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Taking Notes: Generating Twelve-Tone Music with Mathematics

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Taking Notes: Generating Twelve-Tone Music with Mathematics

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by

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ABSTRACT

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There has often been a connection between music and mathematics. The world of musical composition is full of combinations of orderings of different musical notes, each of which has different sound quality, length, and emphasis. One of the more intricate composition styles is twelve-tone music, where twelve unique notes (up to octave isomorphism) must be used before they can be repeated. In this thesis, we aim to show multiple ways in which mathematics can be used directly to compose twelve-tone musical scores.

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1 INTRODUCTION

Mathematics and music have often lived together in a form of symbiosis. Reading and composing music can be broken down into mathematical arrangements. Composers must choose what tempo to keep, how long a note is sustained, and what should traditionally follow that note in sequence. Even Pythagoras was a known musician, and is said to have discovered the ratios between musical harmonies [8] such as the major third and fifth, which are how common chords are constructed. Notes themselves are broken down into weights of $\frac{1}{2^n}$, where a whole note is held for an entire measure (i.e., $\frac{1}{2^0}$). Compositions can be as simple as using the same three or four chords (arrangements of three or more notes played simultaneously) played in different or repeated progressions. There are, however, more non-traditional styles of composition that offer more of a challenge. One of these styles is called *twelve-tone music*.

Twelve-tone music structure was created by Austrian composer Arnold Schoenberg [12], and refers to the arrangement of twelve notes in an octave:

$$C, C\sharp/D\flat, D, D\sharp/E\flat, E, F, F\sharp/G\flat, G, G\sharp/A\flat, A, A\sharp/B\flat, B.$$

"The technique... ensure[s] that all 12 notes of the chromatic scale are sounded as often as one another in a piece of music while preventing the emphasis of any one note through the use of tone rows, orderings of the 12 pitch classes [15].” These notes differ in sound by what is referred to as a “semitone.” As seen above, a sharp note (denoted $\sharp$) has an aurally equivalent flat note (denoted $\flat$). However, in the case of E and F, and B and C, these notes already differ by a semitone. Without loss of generality, we will opt for the sharp notation. These notes must be arranged in some permutation before another set of twelve notes can be introduced, hence the moniker “twelve-tone music.”

There are a few important terms used throughout this paper that may not be commonly known to mathematicians. A note’s **weight** is the length of time over which a note is held. An **octave** is “the whole series of notes [or] tones [...] comprised within [a musical] interval and forming the unit of the modern scale [13].” A **semitone** is “a difference in sound that is equal to $\frac{1}{12}$ of an octave [14].” A **time signature** refers to the number of beats in a measure. For example, 3/4 time denotes three beats per measure, where each beat is worth one quarter-note. Similarly, 9/8 time denotes nine beats per measure, where each beat is worth one eighth-note.

In this thesis, each twelve-set will be assembled and referred to as a single vertex in the path graph that will represent our musical scores. Each of these $479,001,600$ vertices is unique. As this yields a large number of vertices, this thesis aims to mathematically generate twelve-tone musical scores using a series of restrictions such as:

- No six-note arrangement from an ordered 12-set will be repeated in a different 12-set.
- No symbols will be fixed.
- No single transposition will exist between any two 12-sets.
- The number of 12-sets will vary.
- Graphical representations of cycle types will be exemplified.

For the purposes of this thesis, choosing a time signature will not affect the ordering of the notes, as only the arrangement of notes is important to our goals. As such, this thesis will assume the time signatures will yield isomorphic musical scores. For example, arrange the notes in order of ascending semitones from C to B. If the score is composed in 3/4 time, the order will be the same as one composed in 4/4 time. However, at a certain point, time signatures must be considered when determining the number of 12-sets used in musical composition. Additionally, since this thesis only considers the ordering of the elements in each 12-set, this thesis assumes that each note gets the same weight. Also, no arrangements of elements will be removed for the sole purpose of retaining an aurally pleasing sound, as this is a purely theoretical approach, and pleasance is subjective.

2 THE FIRST RESTRICTION: NO SIX-NOTE ARRANGEMENT FROM A 12-SET WILL BE REPEATED IN A DIFFERENT 12-SET

It has been established that there are almost half a billion possible arrangements of twelve notes in a sequence. As such, it is clear that some restrictions must be put into place in an attempt to eliminate as many of these arrangements as possible. In this section, the Principle of Inclusion and Exclusion will be used to eliminate any arbitrary arrangements of six-element sequences that are used more than once across multiple 12-sets. For example:

$$\begin{pmatrix} C & C\sharp & D & D\sharp & E & F & F\sharp & G & G\sharp & A & A\sharp & B \\ B & G & C\sharp & D & D\sharp & E & F & F\sharp & A & A\sharp & C & G\sharp \end{pmatrix}$$

would be ineligible due to the repetition of the notes from $C\sharp$ to $F\sharp$.

The choice of identity vertex is arbitrary, and is isomorphic to any other vertex up to relabeling the elements within. Without loss of generality, this thesis will assume the starting vertex to be the identity vertex, where the elements are arranged in ascending semitones from C to B.

Let U be the set of all permutations on twelve elements. Observe that $|U| = 12!$. Let α_1 be the set of permutations that contains C through F in increasing semitonic order. These six elements are fixed, leaving the remaining six elements to be placed in any order. There are seven starting locations for the first fixed element in the set to appear. Thus, the cardinality of $\alpha_1 = 6! \cdot 7 = 7! = 5040$. Similarly, let α_2 be the set of permutations that contains $C\sharp$ through $F\sharp$ in increasing semitonic order. There are also 5040 ways to arrange these elements. Continuing in this fashion, there are seven total subsets of U of order 5040, call these $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7 \subset U$. The sum of the cardinalities of these sets is $7(7!) = 35,280$.

Now consider $\alpha_1 \cap \alpha_2$. This is the set containing both C to F and $C\sharp$ to $F\sharp$ in ascending semitonic order. The only way this is possible is if the permutation contains both α_1 and α_2. There are 5! ways to arrange the variable elements, and six starting locations for the first fixed element in the set to appear. Therefore $|\alpha_1 \cap \alpha_2| = 6! = 720$.

Similarly, $\alpha_1 \cap \alpha_3$, the set containing C to G, has cardinality $5! = 120$; $\alpha_1 \cap \alpha_4$, the set containing C to $G\sharp$, has cardinality $4! = 24$; $|\alpha_1 \cap \alpha_5| = 3! = 6$, $|\alpha_1 \cap \alpha_6| = 2$, and $|\alpha_1 \cap \alpha_7| = 1$.

Continuing this trend, the intersection of two sets is given in Table 1.

From this, it follows that the sum of the cardinalities of all sets containing two intersecting six-sets is $6(6!) + 5(5!) + 4(4!) + 3(3!) + 2(2) + 1 = 5039$.

It can be observed that for the intersection of three or more sets, the intersection of the left and rightmost sets considered in the intersection are

Table 1: Intersection of Two Sets

Intersection	# possible arrangements	# occurrences
$\lvert\alpha_i \cap \alpha_{i+1}\rvert, i \in [1,6]$	6!	6
$\lvert\alpha_i \cap \alpha_{i+2}\rvert, i \in [1,5]$	5!	5
$\lvert\alpha_i \cap \alpha_{i+3}\rvert, i \in [1,4]$	4!	4
$\lvert\alpha_i \cap \alpha_{i+4}\rvert, i \in [1,3]$	3!	3
$\lvert\alpha_i \cap \alpha_{i+5}\rvert, i \in \{1,2\}$	2	2
$\lvert\alpha_i \cap \alpha_{i+6}\rvert, i = 1$	1	1

the only contributing factors in determining the intersection of multiple sets, as the intermediate sets will already exist in the larger set. For example, $|\alpha_1 \cap \alpha_2 \cap \alpha_3 \cap \alpha_4| = |\alpha_1 \cap \alpha_2 \cap \alpha_4| = |\alpha_1 \cap \alpha_3 \cap \alpha_4| = |\alpha_1 \cap \alpha_4|$. As such, the intersection of three sets is given in Table 2.

Table 2: Intersection of Three Sets

Intersection	# possible arrangements	# occurrences	# intermediate sets
$\lvert\alpha_i \cap \alpha_{i+2}\rvert, i \in [1,5]$	5!	5	1
$\lvert\alpha_i \cap \alpha_{i+3}\rvert, i \in [1,4]$	4!	4	2
$\lvert\alpha_i \cap \alpha_{i+4}\rvert, i \in [1,3]$	3!	3	3
$\lvert\alpha_i \cap \alpha_{i+5}\rvert, i \in [1,2]$	2	2	4
$\lvert\alpha_i \cap \alpha_{i+6}\rvert, i = 1$	1	1	5

Therefore the sum of the cardinalities of all sets containing three intersecting six-sets is $5(5!) + 2(4)(4!) + 3(3)(3!) + 4(2)(2) + 5(1) = 867$.

The intersection of four sets is given in Table 3.

Table 3: Intersection of Four Sets

Intersection	# possible arrangements	# occurrences	# intermediate sets
$\lvert\alpha_i \cap \alpha_{i+3}\rvert, i \in [1,4]$	4!	4	1
$\lvert\alpha_i \cap \alpha_{i+4}\rvert, i \in [1,3]$	3!	3	3
$\lvert\alpha_i \cap \alpha_{i+5}\rvert, i \in [1,2]$	2	2	6
$\lvert\alpha_i \cap \alpha_{i+6}\rvert, i = 1$	1	1	10

The sum of the cardinalities of all sets containing four intersecting six-sets is $1(4)(4!) + 3(3)(3!) + 6(2)(2) + 10(1)(1) = 136$.

The intersection of five sets is given in Table 4.

Table 4: Intersection of Five Sets

Intersection	# possible arrangements	# occurrences	# intermediate sets
$\lvert\alpha_i \cap \alpha_{i+4}\rvert, i \in [1,3]$	3!	3	1
$\lvert\alpha_i \cap \alpha_{i+5}\rvert, i \in [1,2]$	2	2	4
$\lvert\alpha_i \cap \alpha_{i+6}\rvert, i = 1$	1	1	10

The sum of the cardinalities of all sets containing five elements is $1(3)(3!) + 4(2)(2) + 10(1)(1) = 44$.

The intersection of six sets is given in Table 5.

Thus the sum of the cardinalities of all sets containing six intersecting six-sets is $1(2)(2) + 5(1)(1) = 9$.

Table 5: Intersection of Six Sets

Intersection	# possible arrangements	# occurrences	# intermediate sets
$\lvert\alpha_i \cap \alpha_{i+5}\rvert, i \in [1, 2]$	2	2	1
$\lvert\alpha_i \cap \alpha_{i+6}\rvert, i = 1$	1	1	5

For the intersection of seven sets, there is $|\alpha_1 \cap \alpha_7| = 1$ possible way to arrange these elements, which makes sense due to this being the identity.

Recall the goal of this section was to find the number of twelve-sets that do not contain a repetition of a six-note subset of the chosen identity set. By the Principle of Inclusion and Exclusion, the number of possible twelve-sets satisfying the condition is

$$|U| - \sum |\alpha_i| + \sum |\alpha_i \cap \alpha_j| - \sum |\alpha_i \cap \alpha_j \cap \alpha_k| + \sum |\alpha_i \cap \alpha_j \cap \alpha_k \cap \alpha_l|$$

$$- \sum |\alpha_i \cap \alpha_j \cap \alpha_k \cap \alpha_l \cap \alpha_m| + \sum |\alpha_i \cap \alpha_j \cap \alpha_k \cap \alpha_l \cap \alpha_m \cap \alpha_n|$$

$$- \sum |\alpha_i \cap \alpha_j \cap \alpha_k \cap \alpha_l \cap \alpha_m \cap \alpha_n \cap \alpha_p|$$

$$= 479,001,600 - 35,280 + 5039 - 867 + 136 - 44 + 9 - 1$$

$$= 478,970,592.$$

As such, using this method has only eliminated $31,008$ possible arrangements. To continue this iteratively, the choice of the next element would *not* be arbitrary. The idea, however, is to choose a second 12-set from the remaining $478,970,592$, then determine how many remaining 12-sets do not have any six-note repetitions from either of the two that have already been selected. Our desired restriction is that each 12-set is considered separately; that is, repetitions of six notes will only be considered for each 12-set disjointly (i.e., the last three elements from the first 12-set and the first three elements from the second 12-set can be repeated, provided this string is not also contained in a single 12-set).

With that stated, due to the relatively small amount of 12-sets removed from the original half-billion, it may be more lucrative to build up using different parameters instead of tearing away at the universal set.

3 THE SECOND RESTRICTION: NO 12-SET WILL LEAVE ANY SYMBOLS FIXED

This section will eliminate any use of fixed points in any string. That is, when compared to the identity permutation, no symbol can be mapped to itself when crafting the next string.

As this is a very exhaustive process, it will prove beneficial to examine the conjugacy classes of the group S_{12}, as shown in Figure 1.

The *conjugacy class* of a group is a representation of an equivalence relation acting on equivalence classes, called orbits, in such a way that the group in question is acting on itself. Element a is said to be conjugate to b if, for some $g \in G$, we have $a = g^{-1}bg$. In this case, $G = S_{12}$, and $a, b \in G$. Further, a *cycle type* is a sequence of cycle lengths k_i such that $k_i \geq k_{i+1}$, where the cycle type of α is denoted $(k_1, k_2, ..., k_i, ..., k_n)$ [16]. Theorems 3.1 and 3.2 are given for completeness.

Theorem 3.1. *[9] Let S_n be the symmetric group on n symbols ($n \geq 1$), and let ρ and τ be permutations on S_n. Then ρ and τ are conjugate if and only if they are of the same cycle type.*

Theorem 3.2. *[6] Every nonidentity permutation in S_n is uniquely (up to the order of the factors) a product of disjoint cycles, each of which has length at least 2.*

```
G = SymmetricGroup(12)
```

```
G1 = G.conjugacy_classes_representatives(); G1
```

```
[(), (1,2), (1,2)(3,4), (1,2)(3,4)(5,6), (1,2)(3,4)(5,6)(7,8), (1,2)(3,4)(5,6)
(7,8)(9,10), (1,2)(3,4)(5,6)(7,8)(9,10)(11,12), (1,2,3), (1,2,3)(4,5), (1,2,3)
(4,5)(6,7), (1,2,3)(4,5)(6,7)(8,9), (1,2,3)(4,5)(6,7)(8,9)(10,11), (1,2,3)
(4,5,6), (1,2,3)(4,5,6)(7,8), (1,2,3)(4,5,6)(7,8)(9,10), (1,2,3)(4,5,6)(7,8)
(9,10)(11,12), (1,2,3)(4,5,6)(7,8,9), (1,2,3)(4,5,6)(7,8,9)(10,11), (1,2,3)
(4,5,6)(7,8,9)(10,11,12), (1,2,3,4), (1,2,3,4)(5,6), (1,2,3,4)(5,6)(7,8),
(1,2,3,4)(5,6)(7,8)(9,10), (1,2,3,4)(5,6)(7,8)(9,10)(11,12), (1,2,3,4)(5,6,7),
(1,2,3,4)(5,6,7)(8,9), (1,2,3,4)(5,6,7)(8,9)(10,11), (1,2,3,4)(5,6,7)(8,9,10),
(1,2,3,4)(5,6,7)(8,9,10)(11,12), (1,2,3,4)(5,6,7,8), (1,2,3,4)(5,6,7,8)(9,10),
(1,2,3,4)(5,6,7,8)(9,10)(11,12), (1,2,3,4)(5,6,7,8)(9,10,11), (1,2,3,4)(5,6,7,8)
(9,10,11,12), (1,2,3,4,5), (1,2,3,4,5)(6,7), (1,2,3,4,5)(6,7)(8,9), (1,2,3,4,5)
(6,7)(8,9)(10,11), (1,2,3,4,5)(6,7,8), (1,2,3,4,5)(6,7,8)(9,10), (1,2,3,4,5)
(6,7,8)(9,10)(11,12), (1,2,3,4,5)(6,7,8)(9,10,11), (1,2,3,4,5)(6,7,8,9),
(1,2,3,4,5)(6,7,8,9)(10,11), (1,2,3,4,5)(6,7,8,9)(10,11,12), (1,2,3,4,5)
(6,7,8,9,10), (1,2,3,4,5)(6,7,8,9,10)(11,12), (1,2,3,4,5,6), (1,2,3,4,5,6)(7,8),
(1,2,3,4,5,6)(7,8)(9,10), (1,2,3,4,5,6)(7,8)(9,10)(11,12), (1,2,3,4,5,6)(7,8,9),
(1,2,3,4,5,6)(7,8,9)(10,11), (1,2,3,4,5,6)(7,8,9)(10,11,12), (1,2,3,4,5,6)
(7,8,9,10), (1,2,3,4,5,6)(7,8,9,10)(11,12), (1,2,3,4,5,6)(7,8,9,10,11),
(1,2,3,4,5,6)(7,8,9,10,11,12), (1,2,3,4,5,6,7), (1,2,3,4,5,6,7)(8,9),
(1,2,3,4,5,6,7)(8,9)(10,11), (1,2,3,4,5,6,7)(8,9,10), (1,2,3,4,5,6,7)(8,9,10)
(11,12), (1,2,3,4,5,6,7)(8,9,10,11), (1,2,3,4,5,6,7)(8,9,10,11,12),
(1,2,3,4,5,6,7,8), (1,2,3,4,5,6,7,8)(9,10), (1,2,3,4,5,6,7,8)(9,10)(11,12),
(1,2,3,4,5,6,7,8)(9,10,11), (1,2,3,4,5,6,7,8)(9,10,11,12), (1,2,3,4,5,6,7,8,9),
(1,2,3,4,5,6,7,8,9)(10,11), (1,2,3,4,5,6,7,8,9)(10,11,12),
(1,2,3,4,5,6,7,8,9,10), (1,2,3,4,5,6,7,8,9,10)(11,12), (1,2,3,4,5,6,7,8,9,10,11),
(1,2,3,4,5,6,7,8,9,10,11,12)]
```

Figure 1: A list of conjugacy classes for S_{12} generated in Sage.

So in simpler terms, conjugation is just a switching of symbols in permutations. In other words, think rotations, cycles of up to length k, "flips," and other group actions. In addition, the cycle type is how each conjugacy class is broken down, such that one element in the cycle is mapped to another (one-symbol cycles are fixed points, two-symbol cycles are transpositions).

Now all that must be done is to eliminate any conjugacy class that does not contain all elements of the 12-set. It can be determined that there are twenty-one total conjugacy classes that permute all twelve symbols. Further, these conjugacy classes can be represented by direct products of cyclic groups. Now all that must be done is to start with the selected identity element and permute each item with respect to the operations defined in the conjugacy classes. Tables 6 and 7 shows one possible result.

Table 6: Conjugacy classes applied to the identity permutation

Conjugacy Class	Result (Applied to Identity) (Direct Product Representation)
$(1,2)(3,4)(5,6)(7,8)(9,10)(11,12)$	$(C\sharp, C)(D\sharp, D)(F, E)(G, F\sharp)(A, G\sharp)(B, A\sharp)$ $(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$
$(1,2,3)(4,5,6)(7,8)(9,10)(11,12)$	$(D, C, C\sharp)(F, D\sharp, E)(G, F\sharp)(A, G\sharp)(B, A\sharp)$ $(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$
$(1,2,3)(4,5,6)(7,8,9)(10,11,12)$	$(D, C, C\sharp)(F, D\sharp, E)(G\sharp, F\sharp, G)(B, A, A\sharp)$ $(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3)$
$(1,2,3,4)(5,6)(7,8)(9,10)(11,12)$	$(D\sharp, C, C\sharp, D)(F, E)(G, F\sharp)(A, G\sharp)(B, A\sharp)$ $(\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$
$(1,2,3,4)(5,6,7)(8,9,10)(11,12)$	$(D\sharp, C, C\sharp, D)(F\sharp, E, F)(A, G, G\sharp)(B, A\sharp)$ $(\mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_2)$
$(1,2,3,4)(5,6,7,8)(9,10)(11,12)$	$(D\sharp, C, C\sharp, D)(G, E, F, F\sharp)(A, G\sharp)(B, A\sharp)$ $(\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$
$(1,2,3,4)(5,6,7,8)(9,10,11,12)$	$(D\sharp, C, C\sharp, D)(G, E, F, F\sharp)(B, G\sharp, A, A\sharp)$ $(\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4)$
$(1,2,3,4,5)(6,7,8)(9,10)(11,12)$	$(E, C, C\sharp, D, D\sharp)(G, F, F\sharp)(A, G\sharp)(B, A\sharp)$ $(\mathbb{Z}_5 \times \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$
$(1,2,3,4,5)(6,7,8,9)(10,11,12)$	$(E, C, C\sharp, D, D\sharp)(G\sharp, F, F\sharp, G)(B, A, A\sharp)$ $(\mathbb{Z}_5 \times \mathbb{Z}_4 \times \mathbb{Z}_3)$
$(1,2,3,4,5)(6,7,8,9,10)(11,12)$	$(E, C, C\sharp, D, D\sharp)(A, F, F\sharp, G, G\sharp)(B, A\sharp)$ $(\mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_2)$

Table 7: Conjugacy classes applied to the identity permutation (cont.)

$(1,2,3,4,5,6)(7,8)(9,10)(11,12)$	$(F,C,C\sharp,D,D\sharp,E)(G,F\sharp)(A,G\sharp)(B,A\sharp)$ $(\mathbb{Z}_6\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2)$
$(1,2,3,4,5,6)(7,8,9)(10,11,12)$	$(F,C,C\sharp,D,D\sharp,E)(G\sharp,F\sharp,G)(B,A,A\sharp)$ $(\mathbb{Z}_6\times\mathbb{Z}_3\times\mathbb{Z}_3)$
$(1,2,3,4,5,6)(7,8,9,10)(11,12)$	$(F,C,C\sharp,D,D\sharp,E)(A,F\sharp,G,G\sharp)(B,A\sharp)$ $(\mathbb{Z}_6\times\mathbb{Z}_4\times\mathbb{Z}_2)$
$(1,2,3,4,5,6)(7,8,9,10,11,12)$	$(F,C,C\sharp,D,D\sharp,E)(B,F\sharp,G,G\sharp,A,A\sharp)$ $(\mathbb{Z}_6\times\mathbb{Z}_6)$
$(1,2,3,4,5,6,7)(8,9,10)(11,12)$	$(F\sharp,C,C\sharp,D,D\sharp,E,F)(A,G,G\sharp)(B,A\sharp)$ $(\mathbb{Z}_7\times\mathbb{Z}_3\times\mathbb{Z}_2)$
$(1,2,3,4,5,6,7)(8,9,10,11,12)$	$(F\sharp,C,C\sharp,D,D\sharp,E,F)(B,G,G\sharp,A,A\sharp)$ $(\mathbb{Z}_7\times\mathbb{Z}_5)$
$(1,2,3,4,5,6,7,8)(9,10)(11,12)$	$(G,C,C\sharp,D,D\sharp,E,F,F\sharp)(A,G\sharp)(B,A\sharp)$ $(\mathbb{Z}_8\times\mathbb{Z}_2\times\mathbb{Z}_2)$
$(1,2,3,4,5,6,7,8)(9,10,11,12)$	$(G,C,C\sharp,D,D\sharp,E,F,F\sharp)(B,G\sharp,A,A\sharp)$ $(\mathbb{Z}_8\times\mathbb{Z}_4)$
$(1,2,3,4,5,6,7,8,9)(10,11,12)$	$(G\sharp,C,C\sharp,D,D\sharp,E,F,F\sharp,G)(B,A,A\sharp)$ $(\mathbb{Z}_9\times\mathbb{Z}_3)$
$(1,2,3,4,5,6,7,8,9,10)(11,12)$	$(A,C,C\sharp,D,D\sharp,E,F,F\sharp,G,G\sharp)(B,A\sharp)$ $(\mathbb{Z}_{10}\times\mathbb{Z}_2)$
$(1,2,3,4,5,6,7,8,9,10,11,12)$	$(B,C,C\sharp,D,D\sharp,E,F,F\sharp,G,G\sharp,A,A\sharp)$ $(\mathbb{Z}_{12})$

Notice that this is arguably the simplest way to apply the operations induced by the conjugacy classes. Further, for any cycle, any notes can be chosen (order matters). For example, in the first conjugacy class from the table above, any two notes can be permuted, not just if they are sequentially adjacent. For $(1, 2)$, the two notes could have been chosen to be F and $C\sharp$. The only important factor is that if two notes have been switched (or rotated, in the case of larger subclasses), they cannot be moved by another subclass of the same conjugacy class. In general, each conjugacy class essentially states that for each cycle of length k of the conjugacy class, $k \leq 12$, choose k notes from C to B to be elements of the subclass. Then apply a permutation on those elements, or a transposition if only two elements.

4 THE THIRD RESTRICTION: NO SINGLE TRANSPOSITIONS BETWEEN 12-SETS

This section will eliminate any instances of a single transposition between two 12-sets. For example,

$$\begin{pmatrix} C & C\sharp & D & D\sharp & E & F & F\sharp & G & G\sharp & A & A\sharp & B \\ C & C\sharp & D & D\sharp & E & F & F\sharp & G & G\sharp & A & B & A\sharp \end{pmatrix}$$

would not be an eligible 12-set, since it only differs by a single transposition.

Similarly to Section 3, we consider the conjugacy classes seen in Figure 1. However, this time, fixed points are included, but any conjugacy class containing transpositions are eliminated. From Tables 8 and 9, we see that there are thirty-four conjugacy classes that do not contain single transpositions.

Table 8: Conjugacy Classes - No Transpositions.

$(1,2,3)$
$(1,2,3)(4,5,6)$
$(1,2,3)(4,5,6)(7,8,9)$
$(1,2,3)(4,5,6)(7,8,9)(10,11,12)$
$(1,2,3,4)$
$(1,2,3,4)(5,6,7)$
$(1,2,3,4)(5,6,7)(8,9,10)$
$(1,2,3,4)(5,6,7,8)$
$(1,2,3,4)(5,6,7,8)(9,10,11)$
$(1,2,3,4)(5,6,7,8)(9,10,11,12)$
$(1,2,3,4,5)$
$(1,2,3,4,5)(6,7,8)$

Table 9: Conjugacy Classes - No Transpositions (cont.).

$(1,2,3,4,5)(6,7,8)(9,10,11)$
$(1,2,3,4,5)(6,7,8,9)$
$(1,2,3,4,5)(6,7,8,9)(10,11,12)$
$(1,2,3,4,5)(6,7,8,9,10)$
$(1,2,3,4,5,6)$
$(1,2,3,4,5,6)(7,8,9)$
$(1,2,3,4,5,6)(7,8,9)(10,11,12)$
$(1,2,3,4,5,6)(7,8,9,10)$
$(1,2,3,4,5,6)(7,8,9,10,11)$
$(1,2,3,4,5,6)(7,8,9,10,11,12)$
$(1,2,3,4,5,6,7)$
$(1,2,3,4,5,6,7)(8,9,10)$
$(1,2,3,4,5,6,7)(8,9,10,11)$
$(1,2,3,4,5,6,7)(8,9,10,11,12)$
$(1,2,3,4,5,6,7,8)$
$(1,2,3,4,5,6,7,8)(9,10,11)$
$(1,2,3,4,5,6,7,8)(9,10,11,12)$
$(1,2,3,4,5,6,7,8,9)$
$(1,2,3,4,5,6,7,8,9)(10,11,12)$
$(1,2,3,4,5,6,7,8,9,10)$
$(1,2,3,4,5,6,7,8,9,10,11)$
$(1,2,3,4,5,6,7,8,9,10,11,12)$

Proceeding with the same mindset in Sections 3 and 4, consider the conjugacy subclasses seen in the above sections. The only thing that must be done to find subgroups that do not contain any fixed points *or* transpositions, is to check where there are overlapping conjugacy classes between both lists. Fortunately, Table 10 gives that there are only eight total conjugacy classes:

Table 10: Combining the Second and Third Restrictions

Conjugacy Class	Result (Applied to Identity) (Direct Product Representation)
$(1,2,3)(4,5,6)(7,8,9)(10,11,12)$	$(D, C, C\sharp, F, D\sharp, E, G\sharp, F\sharp, G, B, A, A\sharp)$ $(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3)$
$(1,2,3,4)(5,6,7,8)(9,10,11,12)$	$(D\sharp, C, C\sharp, D, G, E, F, F\sharp, B, G\sharp, A, A\sharp)$ $(\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4)$
$(1,2,3,4,5)(6,7,8,9)(10,11,12)$	$(E, C, C\sharp, D, D\sharp, G\sharp, F, F\sharp, G, B, A, A\sharp)$ $(\mathbb{Z}_5 \times \mathbb{Z}_4 \times \mathbb{Z}_3)$
$(1,2,3,4,5,6)(7,8,9)(10,11,12)$	$(F, C, C\sharp, D, D\sharp, E, G\sharp, F\sharp, G, B, A, A\sharp)$ $(\mathbb{Z}_6 \times \mathbb{Z}_3 \times \mathbb{Z}_3)$
$(1,2,3,4,5,6,7)(8,9,10,11,12)$	$(F\sharp, C, C\sharp, D, D\sharp, E, F, B, G, G\sharp, A, A\sharp)$ $(\mathbb{Z}_7 \times \mathbb{Z}_5)$
$(1,2,3,4,5,6,7,8)(9,10,11,12)$	$(G, C, C\sharp, D, D\sharp, E, F, F\sharp, B, G\sharp, A, A\sharp)$ $(\mathbb{Z}_8 \times \mathbb{Z}_4)$
$(1,2,3,4,5,6,7,8,9)(10,11,12)$	$(G\sharp, C, C\sharp, D, D\sharp, E, F, F\sharp, G, B, A, A\sharp)$ $(\mathbb{Z}_9 \times \mathbb{Z}_3)$
$(1,2,3,4,5,6,7,8,9,10,11,12)$	$(B, C, C\sharp, D, D\sharp, E, F, F\sharp, G, G\sharp, A, A\sharp)$ $(\mathbb{Z}_{12})$

This table gives the list of conjugacy classes that contain neither transpositions nor fixed points.

Now that we have reduced the number of conjugacy classes, we can start setting new restrictions. Some ideas worth exploring include:

- Restricting the ordering of sequential 12-sets, so a certain number of unique cycle types must be used before repeating one, for all cycle types used.
- Selecting a cycle type and using a graph representation of the distinct cycles (see Section 7).

6 THE FIFTH RESTRICTION: VARYING THE NUMBER OF 12-SETS

For this section, time signatures will play a significant part in choosing the number of 12-sets used in composing any musical scores. We will be composing songs in either 3/4 or 4/4 time. As such, consider a three-minute song, composed in 4/4 time. Assume the tempo to be 120 beats per minute. This yields a total of 360 beats. Assuming each note gets exactly one beat, this gives 360 notes. Dividing this by twelve, which comes from the order of the original set, gives a required thirty total 12-sets. Similarly, a four-minute song will require forty total 12-sets.

It is useful to observe that the order of the symmetric group S_{12} is 12!, so the set of all permutations on any 12-set is isomorphic to S_{12}. As such, instead of listing all possible combinations of 12-sets and choosing thirty to forty of them, it may be more beneficial to determine how many subgroups of the symmetric group S_{12} have order between thirty and forty.

Another useful idea is to choose a cycle type from Figure 1, then choose an element π from this cycle type. Find an element of some subgroup of S_{12} that has order between thirty and forty (as opposed to searching for all of them), call this element ρ. As a rule of generation, let $\pi = \pi_0$ be our first 12-string. Then, take $\rho\pi_0\rho^{-1} = \pi_1$ to be our next 12-string. Continuing this process, wherein $\pi_n = \rho\pi_{n-1}\rho^{-1}$, given $|\rho| = 35$, this will generate thirty-five unique 12-strings, which can then form a three minute, thirty second song, assuming 120 beats per minute. Similarly, if we choose our element ρ such that $|\rho| = 40$, then it will generate forty unique 12-strings, giving us a four minute song (at 120 beats per minute).

We will let $\pi_0 = (1, 2, 3, 4)(5, 6, 7, 8)(9, 10, 11, 12)$. We chose an element of the cycle type $[4, 4, 4]$ to make it easier to see how each π_i is placed into the musical score. Let $\rho = (4, 6, 5, 9, 12, 7, 1)(11, 2, 10, 8, 3)$. This implies $\rho^{-1} = (3, 8, 10, 2, 11)(1, 7, 12, 9, 5, 6, 4)$, and $|\rho| = 35$ for this example. Table 11 gives each iteration given by the equation $\pi_i = \rho\pi_{i-1}\rho^{-1}$, starting at π_0 :

Table 11: Musical Score Iterations

π_i	$\rho\pi_{i-1}\rho^{-1}$	π_{17}	$(1, 3, 10, 4)(2, 9, 5, 8)(6, 11, 7, 12)$
π_0	$(1, 2, 3, 4)(5, 6, 7, 8)(9, 10, 11, 12)$	π_{18}	$(1, 7, 5, 2)(3, 10, 12, 9)(4, 11, 8, 6)$
π_1	$(1, 3, 9, 5)(2, 7, 12, 8)(4, 10, 11, 6)$	π_{19}	$(1, 9, 10, 4)(2, 3, 5, 6)(7, 12, 11, 8)$
π_2	$(1, 7, 3, 10)(2, 5, 6, 8)(4, 11, 12, 9)$	π_{20}	$(1, 7, 2, 3)(4, 12, 8, 6)(5, 10, 11, 9)$
π_3	$(1, 11, 8, 4)(2, 7, 12, 6)(3, 10, 9, 5)$	π_{21}	$(1, 10, 11, 4)(2, 12, 9, 8)(3, 5, 6, 7)$
π_4	$(1, 7, 5, 10)(2, 3, 6, 4)(8, 12, 9, 11)$	π_{22}	$(1, 11, 9, 5)(2, 6, 4, 8)(3, 10, 7, 12)$
π_5	$(1, 9, 8, 4)(2, 3, 7, 12)(5, 6, 10, 11)$	π_{23}	$(1, 7, 11, 8)(2, 12, 9, 4)(3, 10, 5, 6)$
π_6	$(1, 7, 10, 11)(2, 9, 5, 8)(3, 6, 4, 12)$	π_{24}	$(1, 2, 3, 4)(5, 11, 8, 9)(6, 10, 7, 12)$
π_7	$(1, 8, 2, 4)(3, 10, 12, 9)(5, 6, 7, 11)$	π_{25}	$(1, 7, 5, 8)(2, 3, 12, 9)(4, 10, 11, 6)$
π_8	$(1, 2, 9, 5)(3, 10, 6, 4)(7, 12, 11, 8)$	π_{26}	$(1, 9, 3, 4)(2, 5, 6, 8)(7, 12, 10, 11)$
π_9	$(1, 7, 2, 3)(4, 10, 12, 9)(5, 6, 11, 8)$	π_{27}	$(1, 7, 8, 2)(3, 10, 9, 5)(4, 12, 11, 6)$
π_{10}	$(1, 10, 11, 4)(2, 3, 9, 5)(6, 8, 7, 12)$	π_{28}	$(1, 3, 10, 4)(2, 5, 6, 7)(8, 12, 9, 11)$
π_{11}	$(1, 7, 5, 3)(2, 6, 4, 8)(9, 10, 11, 12)$	π_{29}	$(1, 10, 9, 5)(2, 3, 7, 12)(4, 11, 8, 6)$
π_{12}	$(1, 9, 11, 4)(2, 7, 12, 8)(3, 10, 5, 6)$	π_{30}	$(1, 7, 10, 11)(2, 3, 5, 6)(4, 8, 12, 9)$
π_{13}	$(1, 7, 3, 10)(2, 6, 4, 12)(5, 11, 8, 9)$	π_{31}	$(1, 8, 2, 4)(3, 7, 12, 6)(5, 10, 11, 9)$
π_{14}	$(1, 11, 8, 4)(2, 3, 12, 9)(5, 6, 7, 10)$	π_{32}	$(1, 7, 5, 11)(2, 12, 9, 8)(3, 10, 6, 4)$
π_{15}	$(1, 8, 9, 5)(2, 3, 6, 4)(7, 12, 10, 11)$	π_{33}	$(1, 9, 2, 4)(3, 10, 7, 12)(5, 6, 11, 8)$
π_{16}	$(1, 7, 8, 2)(3, 12, 9, 4)(5, 6, 10, 11)$	π_{34}	$(1, 7, 11, 8)(2, 3, 9, 5)(4, 12, 10, 6)$

Once each symbol is mapped to a musical note, transposing these symbols in this order will generate a song three and a half minutes long, assuming 120 BPM and each note having a weight of one quarter-note. Also, notice that this process is modular, that is, $\pi_{|\rho|} = \pi_0$. This means we can change the length of the song simply by choosing a ρ of different order.

This process can also be generated by a series of Sage commands. Figure 2 shows how Table 11 can be generated exactly as if done by hand. This process can help to easily change starting values for ρ and π in order to create new compositions. One thing to notice, however, is that when done by hand, we multiplied from right to left, whereas Sage multiplies from left to right. The resulting set is equivalent, but the order will differ when multiplying from opposite directions. As such, the Sage code's multiplying function reflects $\rho^{-1}\pi\rho$ instead of our choice of $\rho\pi\rho^{-1}$.

```
S12=SymmetricGroup(12)
p=S12("(4,6,5,9,12,7,1)(11,2,10,8,3)")
s=S12("(1,2,3,4)(5,6,7,8)(9,10,11,12)")
for i in range(35):
  s=p^(-1)*s*p;
  s
```

```
(1,3,9,5)(2,7,12,8)(4,10,11,6)
(1,7,3,10)(2,5,6,8)(4,11,12,9)
(1,11,8,4)(2,7,12,6)(3,10,9,5)
(1,7,5,10)(2,3,6,4)(8,12,9,11)
(1,9,8,4)(2,3,7,12)(5,6,10,11)
(1,7,10,11)(2,9,5,8)(3,6,4,12)
(1,8,2,4)(3,10,12,9)(5,6,7,11)
(1,2,9,5)(3,10,6,4)(7,12,11,8)
(1,7,2,3)(4,10,12,9)(5,6,11,8)
(1,10,11,4)(2,3,9,5)(6,8,7,12)
(1,7,5,3)(2,6,4,8)(9,10,11,12)
(1,9,11,4)(2,7,12,8)(3,10,5,6)
(1,7,3,10)(2,6,4,12)(5,11,8,9)
(1,11,8,4)(2,3,12,9)(5,6,7,10)
(1,8,9,5)(2,3,6,4)(7,12,10,11)
(1,7,8,2)(3,12,9,4)(5,6,10,11)
(1,3,10,4)(2,9,5,8)(6,11,7,12)
(1,7,5,2)(3,10,12,9)(4,11,8,6)
(1,9,10,4)(2,3,5,6)(7,12,11,8)
(1,7,2,3)(4,12,8,6)(5,10,11,9)
(1,10,11,4)(2,12,9,8)(3,5,6,7)
(1,11,9,5)(2,6,4,8)(3,10,7,12)
(1,7,11,8)(2,12,9,4)(3,10,5,6)
(1,2,3,4)(5,11,8,9)(6,10,7,12)
(1,7,5,8)(2,3,12,9)(4,10,11,6)
(1,9,3,4)(2,5,6,8)(7,12,10,11)
(1,7,8,2)(3,10,9,5)(4,12,11,6)
(1,3,10,4)(2,5,6,7)(8,12,9,11)
(1,10,9,5)(2,3,7,12)(4,11,8,6)
(1,7,10,11)(2,3,5,6)(4,8,12,9)
(1,8,2,4)(3,7,12,6)(5,10,11,9)
(1,7,5,11)(2,12,9,8)(3,10,6,4)
(1,9,2,4)(3,10,7,12)(5,6,11,8)
(1,7,11,8)(2,3,9,5)(4,12,10,6)
(1,2,3,4)(5,6,7,8)(9,10,11,12)
```

Figure 2: Generation of Sets using $\rho\pi\rho^{-1}$ Function in Sage

7 BUILDING A SONG GRAPHICALLY

After determining which and how many vertices we wish to use, with each vertex representing some permutation of S_{12}, we can craft our song by setting restrictions on the edge set of the graph induced by these vertices. For example, if repetition of vertices is allowed, we can set a restriction such as "no vertex can be revisited before $k \leq n-1$ vertices have been visited," where k is the number of intermediate vertices we must "visit" before returning to a vertex, and n is the number of vertices in the graph.

For this section, we will be considering the cycle type $[4, 4, 4]$ taken from Figure 1 and considering the Cayley graph (a graph with vertices representing a "flip," "rotation," or a combination of the two with respect to its generating element) representations of three groups of order four.

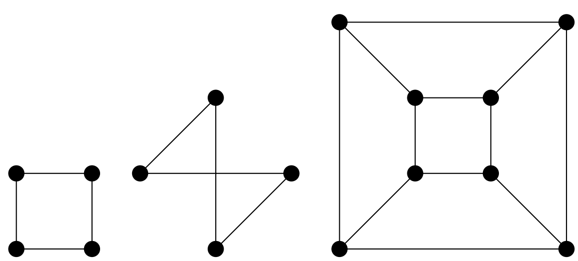
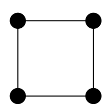
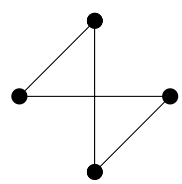
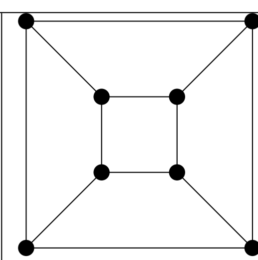


Figure 3: Cayley graphs of $\mathbb{Z}_4$, V_4, and D_4

We want to find transposition representation commonalities amongst all three groups. The idea is to find a way to assign each vertex a note, then allow movement specifically along the edges induced by the graph itself. That is, for the resulting graph, each note must be followed by an adjacent note, and the only adjacencies between subgraphs (which are isomorphic to $\mathbb{Z}_4, V_4$ and D_4) are those that share transposition structure.

Table 12: Cayley graphs of groups of order four, along with their cycle notation

 $\mathbb{Z}_4$	 V_4	 D_4
$v_1 = (1)(2)(3)(4) = e$	$v_5 = (1)(2)(3)(4)$	$v_9 = (1)(2)(3)(4)$
$v_2 = (1,2,3,4)$	$v_6 = (1,3)(2)(4)$	$v_{10} = (1,4)(2,3)$
$v_3 = (1,3)(2,4)$	$v_7 = (1)(2,4)(3)$	$v_{11} = (1,2,3,4)$
$v_4 = (1,4,3,2)$	$v_8 = (1,3)(2,4)$	$v_{12} = (1,3)(2,4)$
		$v_{13} = (1,3)(2)(4)$
		$v_{14} = (1)(2,4)(3)$
		$v_{15} = (1,4,3,2)$
		$v_{16} = (1,2)(3,4)$

We see that there are two induced C_3 graphs between the three subgraphs; these represent a shared representation of permutations. Also, no notes have been assigned at this point, so simply creating the graph will not be enough to compose the music. Before assigning a note to each vertex, we must address the issue that stems from the fact that the order of the vertex set $|V(G)| = 16$. The Pigeonhole Principle states that, given n labels and k vertices to assign them, if $k > n$, then at least one of the n labels must be used at least twice. This creates a problem with our structure, as we cannot construct our musical score unless twelve unique notes have been used. We must tread carefully in our selection; we must still adhere to the restriction of using all twelve notes before repeating any. We must find a way to avoid or otherwise solve the issue created here.

First, we must figure out where each edge connects between these vertices. Referring to Table 12, we can see where the C_3 graphs are created when adding edges between vertices of similar labeling. In order to avoid repeating certain edges from $\mathbb{Z}_4$ to any other graph can simply serve as a bridge to get to another graph. That is, think of specific edges from $\mathbb{Z}_4$ to V_4 or D_4 as lateral transitions instead of paths, such that when traveling on these edges, we select a different note unless the destination vertex completes a C_3 subgraph. Depending on how many times this happens, this could eliminate the issue of having too many vertices to label with a note.

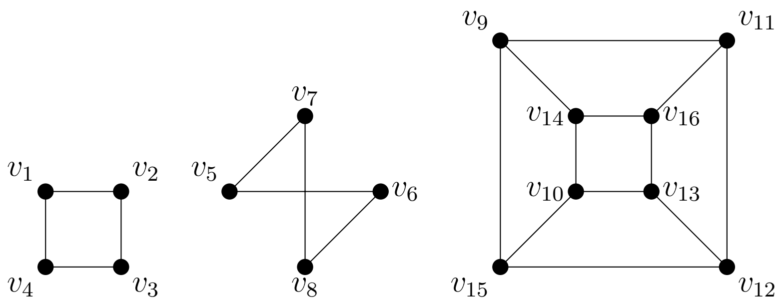


Figure 4: Cayley Graphs with Labeled Vertices

For the graph combining these three subgraphs, the edge set

$$E(G) = \{v_i v_j \in G \ : \ v_i = v_j, \ i, j \in [16], \ i \neq j\} \cup E(\mathbb{Z}_4) \cup E(V_4) \cup E(D_4).$$

While it is true that we used group theory to construct this graph, let us only follow the graph to compose our score, ignoring for the moment the permutations induced by the vertex labeling. Consider the problem given by the Pigeonhole Principle above. By construction of our edge set, notice that $v_1 = v_5 = v_9$, and $v_3 = v_8 = v_{12}$. Instead of assigning a unique note to each of these vertices, let the equivalence also carry over to the assignment of the note. This reduces our required number of labels to exactly twelve. We can now assign a note to each vertex. For example, let $v_1 = v_5 = v_9 = C, v_2 = C\sharp, v_3 = v_8 = v_{12} = D, v_4 = D\sharp, v_6 = E, v_7 = F, v_{10} = F\sharp, v_{11} = G, v_{13} = G\sharp, v_{14} = A, v_{15} = A\sharp, v_{16} = B$. Now let the score be constructed such that, given a vertex v_i, $v_j \in N(v_i)$. Further, we must carefully choose each vertex such that no v_i appears more than once in the same string of twelve symbols.

8 CONCLUSION

We have demonstrated three methods that can be used in order to compose musical scores using Schoenberg's twelve-tone technique. The most promising method was the variation of vertices, in which we used an element of specific order, multiplied on either side by either another element of S_{12} or that element's inverse. The maximum order of the chosen middle element is sixty (this is left to the reader to verify) by using $\mathbb{Z}_5 \times \mathbb{Z}_3 \times \mathbb{Z}_2$, which would yield a six-minute song if our parameters were otherwise unchanged.

There seems to be promising potential with using the principle of inclusion and exclusion as found in the first restriction; however, we must first determine what kind of effect our choice in the second 12-set has on the iterative method we chose.

Using the Cayley graphs for different cycle types shows promise, as we can use different algorithms to construct path subgraphs in order to construct different strings of twelve symbols. For example, by assigning weights to each edge, we can use Kruskal's algorithm to determine a minimum spanning tree to choose our first twelve unique symbols, then reorder the weights on each edge.

Some open problems include:

For the element of maximum order in S_{12}, use Cayley graphs to construct a minimum spanning tree to determine potential strings of twelve symbols.

Change the first restriction in this thesis to disallow *four* symbols to be repeated. Use Sage to choose a remaining element, then continue the process.

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APPENDIX

Composition for Thesis

Nathan Molder


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VITA

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