5-2018

A Study of Topological Invariants in the Braid Group B2

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A Study of Topological Invariants in the Braid Group B_2

A thesis

presented to

the faculty of the Department of Mathematics

East Tennessee State University

In partial fulfillment

of the requirements for the degree

Master of Science in Mathematical Sciences

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by

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May 2018

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Keywords: Jones polynomial, ambient isotopy, B_2, Temperley-Lieb algebra

ABSTRACT

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The Jones polynomial is a special topological invariant in the field of Knot Theory. Created by Vaughn Jones, in the year 1984, it is used to study when links in space are topologically different and when they are topologically equivalent. This thesis discusses the Jones polynomial in depth as well as determines a general form for the closure of any braid in the braid group B_2 where the closure is a knot. This derivation is facilitated by the help of the Temperley-Lieb algebra as well as with tools from the field of Abstract Algebra. In general, the Artin braid group B_n is the set of braids on n strands along with the binary operation of concatenation. This thesis also shows results of the relationship between the closure of a product of braids in B_2 and the connected sum of the closure of braids in B_2. Results on the topological invariant of tricolorability of closed braids in B_2 and (2,n) torus links along with their obverses are presented as well.

ACKNOWLEDGEMENTS

I would like to thank my family for their support throughout this entire process. I appreciate all of the times you were there for me. I would also like to thank all of my friends that I have met here at ETSU. You guys have been amazing and I am thankful for you all. I would like to thank the department for giving me all of the opportunities that I have been fortunate enough to receive while I have been here. I would like to thank Dr. Debra Knisley for teaching me how to use LaTex. I would like to thank Dr. Norwood for taking me on for this thesis as well as thank Dr. Keaton and Dr. Gardner for agreeing to be members in my committee. I would lastly like to thank Mrs. Apple. You're the reason why I majored in Math. Thank you. YNWA.

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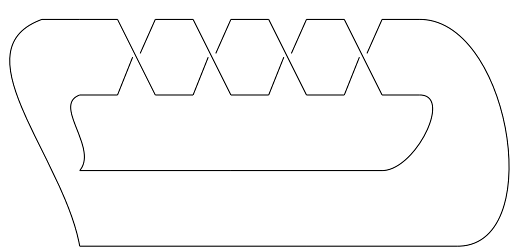
1 THE JONES POLYNOMIAL

The *Jones polynomial* is an example of what is called a *knot invariant*. A *knot invariant* is a function from the set of all knots to any other set such that the function does not change as the knot is changed up to isotopy [3]. Before we can talk about the *Jones polynomial* we first need a few definitions.

1.1 Introductory Definitions

A *knot* is defined as a closed, non-self intersecting curve that is embedded in three dimensions that cannot be untangled to produce a single loop [2]. The single loop, in the above definition, refers to a basic circle which is called the *unknot* [2]. Any knot that can be continuously deformed to the unknot is said to be *unknotted* [2]. Continuously deforming a knot means you are not allowed to tear, rip, or glue the knot back together at any point during the deformation [2]. A *link* is defined as a knotted collection of one or more closed strands [2]. A knot is said to be *chiral* if it is topologically distinct from its mirror image, and *achiral* if it can be deformed into its mirror image [2]. The smallest example of a nontrivial knot is the trefoil knot, and an example of a link is the (2,4) torus link shown below:

$(2, 4)$:



How exactly do you continuously deform a knot or a link? The answer is through a sequence of what are known as *Reidemeister moves* [2]. Reidemeister proved that every ambient isotopy can be carried out with the following moves in figure 1 [1]:

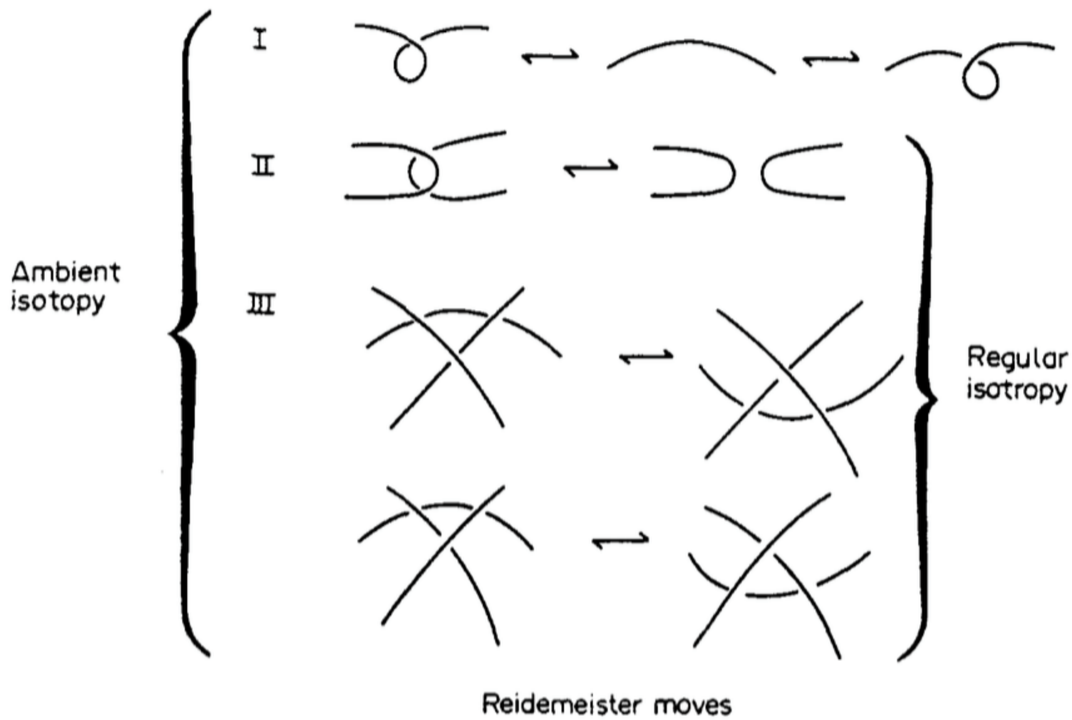


Figure 1: Ambient Isotopy vs. Regular Isotopy

The above moves have been drawn by Louis Kauffman [1]. They demonstrate when two links are *regular isotopic* and when they are *ambient isotopic*.

In order to study knots, we continuously deform any knot or link diagram until all crossings have been untangled or *spliced*. If the knot or link diagram has n crossings, where n $\in (\mathbb{N} \cup \{0\})$, then the original diagram will have a total of 2n states. A *state* is a diagram that can be used to reconstruct our original knot or link [2]. An example of a state diagram and its reconstruction, given by Kauffman, follows [2]:

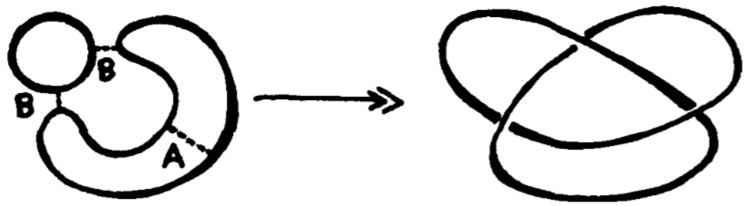
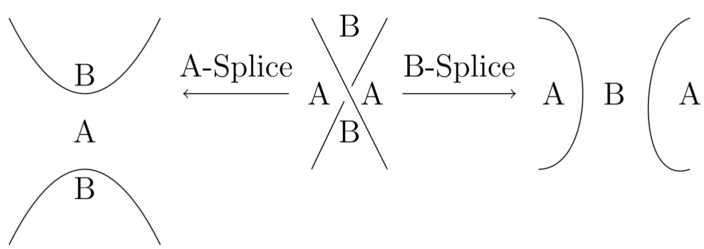


Figure 2: State Diagram

On the right we see the original diagram and on the left we see a particular state of the original diagram. In our state we see 2 Bs and 1 A. These letters refer to the kind of splices we have performed on the three crossings. The two types of splices at our disposal are *Type A* and *Type B* splices [2]. Kauffman explains, "The regions labelled A are those that appear on the left to an observer walking toward the crossing along the undercrossing segments. The B-regions appear on the right for this observer" [2]. Below is a graphical representation of the two splices:



We can think of the splices as a *Type A* splice is joining both areas labeled A together and a *Type B* splice is joining both areas labeled B together [2].

1.2 The Bracket Polynomial

Now we let $B = A^{-1}$ and $d = -A^2 - A^{-2}$. An important polynomial in the field of Knot Theory is called *The Bracket Polynomial.* This polynomial is defined by

$\langle K \rangle = \langle K \rangle(A) = \sum_\sigma \langle K|\sigma \rangle d^{\|\sigma\|}$, where $\|\sigma\|$ is the total number of loops minus one in the state σ, $\langle K|\sigma\rangle$ is the product of the labels attached to σ, and the sum is taken over all states of K [2]. The bracket polynomial follows the following three rules [3]:

1. $\langle \bigcirc \rangle = 1$
2. $\langle L \cup \bigcirc \rangle = (-A^2 - A^{-2})\langle L \rangle$
3. $\langle \times \rangle = A \langle \asymp \rangle + A^{-1} \langle)(\rangle$

- Rule 1 states that the Bracket Polynomial of the unknot is equal to 1.
- Rule 2 states that adding a disjoint circle to any link or knot diagram changes the bracket for that link or knot by a factor of $-A^2 - A^{-2}$.
- Rule 3 states that $B = A^{-1}$

Two examples of bracket polynomials are the brackets for the Hopf link and the trefoil. The Hopf link can be thought of as two circles that are linked together:

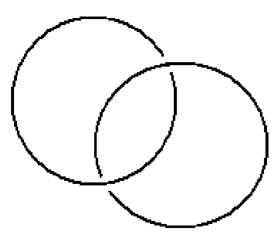


Figure 3: Hopf Link

Let H be the Hopf link and T be the trefoil. Their brackets are as follows:

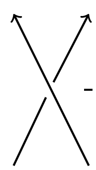
$$\langle H \rangle = -A^4 - A^{-4}$$

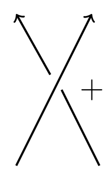
$$\langle T \rangle = -A^5 - A^{-3} + A^{-7}$$

However, the bracket polynomial is not a knot invariant under all three Reidemeister moves [2]. When $B = A^{-1}$ and $d = -A^2 - A^{-2}$ the bracket polynomial is an invariant under Reidemeister moves 2 and 3 [2]. In order to define a polynomial that is invariant under all three Reidemeister moves we need some definitions.

1.3 The Normalized Bracket

An oriented link or knot is simply a link or knot with an orientation [2]. We define the *sign* of a crossing as follows [2]:





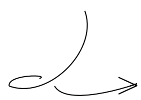
When the orientations both point either up or down then the diagram above on the left is a negative crossing and the diagram above on the right is a positive crossing [2]. Next, define the *writhe* of a link or knot to be the summation of all the signs of all of the crossings of the link or knot [3]. For example, the trefoil has writhe equal to 3 because all 3 of its crossings are positive. Let K be an oriented link diagram and define the *Normalized Bracket* of K, denoted as $\mathscr{L}_K$, as

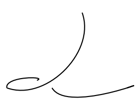
$$\mathscr{L}_K = (-A^3)^{w(K)} \langle K \rangle$$

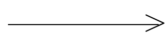
Where $w(K)$ denotes the writhe of K. The problem is now to show that this normalized bracket is an invariant under all three Reidemeister moves. Another way to say this is that the normalized bracket is an invariant of ambient isotopy.

Theorem 1.1. [2] *The normalized bracket polynomial $\mathscr{L}_K$ is an invariant of ambient isotopy.*

Proof. [2] Let K be any oriented link diagram. Because $w(K)$ and $\langle K \rangle$ are both invariants of regular isotopy, meaning they are invariants under Reidemeister moves of types two and three, then $\mathscr{L}_K$ is also an invariant of regular isotopy. In order to show this normalized bracket is an invariant of ambient isotopy we need to show that it is invariant under type one moves as well.

Let J = 

M = 

N = 

P =

We first note that $\langle J \rangle = \langle M \rangle$ and $\langle N \rangle = \langle P \rangle$ because the bracket does not depend on orientation.

$$\begin{aligned} \mathscr{L}_J &= (-A^3)^{-w(J)} \langle M \rangle \\ &= (-A^3)^{-[1+w(N)]} (-A^3) \langle P \rangle \\ &= (-A^3)^{-w(N)} \langle P \rangle \\ &= \mathscr{L}_N \end{aligned}$$

■

This proof demonstrates that the normalized bracket is an invariant under all three Reidemeister moves. Now that the normalized bracket polynomial has been introduced we are ready to discuss the *Jones polynomial*.

1.4 Formal Definition of Jones Polynomial

The 1-variable *Jones polynomial*, $V_K(t)$, is a Laurent polynomial in the variable $t^{1/2}$ assigned to an oriented link K [3]. A Laurent polynomial, in one variable over a field $\mathbb{F}$, is defined as a linear combination of both negative and positive powers of the variable with coefficients as elements of the field $\mathbb{F}$. The Jones polynomial satisfies the following three properties [3]:

- If K and K' are ambient isotopic then $V_K(t) = V_{K'}(t)$
- The Jones polynomial of an oriented unknot is equal to 1.
- $t^{-1}V_{\times} - tV_{\times} = (t^{1/2} - t^{-1/2})\, V_{\rightrightarrows}$

The next theorem will prove vital in the coming pages of results.

Theorem 1.2. [2] *Let* $\mathscr{L}_K(A) = (-A^3)^{-w(K)}\langle K\rangle$. *Then* $\mathscr{L}_K(t^{-1/4}) = V_K(t)$

Proof. [2] Let $B = A^{-1}$. We know the following formulas from the Bracket Polynomial:

So,

Normalization gives us

$A \left\langle \diagdown\!\!\!\!\diagup \right\rangle - A^{-1} \left\langle \diagup\!\!\!\!\diagdown \right\rangle = (A^2 - A^{-2}) \langle \rightrightarrows \rangle$

Set $c = w(\rightrightarrows)$ so we have $w(\diagdown\!\!\!\!\diagup) = c + 1$ and $w(\diagup\!\!\!\!\diagdown) = c - 1$. Setting $\eta = -A^3$ and multiplying on both sides of the above equation by η^{-c} we get

$A \left\langle \diagdown\!\!\!\!\diagup \right\rangle \eta^{-c} - A^{-1} \left\langle \diagup\!\!\!\!\diagdown \right\rangle \eta^{-c} = (A^2 - A^{-2}) \langle \rightrightarrows \rangle \eta^{-c}$

Factoring on the left side of the equation gives us,

$A\eta \left\langle \diagdown\!\!\!\!\diagup \right\rangle \eta^{-(c+1)} - A^{-1}\eta^{-1} \left\langle \diagup\!\!\!\!\diagdown \right\rangle \eta^{-(c-1)} = (A^2 - A^{-2}) \langle \rightrightarrows \rangle \eta^{-c}$

Now we can simplify this expression such that we get,

$A\eta \mathscr{L}_{\diagdown\!\!\!\!\diagup} - A^{-1}\eta^{-1} \mathscr{L}_{\diagup\!\!\!\!\diagdown} = (A^2 - A^{-2})\ \mathscr{L}_{\rightrightarrows}.$

Simplifying further we get,

$-A^4 \mathscr{L}_{\diagdown\!\!\!\!\diagup} + A^{-4} \mathscr{L}_{\diagup\!\!\!\!\diagdown} = (A^2 - A^{-2})\ \mathscr{L}_{\rightrightarrows}.$

Now if we allow $A = t^{-1/4}$ then by substitution we get the following:

$t^{-1} \mathscr{L}_{\diagdown\!\!\!\!\diagup} - t \mathscr{L}_{\diagup\!\!\!\!\diagdown} = (t^{1/2} - t^{-1/2})\ \mathscr{L}_{\rightrightarrows}$

This proves the third property of the three properties listed earlier. Property two is shown from the fact that the Jones Polynomial of any oriented unknot is equal to 1. So let U be any oriented unknot. We know from properties of the normalized bracket that $\mathscr{L}_U = 1 = V_U$. So, $\mathscr{L}_U(t^{-1/4}) = 1 = V_U(t)$. This proves property two. Property 1 is also shown through properties of the normalized bracket.

■

2 THE ARTIN BRAID GROUP B_n

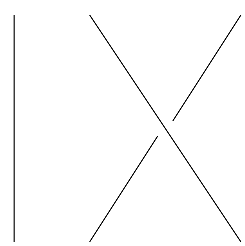
A *group* G is defined as a set S paired with an operation $*$ such that [4]

1. $*$ is associative

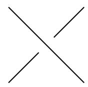
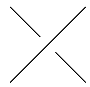
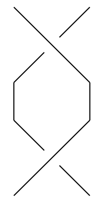
2. $\exists$ e $\in S$ such that $\forall$ s $\in S$ $s * e = s$ and $e * s = s$

3. $\exists$ $s^{-1} \in S$ for each $s \in S$ such that $s * s^{-1} = e$ and $s^{-1} * s = e$.

In this thesis we are concerned with groups where the elements of the groups are braids. A *braid* is formed by taking n points in a plane and attaching strands to these points so that parallel planes intersect the strands in n points [2]. A simple example of a braid is a braid on three strands with only one crossing.

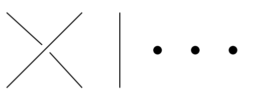


We can create a group where the elements of the group are braids and the operation $*$ is defined as concatenation. Let B_n be the set of all braids on n strands where $n \in \mathbb{N}$ [2]. Concatenation is defined as taking two braids in the group B_n, stacking one on top of another, and creating a new braid that is also an element of B_n [2]. Let b_1 and b_2 be elements of the braid group B_2. The following is an example of the operation of concatenation:

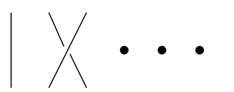
b_1 :  b_2 :  $b_1 * b_2$: 

The group $(B_n, *)$ is called *The Artin Braid Group* [2]. Every braid in B_n can be produced by the set of generators $\sigma_1, \sigma_2, \sigma_3, ..., \sigma_{n-1}$ where these generators are defined as follows [1]:

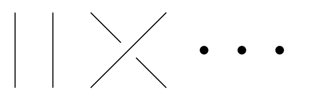
σ_1 :



σ_2 :

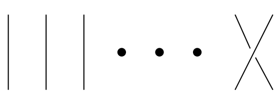


σ_3 :

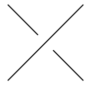


all the way until we reach the final generator

σ_{n-1} :



The first generator is a braid on n strands with one positive crossing where strands 1 and 2 have been crossed. The second generator is a braid on n strands with one positive crossing where strands 2 and 3 have been crossed. The number of generators a given braid group has depends on the particular braid group being studied. B_2, the group of all braids on two strands, has one generator. This generator is

σ_1 : 

The inverse of any one of these generators is simply the generator where instead of having one positive crossing there is one negative crossing, where the strand on the left crosses over the strand on the right, rather than under. Now, define a *braid word* to be a sequence of these generators and their inverses [2]. For example, $\sigma_1\sigma_1\sigma_1$ where $\sigma_1 \in B_2$ is a *braid word*. The Artin Braid Group can be expressed with all of these generators along with the relations below: [2]

1.) $\sigma_i\ \sigma_i^{-1} = 1$ for $i = 1, ..., n-1$

2.) $\sigma_i\ \sigma_{i+1}\ \sigma_i = \sigma_{i+1}\ \sigma_i\ \sigma_{i+1}$ for $i = 1, ...n-2$

3.) $\sigma_i\ \sigma_j = \sigma_j\ \sigma_i$ for $|i-j| > 1$.

The closure of any braid b is simply joining the strands together in a way such that none of these connections intersect each other. Kauffman gives an example, [2]

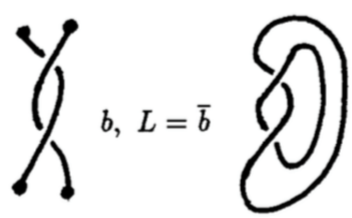


Figure 4: Closure

Another example would be if we were to take the *braid word* $\sigma_1\ \sigma_1\ \sigma_1$ and take its *closure* we obtain the Trefoil Knot. One of the results of this thesis will be a general form of the *Jones Polynomial* of the *closure* of any braid in B_2. An important theorem that relates braids to knots and links is *Alexander's Theorem*.

Alexander's Theorem [5] *Each link in three-dimensional space is ambient isotopic to a link in the form of a closed braid.*

Proof. Alexander proved this theorem by the following: [5] "Suppose we have a link L and its projection. We orient the components of L. We then choose a point P on the projection such that P does not intersect the knot. Though P is a point in the projection, it is helpful to think of P as an axis extending through the projection plane. The goal will be to manipulate L so that every component is oriented in a particular direction around this axis. Fix an orientation about P. We consider a finite number of subarcs of L such that each subarc is either oriented in our orientation or in the reverse orientation, not both. If there are no subarcs oriented in the reverse direction, then were done. Otherwise, choose some subarc S which is oriented opposite our chosen orientation. We divide up S further into a finite number of subarcs S_i, such that each subarc contains at most one crossing. Now consider an arbitrary S_i. Keeping the endpoints of the subarc fixed, we can pull the subarc across our axis P to give it the correct orientation. We pull it over all other parts of the knot, except possibly the single crossing on S_i. Note that we can avoid adding another crossing to any part of S which is still oriented incorrectly, which ensures that S is re-oriented in a finite number of steps. To every subarc oriented in the reverse direction, we can apply the same procedure, splitting it up into subarcs with at most one crossing and then pulling each piece over the knot and across the axis. We can continue doing this until we have a link which is all oriented around the axis. But this is exactly what we want, because a link that is entirely oriented around an axis is braided around that axis. If we take a half-plane through the axis going out to infinity, this half-plane necessarily passes through every strand, and can be regarded as the plane of the braid closure."

■

This theorem guarantees that for every knot or link there is a closed braid that the knot or link is ambient isotopic to. We will see later that (2,n) torus links are ambient isotopic to the closures of braids in the braid group B_2. This will be shown through the fact that B_2 is a cyclic group with generator σ_1.

3 DETERMINATION OF THE FIRST RESULT

In this section I will derive a general form for the closure of any braid in B_2. I will do this with the help of the *Temperley-Lieb Algebra*. *Alexander's Theorem* relates some knot or link to the closure of these braids. To begin, B_2 is the group of all braids on two strands. B_2 is cyclic, meaning the group can be *generated* by a single element of B_2 [4]. Every braid group B_n, for $n \in \mathbb{N}$, is infinite meaning that there are an infinite number of elements in the group.

3.1 The Braid Group B_2

Below, I prove some properties of B_2.

Theorem 3.1. $B_2 \cong (\mathbb{Z}, +)$

Proof. From Group Theory, we know that $(\mathbb{Z}, +)$ is an infinite cyclic group [4]. B_2 is also an infinite cyclic group. Also from Group Theory, we know that an infinite cyclic group is *isomorphic* to the integers [4]. Therefore, $B_2 \cong (\mathbb{Z}, +)$.

■

Theorem 3.2. *B_2 is an abelian, nilpotent, solvable group*

Proof. Because B_2 is isomorphic to the integers it is an abelian group. Because it is abelian it is nilpotent [4]. Lastly, because it is nilpotent it is solvable [4]. ■

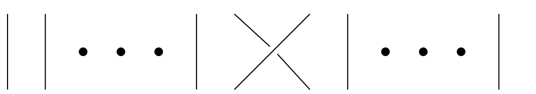
These theorems give us some characteristics about the group B_2. The braid group B_2 will prove important when deriving the Jones polynomial for braids in B_2.

3.2 The Bracket for Braids along with the Temperley-Lieb Algebra

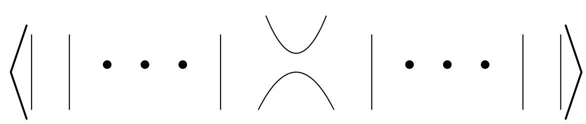
Just like with knots and links, braids also have *writhe*. Define the *writhe* of a braid as the summation over all crossings of a braid b [2]. When doing this, we need to keep in mind which crossings are *positive* and which crossings are *negative*. The same rule applies for braids as it did for knots and links when determining the *sign* of a crossing in a braid b [2]. We are also able to determine the bracket for braids just like we were able to determine the bracket for knots and links.

Let σ_i be an arbitrary generator of B_n [2]

σ_i

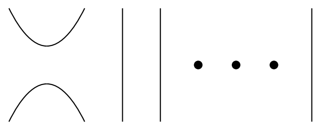


Taking the bracket of this generator gives us the following: [2]

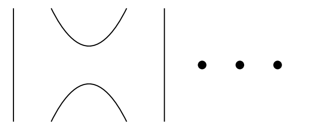
$\langle \sigma_i \rangle = A\langle 1_n \rangle + A^{-1}$ 

Simplifying we have $\langle \sigma_i \rangle = A\ \langle 1_n \rangle + A^{-1}\ \langle U_i \rangle$ [2]. We can repeat the same process as above and derive the formula $\langle \sigma_i^{-1} \rangle = A^{-1}\langle 1_n \rangle + A\langle U_i \rangle$. Here 1_n is the identity element of B_n and U_i is an element of what is called the *Temperley-Lieb Algebra* or $\mathscr{A}_n$ [2]. An *algebra* is defined as a vector space with multiplication [2]. Let R be a commutative ring and let $\delta \in R$ where $\delta = -A^2 - A^{-2}$ [3]. *The Temperley-Lieb Algebra*, sometimes denoted as $TL_n(\delta)$, is the algebra generated by the elements [3]

U_1

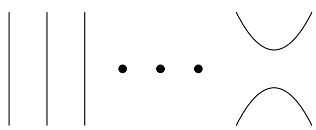


U_2



Culminating with the generator

U_{n-1}



The bracket for braids is similar to the bracket for knots and links but will utilize the *Temperley-Lieb Algebra*. First, let $\mathscr{U}(b) \equiv b$, where b is a braid word and $\mathscr{U}(b)$ is a product of elements U_i [2]. Every product when closed gives a collection of loops [2]. If we set U as such a product then $\langle U \rangle = \langle \overline{U} \rangle = \delta^{\|U\|}$ where $\|U\|$ denotes the number of loops minus one in the closure of U [2].

$$\textstyle\sum_s \langle b|s \rangle U_s$$

here s is indexed over all of the states of the braid b [2], $\langle b|s \rangle$ is a product of powers of A [2], and U_s is the *U-product* [2].The bracket for b is defined as [2]

$$\langle b \rangle = \langle \mathscr{U}(b) \rangle = \textstyle\sum_s \langle b|s \rangle \langle U_s \rangle$$

$$= \textstyle\sum_s \langle b|s \rangle \delta^{\|s\|}$$

It is also important to note that the *Temperley-Lieb Algebra* can be expressed with the above generators along with the following relations [3]:

1.) $U_iU_{i\pm1}U_i = U_i$

2.) $U_iU_i = \delta U_i$

3.) $U_iU_j = U_jU_i$ for $|i-j| > 1$.

An example of computing the bracket for $b = \sigma_1^3$ follows:

$\mathscr{U}(b) = (A + A^{-1}U_1)^3$

$= A^3 + 3AU_1 + 3A^{-1}U_1^2 + A^{-3}U_1^3.$

$\langle b\rangle = \langle \mathscr{U}(b)\rangle = A^3\langle 1_2\rangle + 3A\langle U_1\rangle + 3A^{-1}\langle U_1^2\rangle + A^{-3}\langle U_1^3\rangle$

If we want to translate this bracket for b to the bracket for $\overline{b}$, noticing that the closure of b is in fact the trefoil, we have that

$\langle 1_2\rangle = \delta$

$\langle U_1\rangle = 1$

$\langle U_1^2\rangle = \delta$

$\langle U_1^3\rangle = \delta^2$ where $\delta = -A^2 - A^{-2}$

So, $\langle \overline{b}\rangle = A^3(-A^2 - A^{-2}) + 3A + 3A^{-1}(-A^2 - A^{-2}) + + A^{-3}(A^4 + 2 + A^{-4})$

$= -A^5 - A^{-3} + A^{-7}$

This is the bracket polynomial of the trefoil. It is the same as if we were to calculate the bracket of the trefoil by splicing the crossings one by one. A smaller example is if $b = \sigma_1$ for $\sigma_1 \in B_2$ then $\mathscr{U}(b) = A + A^{-1}U_1$. This mean that $\langle b\rangle = A\langle 1_2\rangle + A^{-1}\langle U_1\rangle$. By substitution we have that $\langle \overline{b}\rangle = -A^3 - A^{-1} + A^{-1}$ which simplifies to $-A^3$. We can see that calculating the brackets of closed braids is facilitated with the help of the *Temperley-Lieb Algebra*.

3.3 Determination

With this derivation we will only be interested in the braids with a finite number of crossings. We start with the group B_2. As mentioned, B_2 is a cyclic group with the generator σ_1. Because any braid $b \in B_2$ can be generated from σ_1 the problem of determining the Jones polynomial for the closures of these braids becomes a problem of looking at braids with all negative crossings or all positive crossings. For example, the braid that is generated by the word $\sigma_1\ \sigma_1\ \sigma_1\ \sigma_1\ \sigma_1^{-1}\ \sigma_1$ has a closure that is ambient isotopic to the closure of the braid word generated by $\sigma_1\ \sigma_1\ \sigma_1\ \sigma_1$. B_2 is a group implying that its operation $*$, which in this case is concatenation, is associative. So, $\sigma_1\ \sigma_1\ \sigma_1\ \sigma_1\ \sigma_1^{-1}\ \sigma_1 = \sigma_1\ \sigma_1\ \sigma_1\ \sigma_1\ (\sigma_1^{-1}\ \sigma_1) = \sigma_1^4$. So, the answer to the problem requires us to look at the writhes of the braids in question. Start with braids whose writhes are equal to 1 or greater than one. So we are looking at braids of the form $b = \sigma_1^n$ where $n \in \mathbb{N}$. We know that $\mathscr{U}(\sigma_1^n) = (A + A^{-1}U_1)^n$ where $U_1 \in \mathscr{A}_2$ [2]. Here, $\mathscr{A}_2$ is the particular Temperley-Lieb Algebra we will be using in this derivation. Using binomial expansion we get the following:

$(A + A^{-1}U_1)^n =$

$\binom{n}{0}A^n(A^{-1}U_1)^0 + \binom{n}{1}A^{n-1}(A^{-1}U_1)^1 + \binom{n}{2}A^{n-2}(A^{-1}U_1)^2 + \ldots + \binom{n}{n}A^0\ (A^{-1}U_1)^n$

Simplifying this expression gives us,

$A^n + \binom{n}{1}A^{n-1}A^{-1}U_1 + \binom{n}{2}A^{n-2}(A^{-1}U_1)^2 + \ldots + (A^{-1}U_1)^n =$

$A^n + \binom{n}{1}A^{n-2}U_1 + \binom{n}{2}A^{n-4}U_1^2 + \ldots + A^{-n}U_1^n.$

From the section on the bracket for braids we know that if b is a braid word then $\langle b\rangle = \langle \mathscr{U}(b)\rangle$. This means that

$\langle \mathscr{U}(b)\rangle = A^n\langle 1_2\rangle + \binom{n}{1}A^{n-2}\langle U_1\rangle + \binom{n}{2}A^{n-4}\langle U_1^2\rangle + \ldots + A^{-n}\langle U_1^n\rangle$

$$= A^n\delta + \binom{n}{1}A^{n-2} + \binom{n}{2}A^{n-4}\delta + \ldots + A^{-n}\ \delta^{n-1}$$

This is the general form for the bracket of any braid with positive writhe in the group B_2. This equals the bracket for the closure of any braid in B_2. Let $b \in B_2$ be a braid with positive writhe and let $\bar{b}$ be the closure of b. We have the following:

$$\langle\overline{\sigma_1^n}\rangle = A^n(-A^2 - A^{-2}) + \binom{n}{1}A^{n-2} + \binom{n}{2}A^{n-4}(-A^2 - A^{-2}) + \ldots + A^{-n}(-A^2 - A^{-2})^{n-1}$$

$$= -A^{n+2} - A^{n-2} + \binom{n}{1}\ A^{n-2} + \binom{n}{2}\ (-A^{n-2} - A^{n-6}) + \ldots + A^{-n}(-A^2 - A^{-2})^{n-1}$$

This is the general form of the bracket polynomial of the closure of the braid word σ_1^n. The next question is what is the bracket polynomial of the closure of the braid word σ_1^{n+1}?

We know a recursion formula for closed braids of this form [2]. It is the following: $\langle\overline{\sigma_1^{n+1}}\rangle = A\langle\overline{\sigma_1^n}\rangle + (-1)^n A^{-3n-1}$. This formula allows us to see how the bracket changes as we add positive crossings to an already existing closed braid with all positive crossings. The new bracket is simply a sum of two terms. The first term is the previous bracket multiplied by A, and the second term is namely $(-1)^n A^{-3n-1}$. The sign of this second term is dependent on the parity of n. For example, if we start with n = 1 we have that $\langle\overline{\sigma_1}\rangle = -A^3$. We then can calculate $\langle\overline{\sigma_1^2}\rangle$. In fact, we can calculate all of them. Below are a few examples,

$\langle\overline{\sigma_1}\rangle = -A^3$

$\langle\overline{\sigma_1^2}\rangle = -A^4 - A^{-4}$

$\langle\overline{\sigma_1^3}\rangle = -A^5 - A^{-3} + A^{-7}$

$\langle\overline{\sigma_1^4}\rangle = -A^6 - A^{-2} + A^{-6} - A^{-10}$

$\langle\overline{\sigma_1^5}\rangle = -A^7 - A^{-1} + A^{-5} - A^{-9} + A^{-13}$

What we can see from this recursion is that the number of crossings in our braid, where these crossings are all positive, is equal to the number of terms in the bracket polynomial for that braid. This is what the brackets for particular closures look like. To return to the general form of the bracket for $\overline{\sigma_1^n}$ for $n \in \mathbb{N}$ we have the formula

$\langle\overline{\sigma_1^n}\rangle = -A^{n+2} - A^{n-2} + \binom{n}{1} A^{n-2} + \binom{n}{2} (-A^{n-2} - A^{n-6}) + \ldots + A^{-n}(-A^2 - A^{-2})^{n-1}$.

Now we need the general formula for the normalized brackets of these closed braids. We know that

$\mathscr{L}\overline{\sigma_1^n} = (-A)^{-3n} \langle\overline{\sigma_1^n}\rangle$

From this formula and simplifying we get that $\mathscr{L}\overline{\sigma_1^n}$ equals

$A^{2-2n} + (1-n)A^{-2-2n} + \frac{n^2-n}{2} (A^{-2-2n} + A^{-2n-6}) + \ldots + A^{-4n}(A^{2-3n} + A^{-2-3n})^{n-1}$

The signs of these normalized polynomials follow a somewhat alternating pattern. We see the pattern from calculating the normalized brackets for $\langle\overline{\sigma_1}\rangle$ to $\langle\overline{\sigma_1^5}\rangle$.

$$\mathscr{L}_{\overline{\sigma_1}} = 1$$

$$\mathscr{L}_{\overline{\sigma_1^2}} = -A^{-2} - A^{-10}$$

$$\mathscr{L}_{\overline{\sigma_1^3}} = A^{-4} + A^{-12} - A^{-16}$$

$$\mathscr{L}_{\overline{\sigma_1^4}} = -A^{-6} - A^{-14} + A^{-18} - A^{-22}$$

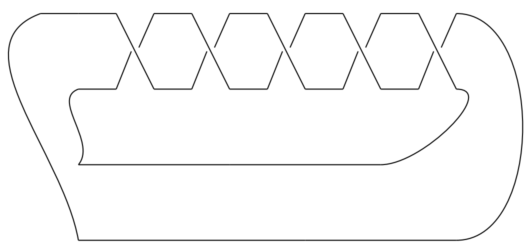
$$\mathscr{L}_{\overline{\sigma_1^5}} = A^{-8} + A^{-16} - A^{-20} + A^{-24} - A^{-28}$$

As we increase the number of positive crossings in our braids the more terms we have in both the normalized and unnormalized brackets. In the normalized brackets, we see that the first two coefficients will always be equivalent. Then starting with term two there is an alternating pattern of the signs of therms of the polynomials. We have shown that the normalized bracket is an invariant under all three Reidemeister moves.

These normalized bracket polynomials are equivalent to the normalized bracket polynomials for (2,n) torus links where $n \in \mathbb{N}$. For example, we have shown that $\mathscr{L}_{\overline{\sigma_1^3}} = A^{-4} + A^{-12} - A^{-16}$. This is the same normalized bracket for the trefoil. The trefoil is ambient isotopic to the closure of the braid $b \in B_2$ where $w(b) = 3$.

The closure of the braid word σ_1^n is of the form of the (2,n) torus link. For now we are dealing with $n \in \mathbb{N}$. Any torus link or knot can be formed through a sequence of loops through the torus and revolutions around the torus with one or more strands [2]. The following example is the (2,5) torus knot:

$(2,5)$:



This also means that the (2,n) torus link, for n $\in \mathbb{N}$, is ambient isotopic to the closure of any braid $b \in B_2$ where $w(b) = n$. So far we have only shown this to be true for values of n that are natural numbers but it will be shown that this is true for all integers. Using the formula $\mathscr{L}_{\overline{\sigma_1^n}}(t^{-1/4}) = V_{\overline{\sigma_1^n}}(t)$ we can calculate some examples of Jones polynomials of these braids.

$V_{\overline{\sigma_1}} = 1$

$V_{\overline{\sigma_1^2}} = -t^{1/2} - t^{5/2}$

$V_{\overline{\sigma_1^3}} = t^1 + t^3 - t^4$

$V_{\overline{\sigma_1^4}} = -t^{3/2} - t^{7/2} + t^{9/2} - t^{11/2}$

$V_{\overline{\sigma_1^5}} = t^2 + t^4 - t^5 + t^6 - t^7$

The braids with *positive* writhe in B_2 are ambient isotopic to torus links of type $(2, n)$ for $n \in \mathbb{N}$. Denote $L = K(2, n)$ where $K(2, n)$ denotes the torus link on 2 strands with n crossings. We know that the Jones polynomials of these torus knots, for when n is odd, are equal to the following: [6]

$$V_L(t) = t^{\frac{n-1}{2}} + \sum_{k=1}^{n-1} (-1)^{k+1} t^{(k+\frac{n+1}{2})}$$

So for $n \geq 1$, $V_{\overline{\sigma_1^n}}(t) = t^{\frac{n-1}{2}} + \sum_{k=1}^{n-1} (-1)^{k+1} t^{(k+\frac{n+1}{2})}$ for $\sigma_1 \in B_2$. Now, what about for $n \leq -1$? In other words, what about the braids with all negative crossings? We will use the following theorem to determine the Jones polynomial of these braids:

Theorem 3.3. [2] *If K^* is the mirror image of an oriented link diagram K then $\mathscr{L}_{K^*}(A) = \mathscr{L}_K(A^{-1})$.*

Proof. [2] *Reversing all crossings exchanges the roles of A and A^{-1} in the definition of $\langle K \rangle$ and $\mathscr{L}_K$.*

■

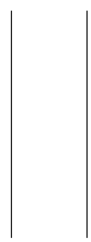
Consequently, a braid with n negative crossings is the mirror image of a braid with n positive crossings. This means that the Jones polynomial of braids with negative writhes is equivalent to the Jones polynomial of braids with the same number of crossings (where these crossings are positive) except the signs of the exponents are the opposite sign of its mirror image counterpart.

For odd n we know the following:

$$V_{\overline{\sigma_1^{-n}}}(t) = t^{-\frac{n-1}{2}} + \sum_{k=1}^{n-1} (-1)^{k+1} t^{-(k+\frac{n+1}{2})}$$

Lastly, what about the braids whose writhes are equal to zero? For example, The braid b generated by the word $\sigma_1\ \sigma_1^{-1}$ has writhe equal to zero. We know that $\sigma_1\ \sigma_1^{-1} = 1_2$ which means that the closure of the word $\sigma_1\ \sigma_1^{-1}$ is ambient isotopic to the closure of 1_2. The identity braid of B_2 is the braid:

1_2



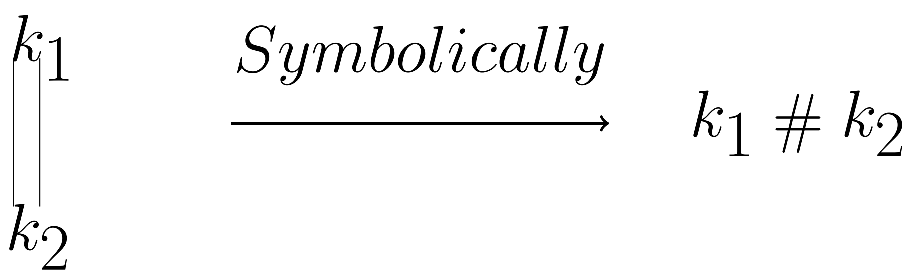
The closure of this braid yields two disjoint circles. We know the bracket polynomial of one disjoint circle is equal to zero. We also know that adding a disjoint circle to a knot or link diagram changes the bracket by a factor of $-A^2 - A^{-2}$. This means that $\langle \overline{1_2} \rangle = -A^2 - A^{-2}$. The normalized bracket of $\overline{1_2}$ is equal to $(-A)^0 \langle \overline{1_2} \rangle = \langle \overline{1_2} \rangle$. Calculating the Jones polynomial of this closure requires us to use the formula $\mathscr{L}_{\overline{1_2}}(t^{-1/4}) = V_{\overline{1_2}}(t)$. So, $V_{\overline{1_2}}(t) = -t^{-1/2} - t^{1/2}$. We note that $\overline{1_2}$ is *achiral* meaning that it is not topologically different from its mirror image meaning that the closure of this identity element will have a mirror image with an identical Jones polynomial.

4 DETERMINING POLYNOMIALS FOR $\overline{b_1} \# \overline{b_2}$ FOR $b_1, b_2 \in B_2$

Next, we will define what the *connected sum* of two knots is. This operation is only defined for knots so what happens if we want to connect a knot to a link? Which component of the link do we connect the knot to? This problem does not occur in this section's results because our links $\overline{\sigma_1^{2k}}$ have two components which are identical. So it will not matter which component of the link we connect the knot to.

4.1 The Connected Sum of Knots

Given two knot diagrams, let's call these diagrams k_1 and k_2, the connected sum of k_1 and k_2 is defined as the following:



We are going to be interested in the connected sum of the closures of two braids $b_1, b_2 \in B_2$. The question is what is the bracket polynomial of this sum? How do we calculate the bracket polynomial? The answer to this question relies on the fact that any braid in B_2 with writhe equal to n or -n for n $\in (\mathbb{N} \cup \{0\})$ is going to have a closure that is ambient isotopic to either $\overline{\sigma_1^n}$ or $\overline{\sigma_1^{-n}}$ respectively. So again, we need only look at braids with all positive crossings, all negative crossings, or zero crossings. The braid with zero crossings is the identity braid 1_2. We now calculate the bracket polynomial for $\overline{\sigma_1}\ \overline{\sigma_1^n}$ for $n \in (\mathbb{Z} - \{0\})$. If n = 0 then we have $\langle \overline{\sigma_1}\ \overline{1_2} \rangle = \langle\ \overline{\sigma_1}\ \bigcirc \rangle =$

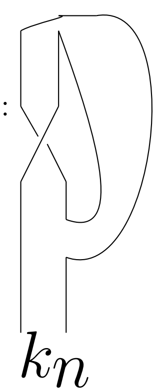
$(-A^2 - A^{-2})\ \langle\overline{\sigma_1}\rangle = \langle\overline{\sigma_1}\rangle\ \langle\overline{1_2}\rangle$. So the bracket of the sum is the product of the brackets.

4.2 $\langle\overline{\sigma_1}\#\ \overline{\sigma_1^n}\rangle = \langle\overline{\sigma_1}\rangle\ \langle\overline{\sigma_1^n}\rangle$ for $n \in (\mathbb{Z} - \{0\})$

Theorem 4.1. $\langle\overline{\sigma_1}\#\ \overline{\sigma_1^n}\rangle = \langle\overline{\sigma_1}\rangle\ \langle\overline{\sigma_1^n}\rangle$ for $n \in (\mathbb{Z} - \{0\})$

Proof. We note the closure of a power of σ_1^n is a knot if n is odd. If n is even, it is a two component link. Since the two components are identical, it does not matter which component we join $\overline{\sigma_1}$ to, and so we can extend the idea of connected sum of knots to the connected sum of links in this special case. Let $k_n = \overline{\sigma_1^n}$. Consider the following:

$\overline{\sigma_1}\ \#\ \overline{\sigma_1^n}$:



We also note that any braid in B_2 with positive or negative writhe can be reduced to a braid with all positive or all negative crossings respectively. This means the result will hold for any $b \in B_2$. To calculate the bracket of $\overline{\sigma_1}\ \#\ \overline{\sigma_1^n}$: we first have that

$\langle\overline{\sigma_1}\ \#\ \overline{\sigma_1^n}\rangle = A \left\langle\ k_n \bigcirc \right\rangle + A^{-1}\langle k_n\rangle$

$\langle\overline{\sigma_1}\ \#\ \overline{\sigma_1^n}\rangle = A\delta\langle k_n\rangle + A^{-1}\langle k_n\rangle.$

We also know that $\delta = -A^2 - A^{-2}$. So by substitution we know the following:

$\langle\overline{\sigma_1}\ \#\ \overline{\sigma_1^n}\rangle = A(-A^2 - A^{-2})\langle k_n\rangle + A^{-1}\langle k_n\rangle.$

$= (-A^3 - A^{-1})\ \langle k_n\rangle + A^{-1}\langle k_n\rangle.$

$= -A^3 \, \langle k_n \rangle \, - \, A^{-1} \, \langle k_n \rangle \, + \, A^{-1} \, \langle k_n \rangle.$

$\langle \overline{\sigma_1} \, \# \, \overline{\sigma_1^n} \rangle = -A^3 \, \langle k_n \rangle.$

We know that $\langle \overline{\sigma_1} \rangle = -A^3$ and we set $\overline{\sigma_1^n} = k_n$. So, $\langle \overline{\sigma_1} \, \# \, \overline{\sigma_1^n} \rangle = \langle \overline{\sigma_1} \rangle \, \langle \overline{\sigma_1^n} \rangle$ as desired.

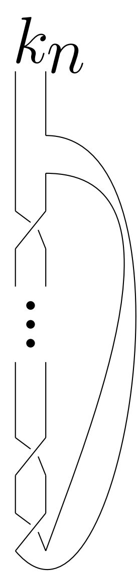
■

It is important to note that the above result holds for the bracket of $\sigma_1^{-1} \, \sigma_1^n$ for $n \in (\mathbb{Z} - \{0\})$ as well. This is true by the same reasoning as above except for the fact that $\langle \overline{\sigma_1^{-1}} \rangle = -A^{-3}$ as opposed to $-A^3$. Now is this result true for $\sigma_1^n \, \sigma_1^m$ for $m, n \in \mathbb{N}$? It turns out the answer is yes.

Theorem 4.2. $\langle \overline{\sigma_1^n} \# \overline{\sigma_1^m} \rangle = \langle \overline{\sigma_1^n} \rangle \, \langle \overline{\sigma_1^m} \rangle$ for $m, n \in \mathbb{N}$.

Proof. Let $m, n \in \mathbb{N}$. We will prove this theorem by induction. Let m = 1. By theorem 4.1 $\langle \overline{\sigma_1^n} \# \overline{\sigma_1^1} \rangle = \langle \overline{\sigma_1^n} \rangle \, \langle \overline{\sigma_1^1} \rangle$. Now assume that $\langle \overline{\sigma_1^n} \# \, \overline{\sigma_1^m} \rangle = \langle \overline{\sigma_1^n} \rangle \, \langle \overline{\sigma_1^m} \rangle$ for $m, n \in \mathbb{N}$. Consider the following diagram of $\overline{\sigma_1^n} \, \# \, \overline{\sigma_1^{m+1}}$. Denote $\overline{\sigma_1^n}$ as k_n.

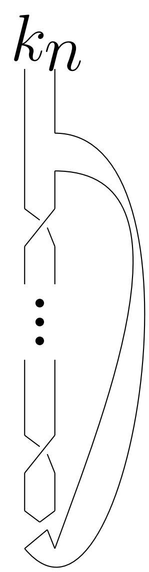
$\overline{\sigma_1^n} \# \, \overline{\sigma_1^{m+1}}$:



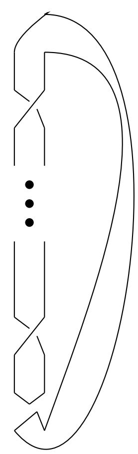
In the previous diagram there are a total of $n+m+1$ positive crossings. We first want to unravel the crossing at the bottom of the diagram. When we splice this crossing we will have the following equation:

$\langle \overline{\sigma_1^n} \# \overline{\sigma_1^{m+1}} \rangle = A\langle k_n k_m \rangle + A^{-1}\langle P \rangle$ where P is

$P:$



From our inductive hypothesis we know that $\langle k_n \# k_m \rangle = \langle k_n \rangle \ \langle k_m \rangle$. So, $\langle \overline{\sigma_1^n} \# \overline{\sigma_1^{m+1}} \rangle = A\langle k_n \rangle \langle k_m \rangle + A^{-1}\langle P \rangle$. But what is $\langle P \rangle$? If we examine P we see that it has n + m positive crossings. It has n crossings in k_n and m crossings in the remaining part of the structure. Let L be the part of P such that $P = k_n \# L$. One way we could calculate $\langle P \rangle$ is to go crossing by crossing and calculate its bracket manually. That would be a valid procedure. Another way of calculating $\langle P \rangle$ is to recognize that we know what the bracket of L is equal to. We know that L has m positive crossings and is of the form:

L : 

We know for this form that $\langle L\rangle = (-A^{-3})^m$ [2]. So, $\langle P\rangle = \langle k_n\rangle(-A^{-3})^m$ and by substitution we have $\langle \overline{\sigma_1^n} \# \overline{\sigma_1^{m+1}}\rangle = A\langle k_n\rangle\langle k_m\rangle + A^{-1}\langle k_n\rangle(-A^{-3})^m$.

$$= \langle k_n\rangle \left(A\langle k_m\rangle + A^{-1}(-A^{-3})^m\right)$$

Recognizing that $(A\langle k_m\rangle + A^{-1}(-A^{-3})^m) = \langle \overline{\sigma_1^{m+1}}\rangle$ and that we set $k_n = \overline{\sigma_1^n}$ we have that $\langle \overline{\sigma_1^n} \# \overline{\sigma_1^{m+1}}\rangle = \langle \overline{\sigma_1^n}\rangle \langle \overline{\sigma_1^{m+1}}\rangle$ for $m, n \in \mathbb{N}$.

■

We note that this theorem shows that when you take two different braids, both of which have positive writhes, or that both of the braids have negative writhes due to mirror images, then the bracket of the connected sum of their closures is equal to the product of the brackets of the closures. It is still to be shown that this is also true for braids with negative writhes. The proof to show this would more than likely follow a similar procedure as to the previous proof for positive writhes.

4.3 The Normalized Bracket of $\overline{b_1}$ # $\overline{b_2}$

We can use this result and expand it to the normalized bracket of $\overline{b_1}$ # $\overline{b_2}$. We first note that the following two results hold for braids in which we have proven that $\langle\overline{b_1}\#\overline{b_2}\rangle = \langle\overline{b_1}\rangle\ \langle\overline{b_2}\rangle$

Theorem 4.3. $\mathscr{L}_{\overline{b_1}\#\overline{b_2}}(A) = \mathscr{L}_{\overline{b_1}}(A)\mathscr{L}_{\overline{b_2}}(A)$ for $b_1, b_2 \in B_2$.

Proof. Let $b_1, b_2 \in B_2$ where $w(b_1) = n$ and $w(b_2) = m$ for $n, m \in \mathbb{Z}$. We know that $\langle\overline{b_1}\ \#\ \overline{b_2}\rangle = \langle\overline{b_1}\rangle\ \langle\overline{b_2}\rangle$, $\mathscr{L}_K = (-A^3)^{-w(K)}\langle K\rangle$, and $w(\overline{b_1}\ \#\ \overline{b_2}) = n + m$. So, we have that

$$\begin{aligned}\mathscr{L}_{\overline{b_1}\#\overline{b_2}}(A) &= (-A^3)^{-(n+m)}\langle\overline{b_1}\ \#\ \overline{b_2}\rangle \\ &= (-A^3)^{-n}\ (-A^3)^{-m}\ \langle\overline{b_1}\rangle\ \langle\overline{b_2}\rangle. \\ &= \mathscr{L}_{\overline{b_1}}(A)\ \mathscr{L}_{\overline{b_2}}(A)\end{aligned}$$

■

4.4 The Jones Polynomial of $\overline{b_1}$ # $\overline{b_2}$

The question now is if the Jones Polynomial of $\overline{b_1}$ # $\overline{b_2}$ is equal to the product of $\overline{b_1}$ and $\overline{b_2}$'s respective Jones Polynomials. The answer is yes. We will show this with a proof.

Theorem 4.4. $V_{\overline{b_1}\#\overline{b_2}}(t) = V_{\overline{b_1}}\ (t)V_{\overline{b_2}}(t)$

Proof. $V_{\overline{b_1}\#\overline{b_2}}(t) = \mathscr{L}_{\overline{b_1}\#\overline{b_2}}(t^{-1/4})$ by definition

$$= \mathscr{L}_{\overline{b_1}}(t^{-1/4})\ \mathscr{L}_{\overline{b_2}}(t^{-1/4}) \text{ by Theorem 4.3}$$

$$= V_{\overline{b_1}}(t)\ V_{\overline{b_2}}(t)$$

$$\blacksquare$$

4.5 Conclusion on Jones Polynomials

Our first result was based on the fact that B_2 is a cyclic group and the fact that any braid word can be reduced to a power of σ_1, a power of σ_1^{-1}, or the identity element 1_2. These facts allowed us to show that braids with positive writhe n odd have closures with a Jones Polynomial of $V(t) = t^{\frac{n-1}{2}} + \sum_{k=1}^{n-1}(-1)^{k+1}t^{(k+\frac{n+1}{2})}$. Their mirror images, which would be braids with negative writhe m where $m = -n$, have a Jones Polynomial of $V(t) = t^{-\frac{n-1}{2}} + \sum_{k=1}^{n-1}(-1)^{k+1}t^{(k-\frac{n+1}{2})}$. It was also shown that $V_{\overline{1_2}}(t) = -t^{-1/2} - t^{1/2}$. Our second result showed that for a particular set braids in B_2, these braids were enumerated in previous sections, that the Jones polynomial of the connected sum of the closures of these braids was equal to to the product of the brackets themselves. Notationally, $\langle \overline{b_1} \# \overline{b_2} \rangle = \langle \overline{b_1} \rangle \langle \overline{b_2} \rangle$. It is also worth noting that we showed that the product of brackets for these braids was equal to the bracket of products as well as the product of normalized brackets is equal to the normalized bracket of products of the form $\overline{b_1}\ \#\ \overline{b_2}$ for $b_1, b_2 \in B_2$. The Jones Polynomial is just one example of a topological invariant in the field of Knot Theory. The second invariant this thesis discusses is the invariant known as *tricolorability*.

5 TRICOLORABILITY

Tricolorability is another example of a knot invariant. It can be used to determine when a knot is **not** topologically equivalent to the unknot [7]. However, if two knots are both tricolorable that does not mean they are ambient isotopic [7]. This will be seen when determining which braids in B_2 have closures that are tricolorable.

5.1 Definition of Tricolorability

A knot is tricolorable if each strand on the diagram can be colored using the following two rules [7]:

- Three colors must be used
- At each crossing, the three incident strands are either all the same color or all different colors.

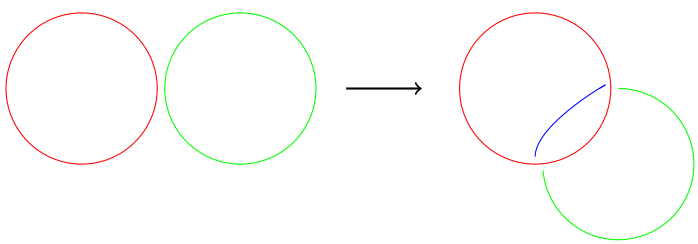
The following theorem will allow us to draw our conclusions on the tricolorability of braids in B_2.

Theorem 5.1. [7] *Tricolorability is a Knot Invariant*

If two knots are ambient isotopic to each other then they will both be either tricolrable or not tricolorable. *Tricolorability* is a knot invariant which allows us to distinguish when knot diagrams are not ambient isotopic to the diagram of the unknot. The unknot is not tricolorable so any diagram that is tricolorable is not topologically equivalent to the unknot.

5.2 Which $b \in B_2$ have tricolorable closures?

We can see here that the unknot is not tricolorable as only one color can be used to color a circle. We determined that if $w(\overline{b}) = \pm 1$ then $\overline{b}$ was ambient isotopic to the unknot. So, for $b \in B_2$, if $w(b) = \pm 1$ then $\overline{b}$ is not tricolorable. Also, $\overline{1_2}$ is not tricolorable because the closure of $1_2 \in B_2$ yields two circles. We can only use two colors to color two circles. This means that for $b \in B_2$ if $w(b) = 0$ then $\overline{b}$ is not tricolorable. Consider $\overline{b}$ for $b \in B_2$ such that $w(\overline{b}) = \pm 2$. From the section on the Jones polynomial we determined that if $w(b) = \pm 2$ for $b \in B_2$ then $\overline{b}$ was a link. If this closure is tricolorable and is ambient isotopic to another link does that mean the second link is also tricolorable? This answer is not necessarily. We know that tricolorability can only be used to determine if two knots are ambient isotopic and one is tricolorable then the other is most certainly tricolorable. Theorem 5.1 demonstrated that. However, this is untrue for links. Consider the following diagram [2]:



These two links are ambient isotopic to one another. However, the diagram on the left is not tricolorable but the diagram on the right is. So in general, if two links are ambient isotopic to one another and one is tricolorable we cannot automatically assume that the second is also tricolorable. We note that in the previous section on the Jones polynomial we determined a general form for the closure of any element of B_2. The results showed that if b had even writhe then the closure of b was a link because of the facts that $\overline{b}$ had two components and $V_{\overline{b}}(t)$ had terms with noninteger exponents indicating that they were links. There were closures of braids that were not links but knots. For example $\overline{\sigma_1^3}$ was the trefoil. The mirror images of the links and knots are also links and knots respectively.

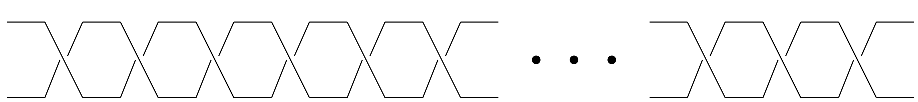
5.3 Results on Tricolorability

We first prove that (2, 3n) torus links are tricolorable for $n \neq 0$

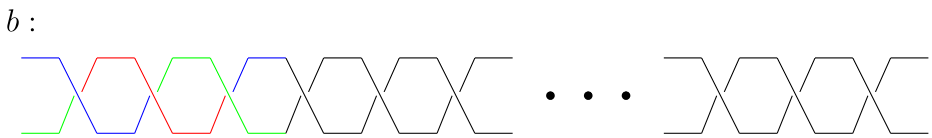
Theorem 5.2. *(2,3n) torus links are tricolorable for* $n \in (\mathbb{Z} - \{0\})$

Proof. We have shown through the Jones polynomial that the closure of any braid in B_2 whose writhe equals three is ambient isotopic to the trefoil and consequently braids whose writhes are equal to -3 have closures that are ambient isotopic to the mirror image of the trefoil. So, if $w(b) = \pm 3$ then $\overline{b}$ is tricolorable. Assume that k is a (2,3n) torus link where $w(k) = 3n$ for $n \in \mathbb{Z}_{>0}$.

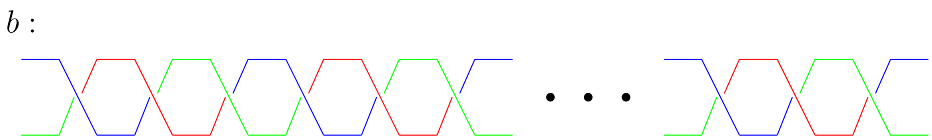
b :



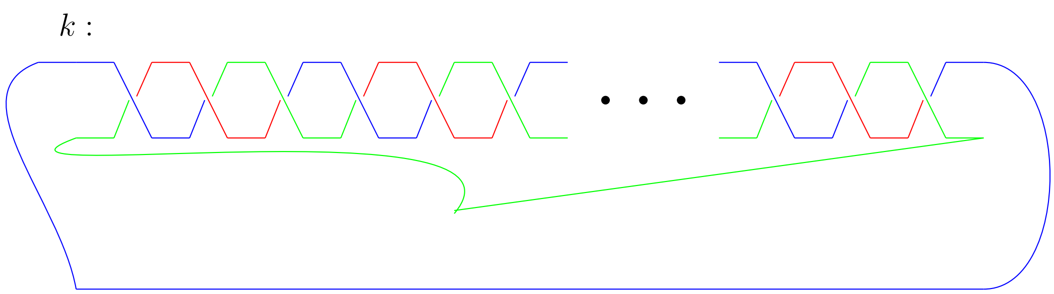
In the above diagram $b \in B_2$ has a writhe that is a positive multiple of 3. The diagram has been turned 90° to facilitate the process of coloring. We can now begin coloring b. because any braid word in B_2 to all positive, negative, or no crossings, b is a braid with all positive crossings. The traditional colors we use to color knots are red, blue, and green [7]. Color the first three strands of b.



Note that what occurs in the ellipsis in the above diagram is a sequence of positive crossings where the total number of positive crossings is equal to some positive multiple of three. If we repeat the same coloring scheme for the remaining strands we will have



Taking the closure of this braid will give us

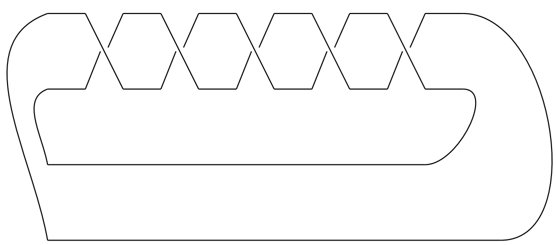


We see here that k is tricolorable. Let $3n$ be any positive multiple of 3. Then if $w(k) = 3n$, k is tricolorable. The mirror image of k is also tricolorable. This proves the theorem.

■

This theorem gives us two results on tricolorability. The first result gives us a set of torus links that are tricolorable. What this theorem also tells us is that braids with writhes that are odd multiples of three have closures that are tricolorable as well as are their mirror images. This is true because the closures of these braids are knots and not links. Braids that have writhes that are even multiples of three have closures that are links and not knots. The following two proofs will give us braids whose closures are **not** tricolorable. Consider the (2,5) torus knot:

$(2,5)$:

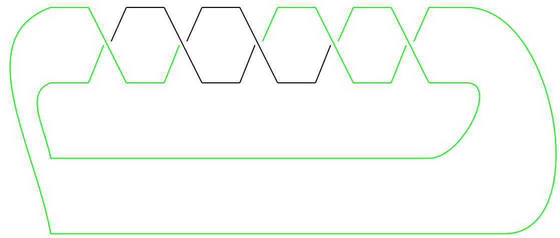


Theorem 5.3 *The (2,5) torus knot is not tricolorable.*

Proof. To show this we need to examine two cases. The first case will be if we want to color the strands of the first crossing all the same color. The second case will be if we want to color the strands different colors. We begin with case 1.

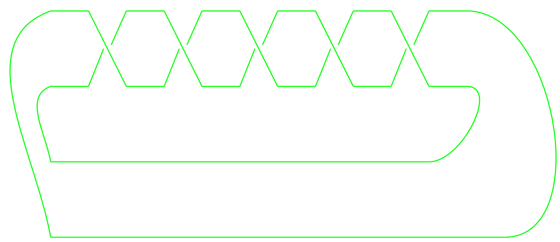
Case 1: If we want to color the first strands of the first crossing of this knot all the same color, let's use the color green, we will have a diagram that looks like the following:

(2, 5) :



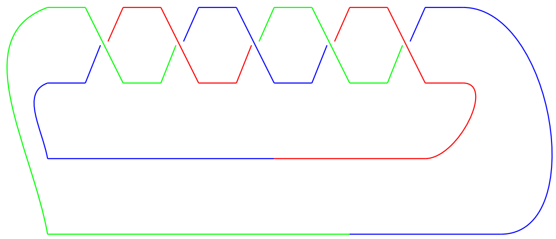
Notice that in order to abide by the rules of tricolorability the strands of the next crossing must be all green as well. This is also true for the third, fourth, and fifth crossings meaning that if we color one crossing all of the same color, green, then we will have the following:

(2, 5) :



Case 2: Now we need to color the first strands of the first crossing all three different colors. If we color the crossing in such a way, abiding by the rules of tricolorability, then coloring the next strands until we reach the final crossing, we will get a diagram that looks like the following noting that the roles of the colors are interchangeable.

$(2,5)$:



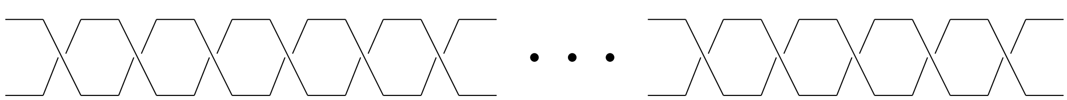
We get a clashing of colors in the diagram which means that if we color the first crossing all three different colors then the (2,5) torus knot is not tricolorable. We have shown this is true for both cases which means that the (2,5) torus knot is not tricolorable. ■

Consequently, because of this proof the mirror image of the (2,5) torus knot is also not tricolorable by the same procedure. Another consequence to this theorem is that for $b \in B_2$ if $w(b) = \pm 5$ then $\bar{b}$ is not tricolorable. This is true due to the fact that tricolorability is a knot invariant and that if $w(b) = \pm 5$ then $\bar{b}$ will be ambient isotopic to the (2,5) torus knot. There are other braids in B_2 whose closures are not tricolorable. First, we will prove the following on torus knots:

Theorem 5.4. *(2,3n + 2) torus links for n a natural number are not tricolorable.*

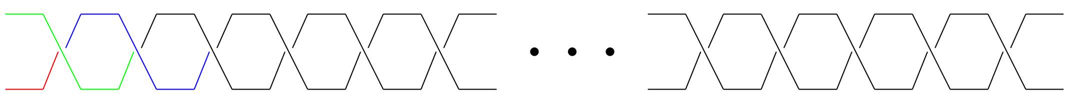
Proof. First, let $b \in B_2$ such that $w(b) = 3n + 2$ for n a natural number. We have already shown the case for $n = 1$. Consider the following diagram of b for $n > 1$

b :



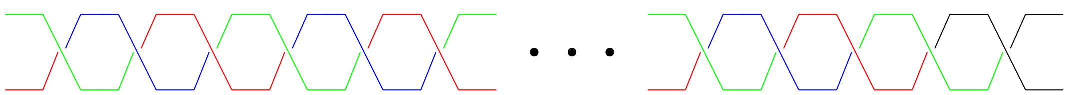
We note what occurs in the above ellipsis is a sequence of m positive crossings where m is a positive multiple of 6 or equal to zero. We also note that we have a braid with all positive crossings because any word in B_2 can be reduced to a word with all positive, negative, or no crossings. If we were to color this braid, where the first crossing was all the same color, we would get a 1-coloring of the braid and so its closure would also be a 1-coloring. This is seen through a process similar to that in Theorem 5.3. Now if we were to begin coloring this braid, where all three colors are used to color the strands of the first crossing, we would have the following diagram:

$b:$



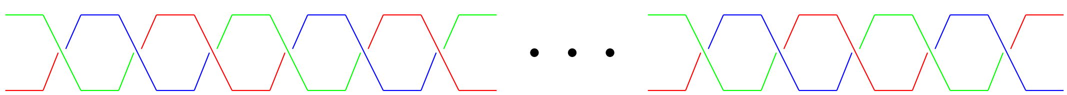
Following the rules of tricolorability, if we were to color these crossings up until the final two crossings we would have the following:

$b:$

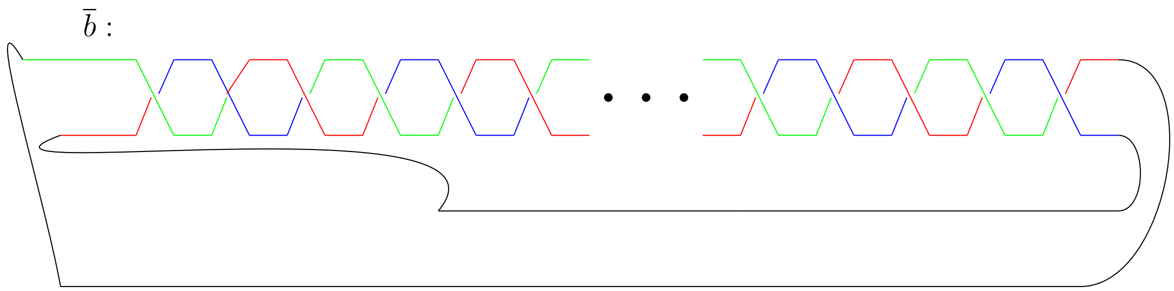


Before continuing to color we note that what we have colored r positive crossings, in accordance with the rules of tricolorability, where r is a positive odd multiple of three. We now complete the coloring of the braid:

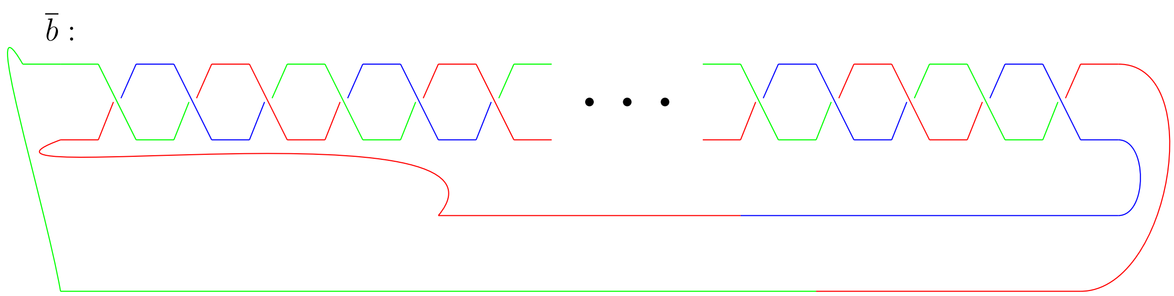
$b:$



Taking the closure of b we have a (2, 3n + 2) torus link



Coloring the closure, while still abiding by the rules of tricolorability, gives us the following diagram:



Noting that the roles of the colors are interchangeable this completes the proof.

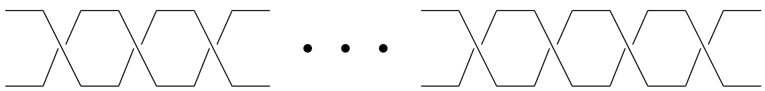
■

We can see here, by similar reasoning, because these links are not tricolorable their mirror images are also not tricolorable. This theorem also gives us information about a collection of braids in B_2. Any braid that has a writhe equal to $\pm(3n+2)$ for n an odd integer, as these closures are knots and not links, have closures that are tricolorable. We know that if two knots are ambient isotopic to one another and one is tricolorable then the other is tricolorable as well [7]. We prove one more theorem on the topic of tricolorability.

Theorem 5.5. *(2,3n + 1) torus links for n a natural number are not tricolorable*

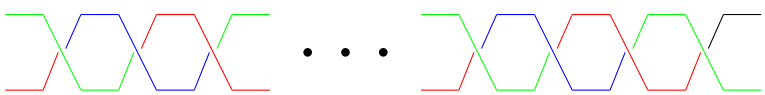
Proof. Let $b \in B_2$ and let $w(b) = 3n + 1$ for n a natural number. Examples of these numbers are 4, 7, 10, 13, 16, ... Consider the following diagram of b:

b :



We note what occurs in the above ellipsis is a sequence of m positive crossings where m is a positive multiple of 3 or equal to zero. We also note that we have a braid with all positive crossings because any word in B_2 can be reduced to a word with all positive, negative, or no crossings. Following the same coloring scheme as in theorem 5.4. we color all of the crossings b up until the final crossing:

b :



We see that we are forced to color the final strand blue. Taking the closure of this braid will give us a (2, 3n+1) torus link, and completing the coloring we see that we have a collision of colors on the strands that complete the closure of the braid. This means that $\bar{b}$ is not tricolorable when we want to color the first crossing with three different colors. If we want to color the first crossing the same color, we get a situation where we are forced to color $\bar{b}$ with only one color. So $\bar{b}$ is not tricolorable.

■

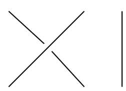
Just like for theorem 5.4. the mirror image of these links are also not tricolorable by similar reasoning. Consequently, this theorem gives us another collection of braids whose closures are not tricolorable. Namely, braids whose writhes are equal to $\pm(3n+1)$ for n an odd integer.

To conclude tricolorability it is important to reiterate the fact that tricolorability is a knot invariant. We have been concerned with braids that have odd writhe because these are the braids whose closures are knots and not links. This was shown in the section on the Jones polynomial. We have determined two different set of braids one whose closures are not tricolorable and the other whose closures are tricolorable.

6 FURTHER RESEARCH

What remains to be done is to extend these results to the braid group B_3 and beyond. The group B_3 is the set of all braids on three strands with the operation of concatenation. B_3 is not a cyclic group as it takes two elements to generate the entire group. Namely, these elements are σ_1 and σ_2 :

σ_1 :



σ_2 :

What is to be done is to determine a general form of the *Jones* polynomial for the closure of any braid in B_3. The results on tricolorability can also be extended to B_3. Another invariant is what is called the *Alexander-Conway* polynomial. More research to be conducted is to determine the general form of the *Alexander-Conway* polynomial for the closure of any braid in B_2 as well as for B_3, B_4, up to a general case for B_n. Another open problem is to find a knot whose Jones polynomial is 1 but is not ambient isotopic to the unknot.

BIBLIOGRAPHY

[1] Kauffman, Louis H. *State Models and the Jones Polynomial.* Pergamon Journals Ltd. , vol. 26, no. 3, 1 Sept. 1986, pp. 395-407.

[2] Louis H. Kauffman. *Knots and Physics* World Scientific Publishing Co. Re. Ltd 2001.

[3] W. B. Raymond Lickorish. *An Introduction to Knot Theory* Springer-Verlag New York, Inc. 1997.

[4] Charles C. Pinter. *A Book of Abstract Algebra* McGraw-Hill Publishing Company, Inc., New York 1982.

[5] Alexander, James, *A lemma on a system of knotted curves.* Proc. Natl. Acad. Sci. USA. 9: pp. 93-95 1923. References

[6] Abdullah Kopuzlu, Abdulgani Sahin, and Tamer Ugur, *On Polynomials of (2,n) Torus Knots.* Applied Mathematical Sciences, vol. 3, no.59, May 2009 pp. 2899-2910.

[7] Gilbert, N. D., and Timothy Porter. *Knots and Surfaces.* Oxford University Press, 2002.

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