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The Complete Structure of Linear and Nonlinear Deformations of Frames on a Hilbert Space

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The Complete Structure of Linear and Nonlinear Deformations

of Frames on a Hilbert Space

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ABSTRACT

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A frame is a possibly linearly dependent set of vectors in a Hilbert space that facilitates the decomposition and reconstruction of vectors. A Parseval frame is a frame that acts as its own dual frame. A Gabor frame comprises all translations and phase modulations of an appropriate window function. We show that the space of all frames on a Hilbert space indexed by a common measure space can be fibrated into orbits under the action of invertible linear deformations and that any maximal set of unitarily inequivalent Parseval frames is a complete set of representatives of the orbits. We show that all such frames are connected by transformations that are linear in the larger Hilbert space of square-integrable functions on the indexing space. We apply our results to frames on finite-dimensional Hilbert spaces and to the discretization of the Gabor frame with a band-limited window function.

DEDICATION

This thesis marks a pivotal point in my life, and it therefore stands as a monument to all the years that have led up to this moment. It is my family with whom I have shared these years and all experiences – both joyous and trying – therein. I dedicate this thesis to them.

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1 INTRODUCTION AND BACKGROUND

A frame is a possibly uncountable set of vectors in a Hilbert space that generalizes the notion of a basis. In particular, the elements of a frame are not required to be linearly independent even if the frame is countable. Nevertheless, a frame provides a sufficient condition for the reconstruction of a vector given its projections on the frame elements. The reconstruction is performed by a dual frame, which is analogous to a dual basis. If a frame acts as its own dual, then it is called a Parseval frame. A Parseval frame is therefore a generalization of an orthonormal basis [1].

Frames have important applications to machine learning. Every frame on a finite-dimensional Hilbert space can be viewed as a matrix whose columns are the frame elements. Parseval frames on a finite-dimensional Hilbert space are characterized by the singular value decompositions of their matrix representations; all singular values of any such Parseval frame are 1 [2]. But such a singular value decomposition can be understood as a feed-forward neural network with a linear activation function. Parseval frames are therefore examples of linear neural networks and hence provide a starting point for neural networks with more general activation functions.

A second application of frames to machine learning is founded on a deep connection between frames and what are called reproducing kernel Hilbert spaces. A reproducing kernel Hilbert space is a Hilbert space of functions such that any function can be evaluated (or reproduced) by integrating it against a certain kernel function. The reproducing property of such kernel functions is closely related to the reconstruction property of frames [1]. Because frames are more general than bases, then frames can be used to construct a variety of kernel functions that are useful for kernel methodbased machine learning algorithms such as support vector machines [9].

The flexibility and generality of frames comes at the price of structure and tractability. In particular, there are more frames on a Hilbert space than there are bases. For example, all orthonormal bases in a Hilbert space are connected by unitary transformations. In contrast, it is possible to have two Parseval frames in a Hilbert space that are not connected by any linear transformation at all. It seems that frames are instead connected by nonlinear transformations that are not yet fully understood. Much effort has been devoted to discovering ways to obtain new frames from old ones. For example, a square-integrable perturbation of a frame results in another frame [3]. A second example is the discretization of a frame, by which we mean the extraction of a countable "subframe" from an uncountable frame. Frame discretization is of course important for computational applications [1]. Both frame perturbation and frame discretization are processes that map frames to frames nonlinearly.

One way to make frames more tractable is to equip them with additional structure. A frame that is generated by a transversal of a square-integrable unitary irreducible representation of a group is called a frame of coherent states [1]. A prime example of a frame of coherent states is the frame of Gabor wavelets or the Gabor frame. The Gabor frame is the collection of all translations and phase modulations of some window function such as the Gaussian. The Gabor frame is therefore intimately related with the Fourier transform and is thus rich with structure [1, 6]. Owing to its structure, the Gabor frame has under certain conditions been successfully discretized [4]. Deformations of such discrete Gabor frames have also been studied. For example, the continuous Gabor frame is indexed by a symplectic phase space, and it has been shown that symplectomorphisms on the indexing space correspond to unitary transformations that map discrete Gabor frames to new frames [5]. It has also been shown that homotopic deformations of the window function can lead to deformations of discrete Gabor frames [7].

While there are examples of nonlinear mappings from frames to frames, we believe the exact structure that connects all frames in a Hilbert space has never been revealed explicitly. All orthonormal bases in a Hilbert space are connected by the structure of unitary transformations. What is the analogous structure connecting frames? In different terms, what structure describes the nonlinear transformations that map frames to frames? We believe knowledge of this structure is important because it could lead to new examples of frames and could also provide new insight into examples of frame deformations already known. For example, the discretization of the Gabor frame given by [4] is a bottom-up construction that makes no direct reference to the continuous Gabor frame. In other words, discretization is viewed as a constructive procedure and not as a true frame deformation. A deeper understanding of the transformations connecting all frames could provide a context for viewing discretization as an actual transformation of frames.

In this thesis, we present a top-down approach to frames. We believe that the key is the correspondence between frames and reproducing kernel Hilbert spaces. We show that there is an accompanying correspondence between nonlinear deformations of frames and linear maps between reproducing kernel Hilbert spaces. In particular, we show that all Parseval frames in a Hilbert space are connected by transformations that are unitary between reproducing kernel Hilbert spaces. We therefore establish the structure that connects all frames on a Hilbert space – namely, transformations that are linear in a larger space. We also provide conditions under which a linear transformation between reproducing kernel Hilbert spaces may be pulled back directly to a deformation of frames.

The thesis is organized as follows: In the remainder of Chapter 1, we provide detailed background that is necessary for later chapters. In Chapter 2, we show that the space of all frames on a Hilbert space indexed by a given measure space is fibrated into orbits under the action of invertible linear transformations and that a transversal of this orbit space is a set of nonlinearly connected Parseval frames (Theorem 2.11). Furthermore, the orbit space of frames has under certain conditions the structure of a principle fiber bundle whose base space is a maximal set of unitarily inequivalent Parseval frames (Theorem 2.22). The upshot is that the study of nonlinear frame deformations is reduced to Parseval frames. In Chapter 3, we establish the correspondence between deformations of Parseval frames and unitary transformations of reproducing kernel Hilbert spaces, thereby explaining the connection of all frames on a Hilbert space (Corollary 3.5 and Theorem 3.6). We finish Chapter 3 with two examples. In the first example, we construct a base space for the fiber bundle of frames on a finite-dimensional Hilbert space (Proposition 3.11). In the second example, we discretize a Gabor frame with a band-limited window function (Propositions 3.13-3.14). We take a top-down approach to the discretization of the Gabor frame by directly applying a sampling operator and invoking the Petersen-Middleton Sampling Theorem; we therefore view discretization as a frame deformation. Finally, in Chapter 4, we discuss some possible directions for future work.

1.1 Frames

We start by stating the definition and basic properties of frames. For details on Sections 1.1-1.2, see [1].

For the entire thesis, let $(H, \langle\cdot,\cdot\rangle_H)$ be a complex Hilbert space with the inner product linear in the first argument. Let X be a locally compact space with positive Borel measure μ.

Definition 1.1. *A map $f : X \mapsto H$ is a* frame *on H if there exist real constants $0 < a \leq b$ such that for all $\phi \in H$, we have*

$$a\|\phi\|_H^2 \leq \int_X |\langle \phi, f(x)\rangle_H|^2 \, d\mu(x) \leq b\|\phi\|_H^2. \qquad \text{(Frame Condition)}$$

The constants a and b are called frame bounds *of f. If $a = b = 1$, then f is called a* Parseval frame.

Given a frame $f : X \mapsto H$, the set $f(X)$ is a set of vectors in H indexed by the space X. Note that $f(X)$ is not required to be linearly independent even if X is countable. Note also that $f : X \mapsto H$ is not required to be injective.

The frame condition is better understood in terms of the operators that describe the decomposition and reconstruction of vectors with respect to a frame. These operators are introduced in the following proposition. Let $(L^2(X), \langle\cdot,\cdot\rangle_2)$ be the Hilbert space of all square-integrable functions mapping X to $\mathbb{C}$.

Proposition 1.2. *Let $f : X \mapsto H$ be a frame. The map*

$$V : H \mapsto L^2(X), \quad (V\phi)(x) = \langle \phi, f(x)\rangle_H$$

is a bounded linear injection whose inverse on $V(H)$ is bounded as well. Furthermore, the adjoint of V is given by

$$V^* : L^2(X) \mapsto H, \quad V^*\alpha = \int_X \alpha(x) f(x)\, d\mu(x),$$

which is a bounded linear surjection.

The maps V and V^* are respectively called the *analysis map* and *synthesis map* associated to the frame f. The integral in the definition of V^* is defined to converge in the weak sense, by which we mean that for all $\phi \in H$, we have

$$\langle V^*\alpha, \phi\rangle_H = \int_X \alpha(x) \langle f(x), \phi\rangle_H \; d\mu(x).$$

The analysis map V describes the decomposition of a vector $\phi \in H$ by mapping ϕ to a function that gives the projections of ϕ on the frame elements of f. The synthesis map V^* describes the construction of a vector in H from a given function of projections on the frame elements of f. In general, V^* is not one-one, meaning that the representation of a vector in H in the frame f is not unique. Moreover, it is in general not true that $V^*V\phi = \phi$. On the other hand, the key property of V is that it has an inverse V^{-1} defined on the range $V(H)$, and it is V^{-1} that can be used to reconstruct a vector given its projections. It turns out that the expression for V^{-1} requires the understanding of the operator V^*V, which is defined in the following proposition.

Proposition 1.3. *Let $f : X \mapsto H$ be a frame. The map*

$$S : H \mapsto H, \quad S\phi = V^*V\phi = \int_X \langle \phi, f(x)\rangle_H \, f(x)\, d\mu(x)$$

is a positive self-adjoint bounded linear bijection with a bounded inverse.

The map S is called the *frame operator* associated to the frame f. The frame condition can be written in terms of the frame operator as

$$a\|\phi\|_H^2 \leq \langle S\phi, \phi\rangle_H \leq b\|\phi\|_H^2.$$

Thus, $a \leq \frac{1}{\|S^{-1}\|_H}$ and $b \geq \|S\|_H$. It follows that a frame is Parseval if and only if its frame operator is the identity operator. More generally, the importance of the frame operator is better understood after we state a final proposition that tells us that the frame condition is sufficient for the reconstruction of a vector given its projections on a frame.

Proposition 1.4. *Let $f : X \mapsto H$ be a frame. Then, there exists a frame $\tilde{f} : X \mapsto H$ such that for all $\phi \in H$, we have*

$$\phi = \int_X \langle \phi, f(x)\rangle_H \tilde{f}(x)\, d\mu(x). \qquad \text{(Reconstruction Property)}$$

Any such $\tilde{f}$ is called a dual frame *of f. Moreover, if S is the frame operator of f, then $\tilde{f}(x) = S^{-1}f(x)$ is a dual frame of f.*

In general, the dual frame of a frame f is not unique. The dual frame $\tilde{f}(x) = S^{-1}f(x)$ is the canonical choice for the dual frame of f. The canonical dual frame is related to the observation that

$$S^{-1}S = S^{-1}V^*V = I,$$

where I is the identity operator, and hence the left inverse of V is given by

$$V^{-1} = S^{-1}V^*.$$

Also, observe that if f is Parseval, then S is the identity operator so that the frame f can act as its own dual frame.

The above discussion of frames is sufficient for us to proceed. In Chapter 2, we develop further properties of frames in the context of fiber bundles.

1.2 Reproducing Kernel Hilbert Spaces

The key property of frames is reconstruction. The reconstruction of vectors in a Hilbert space is also the defining theme of what are called "reproducing kernel Hilbert spaces". In this section, let $(R, \langle \cdot, \cdot \rangle_R)$ be a Hilbert space of functions mapping X to $\mathbb{C}$.

Definition 1.5. *The space R is called a* reproducing kernel Hilbert space *(RK Hilbert space) if for every $x \in X$, the evaluation functional $L_x : R \mapsto \mathbb{C}$ given by $L_x \alpha = \alpha(x)$ is continuous.*

The Riesz Representation Theorem immediately implies that the action of an evaluation functional L_x can be given by taking an inner product with a unique vector in R [10]. This leads to a more useful characterization of an RK Hilbert space as given in the following proposition.

Proposition 1.6. *Let R be an RK Hilbert space. Then, for every $x \in X$, there exists a unique vector $k_x \in R$ such that for every $\alpha \in R$, we have*

$$\alpha(x) = \langle \alpha, k_x \rangle_R \, . \qquad \text{(Reproducing Property)}$$

Moreover, the function $K : X \times X \mapsto \mathbb{C}$ given by

$$K(x, y) = \langle k_y, k_x \rangle_R$$

satisfies the property that for all $x, y \in X$,

$$K(x,y) = k_y(x) = \overline{k_x(y)} = \overline{K(y,x)}. \qquad \text{(Conjugate Symmetry)}$$

Proof. The existence and uniqueness of the vectors k_x follows directly from applying the Riesz Representation Theorem to the continuous evaluation functionals on R. To prove conjugate symmetry, we use the reproducing property and obtain

$$\begin{aligned} k_y(x) &= \langle k_y, k_x \rangle_R \\ &= \overline{\langle k_x, k_y \rangle_R} \\ &= \overline{k_x(y)}, \end{aligned}$$

and using the definition $K(x,y) = \langle k_y, k_x \rangle_R$, we have

$$k_y(x) = K(x,y) = \overline{K(y,x)} = \overline{k_x(y)}.$$

□

The vectors in the collection $\{k_x : x \in X\}$ are called the *coherent states* associated to the RK Hilbert space R. The coherent states are unique. Given a Hilbert space R, suppose that we are able to find a collection of vectors $\{k_x \in R : x \in X\}$ such that

$$\alpha(x) = \langle \alpha, k_x \rangle_R \,,$$

for all $\alpha \in R$ and $x \in X$. The continuity of the inner product then implies that all evaluation functionals on R are continuous and thus that R is an RK Hilbert space. Moreover, the uniqueness of coherent states implies that the vectors in $\{k_x \in R : x \in X\}$ are precisely the coherent states associated to R. The upshot is that in order to show that a Hilbert space is an RK Hilbert space, it is enough to find a set of coherent states in R that satisfy the reproducing property.

The function K is called the *reproducing kernel* associated to the RK Hilbert space R. The function K is so named due to its role in what is arguably the most important class of examples of RK Hilbert spaces: Suppose that R is a closed subspace of $L^2(X)$; that is, $\langle\cdot,\cdot\rangle_R = \langle\cdot,\cdot\rangle_2$. Letting $\alpha \in R$ and using conjugate symmetry, the reproducing property takes the form

$$\begin{aligned}\alpha(x) &= \langle\alpha, k_x\rangle_2 \\ &= \int_X \alpha(y)\overline{k_x(y)}\,d\mu(y) \\ &= \int_X \alpha(y)k_y(x)\,d\mu(y) \\ &= \int_X K(x,y)\alpha(y)\,d\mu(y).\end{aligned}$$

In words, integration of a vector $\alpha \in R$ against the reproducing kernel K returns or "reproduces" the vector α. The function K is also *positive semidefinite*, by which we mean that for all $\alpha \in R$, we have

$$\begin{aligned}\int_X\int_X K(x,y)\overline{\alpha(x)}\alpha(y)\,d\mu(y)\,d\mu(x) &= \int_X \overline{\alpha(x)}\alpha(x)\,d\mu(x) \\ &= \|\alpha\|_2^2 \geq 0,\end{aligned}$$

where we used the reproducing property in the variable y. A final property of K to mention is that K is *square-integrable*, by which we mean that for all $x, y \in X$, we have

$$\int_X K(x,z)K(z,y)\,d\mu(z) = \int_X k_z(x)k_y(z)\,d\mu(z)$$
$$= \int_X k_y(z)\overline{k_x(z)}\,d\mu(z)$$
$$= \langle k_y, k_x\rangle_2$$
$$= K(x,y).$$

The square integrability property implies that the integral operator $\int_X d\mu(y)\,K(\cdot,y)$ is an orthogonal projection that maps $L^2(X)$ onto R. It follows that $\int_X d\mu(y)\,K(\cdot,y)$ reduces to the identity operator on R, which is simply the reproducing property.

The common theme of reconstruction implies a fundamental connection between frames and RK Hilbert spaces. This connection is realized concretely by the class of RK Hilbert spaces that are subspaces of $L^2(X)$. The following two propositions are examples of how frames and RK Hilbert spaces connect to one another.

Proposition 1.7. *Let $f : X \mapsto H$ be a frame on H with frame operator S and analysis map $V : H \mapsto L^2(X)$. Then, the space defined by*

$$R = \operatorname{ran}(V) = \{\langle \phi, f(\cdot)\rangle_H : \phi \in H\} \subseteq L^2(X)$$

is an RK Hilbert space with reproducing kernel $K : X \times X \mapsto \mathbb{C}$ given by

$$K(x,y) = \left\langle S^{-1}f(y), f(x)\right\rangle_H .$$

Moreover, the map $w : H \mapsto R$ defined by $w = VS^{-\frac{1}{2}}$ is an isometry.

Proof. Since f is a frame, then $V : H \mapsto L^2(X)$ and $V^{-1} : R \mapsto H$ are both continuous and hence uniformly continuous. Since H is a Hilbert space, then it follows that R is a Hilbert space as well. To show that R is an RK Hilbert space, we need only show that K satisfies the reproducing property on R. Let $\alpha \in R$. Then, $\alpha = \langle \phi, f(\cdot) \rangle_H$ for some $\phi \in H$. We have

$$\begin{aligned}\int_X K(x,y)\alpha(y)\, d\mu(y) &= \int_X \left\langle S^{-1} f(y), f(x) \right\rangle_H \langle \phi, f(y) \rangle_H \, d\mu(y) \\ &= \left\langle \int_X \langle \phi, f(y) \rangle_H S^{-1} f(y)\, d\mu(y), f(x) \right\rangle_H .\end{aligned}$$

By the reconstruction property of frames, this becomes

$$\begin{aligned}\int_X K(x,y)\alpha(y)\, d\mu(y) &= \langle \phi, f(x) \rangle_H \\ &= \alpha(x).\end{aligned}$$

Therefore, K is a reproducing kernel on R.

Since $S^{-\frac{1}{2}}$ is a bijection on H and V is invertible on its range R, then $w = VS^{-\frac{1}{2}}$ is a bijection. Letting $\phi, \psi \in H$, we have

$$\begin{aligned}\langle w\phi, w\psi \rangle_2 &= \left\langle VS^{-\frac{1}{2}}\phi, VS^{-\frac{1}{2}}\psi \right\rangle_2 \\ &= \left\langle S^{-\frac{1}{2}} V^* V S^{-\frac{1}{2}} \phi, \psi \right\rangle_H \\ &= \left\langle S^{-\frac{1}{2}} S S^{-\frac{1}{2}} \phi, \psi \right\rangle_H \\ &= \langle \phi, \psi \rangle_H ,\end{aligned}$$

where we used the definition $S = V^*V$. Thus, w is an isometry. □

Therefore, to every frame is associated a reproducing kernel Hilbert space. The second connection between frames and RK Hilbert spaces is given by the following proposition.

Proposition 1.8. *Let $R \subseteq L^2(X)$ be an RK Hilbert space with associated coherent states $\{k_x : x \in X\}$. Then, the coherent states form a Parseval frame on R. That is, the map $x \mapsto k_x$ is a Parseval frame on R.*

Proof. Let $\alpha \in R$. We verify the frame condition directly: By the reproducing property,

$$\int_X |\langle \alpha, k_x \rangle_H|^2 \, d\mu(x) = \int_X |\alpha(x)|^2 \, d\mu(x)$$
$$= \|\alpha\|_2^2.$$

Therefore, the claim holds. □

In particular, the coherent states on the RK Hilbert space R associated to a frame $f : X \mapsto H$ form a Parseval frame. Using the inverse of the isometry w defined in Proposition 1.7, these coherent states can be pulled back to the Parseval frame $S^{-\frac{1}{2}} f(\cdot)$ on H, where S is the frame operator of f.

1.3 Example: Finite Frames

An important class of examples of frames is finite frames. A frame $f : X \mapsto H$ is said to be finite if the set $f(X)$ is finite. Finite frames are characterized by the following theorem.

Theorem 1.9. *Every finite spanning set on a finite-dimensional Hilbert space is a finite frame [2].*

Consider the finite-dimensional space $\mathbb{C}^N$. Let $f : \{1, \ldots, M\} \mapsto \mathbb{C}^N$ be a finite frame on $\mathbb{C}^n$. Note that we necessarily have $M \geq N$. We use the notation $f_m = f(m)$.

We think of the frame elements f_m as column vectors each with N components. The space $L^2(\{1,\ldots,M\})$ is simply $\mathbb{C}^M$. Therefore, the analysis map of f is $V:\mathbb{C}^N \mapsto \mathbb{C}^M$ whose matrix representation is

$$V = \begin{bmatrix} f_1^* \\ \vdots \\ f_M^* \end{bmatrix},$$

where f_m^* is the Hermitian transpose of f_m. Letting $\phi \in \mathbb{C}^N$, we have

$$V\phi = \begin{bmatrix} f_1^* \\ \vdots \\ f_M^* \end{bmatrix} \phi = \begin{bmatrix} f_1^*\phi \\ \vdots \\ f_M^*\phi \end{bmatrix},$$

where $f_m^*\phi$ is the product of a row vector with a column vector. This is consistent with the definition of analysis map given to be

$$(V\phi)(m) = \langle \phi, f_m \rangle_= f_m^*\phi.$$

Now that the analysis map V is given as a matrix, it is then straightforward to construct the synthesis map and frame operator. The synthesis map $V^*:\mathbb{C}^M \mapsto \mathbb{C}^N$ is given by

$$V^* = \begin{bmatrix} f_1^* \\ \vdots \\ f_M^* \end{bmatrix}^* = \begin{bmatrix} f_1 & \ldots & f_M \end{bmatrix},$$

and the frame operator $S:\mathbb{C}^N \mapsto \mathbb{C}^N$ is given by

$$S = V^*V = \begin{bmatrix} f_1 & \ldots & f_M \end{bmatrix} \begin{bmatrix} f_1^* \\ \vdots \\ f_M^* \end{bmatrix} = \sum_{m=1}^{M} f_m f_m^*,$$

where $f_m f_m^*$ is an $N \times N$ matrix for each m. Continuing in this way, the frame operator can be inverted as a matrix, and the inverted frame operator S^{-1} can then be used to construct a dual frame $\tilde{f}_m = S^{-1} f_m$, and so on.

The RK Hilbert space associated to f is a subspace $R \subseteq \mathbb{C}^M$ that is the range of V; that is, R is the column space of the matrix V. The reproducing kernel on R is a map $K : \{1, \ldots, M\} \times \{1, \ldots, M\} \mapsto \mathbb{C}$ given by

$$K_{mn} = K(m,n) = \left\langle S^{-1} f_n, f_m \right\rangle = f_m^* S^{-1} f_n.$$

The kernel K can therefore be viewed as a matrix $K : \mathbb{C}^M \mapsto \mathbb{C}^M$ given by

$$K = \begin{bmatrix} f_1^* \\ \vdots \\ f_M^* \end{bmatrix} S^{-1} \begin{bmatrix} f_1 & \ldots & f_M \end{bmatrix} = V S^{-1} V^*.$$

An important interpretation of finite frames is given by the "singular value decomposition". Recall that V^* is an $N \times M$ matrix with $M \geq N$ and the frame elements f_m as its columns. The *singular value decomposition* of V^* is a factorization

$$V^* = U \begin{bmatrix} \Sigma & 0 \end{bmatrix} \tilde{U}^*,$$

where U is an $N \times N$ unitary matrix, $\tilde{U}$ is an $M \times M$ unitary matrix, and $\begin{bmatrix} \Sigma & 0 \end{bmatrix}$ is an $N \times M$ matrix with $\Sigma = \operatorname{diag}(s_1, \ldots, s_N)$ and $s_i \geq 0$. The non-negative numbers s_i are called the *singular values* of V^*. Assuming $s_1 \geq s_2 \geq \ldots \geq s_N$, the matrix Σ is unique. The unitary matrices U and $\tilde{U}$, however, are not unique. By the *Singular Value Decomposition Theorem*, every matrix such as V^* has a singular value decomposition. We now have the following proposition.

Proposition 1.10. *The map $f : \{1, \ldots, M\} \mapsto \mathbb{C}^N$ is a frame on $\mathbb{C}^N$ if and only if all singular values s_i of the synthesis map V^* are positive. Moreover, if f is a frame, then f has frame bounds s_N^2 and s_1^2 [2].*

Proof. Using the singular alue decomposition of V^* as given above, we have

$$\begin{aligned} S &= V^*V \\ &= U \begin{bmatrix} \Sigma & 0 \end{bmatrix} \tilde{U}^* \left(U \begin{bmatrix} \Sigma & 0 \end{bmatrix} \tilde{U}^* \right) \\ &= U \begin{bmatrix} \Sigma & 0 \end{bmatrix} \tilde{U}^* \tilde{U} \begin{bmatrix} \Sigma^* \\ 0 \end{bmatrix} U^* \\ &= U\Sigma\Sigma^* U^*. \end{aligned}$$

Since Σ is a square diagonal matrix with real entries, then

$$S = U\Sigma^2 U^*.$$

Let $\phi \in H$ and $\psi = U^*\phi$. We have

$$\begin{aligned} \langle S\phi, \phi \rangle &= \phi^* U \Sigma^2 U^* \phi \\ &= \psi^* \Sigma^2 \psi. \end{aligned}$$

Since Σ^2 is diagonal and $s_1^2 \geq \ldots \geq S_N^2$, then it follows that

$$s_M^2 \|\psi\|^2 \leq \langle S\phi, \phi \rangle \leq s_1^2 \|\psi\|^2.$$

But since $\psi = U\phi$ with U a unitary matrix, then

$$s_N^2 \|\phi\|^2 \leq \langle S\phi, \phi \rangle \leq s_1^2 \|\phi\|^2.$$

Observe that f is a frame if and only if $s_N^2 > 0$, in which case f has frame bounds S_N^2 and s_1^2. $\square$

The map f is therefore a Parseval frame if and only if $s_1^2 = s_N^2 = 1$. Moreover, since $s_1 \geq \ldots \geq s_N \geq 0$, then we have the following corollary.

Corollary 1.11. *The map $f : \{1, \ldots, M\} \mapsto \mathbb{C}^N$ is a Parseval frame on $\mathbb{C}^N$ if and only if all singular values of the synthesis map V^* are 1; i.e., if $\Sigma = I$.*

The singular value decomposition of the synthesis map V^* associated to a frame $f : \{1, \ldots, M\} \mapsto \mathbb{C}^N$ provides information not only about the particular frame f but more generally about the set of all frames on $\mathbb{C}^N$ indexed by $\{1, \ldots, M\}$. In particular, the unitary matrices U and $\tilde{U}$ in the singular value decomposition tell us how two frames on $\mathbb{C}^N$ are "connected" to each other (i.e., what transformation maps one frame onto the other). We complete this line of thought in Section 3.3 after we develop the context for studying the set of all frames on a Hilbert space in Sections 2-3.

1.4 The Gabor Frame and the Frame Discretization Problem

An example of a continuously indexed frame is the Gabor frame on the Hilbert space $H = L^2(\mathbb{R})$. The Gabor frame is introduced in the following proposition.

Proposition 1.12. *The map $f : \mathbb{R}^2 \mapsto L^2(\mathbb{R})$ defined by*

$$[f(q, p)](x) = e^{i2\pi px}\psi(x - q)$$

is a Parseval frame for all $\psi \in L^2(\mathbb{R})$ [6].

The map f in Proposition 1.12 is a *Gabor frame*, and the function ψ is called the *window function* of the Gabor frame. The frame elements of f are sometimes called Gabor wavelets. The Gabor frame is generated by translations and phase modulations of the window function, and we therefore expect a fundamental connection between the Gabor frame and the Fourier transform. The analysis map of the Gabor frame is given by

$$\begin{aligned} V\phi &= \langle \phi, f(q,p) \rangle_H \\ &= \int_{\mathbb{R}} \phi(x) e^{-i2\pi px} \overline{\psi(x)}\, dx. \end{aligned}$$

If ψ is a localizing function such as a Gaussian, then V is a "windowed" Fourier transform; i.e., V gives the spectrum of "frequencies" p of ϕ that occur at a "time" near q.

There is considerable interest in the search and construction of discrete Gabor frames. A *discrete Gabor frame* is a discretely indexed subcollection of Gabor wavelets that is itself a frame on $L^2(\mathbb{R})$. The search for discrete Gabor frames is a subset of the more general *frame discretization problem*, which poses the following question: Given a frame $f : X \mapsto H$, can we find a discrete subset of $f(X)$ that is itself a frame on H?

An example of a discrete Gabor frame is a map $g : \mathbb{Z}^2 \mapsto L^2(\mathbb{R})$ of the form

$$g(n,m) = e^{i2\pi mx}\psi(x-n),$$

for an appropriately chosen window function ψ [4]. In Section 3.4, we provide an example of a window function ψ for which g is in fact a discrete Gabor frame. As a final note, observe that because

$$\sum_{n,m\in\mathbb{Z}} |\langle \phi, g(n,m)\rangle_H|^2 = \int_{\mathbb{R}^2} |\langle \phi, g(\lfloor q \rfloor, \lfloor p \rfloor)\rangle_H|^2\, dq\, dp,$$

then $g : \mathbb{Z}^2 \mapsto L^2(\mathbb{R})$ as defined above is a frame if and only if $\tilde{g} : \mathbb{R}^2 \mapsto L^2(\mathbb{R})$ defined by

$$\tilde{g}(q,p) = g(\lfloor q \rfloor, \lfloor p \rfloor) = e^{i2\pi \lfloor p \rfloor x}\psi(x - \lfloor q \rfloor)$$

is a frame as well. The map $\tilde{g}$ has the advantage that it is continuously indexed by the same space that indexes the continuous Gabor frame f. For this reason, we prefer to use the function $\tilde{g}$ as opposed to g in Section 3.4.

1.5 The Fourier Transform and Sampling

The richness of the Gabor frame is due in part to its relation with the Fourier transform. Under certain assumptions, the Fourier transform can be used to sample a function such that the original function can be recovered from the sample. In Section 3.4, we apply this idea to obtain a discretization of the Gabor frame under certain conditions. In this section, we establish some background that is necessary in Section 3.4.

We define the *Fourier transform* to be the map $\mathcal{F} : L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \mapsto L^2(\mathbb{R}^n)$ given by

$$\hat{f}(\hat{x_1}, \ldots, \hat{x_n}) = \mathcal{F}(f)(\hat{x_1}, \ldots, \hat{x_n}) = \int_{\mathbb{R}^n} f(x_1, \ldots, x_n) e^{-i2\pi(\hat{x_1}x_1 + \ldots + \hat{x_n}x_n)}\, dx.$$

The support of $\hat{f}$ is called the *Fourier spectrum* of f. If the Fourier spectrum of f is compact, then we say that f is *band-limited.*

The following theorem establishes an important property of the Fourier transform.

Theorem 1.13 (Plancheral's Theorem)**.** *The Fourier transform* $\mathcal{F} : L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \mapsto L^2(\mathbb{R}^n)$ *is unitary [12].*

An immediate corollary to Plancheral's Theorem is that since $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, then the Fourier transform can be extended uniquely to a unitary operator $\mathcal{F} : L^2(\mathbb{R}^n) \mapsto L^2(\mathbb{R}^n)$.

A concept related to the Fourier transform is the Fourier series of a periodic or compactly supported function. Let $e_0 \in \mathbb{R}^n$ and $\{e_i\}_{i=1}^n$ be the standard orthonormal basis on $\mathbb{R}^n$, and define the rectangular lattice

$$\Omega = \{e_0 + a_1(c_1e_1) + \ldots + a_n(c_ne_n) \in \mathbb{R}^n : a_1, \ldots, a_n \in \mathbb{Z}^n\},$$

where $c_1, \ldots, c_n > 0$ are fixed scalars giving the dimensions of one cell of the lattice. Let C be a rectangular cell of the lattice Ω. We define the *Fourier coefficient operator* as the map $\mathcal{F}_C : L^2(C) \mapsto L^2(\mathbb{Z}^n)$ given by

$$\mathcal{F}_C(f)(m_1, \ldots, m_n) = \frac{1}{\|C\|} \int_C f(x_1, \ldots, x_n) e^{i2\pi\left(\frac{m_1x_1}{c_1} + \ldots + \frac{m_nx_n}{c_n}\right)}\, dx,$$

where $\|C\| = c_1 \ldots c_n$ is the volume of the cell C. The Fourier coefficients of f can be used to construct a periodization f_P of f over the lattice Ω that is given by the *Fourier series*

$$f_P(x_1, \ldots, x_n) = \sum_{m_1, \ldots, m_n \in \mathbb{Z}} \mathcal{F}_C(m_1, \ldots, m_n) e^{-i2\pi\left(\frac{m_1x_1}{c_1} + \ldots + \frac{m_nx_n}{c_n}\right)}.$$

It follows that two functions with equal Fourier coefficients differ only by some periodic translation; if $f_1 \in L^2(C_1) \subset L^2(\mathbb{R}^n)$ and $f_2 \in L^2(C_2) \subset L^2(\mathbb{R}^n)$ where C_1 and C_2 are two cells in the lattice Ω, then $\mathcal{F}_{C_1}(f_1) = \mathcal{F}_{C_2}(f_2)$ implies that $f_1 = Tf_2$, where $T : L^2(C_2) \mapsto L^2(C_1)$ is a translation operator along the lattice Ω.

The following theorem gives a property of the Fourier coefficient operator that is analogous to Plancheral's Theorem.

Theorem 1.14 (Parseval's Theorem)**.** *The Fourier coefficient operator $F_C : L^2(C) \mapsto L^2(\mathbb{Z}^n)$ on a cell C of the lattice Ω is unitary.*

We are now in a position to state the theorem that allows us to sample or discretize band-limited functions in a lossless way.

Theorem 1.15 (Petersen-Middleton Sampling Theorem)**.** *Let $f \in L^2(\mathbb{R}^n)$ be a band-limited function whose Fourier spectrum is a cell C of the lattice Ω defined above. Then, the operator $Z : L^2(\mathbb{R}^n) \mapsto L^2(\mathbb{Z}^n)$ defined by*

$$(Zf)(m_1, \ldots, m_n) = \frac{1}{\|C\|} f\left(\frac{m_1}{c_1}, \ldots, \frac{m_n}{c_n}\right)$$

is also given by $Z = \mathcal{F}_C \circ \mathcal{F}$. Moreover, Z is unitary [8].

The map Z samples the function f with a frequency of c_i in the direction of e_i. The map Z is therefore called a *sampling operator*. Since Z is unitary, then in particular it is invertible. Therefore, it is possible to reconstruct f from its sample. If the Fourier spectrum of f is a unit cube so that $c_i = 1$ for all i, then we simply have

$$(Zf)(m_1, \ldots, m_n) = f(m_1, \ldots, m_n).$$

In Section 3.4, we define the sampling operator somewhat differently so that $Z : L^2(\mathbb{R}^n) \mapsto L^2(\mathbb{R}^n)$ and

$$(Zf)(x_1, \ldots, x_n) = f(\lfloor x_1 \rfloor, \ldots, \lfloor x_n \rfloor).$$

We do this so that f and Zf have the same domain. It is easy to check, however, that the modified definition of Z is equivalent to the original definition.

2 LINEAR DEFORMATIONS OF FRAMES

In this section, we consider the set of all frames on the Hilbert space H. In Section 2.1, we show that this set may be fibrated into orbits under the action of linear deformations. We also show that every frame may be linearly deformed or "projected" to a Parseval frame, just as every basis may be linearly deformed into an orthonormal basis. In Section 2.2, we extend the orbit structure to a fiber structure and conclude that under certain conditions the set of all frames on H is a principal fiber bundle. Our purpose is to establish a basic structure that will provide context for future sections.

2.1 The Orbit Space of Frames

Let $\mathrm{GL}(H)$ be the group of all invertible bounded linear operators on H with bounded inverse. Let $\mathrm{GL}^+(H) \subset \mathrm{GL}(H)$ be the cone of all positive operators in $\mathrm{GL}(H)$. We would like to establish some properties about $\mathrm{GL}^+(H)$ as it relates to $\mathrm{GL}(H)$. In particular, we establish the concept of the polar decomposition of an operator in a way suitable to a fiber bundle context.

First, we need a definition.

Definition 2.1. *For every $A \in \mathrm{GL}(H)$, define the map $\mathrm{ad}_A : \mathrm{GL}(H) \mapsto \mathrm{GL}(H)$ by $\mathrm{ad}_A(B) = ABA^*$. We say that $\mathrm{ad}_A(B)$ is the* adjugation *of B by A.*

Define the relation $\sim$ on $\mathrm{GL}(H)$ by $B \sim B'$ if and only if $\mathrm{ad}_A(B) = B'$ for some $A \in \mathrm{GL}(H)$.

Proposition 2.2. *The relation $\sim$ is an equivalence relation.*

Proof. Let $B \in \mathrm{GL}(H)$. Clearly, $\mathrm{ad}_I(B) = IBI^* = B$, so that $B \sim B$. Thus, $\sim$ is reflexive. Given $B, B' \in \mathrm{GL}(H)$, suppose $B \sim B'$. That is, $ad_A(B) = ABA* = B'$ for some $A \in \mathrm{GL}(H)$. Then,

$$\mathrm{ad}_{A^{-1}}(B') = A^{-1}B'(A^{-1})^* = A^{-1}ABA^*(A^*)^{-1} = B.$$

Thus, $B' \sim B$, and hence $\sim$ is symmetric. Finally, suppose $B \sim B'$ and $B' \sim B''$. Thus, $\mathrm{ad}_A(B) = ABA^* = B'$ and $\mathrm{ad}_{A'}(B') = A'B'(A')^* = B''$ for some $A, A' \in \mathrm{GL}(H)$. Then,

$$\mathrm{ad}_{A'A}(B) = A'AB(A'A)^* = A'(ABA^*)(A')^* = A'B'(A')^* = B''.$$

Thus, $B \sim B''$, and hence $\sim$ is transitive. $\square$

The equivalence classes in $\mathrm{GL}(H)$ induced by $\sim$ are called *adjugacy classes*.

We now have the following result.

Proposition 2.3. *The space $GL^+(H)$ is an adjugacy class in* $\mathrm{GL}(H)$*.*

Proof. Let $A \in \mathrm{GL}(H)$ and $B \in \mathrm{GL}^+(H)$. For any $\phi \in H$, we have

$$\langle \mathrm{ad}_A(B)\phi, \phi\rangle_H = \langle ABA^*\phi, \phi\rangle_H = \langle BA^*\phi, A^*\phi\rangle_H \geq 0,$$

where the inequality holds since $A^*\phi \in H$ and B is positive. Thus, $\mathrm{ad}_A(B)$ is positive, so that $GL^+(H)$ is closed under adjugation. Let $B, C \in \mathrm{GL}^+(H)$. Since B, C are positive, then $B = SS^*$ and $C = TT^*$ for some $S, T \in \mathrm{GL}(H)$. There exists $A \in \mathrm{GL}(H)$ such that $T = AS$. We have

$$\mathrm{ad}_A(B) = ABA^* = ASS^*A^* = (AS)(AS)^* = TT^* = C.$$

Thus, $B \sim C$. Ergo, $\mathrm{GL}^+(H)$ is an adjugacy class. $\square$

Because frame operators are elements of $\mathrm{GL}^+(H)$, the above proposition will be useful in subsequent discussions of frame operators.

Next, we define the projection

$$\rho : \mathrm{GL}(H) \mapsto \mathrm{GL}^+(H), \quad \rho(A) = AA^*. \tag{1}$$

This projection is used in the proof of the following lemma to establish an important relationship between $\mathrm{GL}^+(H)$ and $\mathrm{GL}(H)$.

Proposition 2.4. *Define the action of the unitary group $U(H)$ on $GL(H)$ by right multiplication. Then, the orbit space $\mathrm{GL}(H)/U(H)$ is in one-one correspondence with $GL^+(H)$.*

Proof. Since every positive operator $B \in \mathrm{GL}^+(H)$ can be written in the form $B = AA^*$ with $A \in \mathrm{GL}(H)$, then ρ is surjective. Let $A \in \mathrm{GL}(H)$ and $U \in U(H)$. We have

$$\rho(AU) = (AU)(AU)^* = AUU^*A^* = AIA^* = AA^* = \rho(A).$$

In particular, we have

$$\begin{aligned} \ker(\rho) &= \{A \in \mathrm{GL}(H) : \rho(A) = I\} \\ &= \{A \in \mathrm{GL}(H) : AA^* = I\} \\ &= U(H). \end{aligned}$$

Since $\mathrm{GL}(H)/\ker(\rho)$ is in one-one correspondence with $\rho(\mathrm{GL}(H)) = \mathrm{GL}^+(H)$, the quotient space follows. $\square$

Therefore, $\mathrm{GL}(H) = \mathrm{GL}^+(H)U(H)$, which is to say that every operator in $\mathrm{GL}(H)$ can be factored into a positive operator in $\mathrm{GL}^+(H)$ and a unitary operator in $U(H)$. This is simply the polar decomposition of an operator.

The above discussion of $\mathrm{GL}(H)$ is important because we are interested in linear deformations of frames. We begin by introducing spaces of frames over an index set X. We first define the Banach space J to be

$$J = L^\infty(X, H) = \left\{ f : X \mapsto H : \sup_{x \in X} \|f(x)\|_H < \infty \right\}, \tag{2}$$

equipped with the norm

$$\|f\|_J = \sup_{x \in X} \|f(x)\|_H.$$

In addition, we define the spaces

$$F = \{f \in J : f \text{ is a frame}\} \tag{3}$$

$$F_0 = \{f \in F : f \text{ is Parseval}\}. \tag{4}$$

Note that F is restricted to frames whose frame elements have uniformly bounded norms. Moreover, since H is separable, it follows that F_0 is non-empty. That is, H has at least one Parseval frame.

The fibration of F will be given by the action of $\mathrm{GL}(H)$. The following result establishes that this action is continuous.

Lemma 2.5. *Let $A \in \mathrm{GL}(H)$ and $f \in J$ and define the action of A on f by*

$$(Af)(x) = A[f(x)].$$

Then $\mathrm{GL}(H) \subset \mathrm{GL}(J)$. That is, A and A^{-1} are bounded on J.

Proof. We have the operator norm

$$\|A\|_J = \sup_{f \in J, f \neq 0} \frac{\|Af\|_J}{\|f\|_J} = \sup_{f \in J, f \neq 0} \sup_{x \in X} \frac{\|Af(x)\|_H}{\|f\|_J}.$$

Since $A \in \mathrm{GL}(H)$, then we have

$$\begin{aligned}
\sup_{f\in J, f\neq 0} \sup_{x\in X} \frac{1}{\|A^{-1}\|_H} \frac{\|f(x)\|_H}{\|f\|_J} &\leq \|A\|_J \leq \sup_{f\in J, f\neq 0} \sup_{x\in X} \|A\|_H \frac{\|f(x)\|_H}{\|f\|_J} \\
\frac{1}{\|A^{-1}\|_H} \sup_{f\in J, f\neq 0} \sup_{x\in X} \frac{\|f(x)\|_H}{\|f\|_J} &\leq \|A\|_J \leq \|A\|_H \sup_{f\in J, f\neq 0} \sup_{x\in X} \frac{\|f(x)\|_H}{\|f\|_J} \\
\frac{1}{\|A^{-1}\|_H} \sup_{f\in J, f\neq 0} \frac{\|f\|_J}{\|f\|_J} &\leq \|A\|_J \leq \|A\|_H \sup_{f\in J, f\neq 0} \frac{\|f\|_J}{\|f\|_J} \\
\frac{1}{\|A^{-1}\|_H} &\leq \|A\|_J \leq \|A\|_H.
\end{aligned}$$

Therefore, $A \in \mathrm{GL}(J)$. □

Define the "frame operator map" $S : F \mapsto \mathrm{GL}^+(H)$ such that $S(f)$ is the frame operator of f. The following lemma is of central importance.

Lemma 2.6. *Let $f \in F$ and $A \in \mathrm{GL}(H)$. Then $Af \in F$ and the frame operator of Af is*

$$S(Af) = \mathrm{ad}_A(S(f)) = AS(f)A^*.$$

Proof. Let $\phi \in H$. Then, $A^*\phi \in H$. Since f is a frame, then

$$a\|A^*\phi\|_H^2 \leq \int_X |\langle A^*\phi, f(x)\rangle_H|^2 \, d\mu(x) \leq b\|A^*\phi\|_H^2.$$

Since $\langle A^*\phi, f(x)\rangle_H = \langle \phi, A[f(x)]\rangle_H = \langle \phi, (Af)(x)\rangle_H$, then

$$a\|A^*\phi\|_H^2 \leq \int_X |\langle \phi, (Af)(x)\rangle_H|^2 \, d\mu(x) \leq b\|A^*\phi\|_H^2.$$

Since $A \in \mathrm{GL}(H)$ and since A and A^* have the same norms, then we have

$$\frac{a}{\|A^{-1}\|_H^2}\|\phi\|_H^2 = a\left(\frac{\|\phi\|_H}{\|A^{-1}\|_H}\right)^2 \leq a\|A^*\phi\|_H^2.$$

We also have

$$b\|A^*\phi\|_H^2 \le b(\|A\|_H\|\phi\|_H)^2 = b\|A\|_H^2\|\phi\|_H^2.$$

Therefore, for all $\phi \in H$, we have

$$\frac{a}{\|A^{-1}\|_H^2}\|\phi\|_H^2 \le \int_X |\phi, (Af)(x)\rangle_H^2 \, d\mu(x) \le b\|A\|_H^2\|\phi\|_H^2.$$

Thus, Af is a frame.

The frame operator of Af is given by

$$\begin{aligned} S(Af)\phi &= \int_X \langle \phi, (Af)(x)\rangle_H (Af)(x)\, d\mu(x) \\ &= \int_X \langle \phi, A[f(x)]\rangle_H A[f(x)]\, d\mu(x) \\ &= \int_X A\langle A^*\phi, f(x)\rangle_H f(x)\, d\mu(x). \end{aligned}$$

Since A is bounded and hence uniformly continuous, then

$$\begin{aligned} S(Af)\phi &= A\int_X \langle A^*\phi, f(x)\rangle_H f(x)\, d\mu(x) \\ &= AS(f)A^*\phi. \end{aligned}$$

Therefore, $S(Af) = \mathrm{ad}_A(S)$. □

The set of frames F can therefore be fibrated into orbits under the action of $\mathrm{GL}(H)$. We let $F/\mathrm{GL}(H)$ denote the resulting space of orbits. Note that since H is a complex Hilbert space, the group $\mathrm{GL}(H)$ is topologically connected. As a consequence, the orbits in $F/\mathrm{GL}(H)$ are connected spaces in J.

Because all basis sets in H are connected by linear transformations, then exactly one orbit in $F/\mathrm{GL}(H)$ is the space of all basis sets in H. The elements of a frame in any other orbit are therefore necessarily linearly dependent.

By definition, the action of $\mathrm{GL}(H)$ on each orbit in $F/\mathrm{GL}(H)$ is transitive. But because the elements of each orbit are frames, the action has even more structure, as the following lemma illustrates.

Lemma 2.7. *Consider any $f \in F$ and $A \in \mathrm{GL}(H)$. Then $Af = f$ if and only if $A = I$. In other words, the action of $\mathrm{GL}(H)$ is regular on each orbit in $F/\mathrm{GL}(H)$.*

Proof. The reverse implication is trivial. For the forward implication, suppose $Af = f$. Recall Af is defined by $(Af)(x) = A[f(x)]$ for all $x \in X$. Thus, $Af = f$ implies $A[f(x)] = f(x)$ for all $x \in X$. But since f is a frame on H, then $\{f(x) : x \in X\}$ spans H. Since A is linear on H, then we have $A\phi = \phi$ for all $\phi \in H$. $\square$

Because the action of $\mathrm{GL}(H)$ is regular on each orbit in $F/\mathrm{GL}(H)$, then every orbit is a principal homogeneous space. Therefore, the linear transformation connecting two frames is unique.

Lemma 2.6 implies that the frame operator map $S : F \mapsto \mathrm{GL}^+(H)$ may be thought of as a projection map, as the following proposition states.

Proposition 2.8. *The map S is well-defined and surjective.*

Proof. The frame operator $S(f)$ of a frame f is positive, bounded, and has a bounded inverse. Hence, S is well-defined. Let $B \in \mathrm{GL}^+(H)$. Then, $B = AA^*$ for some $A \in \mathrm{GL}(H)$. Let $f_0 \in F_0$ be a Parseval frame, and define $f = Af_0$. By Lemma 2.6, f is a frame and

$$S(f) = S(Af_0) = \mathrm{ad}_A(S(f_0)) = \mathrm{ad}_A(I) = AA^* = B.$$

Ergo, S is surjective. $\square$

We are now interested in showing that every frame can be transformed into a Parseval frame. Define the projection

$$T : F \mapsto F_0, \quad T(f) = S(f)^{-\frac{1}{2}} f. \tag{5}$$

The following proposition verifies that T can indeed be thought of as a projection map.

Proposition 2.9. *The map T is well-defined and surjective.*

Proof. Let $f \in F$. By Lemma 2.6, observe that

$$S(T(f)) = S(S(f)^{-\frac{1}{2}} f) = S(f)^{-\frac{1}{2}} S(f) S(f)^{-\frac{1}{2}} = I.$$

Thus, $T(f) \in F_0$, and hence T is well-defined. Note T fixes F_0 pointwise: If $f \in F_0$, then $T(f) = I^{-\frac{1}{2}} f = f$. Thus, T is surjective. □

Therefore, every frame can be linearly transformed into a Parseval frame. But we would like this transformation to be unique. In particular, we would like to index the orbits in $F/\operatorname{GL}(H)$ by a set of Parseval frames. We must therefore determine how the Parseval frames in a common orbit in $F/\operatorname{GL}(H)$ are related. We recall that $F_0 \subset F$ is the space of Parseval frames and consider the action of $U(H)$ on F_0.

Lemma 2.10. *Let $f \in F_0$ and $A \in \operatorname{GL}(H)$. Then, $Af \in F_0$ if and only if $A \in U(H)$.*

Proof. First assume $Af \in F_0$. Then,

$$S(Af) = \operatorname{ad}_A(S(f)) = \operatorname{ad}_A(I) = AA^* = I.$$

Hence, $A \in U(H)$. For the converse, suppose $A \in U(H)$. Then, Af is a frame and

$$S(Af) = AA^* = I,$$

so that $Af \in F_0$. $\square$

Therefore, all Parseval frames in a common orbit in $F/\operatorname{GL}(H)$ are unitarily equivalent, and hence it is possible to linearly transform or "project" any frame to a Parseval frame that is unique up to unitary equivalence. Let $\overline{F}_0$ be a fixed transversal of the orbit space $F_0/U(H)$, so that $\overline{F}_0$ is a maximal set of unitarily inequivalent Parseval frames on H. Note $F_0 = U(H)\overline{F}_0$. By Lemma 2.10, the "factorization" of a Parseval frame in F_0 into a unitary operator in $U(H)$ and a Parseval frame in $\overline{F}_0$ is unique. We therefore define the projection maps

$$U : F_0 \mapsto U(H) \text{ and } \sigma : F_0 \mapsto \overline{F}_0 \text{ such that } f = U(f)\sigma(f) \text{ for all } f \in F_0. \tag{6}$$

We observe that for all $A \in U(H)$ and $f \in \overline{F}_0$, we have $U(Af) = A$ and $\sigma(Af) = f$. Thus, U and σ are both surjective.

We are ready to show that $\overline{F}_0$ indexes the orbits of $F/\operatorname{GL}(H)$. First, we define the maps

$$\zeta : \operatorname{GL}(H) \times \overline{F}_0 \mapsto F, \quad \zeta(A, f) = Af \tag{7}$$

$$\zeta^+ : \operatorname{GL}^+(H) \times F_0 \mapsto F, \quad \zeta^+(A, f) = Af, \tag{8}$$

and we establish key properties of ζ and ζ^+ in the following theorem.

Theorem 2.11. *The maps ζ and ζ^+ are continuous bijections.*

Proof. First, we prove ζ is a bijection: Let $f \in F$. Since $T(f) \in F_0$, then $T(f)$ has the unique factorization $T(f) = U(T(f))\sigma(T(f))$. Note that $S(f)^{\frac{1}{2}}U(T(f)) \in \operatorname{GL}(H)$

and $\sigma(T(f)) \in \overline{F}_0$. We have

$$\begin{aligned}\zeta(S(f)^{\frac{1}{2}}U(T(f)), \sigma(T(f))) &= S(f)^{\frac{1}{2}}U(T(f))\sigma(T(f)) \\ &= S(f)^{\frac{1}{2}}T(f) \\ &= S(f)^{\frac{1}{2}}S(f)^{-\frac{1}{2}}f \\ &= f.\end{aligned}$$

Thus, ζ is surjective.

Suppose $\zeta(A_1, f_1) = \zeta(A_2, f_2)$. Then, $A_1 f_1 = A_2 f_2$, and hence $(A_2^{-1}A_1)f_1 = f_2$. Since f_1 and f_2 are Parseval, then Lemma 2.10 implies that $A_2^{-1}A_1$ is unitary. But since $f_1, f_2 \in \overline{F}_0$, then either f_1 and f_2 are unitarily inequivalent or $f_1 = f_2$. Since $A_2^{-1}A_1 \in U(H)$, then we must have $f_1 = f_2$ and hence $A_2^{-1}A_1 = I$ by Lemma 2.7. Thus, $A_1 = A_2$. That is, $(A_1, f_1) = (A_2, f_2)$. Ergo, ζ is injective and therefore bijective.

Now, we prove ζ^+ is a bijection: Let $f \in F$. Note $S(f)^{\frac{1}{2}} \in \mathrm{GL}^+(H)$ and $T(f) \in F_0$. We have

$$\begin{aligned}\zeta^+(S(f)^{\frac{1}{2}}, T(f)) &= S(f)^{\frac{1}{2}}T(f) \\ &= S(f)^{\frac{1}{2}}S(f)^{-\frac{1}{2}}f \\ &= f.\end{aligned}$$

Thus, ζ^+ is surjective.

Suppose $\zeta^+(A_1, f_1) = \zeta^+(A_2, f_2)$. Thus, $A_1 f_1 = A_2 f_2$, so that $(A_2^{-1}A_1)f_1 = f_2$. Since f_1 and f_2 are Parseval, then Lemma 2.10 implies that $A_2^{-1}A_1$ is unitary. Thus,

$$(A_2^{-1}A_1)(A_2^{-1}A_1)^* = A_2^{-1}A_1A_1^*(A_2^{-1})^* = I.$$

Since A_2^{-1} and A_1 are positive and hence self-adjoint, then

$$A_2^{-1}A_1A_1A_2^{-1} = I.$$

$$A_1^2 = A_2^2.$$

Since A_1 and A_2 are positive, then the unique principal square roots of A_1^2 and A_2^2 are precisely A_1 and A_2 respectively. Thus, we have $A_1 = A_2$. This implies $A_2^{-1}A_1 = I$, so that $f_1 = f_2$. That is, $(A_1, f_1) = (A_2, f_2)$. Thus, ζ^+ is injective and hence bijective.

Finally, we prove ζ and ζ^+ are both continuous: Since ζ and ζ^+ are both restrictions of the map $\zeta_* : \mathrm{GL}(H) \times F_0 \mapsto F$, then it suffices to show ζ_* is continuous. Let $\{(A_n, f_n)\}_{n=1}^{\infty}$ be a sequence of points in $\mathrm{GL}(H) \times F_0$., and suppose $(A_n, f_n) \to (A, f)$. This means $A_n \to A$ and $f_n \to f$. Let $\varepsilon > 0$. Then, there exists $N_1 \in \mathbb{N}$ such that $n > N_1$ implies

$$\|A_n - A\|_H, \|f_n - f\|_J < \frac{\varepsilon}{2\|A\|_H + \|f\|_J}.$$

Since $A_n \to A$, then there exists $N_2 \in \mathbb{N}$ such that $n > N_2$ implies $\|A_n\|_H < 2\|A\|_H$.

Assume $n > \max\{N_1, N_2\}$. Then, we have

$$\begin{aligned}
\|\zeta_*(A_n, f_n) - \zeta_*(A, f)\|_J &= \|A_n f_n - Af\|_J \\
&= \|A_n f_n - A_n f + A_n f - Af\|_J \\
&\leq \|A_n f_n - A_n f\|_J + \|A_n f - Af\|_J \\
&= \sup_{x \in X} \|A_n(f_n - f)(x)\|_H + \sup_{x \in X} \|(A_n - A)f(x)\|_H \\
&\leq \|A_n\|_H \sup_{x \in X} \|(f_n - f)(x)\|_H + \|A_n - A\|_H \sup_{x \in X} \|f(x)\|_H \\
&= \|A_n\|_H \|f_n - f\|_J + \|A_n - A\|_H \|f\|_J \\
&< 2\|A\|_H \left(\frac{\varepsilon}{2\|A\|_H + \|f\|_J}\right) + \left(\frac{\varepsilon}{2\|A\|_H + \|f\|_J}\right) \|f\|_J \\
&= \varepsilon.
\end{aligned}$$

Ergo, ζ_* and thus ζ and ζ^+ are continuous. $\square$

Because $\zeta : \mathrm{GL}(H) \times \overline{F}_0 \mapsto F$ is a bijection, the orbit space $F/\,\mathrm{GL}(H)$ is in one-one correspondence with $\overline{F}_0$. In other words, the transversal $\overline{F}_0$ of unitarily inequivalent Parseval frames indexes the orbits in F induced by invertible linear transformations. In particular, because ζ is invertible, we have that

$$F = \mathrm{GL}(H)\overline{F}_0,$$

with every frame having a unique representation in $\mathrm{GL}(H)\overline{F}_0$. Recalling the relationship between $\mathrm{GL}^+(H)$ and $\mathrm{GL}(H)$, the following corollary completes this line of thought.

Corollary 2.12. *We have*

$$F = \mathrm{GL}^+(H)U(H)\overline{F}_0 = \mathrm{GL}(H)\overline{F}_0 = \mathrm{GL}^+(H)F_0.$$

Moreover, the factorization of a frame $f \in F$ *in* $\mathrm{GL}^+(H)U(H)\overline{F}_0$ *as*

$$f = S(f)^{\frac{1}{2}}U(f)\sigma(T(f))$$

is unique.

Finally, we define the continuous projection maps

$$\pi_1 : \mathrm{GL}(H) \times F_0 \mapsto \mathrm{GL}(H), \quad \pi_1(A, f) = A$$

$$\pi_2 : \mathrm{GL}(H) \times F_0 \mapsto F_0, \quad \pi_2(A, f) = f.$$

The relationships presented in this section can then be summarized by the following commuting diagram:

$$\begin{tikzcd}
 & \mathrm{GL}(H)/U(H) \arrow[dr, leftrightarrow] & \\
\mathrm{GL}(H) \arrow[ur] \arrow[rr, "\rho"] & & \mathrm{GL}^+(H) \\
\mathrm{GL}(H) \times \overline{F}_0 \arrow[u, "\pi_1"] \arrow[r, leftrightarrow, "\zeta"] \arrow[d, "\pi_2"'] & F \arrow[r, leftrightarrow, "\zeta^+"] & \mathrm{GL}^+(H) \times F_0 \arrow[u, "S", "\pi_1^2"'] \arrow[d, "T"', "\pi_2"] \\
\overline{F}_0 \arrow[dr, leftrightarrow] & & F_0 \arrow[ll, "\sigma"'] \arrow[dl] \\
 & F/U(H) &
\end{tikzcd}$$

By π_1^2, we mean $\pi_1^2(A, f) = A^2$. Also, we have the following identity.

Corollary 2.13. *For all $f \in F$, we have*

$$f = \zeta(\pi_1(\zeta^{-1}(f)), \pi_2(\zeta^{-1}(f))).$$

In the next section, we extend the orbit structure of frames to that of a principal fiber bundle.

2.2 The Fiber Bundle of Frames

We have seen that the space of frames F may be fibrated into orbits that are principal homogeneous spaces under the action of $\mathrm{GL}(H)$. We have also seen that every orbit may be projected to a unique element in the transversal $\overline{F}_0$ of unitarily inequivalent Parseval frames. We might therefore suspect that F has the structure of a principal fiber bundle. But we cannot conclude this immediately because we do not know if ζ^{-1} is continuous. In this section, we provide sufficient conditions for ζ^{-1} to be continuous and hence for F to be a principal fiber bundle.

We begin by stating the definition of a fiber bundle.

Definition 2.14. *Let E_1 and B be topological spaces. A topological space E is called a* fiber bundle *with base space B and fiber E_1 if there exists a* projection *or continuous surjection $\pi : E \mapsto B$ that satisfies the* local triviality condition*: For every $x \in E$, there is an open neighborhood $U \subseteq B$ about $\pi(x)$ and a homeomorphism $\theta : \pi^{-1}(U) \mapsto U \times E_1$ such that*

$$\pi(x) = (\pi_U \circ \theta)(x), \quad \forall x \in \pi^{-1}(U),$$

where $\pi_U : U \times E_1 \mapsto U$ *is the natural projection from the product space* $U \times E_1$ *to the first factor* B*. If the fiber* E_1 *is a principal homogeneous space under the action of a group* G*, then* E *is called a* principal fiber bundle *with* structure group G*.*

Thus, a fiber bundle is simply a space that is locally a product space. Every product space $E = B \times E_1$ is a fiber bundle with base space either B or E_1. A less trivial example of a fiber bundle is the Möbius strip with base space the circle S^1 and fiber $[0, 1]$. For more information on fiber bundles, see [11].

Proceeding, we fix some Parseval frame $f_{10} \in \overline{F}_0$ and define the space

$$F_1 = \mathrm{GL}(H)f_{10} = \{Af_{10} : A \in \mathrm{GL}(H)\}. \tag{9}$$

This space will ultimately be a fiber of F.

Our first task is to show that F is in one-one correspondence with the product space $F_1 \times \overline{F}_0$. This means we have projection maps from F to each component space F_1 and $\overline{F}_0$. We already know that the map $\sigma \circ T$ projects F onto $\overline{F}_0$. In addition, we define the projection map

$$T_1 : F \mapsto F_1, \quad T_1(f) = \pi_1(\zeta^{-1}(f))f_{10}. \tag{10}$$

The following proposition verifies that T_1 is indeed a projection.

Proposition 2.15. *The map* T_1 *is surjective.*

Proof. Let $f_1 \in F_1$. By Corollary 2.13, we have

$$\begin{aligned} f_1 &= \zeta(\pi_1(\zeta^{-1}(f_1)), \pi_2(\zeta^{-1}(f_1))) \\ &= \zeta(\pi_1(\zeta^{-1}(f_1)), f_{10}) \\ &= \pi_1(\zeta^{-1}(f_1))f_{10} \\ &= T_1(f_1). \end{aligned}$$

□

Since F_1 is a principal homogeneous space under the action of $\mathrm{GL}(H)$, then it follows that F_1 and $\mathrm{GL}(H)$ are in one-one correspondence. Next, we define the map

$$\theta : F_1 \mapsto \mathrm{GL}(H), \quad \theta(f) = \pi_1(\zeta^{-1}(f)). \tag{11}$$

We immediately obtain the following lemma.

Lemma 2.16. *The map θ is a bijection.*

Proof. Let $A \in \mathrm{GL}(H)$. Then, $Af_{10} \in F_1$. By Corollary 2.13, we have

$$\begin{aligned} Af_{10} &= \zeta(\pi_1(\zeta^{-1}(Af_{10})), \pi_2(\zeta^{-1}(Af_{10}))) \\ &= \zeta(\theta(Af_{10})), f_{10}) \\ &= \theta(Af_{10})f_{10}. \end{aligned}$$

But since $\mathrm{GL}(H)$ acts regularly on F_1 (by Lemma 2.7), then $\theta(Af_{10}) = A$. Ergo, θ is surjective.

Suppose $\theta(f_1) = \theta(f_2)$. As above, f_1 and f_2 have the unique factorizations $f_1 = \theta(f_1)f_{10}$ and $f_2 = \theta(f_2)f_{10}$. But since $\theta(f_1) = \theta(f_2)$, then

$$f_1 = \theta(f_1)f_{10} = \theta(f_2)f_{10} = f_2.$$

Thus, θ is injective. $\square$

The bijection θ may be lifted to the map

$$\theta_* : F_1 \times \overline{F}_0 \mapsto F, \quad \theta_*(f_1, f_0) = \zeta(\theta(f_1), f_0). \tag{12}$$

This leads to the following:

Theorem 2.17. *The map θ_* is a bijection and has inverse*

$$\theta^{-1}(f) = (T_1(f), \sigma \circ T(f)).$$

Proof. By Lemma 2.16, θ is bijective. The identity map is obviously bijective. Thus, the map

$$(f_1, f_0) \to (\theta(f_1), f_0)$$

is a bijection from $F_1 \times \overline{F}_0$ to $\mathrm{GL}(H) \times \overline{F}_0$. By Theorem 2.11, ζ is bijective. Ergo, θ_* is a bijection.

To verify that the expression θ_*^{-1} is indeed the inverse of θ_*, let $f \in F$ and consider

$$\begin{aligned}
\theta_*(\theta_*^{-1}(f)) &= \zeta(\theta(T_1(f)), \sigma \circ T(f)) \\
&= \zeta[\pi_1 \circ \zeta^{-1}(\pi_1 \circ \zeta^{-1}(f) f_{10}), \sigma \circ T(f)] \\
&= \zeta(\pi_1 \zeta^{-1}(f), \sigma \circ T(f)) \\
&= \pi_1(\zeta(f)) \sigma(T(f)) \\
&= f.
\end{aligned}$$

The reverse composition proceeds similarly. $\square$

We therefore have the following commuting diagram:

$$\begin{tikzcd}
F_1 \times \overline{F}_0 \arrow[r, leftrightarrow, "\theta_*"] \arrow[d] & F \arrow[dl, "T_1"'] \arrow[d, "\pi_1 \circ \zeta^{-1}"] \\
F_1 \arrow[r, leftrightarrow, "\theta"'] & \mathrm{GL}(H)
\end{tikzcd}$$

In particular, F is in one-one correspondence with $F_1 \times \overline{F}_0$. But to show F is a fiber bundle, we also require continuity. In particular, for F to be a fiber bundle with base space $\overline{F}_0$, the projection $\sigma \circ T = \pi_1 \circ \zeta^{-1}$ mapping F onto $\overline{F}_0$ must be continuous. Since π_1 is continuous, it suffices to have ζ^{-1} be continuous.

Proposition 2.18. *If ζ^{-1} is continuous, then $\theta_* : F_1 \times \overline{F}_0 \mapsto F$ is a homeomorphism and F is a principal fiber bundle with base space $\overline{F}_0$, fiber F_1, and structure group* $\mathrm{GL}(H)$.

Proof. Suppose ζ^{-1} is continuous. Then, $\theta = \pi_1 \zeta^{-1}$ is continuous. By Theorem 2.11, ζ is continuous. Thus, $\theta_*(f_1, f_0) = \zeta(\theta(f_1), f_0)$ is continuous

Since ζ^{-1} is continuous, then $T_1(f) = \pi_1(\zeta^{-1}(f))f_{10}$ and $\sigma \circ T = \pi_1 \circ \zeta^{-1}$ are continuous. Thus, $\theta_*^{-1} = (T_1, \sigma \circ T)$ is continuous. Ergo, θ_* is a homeomorphism.

Since F is homeomorphic to the product space $F_1 \times \overline{F}_0$ (via θ_*^{-1}), then F is trivially a fiber bundle as claimed. $\square$

We therefore proceed to establish conditions that are sufficient for ζ^{-1} to be continuous. We first define the Banach space

$$J_1 = L^1(X, H)$$

equipped with the norm

$$\|f\|_{J_1} = \int_X \|f(x)\|_H \, d\mu(x).$$

Suppose that $F \subset J_1$. That is, suppose that all frames (in J) on H are integrable. We will show that this is sufficient for ζ^{-1} to be continuous and hence for F to be a fiber bundle.

By the commuting diagram in Section 2.1 and the unique factorization granted by Corollary 2.12, it is straightforward to show that $\zeta^{-1} : F \mapsto \mathrm{GL}(H) \times \overline{F}_0$ is given by

$$\zeta^{-1}(f) = (S(f)^{\frac{1}{2}} U(f), \sigma \circ T(f)). \tag{13}$$

The three lemmas that follow show that each term on the right side of this equation is continuous in J_1.

Lemma 2.19. *The map $S : F \mapsto \mathrm{GL}^+(H)$ is continuous in the topology of J_1.*

Proof. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of frames in F and $f \in F$ such that $f_n \to f$ in J_1. Let $\varepsilon > 0$. Then, there exists $N \in \mathbb{N}$ such that $n > N$ implies $\|f_n\|_{J_1} < 2\|f\|_{j_1}$ and

$$\|f_n - f\|_{J_1} < \frac{\varepsilon}{3\|f\|_{J_1}}.$$

Suppose $n > N$. Consider any $\phi \in H$. We have

$$\begin{aligned}
\|S(f_n)\phi - S(f)\phi\|_H &= \left\| \int_X \langle \phi, f_n(x) \rangle_H \, f_n(x) \, d\mu(x) - \int_X \langle \phi, f(x) \rangle_H \, f(x) \, d\mu(x) \right\| \\
&= \left\| \int_X \langle \phi, f_n(x) \rangle_H \, f_n(x) \, d\mu(x) - \int_X \langle \phi, f_n(x) \rangle_H \, f(x) \, d\mu(x) \right. \\
&\quad \left. + \int_X \langle \phi, f_n(x) \rangle_H \, f(x) \, d\mu(x) - \int_X \langle \phi, f(x) \rangle_H \, f(x) \, d\mu(x) \right\| \\
&= \left\| \int_X \langle \phi, f_n(x) \rangle_H \, [f_n(x) - f(x)] \, d\mu(x) + \int_X \langle \phi, f_n(x) - f(x) \rangle_H \, f(x) \, d\mu(x) \right\| \\
&\leq \int_X \|\phi\|_H \|f_n(x)\|_H \|f_n(x) - f(x)\|_H \, d\mu(x) + \int_X \|\phi\|_H \|f_n(x) - f(x)\|_H \|f\|_H \, d\mu(x) \\
&= \|\phi\| \int_X \|f_n(x) - f(x)\|_H (\|f_n(x)\|_H + \|f(x)\|_H) \, d\mu(x).
\end{aligned}$$

By Hölder's Inequality, we have

$$\begin{aligned}
\|S(f_n)\phi - S(f)\phi\|_H &\leq \|\phi\|_H \int_X \|f_n(x) - f(x)\|_H \, d\mu(x) \int_X (\|f_n(x)\|_H + \|f(x)\|_H) \, d\mu(x) \\
&= \|\phi\|_H \|f_n - f\|_{J_1} (\|f_n\|_{J_1} + \|f\|_{J_1}) \\
&< 3\|f\|_{J_1} \|f_n - f\|_{J_1} \|\phi\|_H \\
&< \varepsilon \|\phi\|_H.
\end{aligned}$$

Since this holds for all $\phi \in H$, then $\|S(f_n) - S(f)\|_H < \varepsilon$. Ergo, $S(f_n) \to S(f)$, and hence S is continuous. □

Lemma 2.20. *The map $T : F \mapsto F_0$ is continuous in the topology of J_1.*

Proof. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of frames in F and $f \in F$ such that $f_n \to f$ in J_1. By Lemma 2.19, $S(f_n) \to S(f)$. Since the map sending an operator in $\mathrm{GL}(H)$ to its inverse and the map sending an operator in $\mathrm{GL}^+(H)$ to its principal square root are both continuous in the operator norm, then $S(f_n)^{-\frac{1}{2}} \to S(f)^{-\frac{1}{2}}$. Let $\varepsilon > 0$. Then, there exists $N \in \mathbb{N}$ such that $n > N$ implies $\|f_n\|_{J_1} < 2\|f\|_{J_1}$ and

$$\|f - f_n\|_{J_1}, \|S(f_n)^{-\frac{1}{2}} - S(f)^{-\frac{1}{2}}\|_H < \delta = \frac{\varepsilon}{2\|f\|_{J_1} + \|S(f)^{-\frac{1}{2}}\|_H}.$$

Suppose $n > N$. We have

$$\begin{aligned}\|T(f_n) - T(f)\|_{J_1} &= \|S(f_n)^{-\frac{1}{2}} f_n - S(f)^{-\frac{1}{2}} f\|_{J_1} \\ &= \|S(f_n)^{-\frac{1}{2}} f_n - S(f)^{-\frac{1}{2}} f_n + S(f)^{-\frac{1}{2}} f_n - S(f)^{-\frac{1}{2}} f\|_{J_1} \\ &= \|[S(f_n)^{-\frac{1}{2}} - S(f)^{-\frac{1}{2}}] f_n + S(f)^{-\frac{1}{2}}(f_n - f)\|_{J_1} \\ &\leq \|S(f_n)^{-\frac{1}{2}} - S(f)^{-\frac{1}{2}}\|_H \|f_n\|_{J_1} + \|S(f)^{-\frac{1}{2}}\|_H \|f_n - f\|_{J_1} \\ &< \delta 2\|f\|_{J_1} + \|S(f)^{-\frac{1}{2}}\|_H \delta \\ &= \varepsilon.\end{aligned}$$

Ergo, $T(f_n) \to T(f)$, and hence T is continuous. $\square$

Suppose $\overline{F}_0$ is a continuous transversal of $F/\operatorname{GL}(H)$. That is, suppose that the projection $\sigma : F_0 \mapsto \overline{F}_0$ is continuous in the topology of J_1. Then, we have the following:

Lemma 2.21. *The map $U : F_0 \mapsto U(H)$ is continuous in the topology of J_1.*

Proof. Let $f \in F_0$. Then, f_0 has the unique factorization $f = U(f)\sigma(f)$. For any $\phi \in H$ and for all $x \in X$, we have

$$\langle \phi, f(x) \rangle_H = \langle \phi, U(f)\sigma(f)(x) \rangle_H = \langle U(f)^* \phi, \sigma(f)(x) \rangle_H .$$

Letting V_g and V_g^* denote the analysis and synthesis operators of a frame $g \in F$, we have

$$V_f \phi = V_{\sigma(f)} U(f)^* \phi,$$

or simply $V_f = V_{\sigma(f)}U(f)^*$. Thus, $V_f U(f) = V_{\sigma(f)}$, and hence $V_f^* V_f U(f) = V_f^* V_{\sigma(f)}$. But since f is Parseval, then $V_f^* V_f = S(f) = I$ so that

$$U(f) = V_f^* V_{\sigma(f)}.$$

Thus, for all $\phi \in H$, we have

$$U(f)\phi = \int_X \langle \phi, \sigma(f)(x)\rangle_H \, f(x)\, d\mu(x). \tag{14}$$

To show U is continuous, let $\{f_n\}_{n=1}^{\infty}$ be a sequence of frames in F_0 and $f \in F_0$ with $f_n \to f$ in J_1. Since σ is continuous, then $\sigma(f_n) \to \sigma(f)$. Let $\varepsilon > 0$. Then, there exists $N \in \mathbb{N}$ such that $n > N$ implies $\|f_n\|_{J_1} < 2\|f\|_{J_1}$ and

$$\|f_n - f\|_{J_1}, \|\sigma(f_n) - \sigma(f)\|_{J_1} < \delta = \frac{\varepsilon}{\|\sigma(f)\|_{J_1} + 2\|f\|_{J_1}}.$$

Suppose $n > N$. Let $\phi \in H$. By manipulations similar to those used in the proof of Lemma 2.19 (including the Triangle Inequality, Cauchy-Schwartz Inequality, and Hölder's Inequality), we have

$$\begin{aligned}
\|U(f_n)\phi - U(f)\phi\|_H &= \left\| \int_X \langle \phi, \sigma(f_n)(x)\rangle_H \, f_n(x)\, d\mu(x) - \int_X \langle \phi, \sigma(f)(x)\rangle_H \, f(x)\, d\mu(x) \right\| \\
&\le \|\phi\|_H \|\sigma(f_n) - \sigma(f)\|_{J_1} \|f_n\|_{J_1} + \|\phi\|_H \|\sigma(f)\|_{J_1} \|f_n - f\|_{J_1} \\
&< \|\phi\|_H \delta 2\|f\|_{J_1} + \|\phi\|_H \|\sigma(f)\|_{J_1} \delta \\
&= (2\|f\|_{J_1} + \|\sigma(f)\|_{J_1})\delta \|\phi\|_H \\
&= \varepsilon \|\phi\|_H.
\end{aligned}$$

Since this holds for all $\phi \in H$, then $\|U(f_n) - U(f)\|_H < \varepsilon$. Ergo, $U(f_n) \to U(f)$, and hence U is continuous. □

The lemmas 2.19-2.21 thus lead to the following theorem.

Theorem 2.22. *The map ζ^{-1} is continuous in the topology of J_1. Moreover, F is a principal fiber bundle with base space $\overline{F}_0$, fiber F_1, and structure group* $\mathrm{GL}(H)$ *in the topology of J_1.*

Proof. Recall that ζ^{-1} is given by

$$\zeta^{-1}(f) = (S(f)^{\frac{1}{2}}U(f), \sigma \circ T(f)).$$

By Lemmas 2.19-2.21, the maps S, T, and U are continuous in J_1. Moreover, σ and the square root function are continuous as well. Therefore, ζ^{-1} is continuous. By Proposition 2.18, F is a fiber bundle as claimed. □

A special case occurs when μ is a finite measure on X. In this case, convergence in $J = L^\infty(X, H)$ implies convergence in $J_1 = L^1(X, H)$. We therefore have the following corollary.

Corollary 2.23. *If μ is a finite measure on X, then the space F is a principal fiber bundle in the topologies of both J and J_1.*

A special case of finite measure is the counting measure on a finite set. This leads to the example of finite frames. We discuss the fiber bundle structure of finite frames in Section 3.3 after we develop some understanding of the structure of the base space $\overline{F}_0$ in Chapter 3.

3 GENERAL DEFORMATIONS OF FRAMES

In Chapter 2, we showed that the space F of all frames on a Hilbert space H is fibrated into an orbit space under the action of linear transformations in $\mathrm{GL}(H)$. We also showed that if the frames in F are "integrable", then F is a principal fiber bundle with structure group $\mathrm{GL}(H)$. We therefore understand how frames in a common fiber in F are connected to each other. In this section, we show exactly how frames in different fibers are connected to each other; i.e., we show how one can "move" from fiber to fiber in F.

We maintain the notation used in Section 2.1. In addition, let V_f denote the analysis map of a frame f. We will not assume that the frames in F are integrable, as we do not need the entire fiber bundle structure for our purposes; the orbit space structure is sufficient. In Section 3.1, we establish that any two frames can be connected via their associated RK Hilbert spaces. In Section 3.2, we consider a special case in which such general deformations of frames are simplified in their action. Finally, in Sections 3.3-3.4, we apply our results to the examples of finite frames and the discretization of the Gabor frame.

3.1 Deformations of Frames via Reproducing Kernel Hilbert Spaces

The key to connecting frames in F to each other is to look at their associated RK Hilbert spaces. Define the set

$$\mathcal{R} = \{R \subseteq L^2(X) : R \text{ is an RK Hilbert space isometric to } H\},$$

and define the map

$$\Theta : F \mapsto \mathcal{R}, \quad \Theta(f) = \operatorname{ran}(V_f) = \{\langle \phi, f(\cdot)\rangle_H : \phi \in H\},$$

where $\Theta(f)$ has kernel

$$K(x, y) = \left\langle S(f)^{-1} f(y), f(x)\right\rangle_H.$$

By Proposition 1.7, Θ is a well-defined map in the sense that $\theta(f)$ is in fact an RK Hilbert space isometric to H.

Let us establish some properties of Θ that will allow us to understand the connection between F and $\mathcal{R}$.

Lemma 3.1. *The restricted map $\Theta : F_0 \mapsto \mathcal{R}$ is surjective. In particular, let $R \in \mathcal{R}$ with kernel K and isometry $W : R \mapsto H$. Then, the map $f : X \mapsto H$ given by $f(x) = Wk_x$ is a Parseval frame on H.*

Proof. Define $f : X \mapsto H$ by $f(x) = wk_x$. Let $\phi \in H$ so that $\phi = w\alpha$ for some $\alpha \in R$. Since w is an isometry and since $\{k_x : x \in X\}$ is a Parseval frame on R by Proposition 1.8, then we have

$$\begin{aligned}
\int_X |\langle \phi, f(x)\rangle_H|^2 \, d\mu(x) &= \int_X |\langle w\alpha, wk_x\rangle_2|^2 \, d\mu(x) \\
&= \|\alpha\|_2^2 \\
&= \|w^{-1}\phi\|_2^2 \\
&= \|\phi\|_H^2.
\end{aligned}$$

Thus, f is a Parseval frame on H so that $f \in F_0$. Moreover, by properties of the isometry w and the reproducing property of K,

$$\begin{aligned}
\Theta(f) &= \{\langle \phi, f(\cdot)\rangle_H : \phi \in H\} \\
&= \{\langle w^{-1}\phi, w^{-1}[f(\cdot)]\rangle_2 : \phi \in H\} \\
&= \{\langle \alpha, k_\cdot\rangle_2 : \alpha \in R\} \\
&= \{\alpha : \alpha \in R\} \\
&= R.
\end{aligned}$$

The kernel K satisfies

$$\begin{aligned}
K(x, y) &= \langle k_y, k_x\rangle_2 \\
&= \langle w(k_y), w(k_x)\rangle_H \\
&= \langle f(y), f(x)\rangle_H \\
&= \left\langle S(f)^{-1} f(y), f(x)\right\rangle_H,
\end{aligned}$$

where we used the fact that $S(f) = I$ since $f \in F_0$. Therefore, $\Theta(f)$ is in fact the RK Hilbert space R with kernel K, and hence Θ is surjective. $\square$

Since $\Theta : F_0 \mapsto \mathcal{R}$ is surjective and $F_0 \subset F$, then clearly $\Theta : F \mapsto \mathcal{R}$ is surjective as well. The following lemma is needed to address the injectivity of Θ.

Lemma 3.2. *Let $f, f' \in F$. Let $R = \Theta(f)$ and $R' = \Theta(f')$ with kernels K and K' respectively. Then, $R = R'$ and $K = K'$ if and only if $f' = Af$ for some $A \in \mathrm{GL}(H)$.*

Proof. ($\Leftarrow$). Suppose $f' = Af$ for some $A \in \mathrm{GL}(H)$. For any $\phi \in H$, we have

$$\begin{aligned}(V_{f'}\phi)(x) &= \langle \phi, f'(x) \rangle_H \\ &= \langle \phi, Af(x) \rangle_H \\ &= \langle A^*\phi, f(x) \rangle_H \\ &= (V_f A^*\phi)(x).\end{aligned}$$

Thus, $V_{f'} = V_f A^*$. Since A^* is a bijection on H, then $A^*H = H$. Thus,

$$R' = V_{f'}(H) = V_f(A^*(H)) = V_f(H) = R.$$

By Lemma 2.6, $S(f') = S(Af) = AS(f)A^*$. Thus,

$$\begin{aligned}K'(x, y) &= \left\langle S(f')^{-1} f'(y), f'(x) \right\rangle_H \\ &= \left\langle [AS(f)A^*]^{-1} Af(y), Af(x) \right\rangle_H \\ &= \left\langle (A^*)^{-1} S(f)^{-1} A^{-1} Af(y), Af(x) \right\rangle_H \\ &= \left\langle S(f)^{-1} f(y), A^{-1}Af(x) \right\rangle_H \\ &= \left\langle S(f)^{-1} f(y), f(x) \right\rangle_H \\ &= K(x, y).\end{aligned}$$

Ergo, $R = R'$ and $K = K'$.

($\Rightarrow$). Suppose $R = R'$. Let $a \in R = R'$. Since V_f and $V_{f'}$ are invertible on R and R' respectively, then let $\phi = V_f^{-1}a$ and $\psi = V_{f'}^{-1}a$. Define $A^* = V_f^{-1}V_{f'}$ and note

$A \in \mathrm{GL}(H)$. Note that

$$\psi = V_{f'}^{-1} a.$$

$$V_{f'} \psi = a.$$

$$V_f^{-1} V_{f'} \psi = V_f^{-1} a.$$

$$A^* \psi = \phi.$$

This holds for all $a \in R'$. Since $V_{f'}$ is surjective, then this holds for all $\psi \in H$. Since $V_f \phi = V_{f'} \psi = a$, then we have

$$\langle \phi, f(x) \rangle_H = \langle \psi, f'(x) \rangle_H .$$

$$\langle A^* \psi, f(x) \rangle_H = \langle \psi, f'(x) \rangle_H .$$

$$\langle \psi, Af(x) \rangle_H = \langle \psi, f'(x) \rangle_H .$$

$$\langle \psi, Af(x) - f'(x) \rangle_H = 0.$$

Since this holds for all $\psi \in H$ and for all $x \in X$, then $Af - f' = 0$. That is, $f' = Af$. $\square$

It follows that Θ maps each orbit in $F/\,\mathrm{GL}(H)$ to a unique and distinct RK Hilbert space in $\mathcal{R}$. Since $\overline{F}_0$ is a transversal of $F/\,\mathrm{GL}(H)$, then we expect that Θ gives a one-one correspondence between $\overline{F}_0$ and $\mathcal{R}$, which is verified in the following proposition.

Proposition 3.3. *The restricted map $\Theta : \overline{F}_0 \mapsto \mathcal{R}$ is a bijection.*

Proof. Let $R \in \mathcal{R}$. By Lemma 3.1, there exists $f \in F_0$ such that $\Theta(f) = R$. Let $f' = \sigma(f) \in \overline{F}_0$. By definition of σ, there exists $U \in U(H)$ such that $f' = Uf$. By

Lemma 3.2, $\Theta(f') = \Theta(f) = R$. Therefore, $\Theta : \overline{F}_0 \mapsto \mathcal{R}$ is surjective.

Suppose $\Theta(f) = \Theta(f')$ where $f, f' \in \overline{F}_0$. By Lemma 3.2, $f' = Af$ for some $A \in \mathrm{GL}(H)$. By Lemma 2.10, we have $A \in U(H)$. Thus, f' and f are unitarily equivalent. But since $f, f' \in \overline{F}_0$ means that either f and f' are unitarily inequivalent or $f' = f$, then we must have $f' = f$. Ergo, $\Theta : \overline{F}_0 \mapsto \mathcal{R}$ is injective and hence bijective. □

Recall that our goal is to understand how to connect two frames belonging to different orbits in $F/\,\mathrm{GL}(H)$. Since $\overline{F}_0$ is a transversal of $F/\,\mathrm{GL}(H)$, then it suffices to understand what transformations connect the frames in $\overline{F}_0$ to each other. By Proposition 3.3, we can understand the structure of $\overline{F}_0$ by understanding the structure of $\mathcal{R}$, which we now proceed to do.

First, we make a couple of remarks on notation. Given an operator $A \in \mathrm{GL}(L^2(X))$, we write A_x to denote the action of A on a function in the variable x while holding all other variables fixed. For example, let $R \in \mathcal{R}$ with kernel K. Recall that $K(x, y) = k_y(x)$. Thus, the expression $(A_x K)(x, y)$ is equivalent to $(Ak_y)(x)$; i.e., A acts with respect to the variable x wile y is held fixed. In addition, given an operator $A \in \mathrm{GL}(L^2(X))$, we write $\overline{A}$ to denote the action of A followed by conjugation; e.g., given $\alpha \in L^2(X)$, the expression $(\overline{A}\alpha)(x)$ is equivalent to $\overline{(A\alpha)(x)}$.

The following proposition reveals the structure of the set $\mathcal{R}$.

Proposition 3.4. *Given an RK Hilbert space $R \in \mathcal{R}$ with kernel K and a unitary* *operator $U \in U(L^2(X))$, the space $R' = U(R)$ is an RK Hilbert space in $\mathcal{R}$ with kernel*

$$K'(x,y) = U_x \overline{U}_y K(x,y).$$

Conversely, given $R, R' \in \mathcal{R}$, there exists $U \in U(L^2(X))$ such that $R' = U(R)$.

Proof. The converse is straightforward: Given $R, R' \in \mathcal{R}$, then by definition both R and R' are isometric to H. Consequently, R and R' are isometric to each other. Hence, $R' = U(R)$ for some $U \in U(L^2(X))$.

For the forward implication, let $R \in \mathcal{R}$ with kernel K, and let $U \in U(L^2(X))$. Let $R' = U(R)$, and let K' be as claimed. Note that R' is indeed a Hilbert space isometric to H. Thus, our task is only to show that K' satisfies the reproducing property on R'. Let $\alpha' \in R'$ so that $\alpha = U^*\alpha' \in R$. We have

$$\begin{aligned}
\langle \alpha, k'_x \rangle_2 &= \int_X \overline{k'_x(y)} \alpha'(y)\, d\mu(y) \\
&= \int_X K'(x,y)\alpha'(y)\, d\mu(y) \\
&= \int_X U_x \overline{U}_y K(x,y)\alpha'(y)\, d\mu(y) \\
&= U_x \int_X \overline{U}_y K(x,y)\alpha'(y)\, d\mu(y) \qquad (*) \\
&= U_x \langle \alpha', U k_x \rangle \\
&= U_x \langle U^* \alpha', k_x \rangle \\
&= U_x \langle \alpha, k_x \rangle \\
&= (U\alpha)(x) \\
&= \alpha'(x),
\end{aligned}$$

where in the line (*) we used the fact that the integration is not with respect to x and that U_x is uniformly continuous on $L^2(X)$. We conclude that K' is in fact a reproducing kernel on R'. $\square$

The RK Hilbert spaces in $\mathcal{R}$ are therefore connected by unitary transformations in $U(L^2(X))$. But for the unitary transformations connecting $\mathcal{R}$ to be unique, we must "mod out" the transformations that leave a given space $R \in \mathcal{R}$ invariant. Let $R^\perp$ be the orthogonal complement of R in $L^2(X)$. Since R is isometric to H, then the group of unitary transformations that leave R invariant and fix $R^\perp$ pointwise is isomorphic to $U(H)$. Let $U(H^\perp)$ denote the group of all unitary transformations that leave $R^\perp$ invariant and fix R pointwise. Then, the group of unitary transformations in $U(L^2(X))$ that leave R invariant is isomorphic to the direct sum $U(H) \oplus U(H^\perp)$. Therefore, $\mathcal{R}$ is in one-one correspondence with the left coset space $U(L^2(X))/(U(H) \oplus U(H^\perp))$. By Proposition 3.3, the same can be said for $\overline{F}_0$. Combining this result with the linear deformations that act on the orbits in $F/\operatorname{GL}(H)$, we obtain the following corollary.

Corollary 3.5. *The space of frames F is in one-one correspondence with the left coset space* $\operatorname{GL}(H) \times U(L^2(X))/(U(H) \oplus U(H^\perp))$.

The linear transformations connecting the RK Hilbert spaces in $\mathcal{R}$ can now be pulled back to potentially nonlinear transformations connecting frames in F. Because the orbits in $F/\operatorname{GL}(H)$ are understood, we will focus on pulling back transformations that connect Parseval frames.

Theorem 3.6. *Let $f \in F_0$ be a Parseval frame and $U \in U(L^2(X))$ a unitary operator.*

The map $g : X \mapsto H$ given by

$$g(x) = V_f^* \overline{U}_x V_f f(x)$$

is then a Parseval frame on H as well with analysis map $V_g = UV_f$. Conversely, given any two Parseval frames $f, g \in F_0$, there exists $U \in U(L^2(X))$ such that $V_f^* \overline{U}_x V_f f$ and g are unitarily equivalent (i.e., belong to the same orbit in $F_0/U(H)$).

Proof. By Proposition 3.4, $R' = U(R) \in \mathcal{R}$ with kernel $K'(x, y) = U_x \overline{U}_y K(x, y)$. Since $k'_x \in R'$, then $U^* k'_x \in R$. Define $g : X \mapsto H$ as above. We have

$$\begin{aligned} g(x) &= V_f^* \overline{U}_x V_f f(x) \\ &= \int_X \overline{U}_x \langle f(x), f(y) \rangle f(y) \, d\mu(y) \\ &= \int_X K(y, x) f(y) \, d\mu(y) \\ &= \int_X U_y^* U_y \overline{U}_x K(y, x) f(y) \, d\mu(y) \\ &= \int_X U_y^* K'(y, x) f(y) \, d\mu(y) \\ &= \int_X (U^* k'_x)(y) f(y) \, d\mu(y) \\ &= V_f^* U^* k'_x. \end{aligned}$$

Since f is Parseval, then V_f^* restricted to R is an isometry. Since $U^* k'_x \in R$, then $V_f^* U^* : R' \mapsto H$ is an isometry. In addition, $\{k'_x : x \in X\}$ is a Parseval frame on R' by Proposition 1.8. Thus, by Lemma 3.1, g is a Parseval frame on H. Moreover, since $g(x) = V_f^* U^* k'_x = (UV_f)^* k'_x$, then we see that the analysis map of g is $V_g = UV_f$.

For the converse, let $f, g \in F_0$ be Parseval frames, and let $R = \Theta(f)$ and $R' = \Theta(g)$. By Proposition 3.4, there exists $U \in U(L^2(X))$ such that $R' = U(R)$. By the first part of the proof for this theorem, $V_f^*\overline{U}_x V_f f(x)$ is a Parseval frame with analysis map UV_f. We therefore have

$$
\begin{aligned}
\Theta(V_f^*\overline{U}.V_f f) &= \operatorname{ran}(UV_f) \\
&= \{U\langle \phi, f(\cdot)\rangle : \phi \in H\} \\
&= U(R) \\
&= R' \\
&= \Theta(g).
\end{aligned}
$$

Since $\Theta(V_f^*\overline{U}.V_f f) = \Theta(g)$, then Lemma 3.2 implies that $V_f^*\overline{U}.V_f f$ and g are connected by a linear transformation in $\mathrm{GL}(H)$ (i.e., lie in the same orbit in $F/\mathrm{GL}(H)$. But because $V_f^*\overline{U}.V_f f$ and g are both Parseval, then Lemma 2.10 implies that $V_f\overline{U}.V_f f$ and g are connected by a unitary transformation in $U(H)$. □

In the above proof of Theorem 3.6, the unitary operator U is a transformation between the RK Hilbert spaces R and R' associated to the frames f and g respectively. The map $V_f^*\overline{U}.V_f$ is the "pullback" of U to a transformation between f and g. If $R \neq R'$ (i.e., if U does not map R into itself), then f and g are not linearly connected, in which case $V_f^*\overline{U}.V_f$ is a nonlinear transformation that moves us between the orbits containing f and g in $F/\mathrm{GL}(H)$.

We know how each orbit in $F/\mathrm{GL}(H)$ is held together as discussed in Chapter 2. By Theorem 3.6, we now understand how F_0 is connected. We therefore understand in principle how any two frames F are connected to each other: Given two frames $f, g \in F$, they can be projected to Parseval frames using the map $T : F \mapsto F_0$, and these Parseval frames can then be connected by a possibly nonlinear transformation as given by Theorem 3.6.

But while we now do have a complete understanding of the space of frames F, the transformations $V_f^*\overline{U}.V_f$ as introduced in Theorem 3.6 are too general for specific applications. In the next section, we consider a special case in which such maps reduce to a simpler form and have a more elegant interpretation.

3.2 A Special Case: Frames on a Function Space

In this section, we consider an important example for the Hilbert space H in which the frame transformations appearing in Theorem 3.6 simplify in their appearance and action. Let (W,ν) be a positive Borel measure space, and consider the function space $H = L^2(W)$. Let $f : X \mapsto H$ be a frame on H. Note that for every $x \in X$, $f(x)$ is a function $f(x) : W \mapsto \mathbb{C}$. For every $w \in W$, define the evaluation map

$$f_w : X \mapsto \mathbb{C}, \quad f_w(x) = [f(x)](w),$$

and define the vector space

$$B_0 = \mathrm{span}_{\mathbb{C}}(\{f_w : w \in W\}).$$

Notice that both $L^2(X)$ and B_0 are spaces of functions that map X into $\mathbb{C}$. This is the key to the simplification of nonlinear frame deformations.

For every $x \in X$, define the seminorm

$$\|\cdot\|_x : B_0 \mapsto \mathbb{R}^{\geq 0}, \quad \|h\|_x = |h(x)|,$$

and let B be the completion of B_0 under the collection of seminorms $\{\|\cdot\|_x : x \in X\}$. Therefore, $h \in B$ means that there exists a sequence $\{h_n \in B_0\}_{n=1}^{\infty}$ such that $h_n(x) \to$

$h(x)$ independently for every $x \in X$; the sequence $\{h_n\}_{n=1}^{\infty}$ converges pointwise to h. For this reason, we say that B is endowed with and complete under the *pointwise topology*.

Let $\phi \in H$. By the integral

$$\int_W \phi(w) f_w \, d\nu(w),$$

we mean the limit of a sequence of simple functions whose convergence is taken in the pointwise topology on B. Therefore, the integral is a function in B defined by

$$\left(\int_W \phi(w) f_w \, d\nu(w) \right)(x) = \int_W \phi(w) f_w(x) \, d\nu(w),$$

where the integral on the right side is well-defined in the usual sense. By definition of f_w, we have

$$\begin{aligned} \left(\int_W \phi(w) \overline{f_w} \, d\nu(w) \right)(x) &= \int_W \phi(w) \overline{f_w(x)} \, d\nu(w) \\ &= \int_W \phi(w) \overline{[f(x)](w)} \, d\nu(w) \\ &= \langle \phi, f(x) \rangle_H \, . \end{aligned}$$

The following lemma takes advantage of the fact that both B and $L^2(X)$ are spaces of functions that map X to $\mathbb{C}$. But before the lemma, we make a few remarks on notation: Let A be a continuous linear operator on B. Define $Af : X \mapsto H$ such that

$$[(Af)(x)](w) = (Af_w)(x).$$

In this notation, the operator A acts "directly" on the frame f. In other words, A acts on f with respect to the frame index variable x. Further, we write $\overline{\underline{A}}$ to denote conjugation followed by the action of A followed by conjugation; if A is an operator on $L^2(X)$ and $\alpha \in L^2(X)$, then $(\overline{\underline{A}}\alpha)(x) = \overline{(A\overline{\alpha})(x)}$.

Lemma 3.7. *Let $f \in F$ be a frame on $H = L^2(W)$, and let A be a continuous linear operator on B. Then, for all $\phi \in H$, we have*

$$\langle \phi, (Af)(\cdot) \rangle_H = \overline{\underline{A}} \langle \phi, f(\cdot) \rangle_H .$$

Proof. Let $\phi \in H$. Since A is continuous and linear on B, then

$$\begin{aligned}
\langle \phi, (Af)(\cdot) \rangle_H &= \int_W \phi(w)\overline{[(Af)(\cdot)](w)}\, d\nu(w) \\
&= \int_W \phi(w)\overline{(Af_w)(\cdot)}\, d\nu(w) \\
&= \int_W \phi(w)\overline{A} f_w \, d\nu(w) \\
&= \overline{A} \int_W \overline{\phi(w)} f_w \, d\nu(w) \\
&= \overline{A} \langle f(\cdot), \phi \rangle_H \\
&= \overline{\underline{A}} \langle \phi, f(\cdot) \rangle_H .
\end{aligned}$$

□

Given a unitary operator $U \in U(L^2(X))$, the following proposition shows that the operator $V_f^* U.V_f$ from Theorem 3.6 simplifies significantly in the special case of $H = L^2(W)$. Lemma 3.7 is the heart of the proof.

Proposition 3.8. *Let $f \in F_0$ be a Parseval frame on $H = L^2(W)$. Let $U \in U(L^2(X))$ such that U is continuous on B as well. Then, $V_f^* U_x V_f f(x) = U_x f(x)$, and Uf is a Parseval frame on H.*

Proof. Letting $\phi \in H$, we have

$$\begin{aligned}\left\langle \phi, V_f^* U_x V_f f(x) \right\rangle_H &= \langle V_f \phi, U_x V_f f(x) \rangle_2 \\ &= \int_X \langle \phi, f(y) \rangle_H \overline{U_x \langle f(x), f(y) \rangle_H} \, d\mu(y) \\ &= \int_X \langle \phi, f(y) \rangle_H \overline{U_x} \langle f(y), f(x) \rangle_H \, d\mu(y).\end{aligned}$$

Since U is continuous and linear on B, then Lemma 3.7 implies

$$\begin{aligned}\left\langle \phi, V_f^* U_x V_f f(x) \right\rangle_H &= \int_X \langle \phi, f(y) \rangle_H \langle f(y), U_x f(x) \rangle_H \, d\mu(y) \\ &= \left\langle \int_X \langle \phi, f(y) \rangle_H f(y) \, d\mu(y), U_x f(x) \right\rangle_H .\end{aligned}$$

Since f is Parseval, then the reconstruction property implies

$$\left\langle \phi, V_f^* U_x V_f f(x) \right\rangle_H = \langle \phi, U_x f(x) \rangle_H .$$

But since this holds for all $\phi \in H$, then we conclude that $V_f^* U_x V_f f(x) = U_x f(x)$. It follows that Uf is a Parseval frame by Theorem 3.6. $\square$

Notice that we could not conclude the converse that for any two Parseval frames $f, g \in F_0$, there exists a unitary operator $U \in U(L^2(X))$ such that g and Uf are unitarily equivalent via some operator in $U(H)$. This is because we do not whether such a unitary operator U is also continuous on B. But in examples in which all operators in $U(L^2(X))$ are also continuous on B, then the converse would in fact hold. Such an example in which the converse holds is discussed in Section 3.3.

Proposition 3.8 can be generalized to the acttion of certain non-unitary operators on a frame f. This is given in the following theorem.

Theorem 3.9. *Let $f \in F$ be a frame with frame bounds a and b and associated RK Hilbert space $R = \Theta(f)$. Let A be a continuous linear operator on B such that $\underline{\overline{A}} : R \mapsto L^2(X)$ is a continuous linear injection with a continuous inverse on its range. Then, Af is a frame with frame bounds $\frac{a}{\|\underline{\overline{A}}^{-1}\|_R^2}$ and $b\|\underline{\overline{A}}\|_R^2$.*

Proof. Let $\phi \in H$. By Lemma 3.7, we have $\langle \phi, (Af)(\cdot) \rangle_H = \underline{\overline{A}} \langle \phi, f(\cdot) \rangle_H$. Since $\underline{\overline{A}} : R \mapsto L^2(X)$ is linear, continuous, and has a continuous inverse, then we have

$$\frac{1}{\|\underline{\overline{A}}^{-1}\|_R} \| \langle \phi, f(\cdot) \rangle_H \|_2 \leq \|\underline{\overline{A}} \langle \phi, f(\cdot) \rangle_H \|_2 \leq \|\underline{\overline{A}}\|_R \| \langle \phi, f(\cdot) \rangle_H \|_2,$$

and hence

$$\frac{1}{\|\underline{\overline{A}}^{-1}\|_R} \| \langle \phi, f(\cdot) \rangle_H \|_2 \leq \| \langle \phi, (Af)(\cdot) \rangle_H \|_2 \leq \|\underline{\overline{A}}\|_R \| \langle \phi, f(\cdot) \rangle_H \|_2,$$

where the norm of $\underline{\overline{A}}$ is taken over R since $\langle \phi, f(\cdot) \rangle_2 \in R$. Since f is a frame, then it satisfies the frame condition. Combining this with the above double inequality, we obtain

$$\frac{a}{\|\underline{\overline{A}}^{-1}\|_R^2} \|\phi\|_H \leq \int_X |\langle \phi, (Af)(x) \rangle_H|^2 \, d\mu(x) \leq b\|\underline{\overline{A}}\|_R^2 \|\phi\|_H.$$

Ergo, Af is a frame as claimed. $\square$

Theorem 3.9 admits important special cases. If the operator A commutes with conjugation, then $\underline{\overline{A}} = A$. If in addition $A : R \mapsto L^2(X)$ is unitary and f is Parseval, then Af is a Parseval frame as well.

We will now conclude our discussion on general frame deformations with two examples.

3.3 Example: Finite Frames

Recall the example of finite frames discussed in Section 1.3. Consider the function space $H = L^2(W)$ where $W = \{1, \ldots, N\}$. This function space is simply the finite-dimensional space $H = \mathbb{C}^N$. Let F be the space of all (finite) frames on $\mathbb{C}^N$ indexed by the finite set $X = \{1, \ldots, M\}$ where $M \geq N$. Since the measure on X is finite, then Corollary 2.23 implies that F has the structure of a principle fiber bundle. In this section, we give a complete description of the fiber bundle structure of F including a construction of a base space $\overline{F}_0$ for the fiber bundle.

Let $f \in F$; i.e., $f : \{1, \ldots, M\} \mapsto \mathbb{C}^N$ is a frame. We use the notation $f_m = f(m)$, and we think of the f_m as column vectors in $\mathbb{C}^N$. Define the $N \times M$ matrix

$$\mathbf{f} = \begin{bmatrix} f_1 & \ldots & f_M \end{bmatrix}.$$

We call $\mathbf{f}$ the *frame matrix* of f. Because a finite frame spans the space on which it is a frame, then the column space of $\mathbf{f}$ is precisely $\mathbb{C}^N$. This is in fact the defining property of a frame matrix. Often, we will not distinguish between a frame and its frame matrix; in particular, we will often think of the space of frames F as a space of $N \times M$ frame matrices.

Observe that the space $L^2(X)$ is simply $L^2(\{1, \ldots, M\}) = \mathbb{C}^M$. Each row of the frame matrix $\mathbf{f}$ corresponds to a fixed value in $W = \{1, \ldots, N\}$ and can be thought of as a function mapping $\{1, \ldots, M\}$ to $\mathbb{C}$. Therefore, the row space of $\mathbf{f}$ is the vector space B defined in Section 3.2. If the analysis map $V : \mathbb{C}^N \mapsto \mathbb{C}^M$ is given by

$$V\phi = \phi^\top \overline{\mathbf{f}},$$

where $\phi^\top$ is the transpose of the column vector ϕ and $\overline{\mathbf{f}}$ is the element-wise conjugation of $\mathbf{f}$, then we see that the row space of $\overline{\mathbf{f}}$ is the RK Hilbert space $R = \Theta(f) = \mathrm{ran}(V)$ associated to the frame f. Observe that B and R are both subspaces of $\mathbb{C}^M$. Even though B is by definition endowed with the pointwise topology while R is endowed with the L^2 topology, in finite dimensions these two topologies coincide. Any operator $A \in \mathrm{GL}(\mathbb{C}^M)$ is thus continuous on both B and R, and therefore the direct action of such an operator A on the frame f is well-defined and yields a new frame Af. More generally, the results in Section 3.2 apply to the example of finite frames.

The action of an operator $A \in \mathrm{GL}(\mathbb{C}^N)$ on the frame f can be given by the action of the matrix representation $\mathbf{A}$ of A on the frame matrix $\mathbf{f}$ as follows:

$$Af = \mathbf{A}\mathbf{f} = A \begin{bmatrix} f_1 & \dots & f_M \end{bmatrix}.$$

In words, the matrix $\mathbf{A}$ acts on each frame element f_m separately (strictly speaking, the expression Af is not a frame but rather a frame matrix. But this distinction is not important). In contrast, the action of an operator $A \in \mathrm{GL}(\mathbb{C}^M)$ on f is given by the action of the matrix representation $\mathbf{A}$ of A on the frame matrix $\mathbf{f}$ from the right:

$$Af = \mathbf{f}\mathbf{A} = \begin{bmatrix} f_1 & \dots & f_M \end{bmatrix} \mathbf{A}.$$

This means that $\mathbf{A}$ acts on the frame matrix as a whole with respect to the variable that indexes the frame elements (i.e., the variable that indexes the columns of $\mathbf{f}$). To summarize, “linear frame deformations” act on frame matrices from the left, and “general frame deformations” (i.e., deformations that are not necessarily linear in the space $\mathbb{C}^N$) act on frame matrices from the right.

We are now in a position to interpret Proposition 1.11, which gives a characterization of Parseval frames in terms of the singular value decomposition of their frame matrices. Define $e : \{1, \ldots, M\} \mapsto \mathbb{C}^N$ by $e_m = e(m)$ such that $\{e_m\}_{m=1}^N$ is the standard orthonormal basis on $\mathbb{C}^N$ and $e_m = 0$ for $m > N$. Clearly, e is a Parseval frame. The frame matrix of e is

$$\mathbf{e} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix},$$

where $\mathbf{I}$ is the $N \times N$ identity matrix and $\mathbf{0}$ is the $N \times (M - N)$ zero matrix. By Proposition 1.11, $f : \{1, \ldots, M\} \mapsto \mathbb{C}^N$ is a Parseval frame if and only if its frame matrix is of the form

$$\mathbf{f} = \mathbf{TeU},$$

where $\mathbf{T} \in U(\mathbb{C}^N)$ and $\mathbf{U} \in U(\mathbb{C}^M)$. By Theorem 3.6 and Proposition 3.8, the matrix $\mathbf{U}$ facilitates movement between the different fibers in $F/\operatorname{GL}(\mathbb{C}^N)$. More precisely, given a fiber F_1 in $F/\operatorname{GL}(\mathbb{C}^N)$, there exists $\mathbf{U} \in U(\mathbb{C}^M)$ such that $\mathbf{eU} \in F_1$ (with $\mathbf{eU}$ Parseval as well). By Chapter 2, the matrix $\mathbf{T}$ facilitates movement between the Parseval frames in the fiber containing $\mathbf{eU}$. In this way, the matrices $\mathbf{T}$ and $\mathbf{U}$ allow us to transform $\mathbf{e}$ to any other frame in the space of Parseval frames F_0.

Recall from Chapter 2 that every Parseval frame in a given fiber in $F/\operatorname{GL}(\mathbb{C}^N)$ corresponds to a unique element of $U(\mathbb{C}^N)$. In contrast, it is possible that $\mathbf{eU_1}$ and $\mathbf{eU_2}$ belong to the same fiber in $F/\operatorname{GL}(\mathbb{C}^N)$ for distinct $\mathbf{U_1}, \mathbf{U_2} \in U(\mathbb{C}^M)$. But recalling that a base space $\overline{F}_0$ for the fiber bundle F is a transversal of $F_0/U(\mathbb{C}^N)$, then a construction for $\overline{F}_0$ requires that we find a maximal set of unitarily inequivalent Parseval frames. We must therefore find a subset $Y \subseteq U(\mathbb{C}^M)$ such that for every fiber F_1 in $F/\operatorname{GL}(\mathbb{C}^N)$, exactly one matrix in Y transforms $\mathbf{e}$ into a frame in F_1. To this end, we first need the following lemma.

Lemma 3.10. *Consider the subgroup defined by the direct sum*

$$U(\mathbb{C}^N) \oplus U(\mathbb{C}^{M-N}) = \left\{ \begin{bmatrix} \mathbf{U_{11}} & \mathbf{0} \\ \mathbf{0} & \mathbf{U_{22}} \end{bmatrix} \in U(\mathbb{C}^M) : \mathbf{U_{11}} \in U(\mathbb{C}^N) \text{ and } \mathbf{U_{22}} \in U(\mathbb{C}^{M-N} \right\}.$$

Given $\mathbf{U} \in U(\mathbb{C}^M)$*, we have that* $\mathbf{eU} = \mathbf{Te}$ *for some* $\mathbf{T} \in U(\mathbb{C}^N)$ *if and only if* $\mathbf{U} \in U(\mathbb{C}^N) \oplus U(\mathbb{C}^{M-N})$*.*

Proof. Let $\mathbf{U} = \begin{bmatrix} \mathbf{U_{11}} & \mathbf{0} \\ \mathbf{0} & \mathbf{U_{22}} \end{bmatrix} \in U(\mathbb{C}^N) \oplus U(\mathbb{C}^{M-N})$. We have

$$\begin{aligned} \mathbf{eU} &= \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{U_{11}} & \mathbf{0} \\ \mathbf{0} & \mathbf{U_{22}} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{U_{11}} & \mathbf{0} \end{bmatrix} \\ &= \mathbf{U_{11}} \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix} \\ &= \mathbf{U_{11}e}, \end{aligned}$$

where $\mathbf{U_{11}} \in U(\mathbb{C}^N)$. Therefore, the forward implication holds.

For the converse, let $\mathbf{U} = \begin{bmatrix} \mathbf{U_{11}} & \mathbf{U_{12}} \\ \mathbf{U_{21}} & \mathbf{U_{22}} \end{bmatrix} \in U(\mathbb{C}^M)$ where U_{11} is $N \times N$. Let $\mathbf{T} \in U(\mathbb{C}^N)$, and suppose that

$$\mathbf{eU} = \mathbf{Te}.$$

Expanding this and multiplying the block matrices, we have

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{U_{11}} & \mathbf{U_{12}} \\ \mathbf{U_{21}} & \mathbf{U_{22}} \end{bmatrix} = \mathbf{T} \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix}.$$

$$\begin{bmatrix} \mathbf{U_{11}} & \mathbf{U_{12}} \end{bmatrix} = \begin{bmatrix} \mathbf{T} & \mathbf{0} \end{bmatrix}.$$

Therefore, $\mathbf{U_{11}} = \mathbf{T}$ and $\mathbf{U_{12}} = \mathbf{0}$. But since $\mathbf{U}$ is unitary, then

$$\begin{aligned} \mathbf{U}\mathbf{U}^* &= \begin{bmatrix} \mathbf{T} & \mathbf{0} \\ \mathbf{U_{21}} & \mathbf{U_{22}} \end{bmatrix} \begin{bmatrix} \mathbf{T}^* & \mathbf{U_{21}}^* \\ \mathbf{0}^* & \mathbf{U_{22}}^* \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{I} & \mathbf{T}\mathbf{U_{21}}^* \\ \mathbf{U_{21}}\mathbf{T}^* & \mathbf{U_{22}}\mathbf{U_{22}}^* \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \end{aligned}$$

where $\mathbf{T}\mathbf{T}^* = \mathbf{I}$ since $\mathbf{T}$ is unitary. We therefore see that $\mathbf{U_{21}}\mathbf{T}^* = \mathbf{0}$. But since $\mathbf{T}^*$ is invertible, then $\mathbf{U_{21}} = \mathbf{0}$. Further, $\mathbf{U_{22}}\mathbf{U_{22}}^* = \mathbf{I}$, meaning that $\mathbf{U_{22}} \in U(\mathbb{C}^{M-N})$. Ergo, $\mathbf{U} \in U(\mathbb{C}^N) \oplus U(\mathbb{C}^{M-N})$. □

The group $U(\mathbb{C}^N) \oplus U(\mathbb{C}^{M-N})$ is therefore the subgroup whose right action does not transform $\mathbf{e}$ into a different fiber. We therefore expect that by "modding out" $U(\mathbb{C}^N) \oplus U(\mathbb{C}^{M-N})$ from $U(\mathbb{C}^M)$, we will be left with transformations that connect unitarily inequivalent Parseval frames (i.e., Parseval frames in different fibers) in a unique way. Since the matrices in $U(\mathbb{C}^M)$ act on $\mathbf{e}$ from the right, then let $U(\mathbb{C}^M)/(U(\mathbb{C}^N) \oplus U(\mathbb{C}^{M-N}))$ be a right coset space. Observe that $\mathbb{C}^{M-N}$ is isometric to the orthogonal complement of $\mathbb{C}^N$ in $\mathbb{C}^M$. Recalling that $H = \mathbb{C}^N$ and $L^2(X) = \mathbb{C}^M$, the right coset space is of the form $U(L^2(X))/(U(H) \oplus U(H^\perp))$. By Corollary 3.5 and the discussion preceding it, a base space $\overline{F}_0$ can be placed in one-one correspondence with any transversal of the above right coset space. We therefore have the following proposition.

Proposition 3.11. *Let Y be any transversal of the right coset space $U(\mathbb{C}^M)/(U(\mathbb{C}^N) \oplus U(\mathbb{C}^{M-N}))$. Then, the set*

$$\overline{F}_0 = \{\mathbf{e}\mathbf{U} : \mathbf{U} \in Y\}$$

is a maximal set of unitarily inequivalent Parseval frames on $\mathbb{C}^N$*.*

Proof. First, note that $\{\mathbf{eU} : \mathbf{U} \in Y\}$ is a set of Parseval frames. To show this set is maximal, we need to show that every fiber of F (i.e., every orbit in $F/\operatorname{GL}(\mathbb{C}^N)$) contains at least one frame in $\{\mathbf{eU} : \mathbf{U} \in Y\}$. To do this, let $\mathbf{f} \in F_0$ be a Parseval frame. We will show that there exists $\mathbf{U} \in Y$ such that $\mathbf{eU}$ belongs to the same fiber as $\mathbf{f}$, by which we mean $\mathbf{f} = \mathbf{TeU}$ for some $\mathbf{T} \in U(\mathbb{C}^N)$. By Proposition 1.11, we have the decomposition

$$\mathbf{f} = \mathbf{T}'\mathbf{eU}',$$

for some $\mathbf{T}' \in U(\mathbb{C}^N)$ and $\mathbf{U}' \in U(\mathbb{C}^M)$. By definition of Y, $\mathbf{U}'$ can be factored into $\mathbf{U}' = \mathbf{U}''\mathbf{U}$ where $\mathbf{U}'' \in U(\mathbb{C}^N) \oplus U(\mathbb{C}^{M-N})$ and $\mathbf{U} \in Y$. We have

$$\mathbf{f} = \mathbf{T}'\mathbf{eU}''\mathbf{U}.$$

But since $\mathbf{U}'' \in U(\mathbb{C}^N) \oplus U(\mathbb{C}^{M-N})$, then Lemma 3.10 implies that $\mathbf{eU}'' = \mathbf{T}''\mathbf{e}$ for some $\mathbf{T}'' \in U(\mathbb{C}^N)$. We therefore have

$$\mathbf{f} = \mathbf{T}'\mathbf{T}''\mathbf{eU}.$$

Since $\mathbf{T}'\mathbf{T}'' \in U(\mathbb{C}^N)$, then we see that $\mathbf{eU}$ belongs to the same fiber as $\mathbf{f}$, where $\mathbf{U} \in Y$. Therefore, $\{\mathbf{eU} : \mathbf{U} \in Y\}$ contains a Parseval frame from every fiber in F.

We now show that no two distinct frames in $\{\mathbf{eU} : \mathbf{U} \in Y\}$ are unitarily equivalent. Let $\mathbf{U_1}\mathbf{U_2} \in Y$, and suppose that

$$\mathbf{eU_1} = \mathbf{TeU_2},$$

where $\mathbf{T} \in U(\mathbb{C}^N)$. We then have

$$\mathbf{e}(\mathbf{U_1}\mathbf{U_2}^{-1}) = \mathbf{Te}.$$

But by Lemma 3.10, this means that $\mathbf{U_1}\mathbf{U_2}^{-1} \in U(\mathbb{C}^N) \oplus U(\mathbb{C}^{M-N})$ and hence that $\mathbf{U_1}$ and $\mathbf{U_2}$ belong to the same right coset in $U(\mathbb{C}^M)/(U(\mathbb{C}^N) \oplus U(\mathbb{C}^{M-N}))$. But since $\mathbf{U_1}$ and $\mathbf{U_2}$ are also elements of the transversal Y, then we must have $\mathbf{U_1} = \mathbf{U_2}$. Therefore, every frame in $\{\mathbf{eU} : \mathbf{U} \in Y\}$ is unitarily equivalent to no other frame in the set. □

The space $\overline{F}_0$ constructed in Proposition 3.11 is therefore a base space for the fiber bundle F. This construction is consistent with Corollary 3.5. More generally, recalling the polar decomposition $\mathrm{GL}(\mathbb{C}^N) = \mathrm{GL}^+(\mathbb{C}^N)U(\mathbb{C}^N)$, any frame matrix $\mathbf{f} \in F$ has the unique factorization

$$\mathbf{f} = \mathbf{PTeU},$$

where $\mathbf{P} \in \mathrm{GL}^+(\mathbb{C}^N)$, $\mathbf{T} \in U(\mathbb{C}^N)$, and $\mathbf{U} \in Y$.

3.4 Example: Discretization of the Gabor Frame

Let $f : \mathbb{R}^2 \mapsto H$ be the Gabor frame on the space $H = L^2(\mathbb{R})$ with a band-limited window function $\psi \in H$. Recall from Section 1.4 that f is given by

$$[f(q,p)](x) = e^{i2\pi px}\psi(x-q).$$

Suppose that ψ is a band-limited window function such that $\hat{\psi} = \mathcal{F}(\psi)$ is supported on a unit interval $[Q, Q+1] \subset \mathbb{R}$.

In this section, we project the frame f onto certain closed subspaces of H consisting of compactly supported functions. We show that the projected frames admit discrete subframes. We accomplish this by applying a sampling operator directly on the projected frames and showing that the sampling operator satisfies the hypotheses of Theorem 3.9 thanks to the band-limitedness of the window function ψ and the Petersen-Middleton Sampling Theorem.

For every $n \in \mathbb{Z}$, define the subspace

$$H_n = \{\phi \in H : \text{support}(\phi) = [n, n+1]\}.$$

Observe that H_n is a closed subspace of H and is therefore a Hilbert space. In fact, H_n is isometric to $L^2([n, n+1])$. Define the orthogonal projection $P_n : H \mapsto H_n$ by

$$(P_n\phi)(x) = \chi_{[n,n+1]}(x)\phi(x).$$

Since $P_n P_m = \delta_{nm} I$, then we have the orthogonal decomposition

$$H = \bigoplus_{n\in\mathbb{Z}} H_n.$$

Lemma 3.12. *The map $P_n f : \mathbb{R}^2 \mapsto H_n$ defined by*

$$\begin{aligned}(P_n f)(q,p) &= P_n[f(q,p)] \\ &= e^{i2\pi px}\psi(x-q)\chi_{[n,n+1]}(x)\end{aligned}$$

is a Parseval frame on H_n.

Proof. Given any $\phi \in H_n$, we have

$$\begin{aligned}\int_{\mathbb{R}^2} |\langle \phi, P_n f(q,p)\rangle_H|^2\, dq\, dp &= \int_{\mathbb{R}^2} |\langle P_n\phi, f(q,p)\rangle_H|^2\, dq\, dp \\ &= \int_{\mathbb{R}^2} |\langle \phi, f(q,p)\rangle_H|^2\, dq\, dp \\ &= \|\phi\|_H^2.\end{aligned}$$

$\square$

Define the sampling operator $Z : L^2(\mathbb{R}^2) \mapsto L^2(\mathbb{R}^2)$ by

$$(Z\alpha)(q,p) = \alpha(\lfloor q \rfloor, \lfloor p \rfloor).$$

Although $Z\alpha$ has a continuous domain, we think of $Z\alpha$ as discrete since its values are restricted to a countable set.

We proceed to apply Z directly onto $P_n f$ to extract a discrete Parseval frame from $P_n f$ on H_n.

Proposition 3.13. *The map $ZP_n f : \mathbb{R}^2 \mapsto H_n$ given by*

$$[(ZP_n f)(q,p)](x) = e^{i2\pi \lfloor p \rfloor x} \psi(x - \lfloor q \rfloor) \chi_{[n,n+1]}(x)$$

is a Parseval frame on H_n.

Proof. We will show that Z restricted to $\Theta(P_n f)$ satisfies the hypotheses of Theorem 3.9.

First, let $\alpha \in L^2(\mathbb{R}^2)$, and observe that

$$\begin{aligned}
(\overline{Z}\alpha)(q,p) &= \overline{Z\overline{\alpha})}(q,p) \\
&= \overline{\overline{\alpha}(\lfloor q \rfloor, \lfloor p \rfloor)} \\
&= \alpha(\lfloor q \rfloor, \lfloor p \rfloor) \\
&= (Z\alpha)(q,p).
\end{aligned}$$

Therefore, $\overline{Z} = Z$.

Recall the space B from Section 3.2. Clearly, Z is linear on B. Let $\{b_n\}_{n=1}^{\infty}$ be a sequence of functions in B with $b_n \to b \in B$. Since B is endowed with the pointwise topology, then $b_n(q,p) \to b(q,p)$ for all $(q,p) \in \mathbb{R}^2$. In particular, $b_n(q,p) \to b(q,p)$ for all $(q,p) \in \mathbb{Z}^2$. Thus, $Zb_n \to Zb$ in B, and hence Z is a continuous linear operator on B.

Let $\mathcal{F}$ be the Fourier transform on $L^2(\mathbb{R}^2)$. Let $R_n = \Theta(P_n f)$ be the RK Hilbert space associated to $P_n f$. Let $\phi_n \in H_n$ so that $\langle \phi_n, (P_n f)(\cdot,\cdot)\rangle_H \in R_n$, and observe that

$$\begin{aligned}
(\mathcal{F}\langle \phi_n, P_n f(\cdot,\cdot)\rangle_H)(\hat{q},\hat{q}) &= (\mathcal{F}\langle P_n\phi_n, f(\cdot,\cdot)\rangle_H)(\hat{q},\hat{q}) \\
&= (\mathcal{F}\langle \phi_n, f(\cdot,\cdot)\rangle_H)(\hat{q},\hat{q}) \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}} \phi_n(x)e^{-i2\pi px}\overline{\psi}(x-q)e^{-i2\pi q\hat{q}}e^{-i2\pi p\hat{q}}\,dx\,dq\,dp \\
&= \int_{\mathbb{R}^2} \phi_n(x)\left(\int_{\mathbb{R}} \overline{\psi}(x-q)e^{-i2\pi q\hat{q}}\,dq\right)e^{-i2\pi p(\hat{q}+x)}\,dp\,dx \\
&= \int_{\mathbb{R}^2} \phi_n(x)\overline{\left(\int_{\mathbb{R}} \psi(q)e^{i2\pi(x-q)\hat{q}}\,dq\right)}e^{-i2\pi p(\hat{q}+x)}\,dp\,dx \\
&= \int_{\mathbb{R}^2} \phi_n(x)e^{-i2\pi\hat{q}x}\overline{\left(\int_{\mathbb{R}} \psi(q)e^{-i2\pi q\hat{q}}\,dq\right)}e^{-i2\pi p(\hat{q}+x)}\,dp\,dx \\
&= \int_{\mathbb{R}^2} \phi_n(x)e^{-i2\pi\hat{q}x}\overline{\hat{\psi}}(\hat{q})e^{-i2\pi p(\hat{q}+x)}\,dp\,dx \\
&= \int_{\mathbb{R}} \phi_n(x)e^{-i2\pi\hat{q}x}\overline{\hat{\psi}}(\hat{q})\delta(\hat{q}+x)\,dx \\
&= \phi_n(-\hat{q})\overline{\hat{\psi}}(\hat{q})e^{i2\pi\hat{q}\hat{q}},
\end{aligned}$$

which is supported on the unit square $\{(\hat{q},\hat{q}) \in [Q,Q+1]\times[-(n+1),-n]\}$. Thus, the functions in R_n are band-limited with a Fourier spectrum bounded in a unit square. By the Petersen-Middleton Sampling Theorem, the sampling operator Z is invertible on R_n. More precisely, letting $\mathcal{F}_n$ denote the Fourier coefficient operator on functions supported on $[Q,Q+1]\times[-(n+1),-n]$, then $Z : R_n \mapsto L^2(\mathbb{R}^2)$ is given by $Z = \mathcal{F}_n \circ \mathcal{F}$.

By the Plancherel Theorem and Parseval's Identity, $\mathcal{F}$ and $\mathcal{F}_n$ are both unitary, and hence Z is unitary on R_n. By Theorem 3.9, ZP_nf is a frame on H_n. Moreover, since Z is unitary on R_n, then Theorem 3.9 implies that ZP_nf is in fact a Parseval frame on H_n. □

Although the frame ZP_nf is indexed by the continuous set $\mathbb{R}^2$, it is in fact a discrete frame since the set $(ZP_nf)(\mathbb{R}^2)$ is countable. In fact, ZP_nf can equivalently be indexed by the discrete set $\mathbb{Z}^2$. See Section 1.4 for details.

Proposition 3.13 can be extended to larger subspaces of H. For every $N \in \mathbb{Z}^+$, define the subspace

$$\begin{aligned} H^{(N)} &= \bigoplus_{n=-N}^{N} H_n \\ &= \{\phi \in H : \text{support}(\phi) = [-N, N+1]\}. \end{aligned}$$

The space $H^{(N)}$ is a Hilbert space isometric to $L^2([-N, N+1])$. We have the orthogonal projection $P^{(N)} : H \mapsto H^{(N)}$ defined by

$$(P^{(N)}\phi)(x) = \phi(x)\chi_{[-N,N+1]}(x).$$

By mimicking the proof of Lemma 3.12, it is easy to show that the projected frame $(P^{(N)}f) : \mathbb{R}^2 \mapsto H^{(N)}$ is a Parseval frame on $H^{(N)}$. In the next proposition, we discretize the frame $P^{(N)}f$.

Proposition 3.14. *The map $ZP^{(N)}f : \mathbb{R}^2 \mapsto H^{(N)}$ given by*

$$[(ZP^{(N)}f)(q,p)](x) = e^{i2\pi\lfloor p\rfloor x}\psi(x - \lfloor q\rfloor)\chi_{[-N,N+1]}(x)$$

is a frame on $H^{(N)}$.

Proof. We maintain the notation introduced in the proof of Proposition 3.13. We will show that Z restricted to $\Theta(P^{(N)}f)$ satisfies the hypotheses of Theorem 3.9.

Let $R = \Theta(f)$ be the RK Hilbert space associated to the frame f. Let $\phi \in H$ so that $\langle \phi, f(\cdot,\cdot)\rangle_H \in R$. Since H is a direct sum of the H_n, then $\phi = \sum_n \phi_n$ with $\phi_n \in H_n$. Thus,

$$\begin{aligned}
\langle \phi, f(\cdot,\cdot)\rangle_H &= \left\langle \sum_n \phi_n, f(\cdot,\cdot) \right\rangle_H \\
&= \sum_n \langle \phi_n, f(\cdot,\cdot)\rangle_H \\
&= \sum_n \langle P_n phi_n, f(\cdot,\cdot)\rangle_H \\
&= \sum_n \langle \phi_n P_n f(\cdot,\cdot)\rangle_H .
\end{aligned}$$

Note $\langle \phi_n, P_n f(\cdot,\cdot)\rangle_H \in R_n$. We therefore have the decomposition

$$R = \bigoplus_{n\in\mathbb{Z}} R_n.$$

Moreover, since f is Parseval, then the analysis map of f that maps H onto R is an isometry. As a consequence, since H_n and H_m are orthogonal for $n \neq m$, then R_n and R_m are orthogonal as well for $n \neq m$.

We first show that Z is injective on R. It suffices to show that Z maps distinct nonzero elements of R_n and R_m for $n \neq m$ to distinct outputs. Without loss of generality, we assume $m = 0$. Let $\phi, \phi' \in H_0$. Let $\phi'_n \in H_n$ be given by $\phi'_n(x) = \phi'(x-n)$. Then, $\langle \phi, f(\cdot,\cdot)\rangle_H \in R_0$ and $\langle \phi'_n, f(\cdot,\cdot)\rangle_H \in R_n$. Suppose that

$$Z \langle \phi, f(\cdot,\cdot)\rangle_H = Z \langle \phi'_n, f(\cdot,\cdot)\rangle_H .$$

Since $Z = \mathcal{F}_n \circ \mathcal{F}$ on R_n, then

$$\mathcal{F}_0 \circ \mathcal{F} \langle \phi, f(\cdot,\cdot) \rangle_H = \mathcal{F}_n \circ \mathcal{F} \langle \phi'_n, f(\cdot,\cdot) \rangle_H .$$

Define the translation operator $T_n : L^2(\mathbb{R}^2) \mapsto L^2(\mathbb{R}^2)$ by

$$(T_n \alpha)(q,p) = \alpha(q, p-n).$$

The support of functions in both $\mathcal{F}(R_0)$ and $T_n \circ \mathcal{F}(R_n)$ is the unit square $[Q, Q+1] \times [-1, 0]$. Further, since Fourier coefficients are invariant under periodic translations, then $\mathcal{F}_n = \mathcal{F}_0 \circ T_n$. We therefore have

$$\mathcal{F}_0 \circ \mathcal{F} \langle \phi, f(\cdot,\cdot) \rangle_H = \mathcal{F}_0 \circ T_n \circ \mathcal{F} \langle \phi'_n, f(\cdot,\cdot) \rangle_H .$$

since $\mathcal{F}_0$ is unitary on the space of functions supported on $[Q, Q+1] \times [-1, 0]$, then

$$\mathcal{F} \langle \phi, f(\cdot,\cdot) \rangle_H = T_n \circ \mathcal{F} \langle \phi'_n, f(\cdot,\cdot) \rangle_H .$$

More explicitly, we have

$$(\mathcal{F} \langle \phi, f(\cdot,\cdot) \rangle_H)(\hat{q}, \hat{q}) = (\mathcal{F} \langle \phi'_n, f(\cdot,\cdot) \rangle_H)(\hat{q}, \hat{q} - n).$$

By the computation performed in the proof of Proposition 3.13, these Fourier transforms evaluate to

$$\phi(-\hat{q})\overline{\hat{\psi}}(\hat{q})e^{i2\pi\hat{q}\hat{q}} = \phi'_n(-(\hat{q}-n))\overline{\hat{\psi}}(\hat{q})e^{i2\pi(\hat{q}-n)\hat{q}}.$$

Simplifying and using the definition of ψ'_n, we have

$$\phi(-\hat{q}) = \phi'((-\hat{q}+n)-n)e^{-i2\pi n\hat{q}},$$

and hence

$$\phi(-\hat{q}) = \phi'(-\hat{q})e^{-i2\pi n\hat{q}}.$$

Since the left side is independent of $\hat{q}$, then we must have $n = 0$ and hence $\phi = \phi' = \phi'_n$. Therefore,

$$\langle \phi, f(q,p) \rangle_H = \langle \phi'_n, f(q,p) \rangle_H \,,$$

and thus more generally Z maps R_n and R_m for $n \neq m$ to distinct spaces in its range. Ergo, $Z : R \mapsto L^2(\mathbb{R}^2)$ is a linear injection and is therefore invertible on its range.

From Proposition 3.13, we already know that Z is continuous on B. Let $R^{(N)} = \Theta(P^{(N)}f)$ be the RK Hilbert space associated to $P^{(N)}f$. Just as we showed that R can be decomposed into a direct sum of the spaces R_n, it is easy to show that

$$R^{(N)} = \bigoplus_{n=-N}^{N} R_n.$$

Since Z is continuous on the spaces R_n, then by linearity it immediately follows that Z is continuous on $R^{(N)}$. By similar reasoning, the inverse Z^{-1} is continuous on $Z(R^{(N)})$. By Theorem 3.9, we conclude that $ZP^{(N)}f$ is a frame on $H^{(N)}$. □

In generalizing from the space R_n to $R^{(N)}$, we were not able to say whether Z is unitary on $R^{(N)}$ or not. for this reason, we cannot conclude that $ZP^{(N)}f$ is Parseval. Also, we showed that Z is continuous on $R^{(N)}$ but not on all of R. If, however, it held that Z was continuous on R and since Z is injective on all of R, then it would follow that Zf is a frame on the space H. This would be a discretization of the original Gabor frame.

Let us make the observation that

$$\begin{aligned}
[(ZP^{(N)}f)(q,p)](x) &= e^{i2\pi\lfloor p\rfloor x}\psi(x-\lfloor q\rfloor)\chi_{[-N,N+1]}(x) \\
&= \left(e^{i2\pi\lfloor p\rfloor x}\psi(x-\lfloor q\rfloor)\right)\chi_{[-N,N+1]}(x) \\
&= [(P^{(N)}Zf)(q,p)](x).
\end{aligned}$$

For all $\phi \in H^{(N)}$, we have

$$\begin{aligned}
\left\langle \phi, ZP^{(N)}f(\cdot,\cdot)\right\rangle_H &= \left\langle \phi, P^{(N)}Zf(\cdot,\cdot)\right\rangle_H \\
&= \left\langle P^{(N)}\phi, Zf(\cdot,\cdot)\right\rangle_H \\
&= \langle \phi, Zf(\cdot,\cdot)\rangle_H .
\end{aligned}$$

This implies that for all $N \in \mathbb{Z}^+$, the map $Zf : \mathbb{R}^2 \mapsto H$ can be used to decompose and reconstruct elements of $H^{(N)}$ and can therefore be thought of as a type of "pseudoframe"; it cannot be called a frame since the frame elements of Zf are not all contained in $H^{(N)}$. It does not follow that Zf is a pseudoframe on the dense subspace $H^{(\infty)}$ of all compactly supported functions in H; this is because the use of Zf for the reconstruction of vectors in $H^{(N)}$ may require a different frame operator for each N. Indeed, Proposition 3.14 does not provide the frame bounds of $ZP^{(N)}f$, which may vary with N. Therefore, it is possible that in the limit that $N \to \infty$, Zf may fail to be even a pseudoframe on $H^{(\infty)}$.

But it turns out that Zf is a Parseval frame on H if we impose additional assumptions on the window function ψ. In fact, the space of band-limited window functions for which Zf is Parseval is very small, as the next proposition reveals.

Proposition 3.15. *Let f be the Gabor frame with a band-limited window function ψ*

such that $\hat{\psi}$ is supported on $\left[-\frac{1}{2}, \frac{1}{2}\right]$. Then, Zf is a Parseval frame on H if and only if ψ is given by

$$\psi(x) = c\,\mathrm{sinc}(\pi x) = \frac{c \sin(\pi x)}{\pi x},$$

where c is any constant in $\mathbb{C}$.

Proof. We maintain the notation used in the proof of Proposition 3.14. Suppose that Zf is a Parseval frame. Letting $\phi \in H$, the frame condition implies

$$\| \langle \phi, (Zf)(\cdot,\cdot) \rangle_H \|_2 = \|\phi\|_H.$$

Since Z is continuous on the space B, then

$$\|Z \langle \phi, f(\cdot,\cdot) \rangle_H \|_2 = \|\phi\|_H.$$

Since f is Parseval, then we also have

$$\| \langle \phi, f(\cdot,\cdot) \rangle_H \|_2 = \|\phi\|_H,$$

and hence

$$\|Z \langle \phi, f(\cdot,\cdot) \rangle_H \|_2 = \| \langle \phi, f(\cdot,\cdot) \rangle_H \|_2.$$

That is, $\|Z\alpha\|_2 = \|\alpha\|_2$ for all $\alpha \in R$. Thus, $Z : R \mapsto L^2(\mathbb{R}^2)$ is unitary.

Let $\phi \in H_0$ with $\phi \neq 0$. Define $\phi_n \in H_n$ by $\phi_n(x) = \phi(x - n)$. Suppose $n \neq 0$. Then, $\langle \phi, \phi_n \rangle_H = 0$. Since $V_f : H \mapsto R$ is an isometry, then

$$\langle \langle \phi, f(\cdot,\cdot) \rangle_H , \langle \phi_n, f(\cdot,\cdot) \rangle_H \rangle_2 = 0.$$

Since Z is unitary, then

$$\langle Z \langle \phi, f(\cdot,\cdot) \rangle_H , Z \langle \phi_n, f(\cdot,\cdot) \rangle_H \rangle_2 = 0.$$

But by Proposition 3.13, Z is given by $Z = \mathcal{F}_n \circ \mathcal{F}$ on R_n. Thus,

$$\left\langle \mathcal{F}_0 \circ \mathcal{F} \left\langle \phi, f(\cdot,\cdot) \right\rangle_H, \mathcal{F}_n \circ \mathcal{F} \left\langle \phi_n, f(\cdot,\cdot) \right\rangle_H \right\rangle_2 = 0.$$

Recalling that $\mathcal{F}_n = \mathcal{F}_0 \circ T_n$ where T_n is a translation operator defined in the proof of Proposition 3.14, we have

$$\left\langle \mathcal{F}_0 \circ \mathcal{F} \left\langle \phi, f(\cdot,\cdot) \right\rangle_H, \mathcal{F}_0 \circ T_n \circ \mathcal{F} \left\langle \phi_n, f(\cdot,\cdot) \right\rangle_H \right\rangle_2 = 0.$$

But since $\mathcal{F}_0$ is unitary on the spaces of functions supported on $\left[-\frac{1}{2}, \frac{1}{2}\right] \times [-1, 0]$, then

$$\left\langle \mathcal{F} \left\langle \phi, f(\cdot,\cdot) \right\rangle_H, T_n \circ \mathcal{F} \left\langle \phi_n, f(\cdot,\cdot) \right\rangle_H \right\rangle_2 = 0.$$

Expanding this out, we have

$$\int_{\mathbb{R}^2} (\mathcal{F} \left\langle \phi, f(\cdot,\cdot) \right\rangle_H)(\hat{q}, \hat{q}) \overline{(\mathcal{F} \left\langle \phi_n, f(\cdot,\cdot) \right\rangle_H)(\hat{q}, \hat{q} - n)} \, d\hat{q} \, d\hat{q} = 0.$$

Recalling the expression for $\mathcal{F} \left\langle \phi, f(\cdot,\cdot) \right\rangle_H$ obtained in Proposition 3.13, this becomes

$$\int_{\mathbb{R}^2} \phi(-\hat{q}) \overline{\hat{\psi}}(\hat{q}) e^{i2\pi \hat{q}\hat{q}} \overline{\phi_n}(-(\hat{q} - n)) \hat{\psi}(\hat{q}) e^{-i2\pi(\hat{q}-n)\hat{q}} \, d\hat{q} \, d\hat{q} = 0.$$

Simplifying and using the definition of ϕ_n', we obtain

$$\int_{\mathbb{R}^2} \phi(-\hat{q}) \overline{\phi}(-\hat{q}) |\hat{\psi}(\hat{q})|^2 e^{i2\pi n\hat{q}} \, d\hat{q} \, d\hat{q} = 0,$$

which is equivalently

$$\|\phi\|_H \int_{\mathbb{R}} |\hat{\psi}(\hat{q})|^2 e^{i2\pi n\hat{q}} \, d\hat{q} = 0.$$

But since $\phi \neq 0$, then $\|\phi\|_H \neq 0$ so that

$$\int_{\mathbb{R}} |\hat{\psi}(\hat{q})|^2 e^{i2\pi n\hat{q}} \, d\hat{q} = 0.$$

The integral on the left side is the nth Fourier coefficient of the function $|\hat{\psi}(\cdot)|^2$. Since the Fourier coefficients of $|\hat{\psi}(\cdot)|^2$ are 0 for all $n \neq 0$, then we must have that $|\hat{\psi}(\hat{q})|^2 = |c|^2$ for some constant $c \in \mathbb{C}$. Recalling that $\hat{\psi}$ has support $\left[-\frac{1}{2}, \frac{1}{2}\right]$, we find that $\hat{\psi} : \mathbb{R} \mapsto \mathbb{C}$ is given by

$$\hat{\psi}(\hat{x}) = c\chi_{\left[-\frac{1}{2}, \frac{1}{2}\right]}(\hat{x}),$$

where we use the variable $\hat{x}$ since $\hat{\psi}$ is the Fourier transform of ψ. It can be easily verified that the function whose Fourier transform is the characteristic function given above is

$$\psi(x) = c\,\mathrm{sinc}(\pi x) = \frac{c\sin(\pi x)}{\pi x}.$$

For the converse implication, we simply observe that if $\psi(x) = c\,\mathrm{sinc}(\pi x)$ for any $c \in \mathbb{C}$, then reversing the above steps leads to the conclusion that Z is unitary on R and therefore Zf is Parseval by Theorem 3.9. $\square$

4 FUTURE DIRECTIONS

We have seen that the fiber bundle structure of the space of frames on a Hilbert space has the potential to provide an appropriate context for the complete study of frame deformations. Future work will include refinement of the notation used to express nonlinear frame deformations and further investigation of applications and examples of such deformations. In Chapter 3, we denote the action of an operator A followed by conjugation as $\overline{A}$. But conjugation is usually packaged together with transposition in the combined operation of "conjugate transpose". For this reason, while our notation is accurate, it does not elegantly mesh with the usual conventions for the inner product and traditional tensor notation. In future work, we plan to clarify our notation. After that, we will see whether we can extend the example of the discretization of the Gabor frame. In Section 3.4, we describe the discretization of the Gabor frame on Hilbert spaces of functions supported on a fixed compact set. We will investigate whether our approach can be used to discretize the Gabor frame on the entire space $L^2(\mathbb{R})$. Furthermore, because band-limited functions decay slowly, they are not suited for computational applications. For this reason, we will also explore discretization in the case that the window function of the Gabor frame is not band-limited. Finally, we will consider possible applications of our current work on frames to the field of machine learning.

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