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Roman Domination in Complementary Prisms

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Roman Domination in Complementary Prisms

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by

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ABSTRACT

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The complementary prism $G\overline{G}$ of a graph G is formed from the disjoint union of G and its complement $\overline{G}$ by adding the edges of a perfect matching between the corresponding vertices of G and $\overline{G}$. A Roman dominating function on a graph $G = (V, E)$ is a labeling $f : V(G) \mapsto \{0, 1, 2\}$ such that every vertex with label 0 is adjacent to a vertex with label 2. The Roman domination number $\gamma_R(G)$ of G is the minimum $f(V) = \Sigma_{v\in V} f(v)$ over all such functions of G. We study the Roman domination number of complementary prisms. Our main results show that $\gamma_R(G\overline{G})$ takes on a limited number of values in terms of the domination number of $G\overline{G}$ and the Roman domination numbers of G and $\overline{G}$.

DEDICATION

With love I dedicate this dissertation to the three most important people in my life.

My Father

My Father-in-law

My Pretty Wife.

ACKNOWLEDGMENTS

Before anything, I have to say, “Thank God for everything that you gave me.” First, I would like to thank my committee chair, Dr. Teresa Haynes for helping and motivating me. There is no question that she is in love with Graph Theory. I still remember my first class with her in Graph Theory. When the class ended, everyone there was knew that she was in love with Graph Theory. That was my first impression about graph theory and how awesome it was. Second, I would like to thank Dr. Gardner who always supported me. Whenever I had a problem or difficult situation, he was there by my side supporting and advising me. Thank you. Also, he is my academic adviser. Even though he has a wide knowledge and a great position in the math department, he is really humble. Thank you a lot. Thirdly, I would like to thank my friends who gave me support and sheered me up. Also, I have to mention my best friend, Miss. Franklina Samani, who is also in this field for our time and study together. She and I used to study together and whenever I got bored, she was there to encourage me. There is no word kind enough to describe her and what she has done for me. Thank you. Finally, I don’t know what to say to my family. However, one thing that I am pretty sure about is that when you have a nice family like I have, you will be able to accomplish any goal in your life.

GOD BLESS YOU ALL.

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1 INTRODUCTION

Complementary products were introduced in [6] as a generalization of cartesian products. Problems involving domination invariants of cartesian products [9,11] are among the most interesting and well-studied problems in graph theory. In this research, we consider Roman domination in a sub-family of complementary products called complementary prisms.

In graph theory, the complement of a graph G is a graph $\overline{G}$ on the same vertices such that two distinct vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G. That is, to generate the complement of a graph, one fills in all the missing edges required to form a complete graph, and removes all the edges that were previously there.

For a graph $G = (V, E)$, the complementary prism, denoted $G\overline{G}$, is formed from the disjoint union of G and its complement $\overline{G}$ by adding a perfect matching between corresponding vertices of G and $\overline{G}$. For each $v \in V(G)$, let $\overline{v}$ denote the vertex corresponding to v in $\overline{G}$. Formally, the graph $G\overline{G}$ is formed from $G \cup \overline{G}$ by adding the edge $v\overline{v}$ for every $v \in V(G)$. We note that complementary prisms are a generalization of the Petersen graph. In particular, the Petersen graph is the complementary prism $C_5\overline{C}_5$. For another example of a complementary prism, consider the following. The *corona* of a graph G, denoted $G \circ K_1$, is formed from G by adding for each $v \in V(G)$, a new vertex v' and the pendant edge vv'. Thus, the corona $K_n \circ K_1$ is the complementary prism $K_n\overline{K}_n$. See Figure 1 for another example.

The hamiltonicity of complementary prisms is studied in [10] and domination parameters of complementary prisms have been studied in [5,7] and elsewhere. As previously mentioned, our focus is on Roman domination in these graphs.

A *Roman dominating function (RDF)* on a graph G is a vertex labeling $f : V(G) \mapsto \{0, 1, 2\}$ such that every vertex with label 0 is adjacent to at least one vertex with label 2. For any Roman dominating function f of G, and $i \in \{0, 1, 2\}$, let $V_i = \{v \in V(G) \mid f(v) = i\}$. Since this partition determines f, we write $f = (V_0, V_1, V_2)$. The *weight* of a Roman dominating function f is defined as $w(f) = \Sigma_{v \in V} f(v)$, equivalently $w(f) = |V_1| + 2|V_2|$. The *Roman domination number* $\gamma_R(G)$ of G is the minimum weight of a Roman dominating function on the graph G. If a Roman dominating function of G has weight $\gamma_R(G)$, then it is referred to as a γ_R-function of G. Roman domination was introduced by Cockayne et al. [4] in 2004 and has received much attention in the literature, see for example [1,2,3,8].

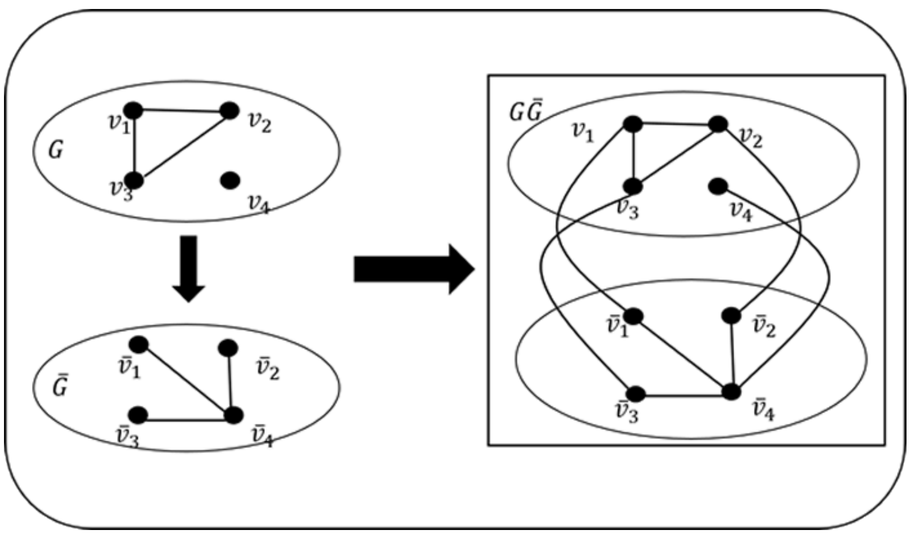

Figure 1: Example of Complementary Prism

To aid in our discussion, we will need some more terminology. For a vertex $v \in V(G)$, the *open neighborhood* of v is $N(v) = \{u \in V(G) \mid uv \in E(G)\}$, and the *closed neighborhood* $N[v] = N(v) \cup \{v\}$. The degree of a vertex v is $deg_G(v) = |N(v)|$. A vertex of degree 0 is an *isolated vertex*.

For two vertices u and v in a connected graph G, the *distance* $d_G(u, v)$ between u and v is the length of a shortest u–v path in G. The maximum distance among all pairs of vertices of G is its *diameter*, which is denoted by $diam(G)$. We say that G is a diameter-k graph if $diam(G) = k$. If G is disconnected, then $diam(G) = \infty$.

A set $S \subseteq V(G)$ is a *dominating set* of G if every vertex of $V(G) \setminus S$ is adjacent to a vertex in S. The *domination number* $\gamma(G)$ is the minimum cardinality of any dominating set of G, and a dominating set of cardinality $\gamma(G)$ is called a γ-*set* of G.

We refer to the complementary prism $G\overline{G}$ as a copy of G and a copy of $\overline{G}$ with a perfect matching between corresponding vertices. For a set $P \subseteq V(G)$, let $\overline{P}$ denote the corresponding set of vertices in $V(\overline{G})$. We also shorten $V(G)$ to V and $V(\overline{G})$ to $\overline{V}$. Further, for any function f on $G\overline{G}$, we let $w(f_V)$ denote the weight of f on G, and $w(f_{\overline{V}})$ denote the weight of f on $\overline{G}$. We note that $G\overline{G}$ is isomorphic to $\overline{G}G$, so our results stated in terms of G also apply to $\overline{G}$ unless otherwise stated.

In this thesis, we show that Roman domination numbers of complementary prisms $G\overline{G}$ take on a limited number of values in terms of the domination number of $G\overline{G}$ and the Roman domination numbers of G and $\overline{G}$. These values are summarized in Table 1 in Section 5. Among other results, we prove the lower bounds of Table 1 in Section 3 and the upper bounds in Section 4.

2 LITERATURE REVIEW

In our literature review, we shall see that domination and total domination in complementary prisms have been previously studied. However, as far as we know, the work in this thesis is the first study at Roman domination in complementary prisms.

First, we will recall some of theorems and properties that relates to Roman domination functions.

Proposition 2.1 *[4] For paths* P_n *and cycles* C_n , $\gamma_R(P_n) = \lceil 2n/3 \rceil = \gamma_R(C_n)$.

Observation 2.2 *[4] For any graph* G, $\gamma_R(G) \leq 2\gamma(G)$.

Observation 2.3 *[4] A graph* G *is called Roman if* $\gamma_R(G) = 2\gamma(G)$. *We say that a graph* G *is almost Roman if* $\gamma_R(G) = 2\gamma(G) - 1$.

Proposition 2.4 *[4] For any graph* G *with no isolated vertices, there exists a* γ_R*-function* $f = (V_0, V_1, V_2)$ *of* G *such that if* $V_1 \neq \emptyset$, *then* V_1 *is a* 2*-packing.*

Theorem 2.5 *[4] For any non-trivial connected graph* G, $\gamma_R(G) = \min\{2\gamma(G-S) + |S| : S \text{ is a 2-packing}\}$.

Proposition 2.6 *[4] A graph* G *is Roman if and only if it has a* γ_R*-function* $f = (V_0, V_1, V_2)$ *where* $V_1 = \emptyset$.

Second, we will recall some of theorems and properties that relates to complementary prisms.

Theorem 2.7 *[6] For the complementary prism* $G\overline{G}$, $diam(G\overline{G}) = 2$ *if* $diam(G) = diam(\overline{G}) = 2$, *else* $diam(G\overline{G}) = 3$.

Finally, there are some results that help us from domination and total domination in complementary prisms.

Proposition 2.8 *[7] Let G be a graph of order n. Then*

1. *If $G = K_n$, then $\gamma(G\overline{G}) = n$.*

2. *If $G = tK_2$, then $\gamma(G\overline{G}) = t + 1$.*

3. *If $G = C_n$, and $n \geq 3$, then $\gamma(G\overline{G}) = \lceil (n + 4)/3 \rceil$.*

4. *If $G = P_n$, and $n \geq 2$, then $\gamma(G\overline{G}) = \lceil (n + 3)/3 \rceil$.*

Proposition 2.9 *[7] Let G be a graph of order n. Then*

1. *If $G = K_n$, then $\gamma_t(G\overline{G}) = n$.*

2. *If $G = tK_n$, then $\gamma_t(G\overline{G}) = 2t = n$.*

3. *If $G \in \{C_n, P_n\}$ with order $n \geq 5$ then*

$$\gamma_t(G\overline{G}) = \begin{cases} \gamma_t(G) & \text{if} \quad n \equiv 2 \ (\text{mod } 4) \\ \gamma_t(G) + 2 & \text{if} \quad G = C_n \text{ and } n \equiv 2 \ (\text{mod } 4) \\ \gamma_t(G) + 1 & 0 \quad \text{otherwise.} \end{cases}$$

3 SMALL VALUES AND LOWER BOUNDS

In our first section containing new results, we characterize the complementary prisms having small Roman domination numbers.

Observe that $\gamma_R(G\overline{G}) \geq 2$ for any graph G. As examples, we determine the complementary prisms $G\overline{G}$ having small Roman domination numbers, namely, those with $\gamma_R(G\overline{G}) \in \{2, 3, 4\}$.

Theorem 3.1 *Let G be a graph of order n. Then*

1. $\gamma_R(G\overline{G}) = 2$ *if and only if* $G = K_1$.

2. $\gamma_R(G\overline{G}) = 3$ *if and only if* $G = K_2$ *or* $\overline{G} = K_2$.

3. $\gamma_R(G\overline{G}) = 4$ *if and only if* $\gamma_R(G) = 3$ *and G has an isolated vertex or* $\gamma_R(\overline{G}) = 3$ *and $\overline{G}$ has an isolated vertex.*

Proof. (1) If $G = K_1$, then $G\overline{G} = K_2$ and $\gamma_R(K_2) = 2$.

Assume that $\gamma_R(G\overline{G}) = 2$. Since a vertex in G (respectively, $\overline{G}$) can Roman dominate at most one vertex in $\overline{G}$ (respectively, G), it follows that any function of weight 2 can Roman dominate at most one vertex in G or at most one vertex in $\overline{G}$. Hence, $G = K_1$. See Figure 2.

(2) If $G = K_2$, then $G\overline{G}$ is isomorphic to the path P_4 and $\gamma_R(P_4) = 3$.

Assume that $\gamma_R(G\overline{G}) = 3$. Then at most one vertex of $G\overline{G}$, say v, is assigned a 2 under any γ_R-function of $G\overline{G}$. It follows that $\overline{G} - \overline{v}$ must be Roman dominated with a weight of 1. Thus, $\overline{G} - \overline{v}$ consists of exactly one vertex, that is, $\overline{G}$, and hence, G has order 2. Thus, $\{G, \overline{G}\} = \{K_2, \overline{K}_2\}$. See Figure 3.

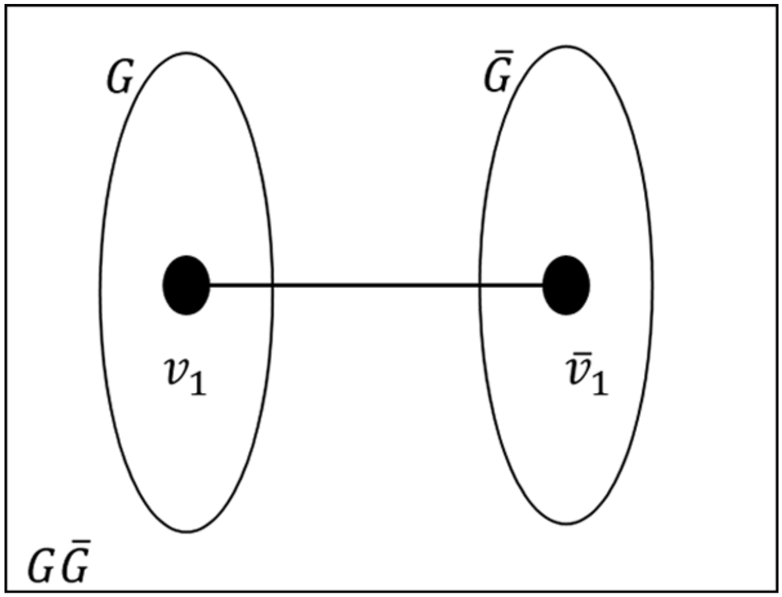

Figure 2: $G = K_1$

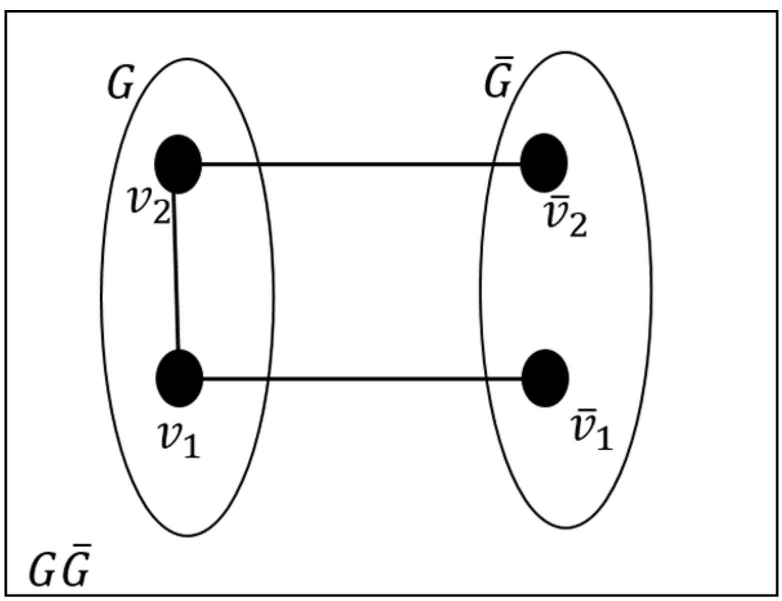

Figure 3: $G = K_2$

(3) Without loss of generality, assume that $\gamma_R(G) = 3$ and G has an isolated vertex v. See Figure 4 for an example of the complementary prism of such a graph, then assigning a 2 to $\overline{v}$ Roman dominates $\overline{V} \cup \{v\}$. Further, since v is an isolate of G and $\gamma_R(G) = 3$, it follows that assigning a total weight of 2 on the vertices of $G - v$ yields an RDF of $G\overline{G}$. Thus, $\gamma_R(G\overline{G}) \leq 2 + 2 = 4$. Equality follows from (1) and (2).

Finally, assume that $\gamma_R(G\overline{G}) = 4$, let f be a γ_R-function of $G\overline{G}$. Note that (1) and (2) imply that G has order at least 3. If no vertex of $G\overline{G}$ is assigned 2, then the order of $G\overline{G}$ is 4, implying that $G = K_2$ or $\overline{G} = K_2$, a contradiction. Thus, we may assume, without loss of generality, that $f(v) = 2$. Moreover, if $w(f_{\overline{V}}) = 0$, then $\overline{G}$ has order at most 2, a contradiction. Hence, we have that $w(f_V) \geq 2$, $w(f_{\overline{V}}) \geq 1$, and $w(f_V) + w(f_{\overline{V}}) = 4$. Further, if $w(f_{\overline{V}}) = 1$, then $w(f_V) = 3$. This implying that at most two vertices of $\overline{G}$ are Roman dominated by f, a contradiction since $\overline{G}$ has order at least 3. Hence, it must be the case that $w(f_V) = w(f_{\overline{V}}) = 2$. If two vertices of $\overline{G}$ are labeled 1, then v dominates G. This implying that $\overline{v}$ is an isolate in $\overline{G}$ and G has order exactly 3. It follows that $\overline{v}$ is assigned 0 under f. Assigning 1 to $\overline{v}$ and $f(\overline{u})$ to each vertex $\overline{u}$ of $\overline{G} - \overline{v}$ gives a RDF of $\overline{G}$, and so, $\gamma_R(\overline{G}) = 3$ and $\overline{G}$ has an isolated vertex. Hence, we may assume that there is a vertex $\overline{u} \in \overline{G}$ for which $f(\overline{u}) = 2$. Thus, v Roman dominates $G - u$ and $\overline{u}$ Roman dominates $\overline{G} - \overline{v}$. Now u and v are adjacent in G or $\overline{u}$ and $\overline{v}$ are adjacent in $\overline{G}$. Hence, either u is an isolate in G or $\overline{v}$ is an isolate in $\overline{G}$. Without loss of generality, let u be an isolate in G. As before, $\gamma_R(G) = 3$, and the result holds. □

Corollary 3.2 *If G and its complement $\overline{G}$ are isolate-free graphs, then $\gamma_R(G\overline{G}) \geq 5$.*

For our next example, we determine the Roman domination number of the com-

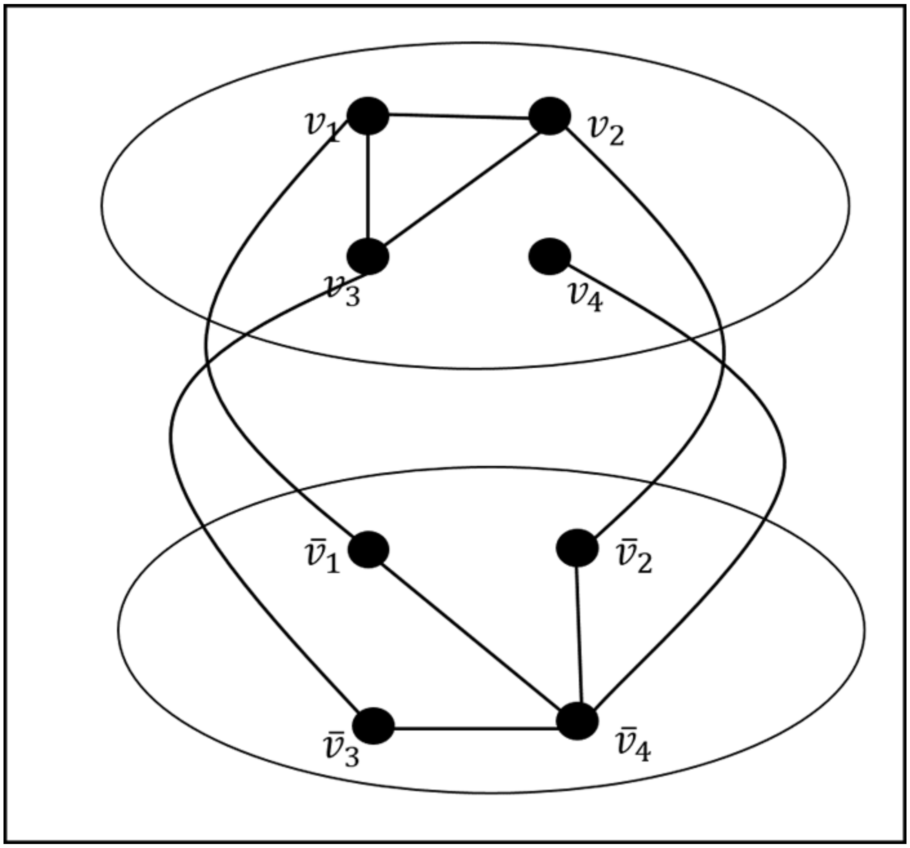

Figure 4: Example of $\gamma_R(G) = 3$ and G has an isolated vertex

plementary prism of a complete graph K_n. See Figure 5.

Proposition 3.3 *If $G = K_n$, then $\gamma_R(G\overline{G}) = n + 1$.*

Proof. Let v be a vertex in G. First note that the function assigning 2 to v, 1 to each vertex in $\overline{V} \setminus \{\overline{v}\}$, and 0 otherwise is an RDF of $G\overline{G}$. Hence, $\gamma_R(G\overline{G}) \leq n + 1$.

To see that $\gamma_R(G\overline{G}) \geq n + 1$, let f be a γ_R-function of $G\overline{G}$. Note that for every vertex $\overline{v} \in \overline{V}$, either $f(\overline{v}) \geq 1$ or $f(v) = 2$, implying that $\gamma_R(G\overline{G}) \geq n$. Further note that if f has weight n, then every vertex of V is assigned 0 under f and every vertex of $\overline{V}$ is assigned 1. But then the vertices of G are not Roman dominated by f, a

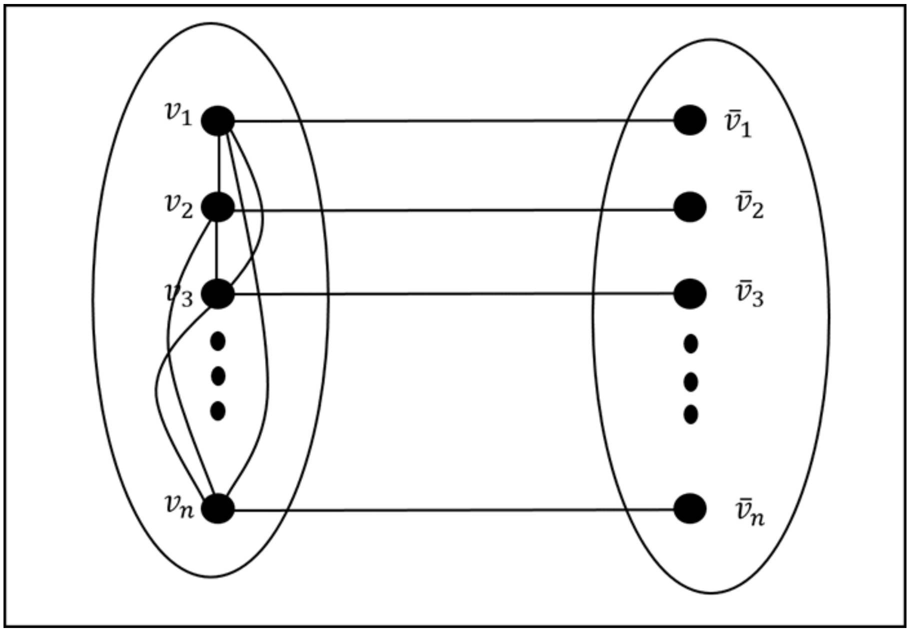

Figure 5: $G = K_n$

contradiction. Hence, $\gamma_R(G\overline{G}) \geq n+1$. □

Notice that from Proposition 3.3, $\gamma_R(G\overline{G}) = n+1 = \gamma_R(\overline{G})+1 = \max\{\gamma_{\mathrm{R}}(\mathrm{G}), \gamma_{\mathrm{R}}(\overline{\mathrm{G}})\}+1$ for $G = K_n$. Next we show that for any graph G, $\gamma_R(G\overline{G}) \geq \max\{\gamma_{\mathrm{R}}(\mathrm{G}), \gamma_{\mathrm{R}}(\overline{\mathrm{G}})\}+1$.

Theorem 3.4 *For any graph G, $\gamma_R(G\overline{G}) \geq \max\{\gamma_{\mathrm{R}}(\mathrm{G}), \gamma_{\mathrm{R}}(\overline{\mathrm{G}})\} + 1$ with equality if and only if G or $\overline{G}$ has an isolated vertex.*

Proof. Let G be a graph of order n. If $n = 1$, then $G\overline{G} = K_1\overline{K}_1 = K_2$ and $\gamma_R(G\overline{G}) = 2 = \max\{\gamma_{\mathrm{R}}(\mathrm{G}), \gamma_{\mathrm{R}}(\overline{\mathrm{G}})\} + 1$. If $n = 2$, then $G\overline{G} = K_2\overline{K}_2 = P_4$ and $\gamma_R(G\overline{G}) = 3 = \max\{\gamma_{\mathrm{R}}(\mathrm{G}), \gamma_{\mathrm{R}}(\overline{\mathrm{G}})\} + 1$. Thus, if $n \leq 2$, the result holds.

Henceforth, we assume that $n \geq 3$, and without loss of generality, that $\gamma_R(G) \geq \gamma_R(\overline{G})$. Clearly, $\gamma_R(G\overline{G}) \geq \gamma_R(G)$. To see this, suppose that $\gamma_R(G\overline{G}) < \gamma_R(G)$. Let f be a γ_R-function of $G\overline{G}$. Since $\gamma_R(G\overline{G}) < \gamma_R(G)$, there exists a set $S \subseteq V$ that are not Roman dominated by the vertices of G. Thus, each vertex in S is dominated by its corresponding vertex in $\overline{S}$, that is, $f(v) = 0$ and $f(\overline{v}) = 2$ for each $v \in S$. Hence, $\gamma_R(G\overline{G}) = w(f_V) + w(f_{\overline{V}}) \geq w(f_V) + 2|S|$. But the function $f^* : V \mapsto \{0, 1, 2\}$ defined by $f^*(x) = f(x)$ if $x \in V \setminus S$ and $f^*(x) = 1$ if $x \in S$, is an RDF of G with weight $w(f^*) = w(f_V) + |S|$ implying that $\gamma_R(G) \leq w(f_V) + |S| < w(f_V) + 2|S| \leq \gamma_R(G\overline{G}) < \gamma_R(G)$, a contradiction.

Now we show that $\gamma(G\overline{G}) \geq \gamma_R(G) + 1$. Suppose, to the contrary, that $\gamma_R(G\overline{G}) = \gamma_R(G)$. Let f be a γ_R-function of $G\overline{G}$. We consider two cases:

Case 1. $w(f_V) = \gamma_R(G)$. Then $w(f_{\overline{V}}) = 0$, implying that every vertex of G is assigned 2 under f. Thus, $w(f_V) = 2n = \gamma_R(G)$, a contradiction since $\gamma_R(G) \leq n$ for all graphs G. See Figure 6.

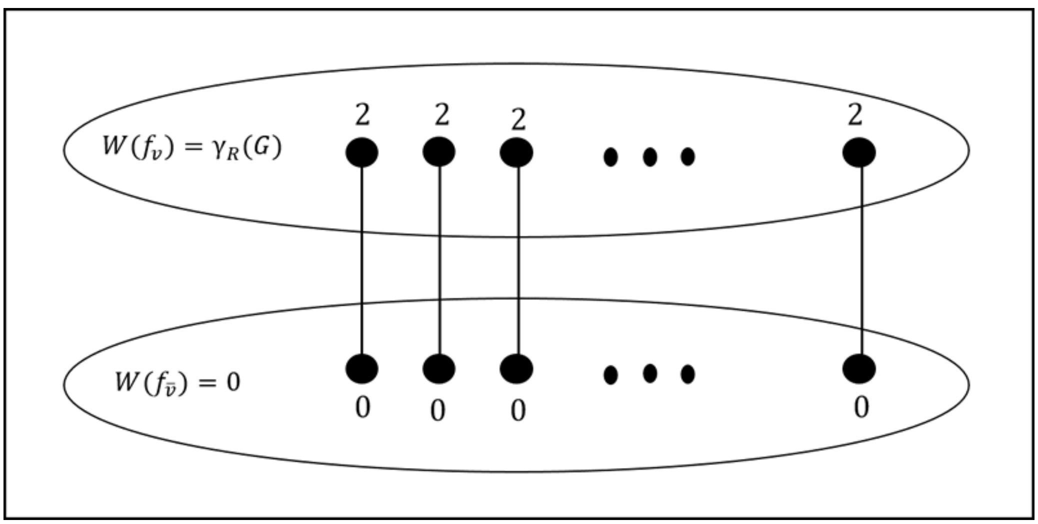

Figure 6: $w(f_V) = 2n$ and $w(f_{\overline{V}}) = 0$

Case 2. $w(f_V) < \gamma_R(G)$. Then $p \geq 1$ vertices of G are Roman dominated by their corresponding vertices in $\overline{G}$. Thus, $\gamma_R(G\overline{G}) \geq w(f_V) + 2p$. On the other hand, assigning the weight of 1 to each of the p vertices in G and $f(v)$ to each other vertex of G is an RDF of G. Hence, $\gamma_R(G) \leq w(f_V) + p$. But then we have $w(f_V) + 2p \leq \gamma_R(G\overline{G}) = \gamma_R(G) \leq w(f_V) + p$, a contradiction since $p \geq 1$. Thus, $\gamma_R(G\overline{G}) \geq \gamma_R(G) + 1$ in both cases.

To complete the proof, we show that equality holds if and only if G or $\overline{G}$ has an isolated vertex. Without loss of generality, assume that G has an isolated vertex v. Then $\bar{v}$ dominates $\overline{G}$, and since $n \geq 3$, it follows that $\gamma_R(\overline{G}) = 2$ and $\gamma_R(G) \geq \gamma_R(\overline{G})$. Moreover, the function assigning a 2 to $\overline{v}$, 0 to the vertices of $\overline{G} - \overline{v}$ and to v combined with a γ_R-function of $G - v$ is an RDF of $G\overline{G}$. Hence, $\gamma_R(G\overline{G}) \leq 2 + \gamma_R(G) - 1 = \gamma_R(G) + 1$, and so, $\gamma_R(G\overline{G}) = \gamma_R(G) + 1$.

Assume that $\gamma_R(G\overline{G}) = \gamma_R(G) + 1$. Suppose to the contrary that neither G nor $\overline{G}$ has an isolated vertex. Thus, G has order $n \geq 4$. Let f be a γ_R-function of $G\overline{G}$. As before, if every vertex of $\overline{G}$ is Roman dominated by the vertices of G, that is, if $w(f_{\overline{V}}) = 0$, then $\gamma_R(G\overline{G}) = 2n = \gamma_R(G) + 1$, implying that $\gamma_R(G) > n$, a contradiction. Similarly, if $w(f_{\overline{V}}) = 1$ and $w(f_V) = \gamma_R(G)$, then the vertices of $\overline{V} \setminus \{\overline{v}\}$, where $f(\overline{v}) = 1$, are Roman dominated by $V \setminus \{v\}$. Hence, each vertex in $V \setminus \{v\}$ is assigned 2 by f, and so, $\gamma_R(G) + 1 = \gamma_R(G\overline{G}) \geq 2(|V \setminus \{v\}|) + 1 = 2n - 1 > \gamma_R(G) + 1$, a contradiction since $\gamma_R(G) \leq n$ and $n \geq 4$. See Figure 7.

Hence, we may assume that $w(f_{\overline{V}}) \geq 2$ and $w(f_V) < \gamma_R(G)$, that is, $p \geq 1$ vertices of G are Roman dominated by the vertices in $\overline{G}$. As before, we deduce that $\gamma_R(G) \leq w(f_V) + p$. In other words, at least $\gamma_R(G) - w(f_V)$ vertices of G must be dominated by their corresponding vertices in $\overline{G}$. Thus, $\gamma_R(G\overline{G}) = w(f_V) + w(f_{\overline{V}}) = \gamma_R(G) + 1 \geq w(f_V) + 2(\gamma_R(G) - w(f_V)) = w(f_V) + 2\gamma_R(G) - 2w(f_V) = 2\gamma_R(G) -$

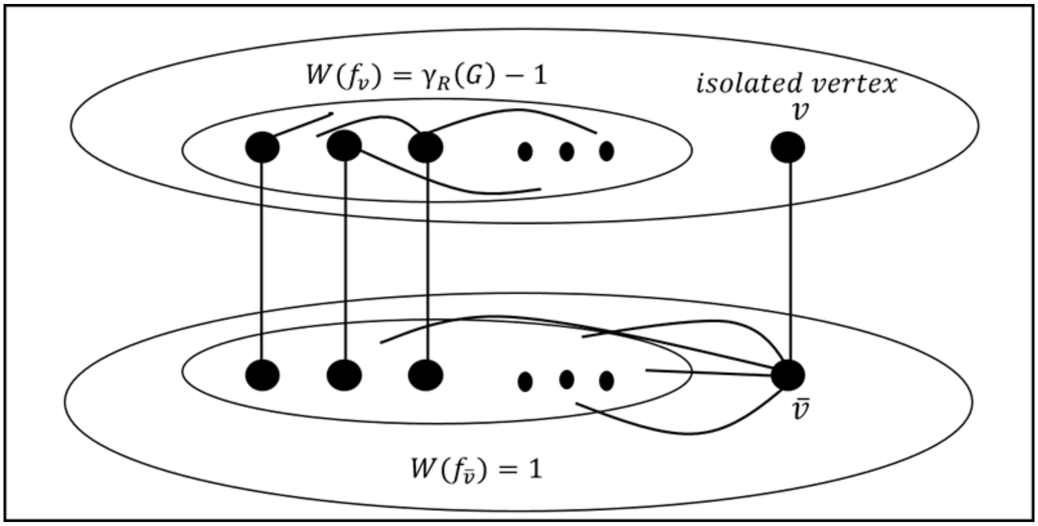

Figure 7: G has an isolated, then $\gamma_R(G\overline{G}) = \gamma_R(G) + 1$

$w(f_V) = 2(\gamma_R(G\overline{G})-1)-w(f_V) = 2w(f_V)+2w(f_{\overline{V}})-2-w(f_V) = w(f_V)+2w(f_{\overline{V}})-2 = \gamma_R(G\overline{G})+w(f_{\overline{V}})-2 = \gamma_R(G)+1+w(f_{\overline{V}})-2 = \gamma_R(G)+w(f_{\overline{V}})-1$. Thus, $w(f_{\overline{V}}) = 2$. Since $\gamma_R(G\overline{G}) = \gamma_R(G)+1$ and $w(f_{\overline{V}}) = 2$, we deduce that there is exactly one vertex of G, say v, that is dominated by a $\overline{v}$ in $\overline{V}$. Thus, $f(\overline{v}) = 2$ and $f(v) = 0$. Let S denote the set of vertices labeled 2 in G. Now, v has no neighbor in S, and so $\overline{v}$ Roman dominates $\overline{S}$ in $\overline{G}$. Moreover, $\overline{v}$ must dominate the vertices of $\overline{V}$ that are not dominated by their corresponding vertices in G, that is, the vertices of $\overline{V} \setminus \overline{S}$. Hence, $\overline{v}$ dominates $\overline{V}$, and so v is an isolate in G. $\square$

Corollary 3.5 *If neither graph G nor its complement $\bar{G}$ has an isolated vertex, then* $\gamma_R(G\overline{G}) \geq \max\{\gamma_{\mathrm{R}}(\mathrm{G}), \gamma_{\mathrm{R}}(\overline{\mathrm{G}})\} + 2$.

Next we show that the bound of Corollary 3.5 is sharp for the complementary prisms of paths. We use the following result from [4].

Proposition 3.6 [4] *For paths P_n, $\gamma_R(P_n) = \lceil 2n/3 \rceil$.*

Theorem 3.7 *For paths $G = P_n$ where $n \geq 3$, $\gamma_R(G\overline{G}) = \lceil \frac{2n}{3} \rceil + 2$.*

Proof. Let $G = P_n = (v_1, v_2, ..., v_n)$ for $n \geq 3$. Let f be a γ_R-function of the path P_n where $f(v_1) = 0$ and $f(v_2) = 2$. See Figure 8, it is straightforward to see that such a function exists. Note that f can be extended to an RDF of $G\overline{G}$ by assigning 2 to $\overline{v}_1$ and 0 to every other vertex of $\overline{G}$. Thus, $\gamma_R(G\overline{G}) \leq \gamma_R(P_n) + 2 = \lceil 2n/3 \rceil + 2$.

For $G = P_3$, $\gamma_R(G\overline{G}) = 4$ and the result holds. Thus, we may assume that $n \geq 4$. Since neither P_n nor $\overline{P}_n$ for $n \geq 4$ has an isolated vertex, by Corollary 3.5, $\gamma_R(G\overline{G}) \geq \gamma_R(G) + 2 = \lceil \frac{2n}{3} \rceil + 2$. $\square$

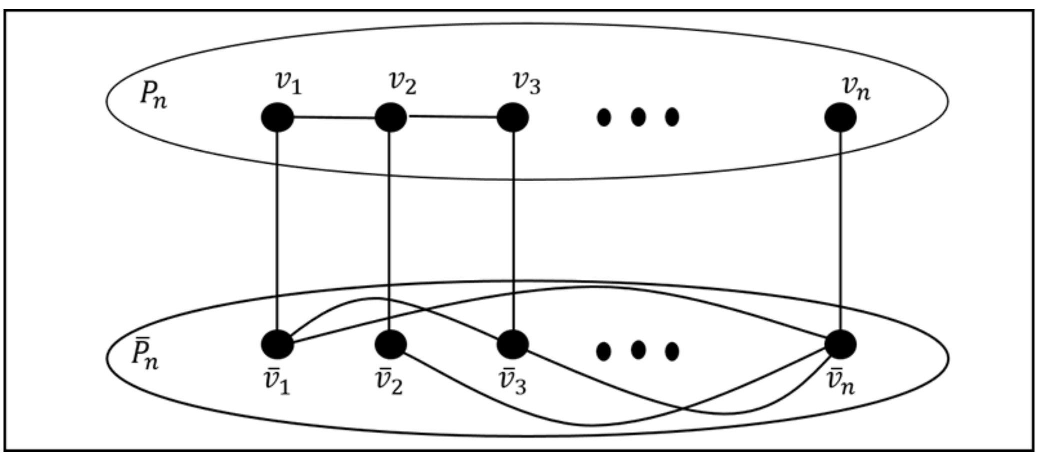

Figure 8: $G = P_n$

4 UPPER BOUNDS

We begin with some results involving general graphs G. Note that assigning a weight of 2 to every vertex of a γ-set S of G and a weight of 0 to the vertices in $V \setminus S$ is an RDF of G. This useful observation was first made in [4] as follows.

Observation 4.1 [4] *For any graph G, $\gamma_R(G) \leq 2\gamma(G)$.*

In [4], a graph G is called *Roman* if $\gamma_R(G) = 2\gamma(G)$. We say that a graph G is *almost Roman* if $\gamma_R(G) = 2\gamma(G) - 1$. Using the following results from [4], we observe that every diameter-2 graph is either Roman or almost Roman.

Proposition 4.2 [4] *For any graph G with no isolated vertices, there exists a γ_R-function $f = (V_0, V_1, V_2)$ of G such that if $V_1 \neq \emptyset$, then V_1 is a 2-packing.*

Theorem 4.3 [4] *For any non-trivial connected graph G, $\gamma_R(G) = \min\{2\gamma(G-S) + |S| : S$ is a 2-packing$\}$.*

Note that if $diam(G) = 2$, then any maximal 2-packing of G contains exactly one vertex. Thus, for diameter-2 graphs, if S is the set in Theorem 4.3, then either $S = \emptyset$ or $|S| = 1$. Since removing a vertex can decrease the domination number of any graph by at most one, we have the following corollaries to Theorem 4.3 .

Corollary 4.4 *If $diam(G) = 2$, then $\gamma_R(G) \in \{2\gamma(G), 2\gamma(G) - 1\}$.*

Corollary 4.5 *If G is a graph of diameter 2, then $\gamma_R(G) = 2\gamma(G) - 1$ if and only if G has a vertex v such that $\gamma(G - v) = \gamma(G) - 1$.*

Now turning our attention back to complementary prisms, we consider the following result from [6].

Theorem 4.6 *[6] For the complementary prism $G\overline{G}$, $diam(G\overline{G}) = 2$ if $diam(G) = diam(\overline{G}) = 2$, else $diam(G\overline{G}) = 3$.*

Corollary 4.4 and Theorem 4.6 now yield the following corollary.

Corollary 4.7 *For any graph G, if $diam(G) = diam(\overline{G}) = 2$, then $\gamma_R(G\overline{G}) \in \{2\gamma(G\overline{G}), 2\gamma(G\overline{G}) - 1\}$.*

In other words, if $diam(G\overline{G}) = 2$, then $G\overline{G}$ is Roman or almost Roman. Now we consider complementary prisms with diameter 3. Clearly, an RDF of G combined with an RDF of $\overline{G}$ forms an RDF of $G\overline{G}$, so we make the following straightforward observation.

Observation 4.8 *For any graph G, $\gamma_R(G\overline{G}) \leq \gamma_R(G) + \gamma_R(\overline{G})$.*

Theorem 4.9 *Let G be a graph with $diam(G) \geq 3$ such that neither G nor $\overline{G}$ has an isolated vertex. Then $\gamma_R(G) + 2 \leq \gamma_R(G\bar{G}) \leq \gamma_R(G) + 4$.*

Proof. The lower bound follows directly from Theorem 3.4 . For the upper bound, let u and v be peripheral vertices of G such that the distance between u and v equals $diam(G) \geq 3$. Since $\{\overline{u}, \overline{v}\}$ dominates $\overline{G}$, it follows that $\gamma(\overline{G}) \leq 2$. Observations 4.1 and 4.8 imply that $\gamma_R(G\overline{G}) \leq \gamma_R(G) + \gamma_R(\overline{G}) \leq \gamma_R(G) + 4$. □

We note that the upper bound of Theorem 4.9 is tight. To see this we consider a family of strong product graphs. The *strong product* $G \boxtimes H$ of two graphs G and H has

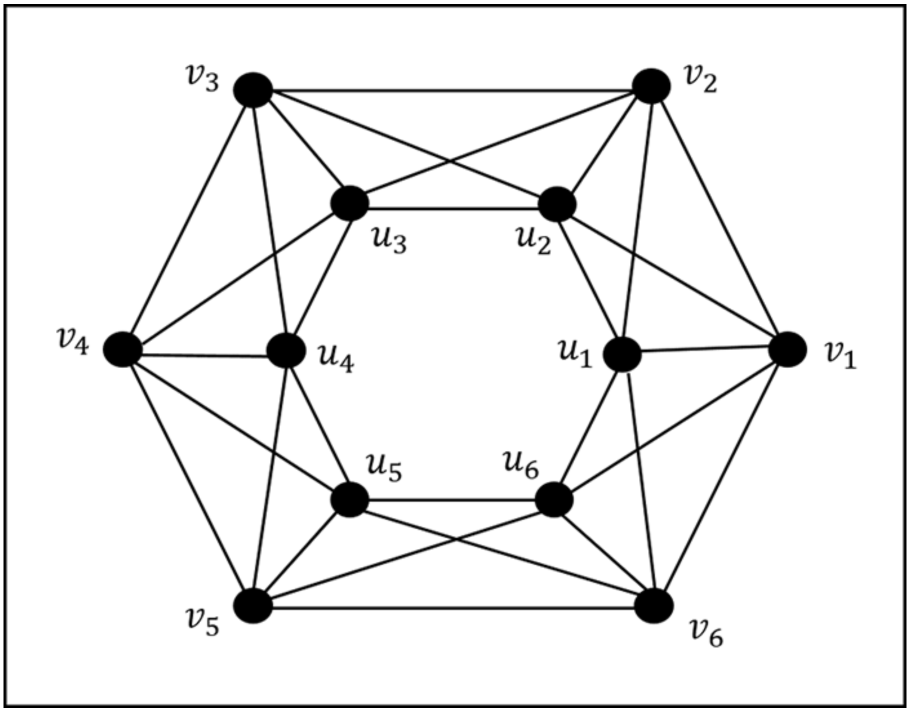

Figure 9: The graph G_k when $k = 2$

vertex set $V(G) \times V(H)$ and any two distinct vertices (u, u') and (v, v') are adjacent in $G \boxtimes H$ if and only if one of the following holds: $uv \in E(G)$ and $u' = v'$, or $u = v$ and $u'v' \in E(H)$, or $uv \in E(G)$ and $u'v' \in E(H)$. For $k \geq 2$, let $G_k = C_{3k} \boxtimes K_2$. For ease of discussion, we label the vertices of one copy of C_{3k} as u_i for $1 \leq i \leq 3k$ and the vertices of the other copy of C_{3k} as v_i for $1 \leq i \leq 3k$. The graph G_2 is illustrated in Figure 9. In our next result, we show that the complementary prisms $G_k\overline{G}_k$ are extremal graphs for the upper bound of Theorem 4.9 .

Proposition 4.10 *For the graph G_k with $k \geq 2$, $\gamma_R(G_k\overline{G}_k) = \gamma_R(G_k) + 4$.*

Proof. Let $G_k = C_{3k} \boxtimes K_2$ with the vertex set described above. Let $A = \{u_i \mid i \equiv 2 \pmod 3\}$. A function assigning a label of 2 to each vertex in A and 0 to each vertex of $V(G_k) \setminus A$ is an RDF of G_k. Hence, $\gamma_R(G_k) \leq 2k$. Any RDF of G_k that assigns a value of 2 to $s < k$ vertices of G_k, must of necessity assign a value of 1 to at least $6(k-s)$ vertices of G_k. Thus, any such function f will have a weight $w(f) = 2s + 6(k-s) = 6k - 4s > 2k$. Hence, $\gamma_R(G_k) = 2k$. Note that $\{u_2, u_5\}$ is a dominating set for $\overline{G}_k$. Therefore, $\gamma_R(\overline{G}_k) \leq 4$. Any RDF of $\overline{G}_k$ that assigns no 2 will have a weight of $n = 6k$ and if it labels exactly one vertex with a 2, it will have a weight of at least 7. Thus, $\gamma_R(G_k) \geq \gamma_R(\overline{G}_k) = 4$.

We note that by Observation 4.8, $\gamma_R(G_k\overline{G}_k) \leq \gamma_R(G_k) + \gamma_R(\overline{G}_k) = 2k+4$. Let f be a γ_R-function of $G_k\overline{G}_k$. We aim to show that $w(f) \geq \gamma_R(G_k) + \gamma_R(\overline{G}_k) = 2k+4$. If f assigns a value of 2 to $s < k$ vertices of $V(G_k)$, then it must either assign a value of 1 to at least $6(k-s)$ vertices of $V(G_k)$ or a value of 2 to their counterparts in $V(\overline{G}_k)$. In either case, $w(f) \geq 2s + 6(k-s) = 6k - 4s \geq 2k+4$. If f assigns a value of 2 to at least $k+2$ vertices of $V(G_k\overline{G}_k)$, then $w(f) \geq 2k+4$. If f assigns a value of 2 to exactly $k+1$ vertices of $V(G_k)$, then in order to Roman dominate the $6k - (k+1) = 5k-1$ vertices of $V(\overline{G}_k)$ not Roman dominated by the vertices of $V(G_k)$, it will also be necessary for $w(f) \geq 2k+2+5k-1 > 2k+4$. Thus, we may assume that exactly k vertices of $V(G_k)$ are assigned a label of 2 by f. If f does not assign a label of 2 to any vertex of $V(\overline{G}_k)$, then $w(f) \geq 2k+5k = 7k > 2k+4$. Hence, we may assume that f assigns a value of 2 to exactly one vertex of $V(\overline{G}_k)$ (if not, then $w(f) \geq 2k+4$ and we would be finished). Without loss of generality, assume that $f(\overline{u}_1) = 2$.

Let S be the set of k vertices of $V(G_k)$ assigned a label of 2 by f. Now $\overline{V} \setminus N[\overline{u}_1] = \{\overline{u}_2, \overline{u}_{3k}, \overline{v}_2, \overline{v}_{3k}, \overline{v}_1\}$. Moreover, if more than one of these vertices is assigned a 1 under

f, then we have the desired result. This implies that at least four of these vertices are dominated by vertices in S. Hence, at least four of the vertices of $\{u_2, u_{3k}, v_2, v_{3k}, v_1\}$ are in S. But then the k vertices of S do not dominate all the vertices $V(G_k) \setminus S$, a contradiction. It follows that $\gamma_R(G_k\overline{G}_k) = \gamma_R(G_k) + 4$. $\square$

As we have seen, the complementary prisms of paths attain the lower bound of Theorem 4.9 . Next we determine two additional families of complementary prisms attaining this lower bound. Note that since $\gamma_R(G) \leq 2\gamma(G)$, it follows that if a graph G is neither Roman nor almost Roman, then $\gamma_R(G) \leq 2\gamma(G) - 2$. Also, we have the following from [4].

Proposition 4.11 [4] *A graph G is Roman if and only if it has a γ_R-function $f = (V_0, V_1, V_2)$ where $V_1 = \emptyset$.*

Theorem 4.12 *If G is a graph that is neither Roman nor almost Roman and diam$(G) \geq 3$, then $\gamma_R(G\overline{G}) \leq \gamma_R(G) + 2 \leq 2\gamma(G)$.*

Proof. Select a γ_R-function $f = (V_0, V_1, V_2)$ of G such that $|V_2|$ is maximized. By Proposition 4.2 and Theorem 4.3, $V_1 = \emptyset$ or V_1 is a 2-packing of G. Since G is not Roman, it follows from Proposition 4.11 that $V_1 \neq \emptyset$.

Assume that $|V_1| = 1$, and let $V_1 = \{v\}$. In this case, $\gamma_R(G) = 2|V_2| + 1$ and V_2 dominates $V \setminus \{v\}$. Since $\gamma_R(G) \leq 2\gamma(G)$, it follows that $2|V_2| + 1 \leq 2\gamma(G)$. Hence, $|V_2| \leq \lfloor \gamma(G) - 1/2 \rfloor = \gamma(G) - 1$. If $|V_2| \leq \gamma(G) - 2$, then $V_2 \cup \{v\}$ is a dominating set of G with cardinality at most $\gamma(G) - 1$, a contradiction. Hence, $|V_2| = \gamma(G) - 1$ and $\gamma_R(G) = 2|V_2| + |V_1| = 2\gamma(G) - 1$, contrary to our assumption that G is not an almost Roman graph.

Thus, we may assume that $|V_1| \geq 2$. Since, V_1 is a 2-packing, there exists vertices u and v in V_1 such that $d(u, v) \geq 3$. Define the function f^* on $G\bar{G}$ as follows. If $x \in V \setminus \{u, v\}$, let $f^*(x) = f(x)$. Let $f^*(u) = f^*(v) = 0$ and $f^*(\overline{u}) = f^*(\overline{v}) = 2$. For all $\overline{x} \in \overline{V} \setminus \{\overline{u}, \overline{v}\}$, let $f^*(\overline{x}) = 0$. We note that $\{\overline{u}, \overline{v}\}$, dominates $\overline{V}$. Thus, f^* is an RDF of $G\overline{G}$, implying that $\gamma_R(G\overline{G}) \leq w(f) = \gamma_R(G) - 2 + 4 = \gamma_R(G) + 2$. Furthermore, since G is not Roman or almost Roman, $\gamma_R(G) \leq 2\gamma(G) - 2$ and the result follows. $\square$

Theorem 3.4 and Theorem 4.12 yield the following corollary.

Corollary 4.13 *Let G be a graph such that both G and $\overline{G}$ are isolate-free. If G is neither Roman nor almost Roman and $diam(G) \geq 3$, then $\gamma_R(G\overline{G}) = \gamma_R(G) + 2$.*

We need the following definition before proceeding. A set $S \subseteq V(G)$ is a *restrained dominating set* if S is a dominating set of G and every vertex $v \in V(G) \setminus S$ has a neighbor in $V(G) \setminus S$. The minimum cardinality of a restrained dominating set of G is called the *restrained domination number* of G and is denoted by $\gamma_r(G)$ (not to be confused with $\gamma_R(G)$).

Theorem 4.14 *If G is a Roman graph such that $\gamma_r(G) > \gamma(G)$ and $\overline{G}$ has no isolated vertices, then $\gamma_R(G\overline{G}) = \gamma_R(G) + 2$.*

Proof. Let S be a γ-set of G. Since $\gamma_r(G) \neq \gamma(G)$, it follows that there exists a vertex $v \in V \setminus S$ such that $N(v) \subseteq S$. Let f be a function $f : V(G\overline{G}) \mapsto \{0, 1, 2\}$ such that $f(u) = 2$ if $u \in S \cup \{\overline{v}\}$ and $f(u) = 0$ otherwise. The function f is an RDF on $G\overline{G}$ with weight $2|S| + 2 = 2\gamma(G) + 2 = \gamma_R(G) + 2$. Hence, $\gamma_R(G\overline{G}) \leq \gamma_R(G) + 2$. Note that an isolated vertex of G would be in V_1. Since G is a Roman graph, Proposition 4.11

implies that G has no isolated vertices. Further, since $\overline{G}$ has no isolated vertices, the result follows from Theorem 3.4. $\square$

5 SUMMARY

Recalling that if G is a graph with no isolates and $diam(G) \geq 3$, then $\gamma(\overline{G}) = 2$, we deduce the following result from the bounds in Sections 3 and 4.

Table 1: Roman Domination Numbers of Complementary Prisms

$diam(G)$	$diam(\overline{G})$	$\gamma_R(G\overline{G})$
2	2	$\{2\gamma(G\overline{G}), 2\gamma(G\overline{G}) - 1\}$
≥ 3	2	$\{\gamma_R(G) + 2, \gamma_R(G) + 3, \gamma_R(G) + 4\}$
≥ 3	≥ 3	≤ 8

Theorem 5.1 *Let G be a graph such that G and $\overline{G}$ are isolate-free graphs and $\gamma(G) \geq \gamma(\overline{G})$. Then Table 1 gives the values of $\gamma_R(G\overline{G})$.*

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