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Results on n-Absorbing Ideals of Commutative Rings

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by

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ABSTRACT
RESULTS ON N-ABSORBING IDEALS OF COMMUTATIVE RINGS

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Let R be a commutative ring with $1 \neq 0$. In his paper *On 2-absorbing ideals of commutative rings*, Ayman Badawi introduces a generalization of prime ideals called 2-absorbing ideals, and this idea is further generalized in a paper by Anderson and Badawi [1] to a concept called n-absorbing ideals. A proper ideal I of R is said to be an n-absorbing ideal if whenever $x_1 \cdots x_{n+1} \in I$ for $x_1, \cdots, x_{n+1} \in R$ then there are n of the x_i's whose product is in I. This paper will provide proofs of several properties in [1] which are stated without proof, and will study how several theorems from Badawi's initial paper on 2-absorbing ideals can be extended to n-absorbing ideals of R. Additionally, Badawi introduces a generalization of primary ideals in his paper *On 2-absorbing primary ideals in commutative rings* [3], and this paper generalizes that idea further by defining n-absorbing primary ideals of R. Let n be a positive integer. A proper ideal I of a commutative ring R is said to be an n -absorbing primary ideal of R if whenever $x_1, \ldots, x_{n+1} \in R$ and $x_1 x_2 \cdots x_{n+1} \in I$ then either $x_1 x_2 \cdots x_n \in I$ or a product of n of the $x_i's$ (other than $x_1 \cdots x_n$) is in $\sqrt{I}$. We will prove several basic properties of n-absorbing primary ideals, including that any n-absorbing primary ideal is m-absorbing for $m \geq n$. We will also show that for $R = \mathbb{Z}$ and $I = n\mathbb{Z}$ if n has k prime factors, then I is a k-absorbing primary ideal, and is not, in fact, a $(k-1)$-absorbing primary ideal of R. This will lead us to a conclusion about the intersection of ideals of this form.

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1 Introduction

Throughout this paper we will assume that R is a commutative ring with unity. The concept of a 2-absorbing ideal of R is first introduced by Badawi in [2], and an n-absorbing ideal of R is introduced by Anderson and Badawi in [1]. We will study these n-absorbing ideals of commutative rings, which are a generalization of prime ideals. A proper ideal I of R is called an n-absorbing ideal if for $x_1, x_2, \ldots, x_{n+1} \in R$ and $x_1 \cdots x_{n+1} \in I$, we have that a product of n of the $x_i's$ are in I. For example, consider the ideal $I = (15)$ of $\mathbb{Z}$, and $a, b, c \in R$ such that $abc \in I$. Then we have that $3 \cdot 5 | abc \in I$. Suppose we know that $3|a$. Then if 5 divides b or c we see that I is 2-absorbing, and if $5|a$ then $a \in I$ and we see that I is 2-absorbing. We will prove several basic properties of n-absorbing ideals. We show that if I is n-absorbing, then $\sqrt{I}$ is also an n-absorbing ideal of R. We also show that any n-absorbing ideal is also an m-absorbing ideal for any $m \geq n$ (Theorem 5). Furthermore, we will prove properties about the intersection of n-absorbing ideals, beginning with the relationship between 2-absorbing ideals and the intersection of prime ideals.

In [3], Badawi introduces a generalization of primary ideals, called 2-absorbing primary ideals. This paper will generalize 2-absorbing primary ideals and study n-absorbing primary ideals of R. A proper ideal I of commutative ring R is called an n-absorbing primary ideal of R if whenever $x_1, x_2, \ldots, x_{n+1} \in R$ with $x_1 x_2 \cdots x_{n+1} \in I$, then $x_1 \cdots x_n \in I$ or $x_1 \cdots \widehat{x_i} \cdots x_{n+1} \in \sqrt{I}$ for some $i \in [1, n]$. For example, consider ideal $I = (18)$ of $\mathbb{Z}$. Since $3 \cdot 3 \cdot 2 \cdot \in I$ and $3 \cdot 3 \notin I$ and $3 \cdot 2 \in \sqrt{I}$ we have that I is a 2-absorbing primary ideal of R. We will show that all n-absorbing ideals are n-absorbing primary ideals, however not all n-absorbing primary ideals are n-absorbing ideals. We show that every n-absorbing primary ideal is an m-absorbing primary ideal of R for $n > m$. Furthermore, if I is an n-absorbing primary ideal, then $\sqrt{I}$ is n-absorbing, and we investigate the relationship between I and $\sqrt{I}$ if we know that

$\sqrt{I}$ is n-absorbing. Finally, we introduce a new concept, that of an n-semi-absorbing ideal, which is a generalization of a radical ideal. We give an example that shows an n-semi-absorbing ideal need not be an intersection of n-absorbing ideals.

2 Literature Review

The first generalization of prime ideals is in Ayman Badawi's paper *On 2-absorbing ideals of commutative rings* [2], where he defines the idea of a 2-absorbing ideal; if $a, b, c \in R$ and $abc \in I$ then I is 2-absorbing if $ab \in I$ or $ac \in I$ or $bc \in I$. Expanding on this definition, Anderson and Badawi introduce n-absorbing ideals in their paper, *On n-absorbing ideals of commutative rings*[1]. They begin by listing basic properties of n-absorbing ideals, such as if I is an n-absorbing ideal, then it is an m-absorbing ideal for $m \geq n$ (Theorem 2.1). However, several of these properties are stated without proof. This paper will provide the proof of those properties. After having defined n-absorbing ideals and articulating their basic properties, it is of interest to consider Badawi's first paper on 2-absorbing ideals, and examine whether his ideas can be extended from the 2-absorbing case to the n-absorbing case. For example, Theorem 2.1 [2] shows that if I is a 2-absorbing ideal, then $\sqrt{I}$ is also a 2-absorbing ideal, and $x^2 \in I$ for every $x \in \sqrt{I}$. This paper will show that this theorem holds for the case when I is an n-absorbing ideal of R. Badawi also shows in [2] (Theorem 2.4) that if I is 2-absorbing ideal of R, then either $\sqrt{I} = P$ is a prime ideal of R such that $P^2 \subseteq I$ or $\sqrt{I} = P_1 \bigcap P_2, P_1P_2 \subseteq I$ and $\sqrt{I}^2 \subseteq I$ where P_1, P_2 are the only distinct prime ideals of R that are minimal over I. The n-absorbing extension of this has proven challenging to show and is left as further research. (We note that Anderson and Badawi show that the theorem holds for the n-absorbing case only if we assume I satisfies a property called strongly n-absorbing (Theorem 6.1, [1]), and this paper will not assume all ideals satisfy such a property). Anderson and Badawi's paper goes on to study the properties of n-absorbing ideals in several special classes of commutative rings, including Dedekind domains and Noetherian integral domains. They prove that all proper ideals of Noetherian rings are n-absorbing ideals for some positive integer n (Theorem 5.3, [1]). Furthermore, the authors explore the converse of this statement, and show in Theorem 5.9 that if we know all proper ideals of a ring are n-absorbing ideals of R, then we must have that $dim(R) = 0$ and R has at most n maximal ideals. In Badawi's most recent publication on this topic, *On 2-absorbing primary ideals in commutative rings* [3] he introduces a generalization of primary ideals in the following way; A proper ideal I of R is called a 2-absorbing primary ideal of R if whenever $a, b, c \in R$ with $abc \in I$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. He then proves basic properties and theorems about 2-absorbing primary ideals. This paper introduces a generalization of 2-absorbing primary ideals called n-absorbing primary ideals, and extends many of the properties in Badawi's paper [3] to the case when I is an n-absorbing primary ideal of R. For example, in Theorem 2.8 of [3] Badawi shows that if $\sqrt{I}$ is a prime ideal of R, then I is a 2-absorbing primary ideal of R. We will show that if $\sqrt{I}$ is an n-absorbing ideal of R, then I is an $(n+1)$-absorbing primary ideal of R. Furthermore, in [3], it is shown (Theorem 2.20) that if f is a homomorphism of commutative rings $R \longrightarrow R^{'}$, and $I^{'}$ is a 2-absorbing primary ideal of $R^{'}$, then $f^{-1}(I^{'})$ is a 2-absorbing primary ideal of R. We show that this theorem holds if $I^{'}$ is an n-absorbing primary ideal of $R^{'}$.

3 Properties of n-absorbing ideals

Definition 1. Let n be a positive integer. A proper ideal I of a commutative ring R is an n **-absorbing ideal of** R if whenever $x_1 \cdots x_{n+1} \in I$ for $x_1, \ldots, x_{n+1} \in R$ then $x_1 \cdots \widehat{x_i} \cdots x_{n+1} \in I$ for some $i \in [1, n+1]$.

Example. Consider the ideal $I = (15)$ of $\mathbb{Z}$, and $a, b, c \in R$ such that $abc \in I$. Then we have that $3 \cdot 5 | abc \in I$. Suppose we know that $3|a$. Then if 5 divides b or c we see that I is 2-absorbing, and if $5|a$ then $a \in I$ and we see that I is 2-absorbing.

Suppose $R = \mathbb{Z}$, or in general let R be any UFD. Then let $I = mR$ for $m \in R$. Suppose $m = p_1^{a_1} \cdots p_k^{a_k}$, where $p_1, \ldots, p_k$ are distinct primes, and $a_1, \ldots, a_k \geq 0$. We will explore what properties of m make I n-absorbing. First we will consider the 2-absorbing case.

Theorem 1. *Let R be a UFD, let $I = Ra$ be a proper nonzero principal ideal, and let n be a positive integer. Then I is n-absorbing if and only if a is the product of at most n irreducible elements (not necessarily distinct).*

Remark. If we write $a = u \prod_{i=1}^{k} p_i^{a_i}$ for non-associate irreducible elements $p_1, \ldots, p_k$, a unit u, and positive integers $a_1, \ldots, a_k$, the hypothesis on a can be written $\sum_{i=1}^{k} a_i \leq n$.

Proof. We proceed by induction on n. The case $n = 1$ says Ra is a prime ideal if and only if a is an irreducible element of R. This is clear.

Now suppose that $n \geq 2$ and that for $m < n$, it is the case that $I = Ra$ is m-absorbing if and only if a is the product of at most m irreducible elements. Let us suppose $a = q_1 \cdots q_k$ for irreducible elements $q_1, \ldots, q_k$.

First suppose I is n-absorbing and $k > n$. Consider the product $q_1 \cdots q_n(q_{n+1} \cdots q_k)$, regarded as a product of $n + 1$ elements of R as indicated by the parentheses. Since I is n-absorbing, some product of less than k irreducible elements is in I and hence is divisible by $q_1 \cdots q_k$. This is impossible and so we must have $k \leq n$.

Next suppose $k \leq n$. If $k = 1$, then a is irreducible and hence I is n-absorbing for any n. Suppose $k > 1$ and $x_1 \cdots x_n x_{n+1} \in I$. Then q_k divides some x_i; without loss of generality, we can assume $x_{n+1} = q_k y_{n+1}$. Set $a' = q_1 \cdots q_{k-1}$ and note that $a' \mid x_1 \cdots x_{n-1}(x_n y_{n+1})$. As $k - 1 \leq n - 1$, the induction hypothesis implies either $a' \mid x_1 \cdots x_{n-1}$ or $a' \mid x_1 \cdots \widehat{x_i} \cdots x_n y_{n+1}$. In the first case we have that

$a = a'q_k \mid x_1 \cdots x_{n-1} q_k \mid x_1 \cdots x_{n-1} x_{n+1}$ and in the second case

$a = a'q_k \mid x_1 \cdots \widehat{x_i} \cdots x_n q_k y_{n+1} = x_1 \cdots \widehat{x_i} \cdots x_n x_{n+1}$. In either case, we see a divides a product of n of the a_i's. This proves $I = Ra$ is n-absorbing.

Corollary 1. *Suppose R is a UFD and $I = Ra$ where a is the product of n irreducible elements (not necessarily distinct). Then I is n-absorbing but not $(n-1)$-absorbing.*

We will now consider several basic properties that are stated without proof in [1]. First, we look at the relationship between an n-absorbing ideal and its radical.

Theorem 2. *Suppose I is an n-absorbing ideal of R. Then $\sqrt{I}$ is an n-absorbing ideal and $x^n \in I$ for all $x \in \sqrt{I}$.*

Proof. Suppose $x \in \sqrt{I}$, so $x^m \in I$ for some m. If $m \leq n$, then $x^n \in I$. If $m > n$, we can repeatedly use the n-absorbing property on products $xx \cdots xx^k$ to conclude that $x^n \in I$. Thus, let $x_1, \ldots, x_{n+1} \in R$ such that $x_1x_2 \cdots x_{n+1} \in \sqrt{I}$. Then $(x_1x_2 \cdots x_{n+1})^n = x_1^n x_2^n \cdots x_{n+1}^n \in I$. We have that I is n-absorbing, thus $x_1^n \cdots \widehat{x_i^n} \cdots x_{n+1}^n \in I$ for some $i \in [1, n+1]$. Thus $(x_1 \cdots \widehat{x_i} \cdots x_{n+1})^n \in I$ which implies that $x_1 \cdots \widehat{x_i} \cdots x_{n+1} \in \sqrt{I}$, and we see that $\sqrt{I}$ is n-absorbing.

Next, we look at what happens when we intersect n-absorbing ideals. The first case we consider is if both ideals are 1-absorbing, or prime, ideals of R.

Theorem 3. *If P and Q are nonzero prime ideals of a ring R, then $P \bigcap Q$ is a 2-absorbing ideal of R.*

Proof. Suppose $(ab)c \in P \bigcap Q$ and $a, b, c \in R$. Then $(ab)c \in P$ and $(ab)c \in Q$. P is prime, so either $ab \in P$ or $c \in P$. If $ab \in P$ then either $a \in P$ or $b \in P$. Similarly, $a \in Q$ or $b \in Q$ or $c \in Q$. Suppose $a \in P \bigcap Q$. Then $ab \in P \bigcap Q$ and $ac \in P \bigcap Q$ since $P \bigcap Q$ is an ideal. Thus $P \bigcap Q$ is 2-absorbing. However, if the statements above lead to different elements in P and Q, we still have that the intersection is 2-absorbing. For example, if $a \in P$ and $b \in Q$, then clearly $ab \in P$ and $ab \in Q$ by definition of an ideal, thus $ab \in P \bigcap Q$, which implies $P \bigcap Q$ is 2-absorbing.

Next we will examine what happens in a more general case, that is, what is the structure of the intersection of m ideals that are each n_j-absorbing ideals of R. This theorem is stated without proof in Theorem 2.1 of [1], thus we will provide the proof.

Theorem 4. *If I_j is an n_j-absorbing ideal of R for each $1 \leq j \leq m$, then $\bigcap_{j=1}^m I_j$ is an n-absorbing ideal, where $n = \Sigma_{j=1}^m n_j$.*

Proof. Suppose $I_1, \ldots, I_m$ are proper ideals of R such that I_j is n_j-absorbing and let $k > n_1 + \cdots + n_m$. Now suppose $x_1 \cdots x_k \in \bigcap_{j=1}^m I_j$. Then we have for all j, that there exists a product of n_j of those k elements in I_j, since each I_j is n_j-absorbing. Let the collection of those elements be denoted A_j. Then let $A = \bigcup_{j=1}^k A_j$. Thus A has at most $n_1 + \cdots + n_m$ elements. Now since I_j is an ideal, the product of all elements of A must be in each I_j. So $\bigcap_{j=1}^m I_j$ contains a product of at most $n_1 + \cdots + n_m$ elements. Thus the intersections of the I_j's is an $n_1 + \cdots + n_m$-absorbing ideal of R.

Theorem 5. *If I is an n-absorbing ideal of R, then I is an m-absorbing ideal of R for all $m \geq n$.*

Proof. We prove by induction on n that if I is n-absorbing, it is $(n+1)$-absorbing. **Induction Base.** Suppose that I is 2-absorbing. We will show that I is 3-absorbing. Let $x_1, x_2, x_3, x_4 \in R$ such that $x_1x_2(x_3x_4) \in I$. I is 2-absorbing, so either $x_1(x_3x_4) \in I$ or $x_2(x_3x_4) \in I$ or $x_1x_2 \in I$. If either $x_1(x_3x_4) \in I$ or $x_2(x_3x_4) \in I$ then I is 3-absorbing and we are done. Thus suppose $x_1x_2 \in I$. Then since $x_3 \in R$, by definition of ideal we know that $x_1x_2x_3 \in I$. Thus I is 3-absorbing. **Induction step.** Suppose I is n-absorbing. We will show that I is $(n+1)$-absorbing. Let $(x_1x_2)x_3 \cdots x_nx_{n+1}x_{n+2} \in I$ for $x_1, \ldots, x_{n+2} \in R$. I is n-absorbing, so either $(x_1x_2) \cdots \widehat{x_i} \cdots x_{n+2} \in I$ for some $i \in [3, n+2]$ or we have that $x_3 \cdots x_{n+2} \in I$. In the first cases it is clear that I is $(n+1)$-absorbing, so suppose only $x_3 \cdots x_{n+2} \in I$ holds (note that this is a product of n terms). Since $x_1, x_2 \in R$, then by definition of ideal we must have that $x_1x_3 \cdots x_{n+2} \in I$ and thus I is $(n+1)$-absorbing.

Example. Let $R = \mathbb{Z}[x]$ and consider the proper ideal $I = (2, x)$ of R. Then elements of I can be expressed as $xp(x) + 2q(x)$ for $p(x), q(x) \in \mathbb{Z}[x]$ that is, every polynomial in I has an even constant term. Clearly, I is a prime ideal (or 1-absorbing). However, we also observe that I is 2-absorbing. For example, for $a, b, c \in R$ and

$a(bc) \in I$, we have that since I is prime, $a \in I$ or $bc \in I$, and thus I is 2-absorbing.

In Anderson and Badawi's paper [1], they make the following conjecture: If I is an n-absorbing ideal of R, then I is a strongly n-absorbing ideal of R, where strongly n-absorbing is defined as follows.

Definition 2. A proper ideal I is a **strongly n-absorbing ideal** of R if whenever $I_1 \cdots I_{n+1} \subseteq I$ for ideals $I_1, \ldots, I_{n+1}$ of R then there are n of the $I_i's$ whose product is in I.

Anderson and Badawi set out to prove this conjecture by showing it implies another conjecture, mainly, if I is n-absorbing, then $\sqrt{I}^n \subseteq I$. This second conjecture has proven challenging to show (without the assumption that I is strongly n-absorbing).

4 Properties of n-absorbing primary ideals

In his most recent publication on this topic, Ayman Badawi introduces a generalization of primary ideals of commutative rings, which he defines as 2-absorbing primary ideals. A proper, nonzero ideal I of commutative ring R is said to be a 2-absorbing primary ideal of R if whenever $a, b, c \in R$ with $abc \in I$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. He notes that 2-absorbing and 2-primary absorbing ideals are different concepts, and provides the following example. Consider the ideal $I = (12)$ of $\mathbb{Z}$. Since $2 \cdot 2 \cdot 3 \in I$, but $2 \cdot 2 \notin I$ and $2 \cdot 3 \notin I$, I is not 2-absorbing. However it is clear that $I = (12)$ is a 2-absorbing primary ideal of $\mathbb{Z}$ [3]. The author goes on to prove basic properties of 2-absorbing primary ideals, and states several useful theorems. However, he does not generalize this idea to an arbitrary n case, thus this paper will proceed to do so. Additionally we will look at expanding various properties and theorems that Badawi states for 2-absorbing primary ideals to this more general definition. We will now generalize Badawi's concept to the arbitrary n-absorbing primary ideal I of R, and we have the following definition.

Definition 3. Let n be a positive integer. A proper ideal I of a commutative ring R is said to be an n **-absorbing primary ideal of** R if whenever $x_1, \ldots, x_{n+1} \in R$ and $x_1 x_2 \cdots x_{n+1} \in I$ then either $x_1 x_2 \cdots x_n \in I$ or a product of n of the $x_i's$ (other than $x_1 \cdots x_n$) is in $\sqrt{I}$.

It is of interest to note that not all n-absorbing primary ideals are n-absorbing ideals of R. In [3], the author provides the following example.

Example. Consider the ideal $I = (12)$ of $\mathbb{Z}$. Then $2 \cdot 2 \cdot 3 \in I$ however, $2 \cdot 2 \notin I$ and $2 \cdot 3 \notin I$ so I is not a 2-absorbing ideal. However, we see that I is a 2-absorbing primary ideal of R.

Equivalently, we can define n-absorbing primary ideals in the following way.

Definition 4. A proper, nonzero ideal I of R is an n**-absorbing primary ideal of** R if and only if $x_1, \ldots, x_{n+1} \in R$ and $x_1 \cdots x_{n+1} \in I$ implies $x_1 \cdots \widehat{x_k} \cdots x_{n+1} \in I$ for some $k \in [1, n+1]$ or $x_1 \cdots \widehat{x_i} \cdots x_{n+1} \in \sqrt{I}$ and $x_1 \cdots \widehat{x_j} \cdots x_{n+1} \in \sqrt{I}$ for $i, j \in [1, n+1], i \neq j$. We see that if there are no products of n elements in I, then we have n products of the form $x_1 \cdots x_{n+1} \in I$, that is, $x_2 \cdots x_{n+1} x_1 \in I$, $x_3 \cdots x_{n+1} x_1 x_2 \in I$, and so forth. Thus we see that if any two products of n elements are in $\sqrt{I}$, then there is a product of n elements in $\sqrt{I}$ for each case.

First, we consider the analog for n-absorbing primary ideals of Theorem 4, that is, if n-absorbing primary ideals are m-absorbing primary ideals where $m > n$.

Theorem 6. *Every primary ideal is a* 2*-absorbing primary ideal.*

Proof. Suppose that ideal I of R is primary. Then for $(ab)c \in I$ we have that either $ab \in I$ or $c \in \sqrt{I}$. Since $\sqrt{I}$ is an ideal, and $a, b \in R$, we have $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. Thus I is a 2-absorbing primary ideal.

Theorem 7. *Every n-absorbing primary ideal of R is an m-absorbing primary ideal for $m > n$.*

Proof. We will show every n-absorbing primary ideal of R is an $n+1$-absorbing primary ideal of R. Suppose that I is an n-absorbing primary ideal. Then suppose for $x_1, x_2, \ldots, x_{n+1}, x_{n+2} \in R$ we have $(x_1x_2)\cdots x_{n+1}x_{n+2} \in I$. Let $(x_1x_2) =: x_{1'}$ Since I is an n-absorbing primary ideal of R, we have that either $x_{1'} \cdots x_{n+1} \in I$ or $x_{1'}x_3 \cdots \widehat{x_i} \cdots x_{n+2} \in \sqrt{I}$ for some $i \in \{1', 3, \ldots, n, n+1\}$. Note that if $i \neq 1'$ then done. Thus suppose that $i = 1'$. Then since $x_1, x_2 \in R$, we have by definition of an ideal that $x_1x_3 \cdots x_{n+2} \in \sqrt{I}$, or $x_2x_3 \cdots x_{n+2} \in \sqrt{I}$. Thus I is an $n+1$-absorbing primary ideal of R.

Thus it is interesting to consider various relationships between n-absorbing ideals and n-absorbing primary ideals. It is clear that any n-absorbing ideal of R is an n-absorbing primary ideal of R. However, the converse is not always true.

Example. Consider the ideal $I = (24)$ of $R = \mathbb{Z}$. We have that $2\cdot 3\cdot 4 \in I$, and clearly $2\cdot 3 \notin I$, and $3\cdot 4 \notin I$ and $2\cdot 4 \notin I$. Thus I is clearly not a 2-absorbing ideal of R. However, we have that $(3\cdot 4)^2 \in I$ and thus I is a 2-absorbing primary ideal of R.

As we did for n-absorbing ideals, we consider $R = \mathbb{Z}$, or in general R is any UFD, and $I = mR$ for $m \in R$. Suppose $m = p_1^{a_1} \cdots p_k^{a_k}$, where $p_1, \ldots, p_k$ are distinct primes, and $a_1, \ldots, a_k \geq 0$. We will explore what properties of m make I an n-absorbing primary ideal of R.

Theorem 8. *Let R be a UFD, let $I = Ra$ be a proper nonzero principal ideal, and let n be a positive integer. Then I is n-absorbing primary if and only if a is divisible by at most n non-associate irreducible elements.*

Remark. If we write $a = u\prod_{i=1}^{k} p_i^{a_i}$ for non-associate irreducible elements $p_1, \ldots, p_k$, a unit u, and positive integers $a_1, \ldots, a_k$, the hypothesis on a can be written $k \leq n$.

Proof. We proceed by induction on n. The case $n = 1$ says Ra is a primary ideal if and only if $a = up^k$ for some irreducible element p and some unit u in R. This is clear.

Now suppose that $n \geq 2$ and that for $m < n$, it is the case that $I = Ra$ is m-

absorbing primary if and only if a is divisible by at most m non-associate irreducible elements. Let us suppose $a = u\prod_{i=1}^{k} p_i^{a_i}$ for the non-associate irreducible elements $p_1, \ldots, p_k$, a unit u, and positive integers $a_1, \ldots, a_k$. Set $b = p_1 \cdots p_k$ and note that $\sqrt{I} = yR$.

First suppose I is n-absorbing primary and $k > n$. Set $x_i = p_i^{a_i}$ for $1 \leq i \leq n$ and set $x_{n+1} = \prod_{i=n+1}^{k} p_i^{a_i}$, so $a = x_1 \cdots x_n x_{n+1}$. Since I is n-absorbing, either $a \mid x_1 \cdots x_n$ or $b \mid x_1 \cdots \widehat{x_i} \cdots x_n x_{n+1}$. In either case, the "divides" statement cannot be true, because k non-associate irreducible elements occur on the left and fewer than k appear on the right.

Next suppose $k \leq n$. If $k = 1$, then a is a unit times the power of an irreducible element and hence I is n-absorbing primary for any n. Suppose $k > 1$ and $x_1 \cdots x_n x_{n+1} \in I$. Then for each $i = 1, \ldots, n+1$, we can write $x_i = p_k^{m_i} y_i$, where $\sum_{i=1}^{n+1} m_i = k_i$. By re-numbering if necessary, we can assume $m_{n+1} \neq 0$. It follows that $a' = \prod_{i=1}^{k-1} p_i^{a_i}$ divides $y_1 \cdots y_{n-1}(y_n y_{n+1})$. Since a' has $k-1$ non-associate irreducible divisors and $k - 1 \leq n - 1$, we can assume by induction that Ra' is $n-1$ absorbing primary. Therefore, either $y_1 \cdots y_{n-1} \in Ra'$ or $y_1 \cdots \widehat{y_i} \cdots y_n y_{n+1} \in \sqrt{Ra'}$ for some $i = 1, \ldots, n-1$. In the first case, $a = a' p_k^{a_k} \mid x_1 \cdots x_{n-1} p_k^{x_k}$, whence $b \mid x_1 \cdots x_{n-1} \widehat{x_n} x_{n+1}$. In the second case, $x_1 \cdots \widehat{x_i} \cdots x_n x_{n+1} \in \sqrt{Ra' p_k} = \sqrt{Ra}$. In either case, an appropriate product is in $\sqrt{I}$, so I is n-absorbing primary.

Corollary 2. *Suppose R is a UFD and $I = Ra$ where the prime factorization of R has n non-associate irreducible elements. Then I is n-absorbing primary but not $(n-1)$-absorbing primary.*

Thus we can consider the intersection of n-absorbing primary ideals. Let $I = I_1 \bigcap I_2 \bigcap \cdots \bigcap I_k$ where each I_j is an n_j-absorbing primary ideal of R. If we let $l = lcm(n_1, \ldots, n_k)$ then we see that from Theorem 8, if m denotes the number of prime factors of l, that I will be an m-absorbing primary ideal of R.

Theorem 9. *If I is an n -absorbing primary ideal of R then $\sqrt{I}$ is an n -absorbing ideal of R.*

Proof. Let $x_1, x_2, \dots, x_{n+1} \in R$ satisfy $x_1x_2\cdots x_{n+1} \in \sqrt{I}$ and suppose we know all products of n of the $x_i's$ except $x_1x_2\cdots x_n$ are not in $\sqrt{I}$. Note that in the other cases we are done, there is nothing to prove. We will show that $x_1x_2\cdots x_n \in \sqrt{I}$. Since $x_1x_2\cdots x_{n+1} \in \sqrt{I}$ then $\exists\ m \in \mathbb{Z}^+$ such that $(x_1x_2\cdots x_{n+1})^m \in I$. Thus we have $x_1^m x_2^m \cdots x_{n+1}^m \in I$. Now, I is an n -absorbing primary ideal and since none of the products of the $x_i's$ are in $\sqrt{I}$, we must conclude that $x_1^m x_2^m \cdots x_n^m = (x_1x_2\cdots x_n)^m \in I$. That is, $x_1x_2\cdots x_n \in \sqrt{I}$. Thus $\sqrt{I}$ is an n-absorbing ideal of R.

Definition 5. Let I be an n-absorbing primary ideal of R. Then $P = \sqrt{I}$ is an n-absorbing ideal. We say that I is a $P-n-$ **absorbing primary ideal of** R.

Theorem 10. *Let $I_1, I_2, \dots, I_m$ be $P-n-$ absorbing primary ideals of R for some n-absorbing ideal P of R. Then $I = \bigcap_{i=1}^m I_i$ is a $P-n-$ absorbing primary ideal of R.*

Proof. Note that $P = \sqrt{I} = \bigcap_{i=1}^m \sqrt{I_i}$. Let $x_1x_2\cdots x_{n+1} \in I$ for some $x_1, \dots, x_{n+1} \in R$, and assume $x_1x_2\cdots x_n \notin I$. (If it is, we are done). Then $x_1x_2\cdots x_n \notin I_i$ for some $i \in [1, m]$. Since each I_i is a $P-n-$ absorbing primary ideal and $x_1x_2\cdots x_n \notin I_i$, we must have that $x_1\cdots\widehat{x_i}\cdots x_{n+1} \in \sqrt{I_i} = P$ for some $i \in [1, n+1]$. Thus I is a $P-n-$ absorbing primary ideal of R.

Theorem 11. *Let $f : R \to R'$ be a homomorphism of commutative rings. Then*

(1) If I' is an n-absorbing primary ideal of R', then $f^{-1}(I')$ is an n-absorbing primary ideal of R.

(2) If f is an epimorphism and I is an n-absorbing primary ideal of R containing $Ker(f)$, then $f(I)$ is an n-absorbing primary ideal of R'.

Proof. (1) Suppose $x_1, \dots, x_n, x_{n+1} \in R$ such that $x_1\cdots x_nx_{n+1} \in f^{-1}(I')$. Then we have that $f(x_1x_2\cdots x_{n+1}) = f(x_1)f(x_2)\cdots f(x_{n+1}) \in I'$. Now I' is an n-absorbing primary ideal of R', so either

$f(x_1)f(x_2)\cdots f(x_n) = f(x_1x_2\cdots x_n) \in I' \Longrightarrow x_1x_2\cdots x_n \in f^{-1}(I')$, or

$f(x_1)\cdots \widehat{f(x_i)}\cdots f(x_{n+1}) = f(x_1\cdots \widehat{x_i}\cdots x_{n+1}) \in \sqrt{I'} \Longrightarrow x_1\cdots\widehat{x_i}\cdots x_{n+1} \in f^{-1}(\sqrt{I'}) = \sqrt{f^{-1}(I')}$ for some $i \in [1,n]$. Thus $f^{-1}(I')$ is n-primary absorbing.

(2) Suppose that $x_1^{'}, x_2^{'}, \ldots, x_n^{'}, x_{n+1}^{'} \in R'$ such that $x_1^{'}\cdots x_n^{'}x_{n+1}^{'} \in f(I)$. Then there are elements in R, say, $x_1, \ldots, x_{n+1}$ such that $f(x_1) = x_1^{'}, \ldots, f(x_n) = x_n^{'}$. Then we have $f(x_1\cdots x_{n+1}) = f(x_1)\cdots f(x_{n+1}) = x_1^{'}\cdots x_{n+1}^{'} \in f(I)$. Since $ker(f) \subseteq I$ we have $x_1 ... x_{n+1} \in I$. I is n-absorbing primary ideal of R, thus we know either $x_1\cdots x_n \in I \Longrightarrow f(x_1\cdots x_n) = f(x_1)\cdots f(x_n) = x_1^{'}\cdots x_n^{'} \in f(I)$, or we have that $x_1\cdots\widehat{x_i}\cdots x_{n+1} \in \sqrt{I} \Longrightarrow f(x_1\cdots\widehat{x_i}\cdots x_{n+1}) = f(x_1)\cdots\widehat{f(x_i)}\cdots f(x_{n+1}) = x_1^{'}\cdots\widehat{x_i^{'}}\cdots x_{n+1}^{'} \in f(\sqrt{I}) \subseteq \sqrt{f(I)}$ for some $i \in [1,n]$. Thus $f(I)$ is an n-absorbing primary ideal of R'.

Theorem 12. *Let I be an ideal of R. If $\sqrt{I}$ is a 2-absorbing ideal of R, then I is a 3-absorbing primary ideal of R.*

Proof. Suppose for $a, b, c, d \in R$ we have that $abcd \in I$ and $abc \notin I$. Now since $abcd \in I$, then $(ad)bc \in I \subseteq \sqrt{I}$ we have that $(ad)bc \in \sqrt{I}$. Since $\sqrt{I}$ is 2-absorbing, we have that either $(ad)b = abd \in \sqrt{I}$ or $(ad)c = acd \in \sqrt{I}$ or $bc \in \sqrt{I} \Longrightarrow bcd \in \sqrt{I}$ since $\sqrt{I}$ is an ideal. Thus I is a 3-absorbing primary ideal of R.

Theorem 13. *If $\sqrt{I}$ is an $(n+1)$-absorbing primary ideal of R, then I is an $(n+1)$-primary ideal of R.*

Proof. Suppose $x_1\cdots x_{n+1}x_{n+2} \in I$ and $x_1\cdots x_{n+1} \notin I$. Then $x_1\cdots x_{n+1}x_{n+2} \in I \subseteq \sqrt{I}$. Thus we have that $x_1x_2\cdots(x_{n+1}x_{n+2}) \in \sqrt{I}$ which is n-absorbing. Let $(x_{n+1}x_{n+2}) = x_0$. Then we know $x_1x_2\cdots\widehat{x_i}\cdots x_nx_0 \in \sqrt{I}$ for some $i \in 0, 1, 2, \ldots, n$ which is a product of $n+1$ elements if $i \in [1,n]$. Thus suppose that $i = 0$. Then $x_1x_2\cdots x_n \in \sqrt{I}$. Since $x_{n+2} \in R$, and $\sqrt{I}$ is an ideal, we have that $x_1\cdots x_nx_{n+2} \in \sqrt{I}$. Thus I is an $(n+1)$-absorbing primary ideal of R.

Theorem 14. *Let $R = R_1 \times \cdots \times R_{n+1}$, and J be a proper nonzero ideal of R. If J is an $n+1$-absorbing primary ideal of R, then $J = I_1 \times \cdots \times I_{n+1}$ for some proper n-*

absorbing primary ideals $I_1, \dots, I_{n+1}$ *of* $R_1, \dots, R_{n+1}$ *and* $I_1 \neq R_1, \dots, I_{n+1} \neq R_{n+1}$.

Proof. Let $a_1, \dots, a_{n+1} \in R$ such that $a_1 \cdots a_{n+1} \in I_1$ and suppose by contradiction that I_1 is not an n-absorbing primary ideal of R. Then define the following elements of R; $x_1 = (a_1, 1, \dots, 1), x_2 = (a_2, 1, \dots, 1), \dots, x_{n+1} = (a_{n+1}, 1, \dots, 1), x_{n+2} = (1, 0, \dots, 0)$. Then we have $x_1 x_2 \cdots x_{n+2} = (a_1 \cdots a_{n+1}, 0, \dots, 0) \in J$ and $x_1 \cdots x_{n+1} = (a_1 \cdots a_{n+1}, 1, \dots, 1) \notin J$, and $x_1 \cdots \widehat{x_i} \cdots x_{n+2} = (a_1 \cdots \widehat{a_i} \cdots a_{n+2}, 0, \dots, 0) \notin \sqrt{J}$ for some $i \in [1, n+1]$. This is a contradiction, since we have that J is $n+1$-absorbing primary ideal. Thus I_1 must be an n-absorbing primary ideal of R. Similarly we can conclude that each I_i is an n-absorbing primary ideal, and thus we are done.

Furthermore, we can extend the idea of strongly n-absorbing ideals to n-absorbing primary ideals, and introduce the following definition.

Definition 6. A proper nonzero ideal I of R is a **strongly n-absorbing primary ideal** of R if whenever $I_1 \cdots I_{n+1} \subseteq I$ for ideals $I_1, \dots, I_{n+1}$ of R then either $I_1 \cdots I_n \subseteq I$ or $I_1 \cdots \widehat{I_i} \cdots I_{n+1} \subseteq I$ for some $i \in [1, n]$.

Proving properties of strongly n-absorbing primary ideals is open to future work in this area.

5 Properties of n-semi-absorbing ideals

Definition 7. A proper, nonzero ideal I of a commutative ring R is said to be **n-semi-absorbing for the positive integer** n if $x^{n+1} \in I$ implies $x^n \in I$ for any $x \in R$.

Clearly I is 1-semi-absorbing if and only if I is a radical, or semiprime, ideal. As far as we know, the concept of n-semi-absorbing ideals for $n \geq 2$ is new. Semi-absorbing ideals have some unexpected properties; the example after the next proposition shows that an n-semi-absorbing ideal need not be an $n+1$-semi-absorbing ideal.

Recall that for a real number r, the ceiling of r, denoted $\lceil r \rceil$, is the smallest integer greater than or equal to r.

Theorem 15. *Let R be a UFD, let $I = Ra$ be a proper nonzero principal ideal, and let n be a positive integer. Then I is n-semi-absorbing if and only if for every irreducible element p, if ℓ is the largest exponent such that $p^\ell \mid a$, then $\lceil \frac{\ell}{n} \rceil = \lceil \frac{\ell}{n+1} \rceil$.*

Remark. If we write $a = u \prod_{i=1}^k p_i^{a_i}$ for non-associate irreducible elements $p_1, \ldots, p_k$, a unit u, and positive integers $a_1, \ldots, a_k$, the hypothesis on a can be written $\lceil \frac{a_i}{n} \rceil = \lceil \frac{a_i}{n+1} \rceil$ for all i.

Proof. Write $a = u \prod_{i=1}^k p_i^{a_i}$ for non-associate irreducible elements $p_1, \ldots, p_k$, a unit u, and positive integers $a_1, \ldots, a_k$. First assume $\lceil \frac{a_i}{n} \rceil = \lceil \frac{a_i}{n+1} \rceil$ for all i and suppose $x^{n+1} \in I$ for some $x \in R$. We can then write $x = \prod_{i=1}^k p_i^{x_i} y$ for some positive integers $x_1, \ldots, x_n$ and some $y \in R$ such that y is relatively prime to each p_i. The fact that $a \mid x^{n+1}$ tells us $a_i \leq (n+1)x_i$ for each i. Thus $\dfrac{a_i}{n+1} \leq x_i$ and our hypothesis implies $\frac{a_i}{n} \leq \lceil \frac{a_i}{n} \rceil = \lceil \frac{a_i}{n+1} \rceil \leq x_i$. This shows $a_i \leq nx_i$ and so $a \mid x^n$. This proves I is n-semi-absorbing.

For the converse, suppose that $\lceil \frac{a_i}{n} \rceil > \lceil \frac{a_i}{n+1} \rceil$ for some i. Set $b_i = \lceil \dfrac{a_i}{n+1} \rceil$ and set $x = p_i^{b_i} \prod j \neq i p_j^{a_j}$. Since $a_i \leq (n+1)b_i$, we have $a \mid x^n$. However, $b_i + 1 \leq \lceil \frac{a_i}{n} \rceil$, and so $nb_i + n \leq n\lceil \frac{a_i}{n} \rceil < a_i + n$. This shows $nb_i < a_i$, so a does not divide x^n. This shows that I is not n-semi-absorbing.

Example. Let $p \in R$ be an irreducible element and let $I = Rp^4$. Since I is not a radical ideal, I is not 1-semi-absorbing. Since $\lceil \frac{4}{2} \rceil = 2 = \lceil \frac{4}{3} \rceil$, the preceding proposition shows that I is 2-semi-absorbing. However, $\lceil \frac{4}{3} \rceil = 2 > 1 = \lceil \frac{4}{4} \rceil$, the preceding proposition implies that I is not 3-semi-absorbing. For $n \geq 4$, the preceding proposition shows that I is n-semi-absorbing. In fact, for $n \geq 4$, we see that I is n-absorbing.

Theorem 5(b) below shows that, although I is 2-semi-absorbing, I cannot be the intersection of any collection of 2-absorbing ideals. This provides a stark contrast with the fact that any radical (i.e., 1-semi-absorbing) ideal is the intersection of prime ideals (1-absorbing ideals).

Theorem 16. *(a) Any intersection of n-semi-absorbing ideals is an n-semi-absorbing ideal.*

(b) Any intersection of n-absorbing ideals is m-semi-absorbing for all $m \geq n$.

Proof. The proof is easy.

6 Further Research

In their paper on n-absorbing ideals of commutative rings [1], Anderson and Badawi conjecture that if I is an n-absorbing ideal of R then it is a strongly n-absorbing ideal of R for any positive integer n. This has yet to be shown and is left as an open question. However, the authors do present some nice results that follow if this statement can be proved. In a paper by Darani and Puczylowski [4], the authors prove this statement in the case that R/I has no additive torsion. It is also an open question whether n-absorbing primary ideals are strongly n-absorbing primary ideals of R. Further research in this area might also include investigating analogs of n-absorbing ideals in noncommutative rings.

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