathbf{0}. \end{array}$$

In this, $y \geq \mathbf{0}$ if and only if $y_i \geq 0$ for $i = 1, \ldots, m$.

Any x that satisfies the constraints of the FLP and y that satisfies the constraints of the FLP* are called feasible solutions for their respective linear programs.

Theorem 2.4.

(i) (Weak Duality) If x and y are feasible solutions for FLP and FLP* respectively, then $c^T x \geq b^T y$.

(i) (Strong Duality) If x^* is an optimal feasible solution of the FLP then there exists y^* which is optimal for FLP* and $c^T x^* = b^T y^*$.

The above definitions and theorem apply to finite dimensional linear programs. We now establish the formulation for infinite-dimensional linear programs since these will be needed in Chapter 4.

Definition 2.5. *Let X be a real vector space. A linear functional on X is a linear map from X to $\mathbb{R}$. The set of all linear functionals, with the operations addition and scalar multiplication is the vector space called the algebraic dual space of X, denoted X^*.*

Definition 2.6. *Let X be real linear vector space and P be a convex cone in X (a convex cone is a set closed under vector addition and multiplication by positive scalars). Define the partial order on X by*

$$x \leq y \ \text{ if } \ y - x \in P, \ (x, y \in X).$$

Notation 2.7. *In each real linear vector space, the null vector will be denoted by θ. The reader is cautioned that θ may be used for different vector spaces.*

Remark 2.8. *Let P be defined as in Definition 2.6. Since it is a tautology that $x \in P$ if and only if $x \geq \theta$, P is called the positive cone.*

Definition 2.9. *Let X and V be two real vector spaces with a bilinear form defined on $X \times V$, that is a function from $X \times V$ to $\mathbb{R}$ which we write as $\langle \cdot, \cdot \rangle$, with $\langle x, v \rangle$ a linear function of x for each fixed v and a linear function of v for each fixed x. If*

(i) *for each $x \neq \theta$ there is some $v \in V$ with $\langle x, y \rangle \neq 0$, and*

(ii) *for each $v \neq \theta$ there is some $x \in X$ with $\langle x, y \rangle \neq 0$,*

then the pair of spaces (X, V) is called a dual pair.

Definition 2.10. *Let (X, V) and (Z, Q) be two dual pairs of vector spaces. Let A be a linear map from X to Z. Then the adjoint of A, denoted by A^*, is the map from Q to X^* defined by the relationship*

$$\langle x, A^*q \rangle = \langle Ax, q \rangle, \text{ for all } x \in X, q \in Q.$$

In this, we have slightly abused notation by using $\langle \cdot, A^*q \rangle$ to denote the linear functional $A^*q \in X^*$. Notice that the linearity of A and bilinearity of $\langle \cdot, \cdot \rangle$ implies that for $q_1, q_2 \in Q$, $x_1, x_2 \in X$ and real constants a_1, a_2, c_1, and c_2

$$\begin{aligned} A^*[c_1 q_1 + c_2 q_2](a_1 x_1 + a_2 x_2) &= a_1 c_1 \langle Ax_1, q_1 \rangle + a_1 c_2 \langle Ax_1, q_2 \rangle \\ &\quad + a_2 c_1 \langle Ax_2, q_1 \rangle + a_2 c_2 \langle Ax_2, q_2 \rangle \\ &= a_1 c_1 A^* q_1(x_1) + a_1 c_2 A^* q_2(x_1) \\ &\quad + a_2 c_1 A^* q_2(x_1) + a_2 c_2 A^* q_2(x_2) \end{aligned}$$

so A^* exists as a linear mapping of Q to X^*.

Remark 2.11. *Let (X, V) be a dual pair of vector spaces. We refer the reader to page 36 of Anderson and Nash (1987) for the definition of the weak topology $\sigma(X, V)$ on the vector space X.*

Proposition 2.12. *A^* maps Q into V if and only if A is continuous with respect to the topologies $\sigma(X,V)$ and $\sigma(Z,Q)$.*

Remark 2.13. *We assume the vector spaces X and Z are endowed with the weak topologies $\sigma(X,V)$ and $\sigma(Z,Q)$, respectively. Then A is a continuous map from X to Z and, further, A^* maps Q into V. A full discussion of this can be found in Anderson and Nash (1987).*

Definition 2.14. *Let (X,V) be a dual pair and let P be the positive cone for X. The dual cone of P is defined by*

$$P^* = \{v \in V |\ \langle x,v\rangle \geq 0 \textit{ for all } \ x \in P\}.$$

Definition 2.15. *Let (X,V) and (Z,Q) be two dual pairs of vector spaces. Let P and W be the positive cones in X and Z respectively. Let our topologies be $\sigma(X,V)$ and $\sigma(Z,Q)$, then A is a continuous map from X to Z. We define an inequality constrained linear program called IP as*

$$\begin{array}{lll} \textit{IP:} & \textit{minimize} & \langle x,c\rangle \\ & \textit{subject to} & Ax \geq b, \\ & & x \in X. \end{array}$$

Given the dual cones P^* and W^* of P and W respectively, we define the resulting dual linear program of IP, IP^*,as

$$\begin{array}{lll} \textit{IP}^*\textit{:} & \textit{maximize} & \langle b,q\rangle \\ & \textit{subject to} & -A^*q + c = \theta, \\ & & q \in W^*. \end{array}$$

Theorem 2.16. *(Weak Duality) If IP and IP* both have feasible solutions, then the value of IP is greater than or equal to the value of IP* and both values are finite.*

2.2 Probability and Measure Theory Background

The definitions and theorems of this section can be found in any measure theoretic probability text, such as Billingsley [2]. For completeness, we give standard definitions. In this section, we follow standard probabilistic notation so P denotes a probability measure, no longer a positive cone, and X will denote a random variable, not a vector space. We also assume that $\Omega \neq \emptyset$.

Definition 2.17. *A σ-algebra $\mathcal{F}$ on a set Ω is a collection of subsets of Ω that is closed under complementation and countably many unions. The pair $(\Omega, \mathcal{F})$ is called a measurable space and the sets of $\mathcal{F}$ are called measurable sets.*

Definition 2.18. *Let Ω be a set and $\mathcal{F}$ be a σ-algebra on Ω. A function $\mu : \mathcal{F} \to \mathbb{R}^+$ is a measure if it satisfies the following property of Countable Additivity: namely, for all countable collections $\{G_i\}_{i\in\mathbb{N}}$ of pairwise disjoint sets in $\mathcal{F}$,*

$$\mu\left(\bigcup_{i\in\mathbb{N}} G_i\right) = \sum_{i\in\mathbf{N}} \mu(G_i).$$

Remark 2.19. *A triple $(\Omega, \mathcal{F}, \mu)$ is called a measure space.*

Definition 2.20. *A probability measure, P, on a measurable space $(\Omega, \mathcal{F})$, is a measure that assigns a mass of 1 to the entire space, $P(\Omega) = 1$.*

Definition 2.21. *A probability space $(\Omega, \mathcal{F}, P)$ is a measurable space $(\Omega, \mathcal{F})$ with a probability measure P defined on $\mathcal{F}$.*

Definition 2.22. *The space of all finite measures on measurable space $(\Omega, \mathcal{F})$ is denoted $\mathcal{M}(\Omega, \mathcal{F})$.*

Definition 2.23. *Let $(\Omega, \mathcal{F})$ be a measurable space and $(\mathbb{R}, \mathcal{B})$ be the real numbers with the Borel σ-algebra. A function $X : \Omega \longrightarrow \mathbb{R}$ is said to be $\mathcal{F}$-measurable if:*

$$X^{-1}(B) = \{\omega \in \Omega | \; X(\omega) \in B\} \in \mathcal{F}, \quad \textit{for all} \;\; B \in \mathcal{B}.$$

Definition 2.24. *The space $L^1(\Omega, \mathcal{F}, \mu)$ is the set of all measurable functions from Ω to $\mathbb{R}$ that satisfies the following condition:*

$$\int_\Omega |f| d\mu < \infty.$$

When $(\Omega, \mathcal{F}, P)$ is a probablity space, a $\mathcal{F}$-measurable function X will be called a random variable.

Definition 2.25. *The space* $L^\infty(\Omega, \mathcal{F}, \mu)$ *is the set of all measurable functions* X *for which there exists some constant* $M < \infty$ *such that* $\mu(\{|X| > M\}) = 0$.

Definition 2.26. *If* $X \in L^1(\Omega, \mathcal{F}, P)$*, then the expected value of* X*, denoted* $E[X]$ *is given by*

$$E[X] = \int_\Omega X dP.$$

Theorem 2.27. *Suppose that* $X \in L^1(\Omega, \mathcal{F}, P)$ *and that* $\mathcal{G}$ *is a sub-*σ*-algebra in* $\mathcal{F}$*. There exists a random variable* $E[X|\mathcal{G}]$ *called the conditional expected value of* X *given* $\mathcal{G}$ *having these two properties:*

(i) $E[X|\mathcal{G}]$ *is measurable and integrable;*

(ii) $E[X|\mathcal{G}]$ *satisfies the equation*

$$\int_G E[X|\mathcal{G}]dP = \int_G X dP, \text{ for all } G \in \mathcal{G}.$$

Definition 2.28. *Let* X *be a random variable and* $\mathcal{B}$ *the Borel* σ*-algebra. Then the* σ*-algebra generated by* X*, denoted* $\sigma(X)$ *is given by*

$$\sigma(X) = \{X^{-1}(S)|\ S \in \mathcal{B}\}.$$

Definition 2.29. *Given a probability space* $(\Omega, \mathcal{F}, P)$ *and* $\mathbb{T} \subset \mathbb{R}^+$*, a stochastic process* X *is a collection*

$$\{X_t|\ t \in \mathbb{T}\},$$

where each X_t *is a random variable on* Ω*.*

Definition 2.30. *Let* $(\Omega, \mathcal{F})$ *be a measurable space and let* $T \subset \mathbb{R}^+$*. A filtration is a sequence of* σ*-algebras,* $\{\mathcal{F}_t\}_{t \in \mathbb{T}}$*, satisfying the following conditions:*

(i) $\mathcal{F}_t \subset \mathcal{F}$ *for each* $t \in \mathbb{T}$*;*

(ii) *for each* t_1*,* $t_2 \in \mathbb{T}$*,* $t_1 \leq t_2$ *implies* $\mathcal{F}_{t_1} \subseteq \mathcal{F}_{t_2}$*.*

Definition 2.31. *The natural filtration* $\{\mathcal{F}_t\}$ *generated by the stochastic process* X *is defined for* $t \in \mathbb{T}$ *by*

$$\mathcal{F}_t = \sigma(\{X_s^{-1}(B) | s \in \mathbb{T},\ s \leq t,\ B \in \mathcal{B}\}).$$

Definition 2.32. *Let* $(\Omega, \mathcal{F}, P)$ *be a probability space with filtration* $\mathcal{F}_t$ *and let* $X = \{X_t |\ t \in \mathbb{T}\}$ *be a stochastic process. Then* X *is a martingale with respect to a filtration* $\{\mathcal{F}_t\}$ *if:*

(i) *for each* t, X_t *is* $\mathcal{F}_t$*-measurable;*

(ii) *for each* t, $X_t \in L^1(\Omega, \mathcal{F}_t, P)$*; and*

(iii) *for* $s < t$, $X_s = E[X_t |\ \mathcal{F}_s]$.

Chapter 3

The Trinomial Asset Pricing Model

We will now develop the finite linear program for the Trinomial Asset Pricing Model in discrete time. For simplicity, we restrict the time horizon to the period $\mathbb{T} = \{0, 1, 2\}$ for this model but will allow a more general time horizon in Chapter 4. The market consists of two assets: a bond and a stock. The value of the bond is fixed throughout, earning interest at the rate r. The interest rate offered in our bond will reflect inflation in the market. For this model, the stock price process $S = \{S_0, S_1, S_2\}$ branches from its current price to three prices in the succeeding time period. We denote the possible prices by S_{nj} in which $n = 0, 1, 2$ denotes the time and $j = 1, \ldots, 3^n$ gives the number of possible values at time n; at time $n = 0$, the price is denoted S_0. The value of the option at the time of maturity is given by C_{2j} with $j = 1, \ldots, 3^2$.

The primal linear program models the actions of the option seller to invest the money paid for the option in the market so as to hedge his risk of paying the contingent claim at the time of maturity. Let $B_0x_0 + S_0y_0$ represent our initial hedging portfolio where B_0 is the fixed bond price, x_0 the number of units of bond, S_0 is the stock price of a single share of stock, y_0 the number of shares of stock. At each time period past 0, there are two different portfolios happening almost simultaneously. For instance, at time 1 there exists the portfolio that has just evolved and has the same number of stock shares and bond units as were initially invested in the market. Then the portfolio is adjusted by the investor, a process we will refer to as re-balancing, and the money in that portfolio is re-invested into a new number of stock shares and bond units. This process of re-balancing the portfolio ensures that it is self-financing. Each re-balancing of the portfolio gives an equality constraint in the primal linear program. The seller's goal is to have the final portfolio be worth at least as much as the contingent claim and therefore contributes inequality constraints on the portfolio that depend on the final stock price. Recalling that an ideal market does not allow for arbitrage, the initial portfolio value $B_0 x_0 + S_0 y_0$ must be as small as possible and thus the primal linear program minimizes this expression as its objective function.

Instead of watching the bond's value accrue by $(1+r)^n$ by time n, we will divide our constraints by this quantity in order to see the entire model from the point of view of present value at time 0. Formally, the non-negative present value stock price process $\tilde{S} = \{\tilde{S}_0, \tilde{S}_1, \tilde{S}_2\}$ takes values $\tilde{S}_{nj} = \frac{S_{nj}}{(1+r)^n}$, $j = 1, ..., 3^n$ and $n = 0, 1, 2$. (In reality, modeling the stock this way has a very limited ability to replicate the actual market where stock prices vary by the penny.) At the time of maturity $N = 2$, the present value of the contingent claim is

$$\tilde{C}_{2j} = \frac{C_{2j}}{(1+r)^2}, \quad j = 1, ..., 9.$$

The primal linear program is now given in Figure 3.1. The first three constraints show the re-balancing of the portfolio at time 1. The rest of the constraints, all in terms of present value, show the final portfolio being chosen greater than or equal to the contingent claim.

We choose dual variables for each constraint in the primal linear program. Since the first three constraints of the primal are equality constraints, the dual variables q_{11}, q_{12}, q_{13} are unrestricted. The inequality constraints result in non-negative dual variables so as a result the final constraints of the primal linear program result in dual variable $q_{2j} \geq 0$. The dual of the primal linear program is given in Figure 3.2.

$$
\left\{
\begin{array}{l}
\text{Minimize} \\
\quad B_0x_0 + S_0y_0 \\
\text{Subject To} \\
\quad B_0x_0 + \tilde{S}_{11}y_0 - B_0x_{11} - \tilde{S}_{11}y_{11} = 0 \\
\quad B_0x_0 + \tilde{S}_{12}y_0 - B_0x_{12} - \tilde{S}_{12}y_{12} = 0 \\
\quad B_0x_0 + \tilde{S}_{13}y_0 - B_0x_{13} - \tilde{S}_{13}y_{13} = 0 \\
\quad B_0x_{11} + \tilde{S}_{21}y_{11} \geq \tilde{C}_{21} \\
\quad B_0x_{11} + \tilde{S}_{22}y_{11} \geq \tilde{C}_{22} \\
\quad B_0x_{11} + \tilde{S}_{23}y_{11} \geq \tilde{C}_{23} \\
\quad B_0x_{12} + \tilde{S}_{24}y_{12} \geq \tilde{C}_{24} \\
\quad B_0x_{12} + \tilde{S}_{25}y_{12} \geq \tilde{C}_{25} \\
\quad B_0x_{12} + \tilde{S}_{26}y_{12} \geq \tilde{C}_{26} \\
\quad B_0x_{13} + \tilde{S}_{27}y_{13} \geq \tilde{C}_{27} \\
\quad B_0x_{13} + \tilde{S}_{28}y_{13} \geq \tilde{C}_{28} \\
\quad B_0x_{13} + \tilde{S}_{29}y_{13} \geq \tilde{C}_{29} \\
x_0, y_0, x_{11}, y_{11}, x_{12}, y_{12}, x_{13}, y_{13} \quad \text{unrestricted.}
\end{array}
\right.
$$

Figure 3.1: The Finite Primal Linear Program for the Trinomial Model in Two Time Periods

$$
\left\{
\begin{array}{llllllllll}
\text{Maximize} & & & & & & & & & \\
\tilde{C}_{21}q_{21} & +\tilde{C}_{22}q_{22}+\tilde{C}_{23}q_{23} & +\tilde{C}_{24}q_{24}+\tilde{C}_{25}q_{25}+\tilde{C}_{25}q_{25} & +\tilde{C}_{27}q_{27}+\tilde{C}_{28}q_{28}+\tilde{C}_{29}q_{29} & & & & & & \\
\text{Subject To} & & & & & & & & & \\
B_0q_{11} & +B_0q_{12}+B_0q_{13} & & & & & & & & =B_0 \\
\tilde{S}_{11}q_{11} & +\tilde{S}_{12}q_{12}+\tilde{S}_{13}q_{13} & & & & & & & & =S_0 \\
-B_0q_{11} & & +B_0q_{21}+B_0q_{22}+B_0q_{23} & & & & & & & =0 \\
-\tilde{S}_{11}q_{11} & & +\tilde{S}_{21}q_{21}+\tilde{S}_{22}q_{22}+\tilde{S}_{23}q_{23} & & & & & & & =0 \\
 & -B_0q_{12} & & +B_0q_{24}+B_0q_{25}+B_0q_{26} & & & & & & =0 \\
 & -\tilde{S}_{12}q_{12} & & +\tilde{S}_{24}q_{24}+\tilde{S}_{25}q_{25}+\tilde{S}_{26}q_{26} & & & & & & =0 \\
 & \quad -B_0q_{13} & & & +B_0q_{27}+B_0q_{28}+B_0q_{29} & & & & & =0 \\
 & \quad -\tilde{S}_{13}q_{13} & & & +\tilde{S}_{27}q_{27}+\tilde{S}_{28}q_{28}+\tilde{S}_{29}q_{29} & & & & & =0 \\
q_{11},q_{12},q_{13} & \text{unresrticted}, \quad q_{2j}\geq 0 & & & & & & & &
\end{array}
\right.
$$

Figure 3.2: The Dual of the Finite Linear Program

Theorem 3.1. *Let Q denote the set of all feasible*

$$q := (q_{11}, q_{12}, q_{13}, q_{21}, q_{22}, q_{23}, q_{24}, q_{25}, q_{26}, q_{27}, q_{28}, q_{29})$$

for the dual linear program. Then for each $q \in Q$

(i) (q_{11}, q_{12}, q_{13}) *is a probability measure on the time* 1 *stock prices* $\{\tilde{S}_{11}, \tilde{S}_{12}, \tilde{S}_{13}\}$*; and*

(ii) $(q_{21}, q_{22}, q_{23}, q_{24}, q_{25}, q_{26}, q_{27}, q_{28}, q_{29})$ *is a probability measure on the time* 2 *stock prices* $\{\tilde{S}_{21}, \tilde{S}_{22}, \tilde{S}_{23}, \tilde{S}_{24}, \tilde{S}_{25}, \tilde{S}_{26}, \tilde{S}_{27}, \tilde{S}_{28}, \tilde{S}_{29}\}$.

Proof. Our first concern is to show that all of the dual variables are positive. The bond constraint

$$-B_0 q_{11} + B_0 q_{21} + B_0 q_{22} + B_0 q_{23} = 0$$

is equivalent to

$$q_{21} + q_{22} + q_{23} = q_{11}. \tag{3.1}$$

Recall the dual variables q_{2j} are positive so the left side of the (3.1) shows that q_{11} is also positive. Similarly, it follows that q_{12}, and q_{13} are positive.

Now by the first constraint is $B_0 q_{11} + B_0 q_{12} + B_0 q_{13} = B_0$ which simplifies to

$$q_{11} + q_{12} + q_{13} \quad = \quad 1. \tag{3.2}$$

Thus the dual variables at time 1 represent probability measures on the outcomes $\{\tilde{S}_{11}, \tilde{S}_{12}, \tilde{S}_{13}\}$ as claimed; that is, $q_{11} = P(\tilde{S}_1 = \tilde{S}_{11})$, $q_{12} = P(\tilde{S}_1 = \tilde{S}_{12})$ and $q_{13} = P(\tilde{S}_1 = \tilde{S}_{13})$.

Next, summing all of the constraints pertaining to bonds from period 2 gives

$$\begin{aligned}
&-B_0 q_{11} + B_0 q_{21} + B_0 q_{22} + B_0 q_{23} \\
&-B_0 q_{12} + B_0 q_{24} + B_0 q_{25} + B_0 q_{26} \\
&-B_0 q_{13} + B_0 q_{27} + B_0 q_{28} + B_0 q_{29} \quad = \quad 0
\end{aligned}$$

or equivalently using (3.2)

$$q_{21} + q_{22} + q_{23} + q_{24} + q_{25} + q_{26} + q_{27} + q_{28} + q_{29} = q_{11} + q_{12} + q_{13} = 1.$$

Since each $q_{2j} \geq 0$, the set $\{q_{21}, q_{22}, q_{23}, q_{24}, q_{25}, q_{26}, q_{27}, q_{28}, q_{29}\}$ represents a probability distribution on $\{\tilde{S}_{21}, \tilde{S}_{22}, \tilde{S}_{23}, \tilde{S}_{24}, \tilde{S}_{25}, \tilde{S}_{26}, \tilde{S}_{27}, \tilde{S}_{28}, \tilde{S}_{29}\}$ as claimed. $\square$

From Theorem 3.1, we can interpret the objective function of the dual linear program as the expectation of the present value of the contingent claim under the feasible probability measure $q \in Q$. Also, by Theorem 2.4, if we have an optimal solution for our primal FLP then we have an optimal solution for our dual program. Further, when these optimal values are entered into their respective objective functions, the solutions are equal.

Remark 3.2. *The analysis in the proof of Theorem 3.1 provides more information that will be useful. Dividing both sides of (3.1) by q_{11}, we get*

$$\frac{q_{21}}{q_{11}} + \frac{q_{22}}{q_{11}} + \frac{q_{23}}{q_{11}} = 1.$$

In this model, the possible values of $\tilde{S}_2$ when $\tilde{S}_1 = \tilde{S}_{11}$ are $\{\tilde{S}_{21}, \tilde{S}_{22}, \tilde{S}_{23}\}$. Hence

$$\begin{aligned}
\frac{q_{21}}{q_{11}} &= P(\tilde{S}_2 = \tilde{S}_{21} | \, \tilde{S}_1 = \tilde{S}_{11}), \\
\frac{q_{22}}{q_{11}} &= P(\tilde{S}_2 = \tilde{S}_{22} | \, \tilde{S}_1 = \tilde{S}_{11}), \\
\frac{q_{23}}{q_{11}} &= P(\tilde{S}_2 = \tilde{S}_{23} | \, \tilde{S}_1 = \tilde{S}_{11}).
\end{aligned}$$

Similar statements can be made about the possible values of $\tilde{S}_2$ when $\tilde{S}_1 = \tilde{S}_{12}$ and $\tilde{S}_1 = \tilde{S}_{13}$ and thus

$$\begin{aligned}
\frac{q_{24}}{q_{12}} + \frac{q_{25}}{q_{12}} + \frac{q_{26}}{q_{12}} &= 1, \\
\frac{q_{27}}{q_{13}} + \frac{q_{28}}{q_{13}} + \frac{q_{29}}{q_{13}} &= 1.
\end{aligned}$$

These equations represent conditional probability measures on the values of $\tilde{S}_2$ given $\tilde{S}_1$.

We now turn to an examination of the dual constraints pertaining to the stock prices.

Theorem 3.3. *For each $q \in Q$, the present value stock price process $\tilde{S}$ is a martingale.*

Proof. Let $q \in Q$ be chosen arbitrarily. First we see that

$$\tilde{S}_{11}q_{11} + \tilde{S}_{12}q_{12} + \tilde{S}_{13}q_{13} = S_0. \tag{3.3}$$

The expected value of the present value stock price at time 1 under the probability measure $q \in Q$ is S_0, $E^q[\tilde{S}_1|S_0] = S_0$. We will continue to build on this fact to show that our two period model satisfies the condition of being a martingale in Definition 2.32.

Next we will examine the constraint that pertains to stock values at time 2 evolving from time 1 stock values being S_{11}.

$$-\tilde{S}_{11}q_{11} + \tilde{S}_{21}q_{21} + \tilde{S}_{22}q_{22} + \tilde{S}_{23}q_{23} = 0$$

which is

$$\tilde{S}_{21}\frac{q_{21}}{q_{11}} + \tilde{S}_{22}\frac{q_{22}}{q_{11}} + \tilde{S}_{23}\frac{q_{23}}{q_{11}} = \tilde{S}_{11}.$$

This is the conditional expectation of the present value stock price at time 2 given $\tilde{S}_1 = \tilde{S}_{11}$, $E^q[\tilde{S}_2|\tilde{S}_1 = \tilde{S}_{11}] = \tilde{S}_{11}$. Similarly we can see that $E^q[\tilde{S}_2|\tilde{S}_1 = \tilde{S}_{12}] = \tilde{S}_{12}$, and $E^q[\tilde{S}_2|\tilde{S}_1 = \tilde{S}_{13}] = \tilde{S}_{13}$.

To finalize our argument that our limited 2 period model satisfies the definition of a martingale, we must show that $E^q[\tilde{S}_2|S_0] = S_0$. Now we will sum all of the constraints pertaining to stock from the second time period to find the conditional expected value of the present value of stock at time period 2 given S_0, $E^q[S_2|S_0] = S_0$. The sum is as follows:

$$\begin{aligned} -\tilde{S}_{11}q_{11} + \tilde{S}_{21}q_{21} + \tilde{S}_{22}q_{22} + \tilde{S}_{23}q_{23} - \tilde{S}_{12}q_{12} + \tilde{S}_{24}q_{24} + \tilde{S}_{25}q_{25} + \tilde{S}_{26}q_{26} \\ - \tilde{S}_{13}q_{13} + \tilde{S}_{27}q_{27} + \tilde{S}_{28}q_{28} + \tilde{S}_{29}q_{29} = 0. \end{aligned}$$

Therefore,

$$\begin{aligned} \tilde{S}_{21}q_{21} + \tilde{S}_{22}q_{22} + \tilde{S}_{23}q_{23} + \tilde{S}_{24}q_{24} + \tilde{S}_{25}q_{25} + \tilde{S}_{26}q_{26} + \tilde{S}_{27}q_{27} + \tilde{S}_{28}q_{28} + \tilde{S}_{29}q_{29} \\ = \tilde{S}_{11}q_{11} + \tilde{S}_{12}q_{12} + \tilde{S}_{13}q_{13}. \end{aligned}$$

The left hand side is $E^q[\tilde{S}_2|S_0]$ and by (3.3) the right hand side is S_0. □

In summary, the strong duality relationship between the primal FLP and the dual FLP* tells us that the value of the minimum hedging portfolio is the maximum of the expectation of the present value contingent claim over all the probability measures $q \in Q$ which make the present value of the stock price a martingale.

In the next chapter, we will investigate a general discrete time stock model. To set the stage, we interpret the primal linear program in Figure 3.1 and the dual linear program in Figure 3.2 using the general framework of dual pairs of vector spaces in Chapter 2. We begin with the primal linear program.

Define the space Ω to be

$$\tilde{S}_0 \times \left(\left(\{\tilde{S}_{11}\} \times \{\tilde{S}_{21}, \tilde{S}_{22}, \tilde{S}_{23}\} \right) \cup \left(\{\tilde{S}_{12}\} \times \{\tilde{S}_{24}, \tilde{S}_{25}, \tilde{S}_{26}\} \right) \cup \left(\{\tilde{S}_{13}\} \times \{\tilde{S}_{27}, \tilde{S}_{28}, \tilde{S}_{29}\} \right) \right)$$

and let $\mathcal{F}$ denote the discrete σ-algebra generated by the individual points. Notice that the natural filtration of the present value stock price process $\tilde{S}$ has $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1$ generated by the sets

$$\begin{array}{c} \{(\tilde{S}_0, \tilde{S}_{11}, \tilde{S}_{21}), (\tilde{S}_0, \tilde{S}_{11}, \tilde{S}_{22}), (\tilde{S}_0, \tilde{S}_{11}, \tilde{S}_{23})\}, \\ \{(\tilde{S}_0, \tilde{S}_{12}, \tilde{S}_{24}), (\tilde{S}_0, \tilde{S}_{12}, \tilde{S}_{25}), (\tilde{S}_0, \tilde{S}_{12}, \tilde{S}_{26})\}, \\ \{(\tilde{S}_0, \tilde{S}_{13}, \tilde{S}_{27}), (\tilde{S}_0, \tilde{S}_{13}, \tilde{S}_{28}), (\tilde{S}_0, \tilde{S}_{13}, \tilde{S}_{29})\} \end{array}$$

and $\mathcal{F}_2 = \mathcal{F}$.

Now the decision variables are the values $x = (x_0, y_0, x_{11}, y_{11}, x_{12}, y_{12}, x_{13}, y_{13})$. Notice that each of the variables is finite and that the pair (x_0, y_0) is based on the initial price S_0 of the stock whereas the pairs (x_{11}, y_{11}), (x_{12}, y_{12}) and (x_{13}, y_{13}) depend on $\tilde{S}_1$ taking values $\tilde{S}_{11}$, $\tilde{S}_{12}$ and $\tilde{S}_{13}$, respectively. Thus the real vector space X is the space of vector-valued functions $x = (x_0, y_0, x_1, y_1)$ mapping Ω to $\mathbb{R}^4$ in which the pair (x_0, y_0) is $\mathcal{F}_0$-measurable and similarly (x_1, y_1) is $\mathcal{F}_1$-measurable. Since Ω is a finite set, x is a bounded function.

The linear transformation A maps X using the matrix

$$\begin{pmatrix} B_0 & \tilde{S}_1 & -B_0 & -\tilde{S}_1 \\ 0 & 0 & B_0 & \tilde{S}_2 \end{pmatrix}$$

so the space Z is $L^1(\Omega, \mathcal{F}_1) \times L^1(\Omega, \mathcal{F}_2)$. The right-hand side vector of the primal problem is the vector $b = (0, \tilde{C}_2)$ and the objective function has coefficients $c = (B_0, S_0, 0, 0)$. With these selections, the primal problem in Figure 3.1 has the form of the primal linear program in Definition 2.15 in which the constraints in the Chapter 2 formulation are required to hold pointwise for each function.

To determine the dual linear program, the space Q of Definition 2.10, distinct from but related to the feasible set Q of Theorem 3.1, is the space

$$Q = \mathcal{M}(\Omega, \mathcal{F}_1) \times \mathcal{M}(\Omega, \mathcal{F}_2)$$

and the bilinear form $\langle \cdot, \cdot \rangle$ is defined for $z = (z_1, z_2)$ and $q = (q_1, q_2)$ by

$$\langle z, q \rangle = \int z_1 \, dq_1 + \int z_2 \, dq_2.$$

The definition of the adjoint A^* in Definition 2.10 now determines the space V. This space will be clearly identified in Chapter 4 in which we analyze the general asset pricing model.

Chapter 4

General Asset Pricing Model

We will now discuss the General Asset Pricing Model in discrete time. The model consists of a present value stock price process $\tilde{S} = \{\tilde{S}_t | t \in \mathbb{N}\}$ defined on some probability space $(\Omega, \mathcal{F}, P)$ and a (present value) contingent claim $\tilde{C}_N$ which matures at time $N \in \mathbb{N}$. The decisions for the hedging portfolio are pairs (x_t, y_t), for $t = 0, 1, \ldots, N-1$, in which x_t denotes the bond units and y_t is the number of shares of stock in the portfolio at time t. The goal of the hedging portfolio is to avoid any loss by the seller of option so the present value of the final portfolio is required to be at least as great as $\tilde{C}_N$.

There are many similarities in the set up between the trinomial and general model. This is the reason why the trinomial model does such a good job at introducing the general model. Again, the initial hedging portfolio, $B_0x_0 + S_0y_0$, of stocks and bonds is equal to the initial price of the option. The portfolio is the seller's insurance against paying the contingent claim at the time of the stock option's maturity. The same process of re-balancing of the portfolio happens at the end of each time period. For the same reasons as proposed earlier, we will price our stocks, bonds, and contingent claims in present value units.

There are some key differences between the two models. Although the linear program for the trinomial can be viewed for more than two time periods, it isn't until this juncture that we see the linear program set up for the contingent claim to mature in N time periods. Whereas before we allowed the stock price to fluctuate to a finite number of possible values, now we will view the present value stock price at each time t, $\tilde{S}_t$, as a non-negative integrable random variable. It is still true that the re-balancing of the portfolio is dictated by the present value stock price, however, now the number of shares of stocks, y_t, and number of units of bonds, x_t, are bounded random variables.

The hedging portfolio leads to the primal linear program given in Figure 4.1. Notice the first $N-1$ constraints show the re-balancing of the portfolio at each time period. The last constraint requires the portfolio's value to be greater than or equal to the present value of the contingent claim, $\tilde{C}_N$, at time of maturity N.

Remark 4.1. *Given a probability space $(\Omega, \mathcal{F}, P)$, random variables are functions mapping elements ω, in Ω, to $\mathbb{R}$ and, although, this cumbersome notation is suppressed they may be represented $S_t(\omega)$, $x_t(\omega)$, $y_t(\omega)$, $C_N(\omega)$.*

Let $\{\mathcal{F}_t\}$ be the natural filtration of the present value stock price process $\tilde{S}$ as given in Definition 2.31. Then $\tilde{S}_t$ is an element of $L^1(\Omega, \mathcal{F}_t, P)$, for each t. Now x_t and y_t depend on $\tilde{S}_t$. Thus for each $t = 0, 1, \dots, N-1$, bounded random variables x_t and y_t are chosen from the set of functions $L^\infty(\Omega, \mathcal{F}_t, P)$. Provided by our LP, elements of the vector space X look like $(x_0, y_0, x_1, y_1, \dots, x_{N-1}, y_{N-1})$ and each pair (x_t, y_t) is a $\mathcal{F}_t$-measurable random vector. Hence, the space X in the definition of the primal linear program in Definition 2.15 is

$$X = \prod_{t=0}^{N-1} L^\infty(\Omega, \mathcal{F}_t, P) \times L^\infty(\Omega, \mathcal{F}_t, P).$$

Referring to the primal linear program in Figure 4.1, the mapping A of the vector space X into the vector space Z is given by the matrix

$$A = \begin{pmatrix} B_0 & \tilde{S}_1 & -B_0 & -\tilde{S}_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & B_0 & \tilde{S}_2 & -B_0 & -\tilde{S}_2 & 0 & 0 & 0 & 0 \\ \vdots & & & & & \ddots & \ddots & & & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & B_0 & \tilde{S}_{N-1} & B_0 & -\tilde{S}_{N-1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_0 & \tilde{S}_N \end{pmatrix}.$$

We examine the first constraint

$$B_0 x_0 + \tilde{S}_1 y_0 - B_0 x_1 - \tilde{S}_1 y_1 = 0.$$

$$
\left\{
\begin{array}{l}
\text{Minimize} \\
\quad B_0 x_0 \quad + S_0 y_0 \\
\text{Subject To} \\
\begin{array}{lllll}
B_0 x_0 + \tilde{S}_1 y_0 - B_0 x_1 - \tilde{S}_1 y_1 & & & = & 0 \\
\qquad B_0 x_1 + \tilde{S}_2 y_1 - B_0 x_2 - \tilde{S}_2 y_2 & & & = & 0 \\
\qquad\qquad \ddots \qquad \ddots & & & \vdots & \vdots \\
& B_0 x_{N-2} + \tilde{S}_{N-1} y_{N-2} - B_0 x_{N-1} - \tilde{S}_{N-1} y_{N-1} & & = & 0 \\
& \qquad B_0 x_{N-1} + \tilde{S}_N y_{N-1} & & \geq & \tilde{C}_N
\end{array} \\
x_0, y_0, x_1, y_1, \ldots, x_{N-1}, y_{N-1} \text{ unrestricted.}
\end{array}
\right.
$$

Figure 4.1: The Primal Linear Program for the General Model in Discrete Time

By the nesting of our natural filtration, x_0 and y_0 are both $\mathcal{F}_0$-measurable and $\mathcal{F}_1$-measurable, while x_1 and y_1 are only $\mathcal{F}_1$-measurable so z_1 is in $L^1(\Omega, \mathcal{F}_1, P)$. Looking at the next bond constraint

$$B_0 x_1 + \tilde{S}_2 y_1 - B_0 x_2 - \tilde{S}_2 y_2 = 0$$

Similarly, x_1 and y_1 are both $\mathcal{F}_1$-measurable and $\mathcal{F}_2$-measurable, while x_2 and y_2 are only $\mathcal{F}_2$-measurable. Then z_2 is in $L^1(\Omega, \mathcal{F}_2, P)$. This can be done for the remaining constraints and thus the space Z is

$$Z = L^1(\Omega, \mathcal{F}_1, P) \times L^1(\Omega, \mathcal{F}_2, P) \times \cdots \times L^1(\Omega, \mathcal{F}_N, P)$$

and elements of Z are of the form $(z_1, z_2, \cdots, z_N)$.

Turning to the derivation of the dual linear program, define

$$Q = \mathcal{M}(\Omega, \mathcal{F}_1) \times \mathcal{M}(\Omega, \mathcal{F}_2) \ldots \times \mathcal{M}(\Omega, \mathcal{F}_N)$$

and for $z = (z_1, z_2, \ldots, z_N) \in Z$ and $q = (q_1, q_2, \ldots, q_N) \in Q$, define the bilinear form on the dual pair (Z, Q) to be

$$\langle z, q \rangle = \int z_1 dq_1 + \int z_2 dq_2 + \cdots + \int z_N dq_N.$$

We will work with the relationship, $\langle Ax, q \rangle = \langle x, A^* q \rangle$, in Definition 2.10 to help the reader understand how we acquired our dual linear program in Figure 4.2. Recall $x \in X$ is $x = (x_0, y_0, x_1, y_1, \ldots, x_{N-1}, y_{N-1})$. To form the dual linear program, we determine the adjoint A^*.

$$\begin{aligned}
\langle Ax, q \rangle \; = \; & \int (B_0 x_0 + S_1 y_0 - B_0 x_1 - S_1 y_1) dq_1 \\
& + \int (B_0 x_1 + S_2 y_1 - B_0 x_2 - S_2 y_2) dq_2 \\
& \;\vdots \\
& + \int (B_0 x_{N-2} + S_{N-2} y_{N-2} - B_0 x_{N-1} - S_{N-1} y_{N-1}) dq_{N-1}) \\
& + \int (B_0 x_{N-1} + S_N y_{N-1}) dq_N)
\end{aligned}$$

$$
\begin{aligned}
&= \int B_0 x_0 dq_1 \\
&\quad + \int S_1 y_0 dq_1 \\
&\quad - \int B_0 x_1 dq_1 + \int B_0 x_1 dq_2 \\
&\quad - \int S_1 y_1 dq_1 + \int S_2 y_1 dq_2 \\
&\quad \vdots \\
&\quad - \int B_0 x_{N-1} dq_{N-1} + \int B_0 x_{N-1} dq_N \\
&\quad - \int S_{N-1} y_{N-1} dq_{N-1} + \int S_N y_{N-1} dq_N \\
&= \langle x, A^* q \rangle
\end{aligned}
$$

We need to identify the image of $A^*q \in V$ so we need to clearly identify the space V. Let $\mathcal{SM}(\Omega, \mathcal{F})$ denote the space of signed measures on $(\Omega, \mathcal{F})$. For this model,

$$V = \prod_{t=0}^{N-1} \mathcal{SM}(\Omega, \mathcal{F}_t) \times \mathcal{SM}(\Omega, \mathcal{F}_t).$$

Thus from the work above, $A^*q = A^*(q_1, q_2, \ldots, q_N)$ has elements (in pairs)

$$
\begin{gathered}
(B_0\, dq_1, S_1\, dq_1), \\
(B_0\, dq_2 - B_0\, dq_1, S_2\, dq_2 - S_1\, dq_1), \\
\vdots \\
(B_0\, dq_N - B_0\, dq_{N-1}, S_N\, dq_N - S_{N-1}\, dq_{N-1}).
\end{gathered}
$$

In this specification of the elements of V, the notation $B_0\, dq_{j+1} - B_0\, dq_j$ represents the potentially signed measure $\mu \in \mathcal{SM}(\Omega, \mathcal{F}_j)$ given by

$$\mu_j(G) = \int_G B_0\, dq_{j+1} - \int_G B_0\, dq_j, \qquad G \in \mathcal{F}_j,$$

and similarly the signed measure $\nu \in \mathcal{SM}(\Omega, \mathcal{F}_j)$ denoted $S_{j+1}\, dq_{j+1} - S_j\, dq_j$ is

$$\nu_j(G) = \int_G S_{j+1}\, dq_{j+1} - \int_G S_j\, dq_j, \qquad G \in \mathcal{F}_j.$$

We define the dual linear program as that given in Figure 4.2.

$$
\left\{
\begin{array}{llllll}
\text{Maximize} & & & & & \\
\int \tilde{C}_N dq_N & & & & & \\
\text{Subject To} & & & & & \\
\int B_0 x_0 dq_1 & & & & & = B_0 x_0 \\
\int \tilde{S}_1 y_0 dq_1 & & & & & = S_0 y_0 \\
-\int B_0 x_1 dq_1 + & \int B_0 x_1 dq_2 & & & & = 0 \\
-\int \tilde{S}_1 y_1 dq_1 + & \int \tilde{S}_2 y_1 dq_2 & & & & = 0 \\
 & -\int B_0 x_2 dq_2 + \int B_0 x_2 dq_3 & & & & = 0 \\
 & -\int \tilde{S}_2 y_2 dq_2 + \int \tilde{S}_3 y_2 dq_3 & & & & = 0 \\
 & & \ddots \quad \ddots & & & \vdots \quad \vdots \\
 & & & -\int B_0 x_{N-2} dq_{N-2} + \int B_0 x_{N-2} dq_{N-1} & & = 0 \\
 & & & -\int \tilde{S}_{N-2} y_{N-2} dq_{N-2} + \int \tilde{S}_{N-1} y_{N-2} dq_{N-1} & & = 0 \\
 & & & & -\int B_0 x_{N-1} dq_{N-1} + \int B_0 x_{N-1} dq_N & = 0 \\
 & & & & -\int \tilde{S}_{N-1} y_{N-1} dq_{N-1} + \int \tilde{S}_N y_{N-1} dq_N & = 0 \\
q_0, q_1, \ldots, q_{N-1} \text{unrestricted}, q_N \geq 0 & & & & &
\end{array}
\right.
$$

Figure 4.2: The Dual Linear Program of the Primal Linear Program

Theorem 4.2. *(Weak Duality) Let* $x = (x_0, y_0, x_1, y_1, \dots, x_{N-1}, y_{N-1})$ *be feasible for the Primal Linear Program and let* $q = (q_1, q_2, \dots, q_N)$ *be feasible for the Dual Linear Program. Then*

$$\int \tilde{C}_N dq_N \leq B_0 x_0 + S_0 y_0.$$

Proof. Refer to Chapter 2 *Weak Duality Theorem.* □

Now that we have identified the dual linear program in Figure 4.2, we utilize the measurability of each x_t and y_t, for each t, to reformulate it. In the first and second constraint, x_0 and y_0 are constants, so they can easily be divided out of the equations. Examining the third equation,

$$-\int B_0 x_1 dq_1 + \int B_0 x_1 dq_2 = 0,$$

must hold for all bounded, $\mathcal{F}_1$-measurable random vectors (x_1, y_1); this condition is equivalently expressed as

$$\int_\Omega x_1 dq_2 = \int_\Omega x_1 dq_1, \qquad \forall x_1 \in L^\infty(\Omega, \mathcal{F}_1).$$

Now let $G_1 \in \mathcal{F}_1$ and take $x_1 = I_{G_1}$. Then we see that

$$\int_{G_1} dq_2 = \int_{G_1} dq_1, \qquad \forall G_1 \in \mathcal{F}_1.$$

This implies that $q_2(G_1) = q_1(G_1)$ and thus q_2, restricted to sets from $\mathcal{F}_1$, is q_1 .

Now we will examine the fourth constraint by integrating over sets from the smaller σ-algebra $\mathcal{F}_1$, to get

$$-\int_{G_1} \tilde{S}_1 dq_1 + \int_{G_1} \tilde{S}_2 dq_2 = 0, \qquad \forall G_1 \in \mathcal{F}_1$$

Remark 4.3. *By Definition 2.31, the elements of our natural filtration are the nested* σ*-algebras* $\mathcal{F}_i \subset \mathcal{F}_{i+1}$ *for* $i = 0, 1, \dots, N$*. When presented with two measures in the same equation, we integrate over the sets from the coarser* σ*-algebra. The argument above allows us to reformulate the constraints as in Figure 4.3.*

$$
\left\{
\begin{array}{llllll}
\text{Maximize} & & & & & \\
\int \tilde{C}_N dq_N & & & & & \\
\text{Subject To} & & & & & \\
\int_\Omega B_0 dq_1 & & & & = B_0 & \\
\int_\Omega \tilde{S}_1 dq_1 & & & & = S_0 & \\
-\int_{G_1} B_0 dq_1 + \int_{G_1} B_0 dq_2 & & & & = 0 & \forall G_1 \in \mathcal{F}_1 \\
-\int_{G_1} \tilde{S}_1 dq_1 + \int_{G_1} \tilde{S}_2 dq_2 & & & & = 0 & \forall G_1 \in \mathcal{F}_1 \\
 & -\int_{G_2} B_0 dq_2 + \int_{G_2} B_0 dq_3 & & & = 0 & \forall G_2 \in \mathcal{F}_2 \\
 & -\int_{G_2} \tilde{S}_2 dq_2 + \int_{G_2} \tilde{S}_3 dq_3 & & & = 0 & \forall G_2 \in \mathcal{F}_2 \\
 & \ddots \quad \ddots & & & \vdots & \vdots \;\vdots \\
 & & -\int_{G_{N-2}} B_0 dq_{N-2} + \int_{G_{N-2}} B_0 dq_{N-1} & & = 0 & \forall G_{N-2} \in \mathcal{F}_{N-2} \\
 & & -\int_{G_{N-2}} \tilde{S}_{N-2} dq_{N-2} + \int_{G_{N-2}} \tilde{S}_{N-1} dq_{N-1} & & = 0 & \forall G_{N-2} \in \mathcal{F}_{N-2} \\
 & & & -\int_{G_{N-1}} B_0 dq_{N-1} + \int_{G_{N-1}} B_0 dq_N & = 0 & \forall G_{N-1} \in \mathcal{F}_{N-1} \\
 & & & -\int_{G_{N-1}} \tilde{S}_{N-1} dq_{N-1} + \int_{G_{N-1}} \tilde{S}_N dq_N & = 0 & \forall G_{N-1} \in \mathcal{F}_{N-1} \\
q_0, q_1, \ldots, q_{N-1} \text{unrestricted}, q_N \geq 0 & & & & &
\end{array}
\right.
$$

Figure 4.3: The Equivalent Dual Linear Program of the Primal Linear Program

Remark 4.4. *In Figure 4.3, notice that the first two constraints are being integrated over the whole space Ω. This is the result of the constant S_0 generating the discrete σ-algebra $\mathcal{F}_0 = \{\Omega, \emptyset\}$. Measuring over the empty set results in a measure of 0, so it is ignored.*

We will now examine all of the dual constraints pertaining to bonds. As in the trinomial model, our first burden of proof is to show that our dual measures are all positive regardless of the primal constraint function being responsible for labeling some of them as unrestricted. Our last constraint pertaining to bonds,

$$-\int_{G_{N-1}} B_0 dq_{N-1} + \int_{G_{N-1}} B_0 dq_N = 0, \text{ for all } \; G_{N-1} \in \mathcal{F}_{N-1},$$

says

$$q_N(G_{N-1}) = q_{N-1}(G_{N-1}), \text{ for all } \; G_{N-1} \in \mathcal{F}_{N-1}.$$

Since q_N is non-negative measure, q_{N-1} is also a non-negative measure. Going to the second to last bond constraint, this argument can be applied to show that q_{N-2} is non-negative. Recursively, it can be shown that all of the dual variables are non-negative measures.

Having that our dual measures are positive, we will continue our examination of the bond constraints to show that the dual measures are actually probability measures. Recall by Definition 2.18, that given a measure space $(\Omega, \mathcal{F}, \mu)$, measures are functions that map elements from the σ-algebra to $\mathbb{R}^+$. By Definition 2.20, a probability measure assigns a mass of 1 to the entire space, Ω.

Proposition 4.5. *Let $q = (q_1, q_2, \ldots, q_N)$ be feasible for the dual linear program. Then, for each t, q_t is a probability measure on $(\Omega, \mathcal{F}_t)$, for $1 \leq s < t \leq N$, and q_t is an extension of q_s to the larger σ-algebra $\mathcal{F}_t$.*

Proof. Looking at the first bond constraint,

$$\begin{aligned} \int_\Omega B_0 dq_1 &= B_0, \\ \int_\Omega dq_1 &= 1. \end{aligned}$$

This implies that $q_1(\Omega) = 1$. Then q_1 is a probability measure on $(\Omega, \mathcal{F}_1)$.

Manipulating the next bond constraint,

$$-\int_{G_1} B_0 dq_1 + \int_{G_1} B_0 dq_2 = 0 \text{ for all } \; G_1 \in \mathcal{F}_1,$$

which implies $q_2(G_1) = q_1(G_1)$ for all $G_1 \in \mathcal{F}_1$.

We will examine the next bond constraint and then propose a pattern. The next bond constraint is

$$-\int_{G_2} B_0 dq_2 + \int_{G_2} B_0 dq_3 = 0 \text{ for all } G_2 \in \mathcal{F}_2.$$

From this we see that $q_3(G_2) = q_2(G_2)$ for all $G_2 \in \mathcal{F}_2$. Continuing this evaluation develops a pattern of equations:

$$\begin{aligned} q_1(\Omega) &= 1; && (4.1) \\ q_2(G_1) &= q_1(G_1), \text{ for all } \; G_1 \in \mathcal{F}_1; && (4.2) \\ q_3(G_2) &= q_2(G_2), \text{ for all } \; G_2 \in \mathcal{F}_2; \\ &\vdots \\ q_N(N-1) &= q_{N-1}(N-1), \text{ for all } \; G_{N-1} \in \mathcal{F}_{N-1}. \end{aligned}$$

The next pattern emerges because of the nesting of our natural filtration. First recall that Ω is in every σ-algebra $\mathcal{F}_t$. Then evaluating the last pattern with the set Ω yields

$$q_N(\Omega) = q_{N-1}(\Omega) = \ldots = q_1(\Omega) = 1,$$

and every measure has been shown to be a probability measure. Now we have exhausted Equation 4.1, as it can only be evaluated at Ω and we will focus on the subsequent equations which can all be evaluated for all G_1 in $\mathcal{F}_1$. This gives us

$$q_N(G_1) = q_{N-1}(G_1) = \ldots = q_1(G_1), \text{ for all } \; G_1 \in \mathcal{F}_1.$$

We are done examining Equation 4.2, as q_1 cannot be evaluated at sets from σ-algebras greater than $\mathcal{F}_1$. We examine the rest of the equations while restricting to sets G_2 from $\mathcal{F}_2$ and get

$$q_N(G_2) = q_{N-1}(G_2) = \ldots = q_2(G_2), \text{ for all } G_2 \in \mathcal{F}_2.$$

To complete our proof, we present our finding as the pattern:

$$\begin{gathered}
q_N(\Omega) = q_{N-2}(\Omega) = \ldots = q_1(\Omega) = 1; \\
q_N(G_1) = q_{N-2}(G_1) = \ldots = q_1(G_1), \text{ for all } G_1 \in \mathcal{F}_1; \\
q_N(G_2) = q_{N-2}(G_2) = \ldots = q_2(G_2), \text{ for all } G_2 \in \mathcal{F}_2; \\
\vdots \\
q_N(G_{N-2}) = q_{N-1}(G_{N-2}) = q_{N-2}(G_{N-2}), \text{ for all } G_{N-2} \in \mathcal{F}_{N-2}; \\
q_N(G_{N-1}) = q_{N-1}(G_{N-1}), \text{ for all } G_{N-1} \in \mathcal{F}_{N-1}.
\end{gathered}$$

□

Now we will turn our focus to the constraints pertaining to stocks. We will use the fact that we are dealing with probability measures in our examination. Now would be a good time for the reader to look at Definition 2.27, as most of the following requires that one understand the definition of *conditional expectation*.

Our first stock constraint is

$$\int_\Omega \tilde{S}_1 dq_1 = S_0.$$

Recall that $\int_\Omega dq_1 = 1$. Then we can rewrite our equation like this

$$\int_\Omega \tilde{S}_1 dq_1 = S_0 \int_\Omega dq_1.$$

Since S_0 is a constant, $\mathcal{F}_0 = \{\emptyset, \Omega\}$ so we have

$$\int_\Omega \tilde{S}_1 dq_1 = \int_\Omega S_0 dq_1.$$

As $\tilde{S}_1$ is not a constant, it is important that we use Theorem 2.27 to explain the integration. Recall from Remark 4.4, when integrating over Ω that we are actually integrating over all sets but one (the empty set) from the σ-algebra $\mathcal{F}_0$. Thus the conditional expected value with respect to feasible measures q of $\tilde{S}_1$ given $\mathcal{F}_0$ is S_0, $E^q[\tilde{S}_1|\mathcal{F}_0] = E^q[S_1] = S_0$. This is an interesting initial finding about our stochastic process $\tilde{S}$.

Now look at the next stock constraint

$$-\int_{G_1} \tilde{S}_1 dq_1 + \int_{G_1} \tilde{S}_2 dq_2 = 0 \text{ for all } \ G_1 \in \mathcal{F}_1.$$

We recall from working with our bond constraints that measures q_1 and q_2 agree with one another when restricted to sets G_1 from $\mathcal{F}_1$. Then we can rearrange the equation and replace dq_2 with dq_1 to get

$$\int_{G_1} \tilde{S}_2 dq_1 = \int_{G_1} \tilde{S}_1 dq_1 \text{ for all } \ G_1 \in \mathcal{F}_1.$$

Now that the measures match, we can proceed to use our definition of *conditional expectation* again to give us $E^q[\tilde{S}_2|\mathcal{F}_1] = \tilde{S}_1$.

In fact, looking at all of the stock constraints in this way gives us the pattern:

$$\begin{aligned} E^q[\tilde{S}_1|\mathcal{F}_0] &= \tilde{S}_0; \\ E^q[\tilde{S}_2|\mathcal{F}_1] &= \tilde{S}_1; \\ &\vdots \\ E^q[\tilde{S}_{N-1}|\mathcal{F}_{N-2}] &= \tilde{S}_{N-2}; \\ E^q[\tilde{S}_N|\mathcal{F}_{N-1}] &= \tilde{S}_{N-1}. \end{aligned}$$

This finding motivates us to look further and see if we can establish the final requirement for our stochastic process, $\tilde{S}$, to be a martingale. Referring back to Definition 2.32, we would need $E^q[\tilde{S}_t|\mathcal{F}_s] = \tilde{S}_s$, for each $0 \leq s < t \leq N$.

Proposition 4.6. *Let $q = (q_1, q_2, \ldots, q_N)$ be feasible for the dual linear program. Then the present value stock price process $\tilde{S}$ is a martingale under q with respect to the natural filtration $\{\mathcal{F}_t\}_{0 \leq t \leq N}$.*

Proof. For $t \leq N-1$, a general form stock constraint is

$$-\int_{G_{t-1}} \tilde{S}_{t-1} dq_{t-1} + \int_{G_{t-1}} \tilde{S}_t dq_{t-1} = 0 \text{ for all } \ G_{t-1} \in \mathcal{F}_{t-1}.$$

Now we will choose some σ-algebra $\mathcal{F}_s$ such that $0 \leq s < t \leq N$. Since we have already covered the matter of s being equal to $t-1$, we will assume that $s < t-1$.

By the work we did with our measures in the bond constraints, we will restrict our general stock constraint to sets from the σ-algebra $\mathcal{F}_s$ to get

$$\int_{G_s} \tilde{S}_t dq_s = \int_{G_s} \tilde{S}_{t-1} dq_s \text{ for all } \; G_s \in \mathcal{F}_s.$$

The left side is $E^q[\tilde{S}_t|\mathcal{F}_s]$ while the right hand side is $E^q[\tilde{S}_{t-1}|\mathcal{F}_s]$. By the recursive nature of our stock constraints, it is true that $E^q[\tilde{S}_{t-1}|\mathcal{F}_s] = E^q[\tilde{S}_{t-2}|\mathcal{F}_s] = \ldots = E^q[\tilde{S}_{s+1}|\mathcal{F}_s]$. But, we have shown that $E^q[\tilde{S}_{s+1}|\mathcal{F}_s] = \tilde{S}_s$ and therefore we can say that $E^q[\tilde{S}_t|\mathcal{F}_s] = \tilde{S}_s$. Thus, our present value stock price process $\tilde{S}$ is a martingale.

$\square$

Corollary 4.7. *The maximal expected present value of the contingent claim $\tilde{C}_N$ over all martingale measures $q \in Q$ is a lower bound on the minimal value of the initial portfolio over all the feasible hedging portfolios.*

Chapter 5

Summary

In conclusion, we summarize our findings and indicate possible further topics of investigation. We now know that we have a *weak duality* relationship between the primal linear program and the dual linear program. By this relationship, the objective function of our dual linear program provides a lower bound for the objective function of the primal linear program, for feasible points of both. For the finite-dimensional linear program arising from the Trinomial Asset Pricing Model, strong duality of linear programming establishes that the optimal values of the two linear programs are equal. This strong result does not necessarily hold for infinite-dimensional linear programs.

The evolution of the present value stock price process, $\tilde{S}$, is captured by its natural filtration. Utilizing the dual pair structure of infinite-dimensional linear programming, the dual space is seen to be a space of measures and the bilinear form is given by the (summation of the) integration of the measurable random variables against these measures. The dual constraints required the feasible dual measures be probability measures and be such that the expected value of $\tilde{S}$ is a martingale.

There are several directions for further investigation on option pricing.

- As indicated above, the issue of strong duality remains to be examined for the infinite-dimensional linear programs. Under what conditions on the model will the optimal values of the primal and dual linear programs be equal?

- The finding of $\tilde{S}$ being a martingale is due to the fact that our hedging portfolio places no hard restrictions on how much money can be borrowed or how many shares of stock can be short-sold. As a result, all of the variables were unrestricted in our primal linear program. How would the placing of such restrictions affect the interpretation of the dual linear program?

Bibliography

[1] E. J. Anderson, P. Nash, *Linear Programming in Infinite Dimensional Spaces*, John Wiley and Sons, New York, 1987.

[2] Billingsley, *Probability and Measure*, Third Ed., John Wiley and Sons, New York, 1995.