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Constructions & Optimization in Classical Real Analysis Theorems

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by

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ABSTRACT

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This thesis takes a closer look at three fundamental Classical Theorems in Real Analysis.

First, for the Bolzano Weierstrass Theorem, we will be interested in constructing a convergent subsequence from a non-convergent bounded sequence. Such a subsequence is guaranteed to exist, but it is often not obvious what it is, e.g., if $a_n = \sin n$.

Next, the Hölder Inequality gives an upper bound, in terms of $p \in [1, \infty]$, for the the integral of the product of two functions. We will find the value of p that gives the best (smallest) upper-bound, focusing on the Beta and Gamma integrals.

Finally, for the Weierstrass Polynomial Approximation, we will find the degree of the approximating polynomial for a variety of functions. We choose examples in which the approximating polynomial does far worse than the Taylor polynomial, but also work with continuous non-differentiable functions for which a Taylor expansion is impossible.

DEDICATION

I love to dedicate this thesis to the soul of my Father Jilali Elallam and to each of my wonderful kids Elias Elallam and Sarah Elallam.

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TABLE OF CONTENTS

1 INTRODUCTION

In the standard upper-level or first year graduate course in Real Analysis, three of the theorems presented are (i) the Bolzano-Weierstrass theorem, which states that any bounded real sequence has a convergent subsequence; (ii) the Hölder inequality, which bounds the integral of the product of two functions by a function of $p \in [0, \infty]$; and (iii) the Weierstrass polynomial approximation theorem of a continuous function.

Little attention is paid in the first case to exhibiting and constructing the convergent subsequence in (i) beyond providing easy examples such as $a_n = (-1)^n$. In this thesis, in Chapter 2, we remedy this by developing techniques for doing so, given "standard" sequences such as $\sin(n)$ and $\langle \sqrt{n} \rangle$, where $\langle a_n \rangle$ denotes the fractional part of a_n. This is a problem of *construction*.

In Chapter 3, we focus on the well-known Beta and Gamma integrals, both of which are the integrals of the product of two functions, and find which values of p yield the best upper bound. This is essentially a problem of *optimization*.

Finally, in Chapter 4, we return to construction, and use lemmas on the modulus of continuity to find the degree of the approximating polynomial in Weierstrass' polynomial. After showing how the Taylor polynomial does significantly better in the case of smooth functions, we find approximating polynomials for nowhere differentiable functions like the original function of Weierstrass, and sample paths of Brownian motion.

The following theorem is an example of a theorem which is proved in Chapter 3:

Theorem 3.2

For any a, denoting the minimum value of p for Hölder's inequality in the incomplete gamma function by $\phi(A, a)$, we have that

$$\lim_{A \to \infty} \phi(A, a) = 1.$$

2 BOLZANO WEIERSTRASS THEOREM

In real analysis, the Bolzano-Weierstrass theorem, named after Bernard Bolzano and Karl Weierstrass, is a fundamental result about convergence in the finite-dimensional Euclidean space $\mathbb{R}^n$. The theorem states that each bounded sequence in $\mathbb{R}^n$ has a convergent subsequence. An equivalent formulation is that a subset of $\mathbb{R}^n$ is sequentially compact if and only if it is closed and bounded, in which case it is sometimes called the sequential compactness theorem. It was actually first proved by Bolzano in 1817 as a lemma in the proof of the intermediate value theorem. Some fifty years later the result was identified as significant in its own right, and proved again by Weierstrass. It has since become an essential theorem of analysis. We show below the proof in one dimension:

Theorem 2.1

Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

Proof. (See, e.g., [9]). Let $\{a_n\}$ be any bounded sequence, so that for some $M > 0$ we have $|a_n| \leq M \quad \forall n \in \mathbb{N}$. Bisect the interval $[-M, M]$ into two closed intervals of the same length. One of these intervals contains infinitely many terms of $\{a_n\}$. Let I_1 be that interval, and let a_{n_1} be any point in I_1. Next we bisect I_1 into two closed intervals, with one of these intervals having infinitely many terms of the sequence $\{a_n\}$. Let I_2 be that interval, and choose a_{n_2} inside this interval, with $n_2 \geq n_1$. In general, we bisect I_{k-1} into two closed intervals, one of which must contain infinitely many terms of $\{a_n\}$. Let I_k be this closed interval, and choose $a_{n_k} \in I_k$ such that $n_k > n_{k-1}$.

Therefore we have obtained a subsequence $(a_{n_1}\ , a_{n_2}, \ldots)$ of $\{a_n\}$ and a sequence of nested intervals $I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots$. By the Nested Interval Theorem, $\bigcap_{n=1}^{\infty} I_n$ is nonempty, and thus contains some element x.

We next prove that $(a_{n_k}) \to x$. Let $\varepsilon > 0$ be arbitrary. Since the length of the interval I_k is $M\left(\frac{1}{2}\right)^{k-1}$, the sequence $\{M\left(\frac{1}{2}\right)^{k-1}\}$ converges to 0. Hence, there exists some $N \in \mathbb{N}$ such that if $k \geq \mathrm{N}$, then $|M\left(\frac{1}{2}\right)^{k-1}| < \varepsilon$. Now since $a_{n_k}, x \in I_k$, we have that

$$|a_{n_k} - x| < \varepsilon,$$

and thus $(a_{n_k}) \to x$. ■

2.1 Example

(1). Consider the sequence

$$\{(-1)^n\} = (-1, 1, -1, 1, \ldots)$$

This sequence does not converge, but the subsequence

$$\{(-1)^{2k}\} = (1, 1, 1, ...)$$

converges to 1. Notice that if the sequence is unbounded, then all bets are off; the sequence may have a convergent subsequence or it may not. The sequences $\{((-1)^n + 1)n\}$ and {n} represent these possibilities as the first has, for example,

$$\{((-1)^{2k+1} + 1)(2k+1)\} = (0, 0, 0 \ldots)$$

as a convergent subsequence, and the second one has none. The Bolzano-Weierstrass Theorem says that no matter how “random” the sequence $\{x_n\}$ may be, as long as it is bounded then some part of it must converge. This is very useful when one has some process which produces a "random" sequence. If, for example, x_n is a sequence of uniform random variables, we can let x_{n_1} be the first number in the interval $[0, 1/2]$, x_{n_2} the first number in $[1/2, 3/4]; n_2 > n_1$, and x_{n_k} the first number, with $n_k > n_{k-1}$, in the interval $[1 - (1/2^{k-1}), 1 - (1/2^k)]$. Such numbers exist by the randomness of the sequence and it is evident that $x_{n_k} \to 1$ as $k \to \infty$.

Another interesting sequence is the sequence which results from enumerating all rationals as Calkin and wilf "Recounting the rationals".

2.2 Weyl's Equidistribution Theorem, see [1].

We define the set $\mathbb{T}$ the circle $\mathbb{R}/2\pi\mathbb{Z}$, e.g, the set of real numbers mod 2π. So if $\theta \in \mathbb{T}$ we have $\theta + 2\pi = \theta$. (again, see [1]).

Weyl's Criterion

A sequence $(x_n)_{n=1}^{\infty}$ is equidistributed iff

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n<N} e^{2\pi i m x_n} = 0$$

for each $m \in \mathbb{N}$. A consequence of this result is that the sequence $(\langle nx \rangle)$ is equidistributed, and then dense, $x \in [0,1] \cap (\mathbb{R}/\mathbb{Q})$, such that n is a natural number and $\langle x \rangle$ is the fractional part of x.

Theorem 2.2

Let's $g : \mathbb{T} \to \mathbb{C}$ be a continuous function and a given $\varepsilon > 0$, then there exists a trigonometric polynomial P where $\sup_{t\in\mathbb{T}} |P(t) - g(t)| \leq \varepsilon$.

Theorem 2.3

If γ is irrational then for all α, β such that $0 \leq \alpha \leq \beta \leq 1$ we get $m^{-1} card\{1 \leq k \leq m : \alpha \leq \langle k\gamma \rangle \leq \beta\} \to \beta - \alpha$ as $n \to \infty$. See [1]

The reviews of papers springing from Weyl's proof of his theorem fill over 100 pages of Mathematics Reviews In Number Theory 1940-72.[1]

Theorem 2.3 is a simple restatement of part (ii) of the following result for $\mathbb{T}$ which we have to prove.

Theorem 2.4

(1) Assume $\gamma \in \mathbb{R}/\mathbb{Q}$. Then if the function $g\colon \mathbb{R} \to \mathbb{C}$ is continuous we get

$$m^{-1}\sum_{k=1}^{m} g(2\pi k\gamma) \to \frac{1}{2\pi}\int_{x\in\mathbb{T}} g(x)dx \quad as\ m \to \infty.$$

(2) Assume $\gamma \in \mathbb{R}/\mathbb{Q}$. Then if $0 \le \alpha \le \beta \le 1$,

$$m^{-1}card\{1 \le k \le m : 2\pi r\gamma \in [2\pi\alpha, 2\pi\beta]\} \to (\beta - \alpha) \quad \text{as } m \to \infty.[1]$$

Proof. Let's define

$$F_m(g) = n^{-1}\sum_{k=1}^{m} g(2\pi k\gamma) - \frac{1}{2\pi}\int_{\frown\in\mathbb{T}} f(x)dx$$

To show (1), we instead can only prove that $F_m(g) \to 0$ as $m \to \infty$. We will prove that into steps.

1.

$$F_m(1) = m^{-1}\sum_{k=1}^{m} 1 - \frac{1}{2\pi}\int_{x\in\mathbb{T}} 1dx = 1 - 1 = 0.$$

2. Let $e_r(x) = e^{irx}$ where $x \in \mathbb{T}$ and $r \in \mathbb{Z}$, then (for $r \neq 0$)

$$\begin{aligned}
\left|F_m(e_r)\right| &= \left|m^{-1}\sum_{k=1}^{m} e^{(2\pi ikr\gamma)} - \frac{1}{2\pi}\int_{x\in\mathbb{T}} e^{(irx)}dx\right| \\
&= \left|m^{-1}e^{(2\pi ir\gamma)}\sum_{k=0}^{m-1} e^{(2\pi ikr\gamma)} - 0\right| \\
&= \left|\frac{1}{m} \times e^{(2\pi ir\gamma)} \times \frac{1 - e^{(2\pi imr\gamma)}}{1 - e^{(2\pi ir\gamma)}}\right| \\
&= \frac{1}{m}\left|\frac{1 - e^{(2\pi imr\gamma)}}{1 - e^{(2\pi ir\gamma)}}\right| \\
&\le \frac{1}{m}\frac{2}{|1 - e^{(2\pi ir\gamma)}|}
\end{aligned}$$

$$\frac{1}{m}\frac{2}{|1-e^{(2\pi is\gamma)}|} \to 0 \; as \; m \to \infty$$

3. If

$$P = \sum_{r=-n}^{n} a_r e_r$$

(which means, if P is a trigonometric polynomial) then, using linearity and the result of

Steps 1 and 2,

$$F_m(P) = \sum_{r=-n}^{n} a_r F_m(e_r) \to 0 \; as \; m \to \infty.$$

4. Let $g, h : \mathbb{T} \to \mathbb{C}$ be a continuous functions then

$$|g(x) - h(x)| \leq \varepsilon \; for \; every \; x \in \mathbb{T}$$

then

$$|F_m(g) - F_m(h)| \leq m^{-1} \sum_{k=1}^{m} |g(2\pi s\gamma) - h(2\pi s\gamma)| + \frac{1}{2\pi}\int_0^{2\pi} |g(x) - h(x)|\, dx$$

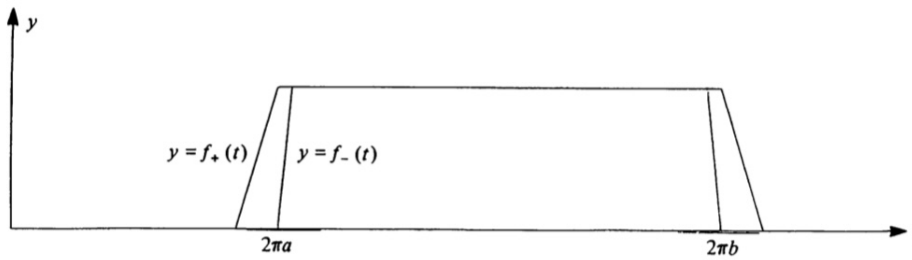
$$\leq \varepsilon + \varepsilon = 2\varepsilon \text{ for every m} \geq 0.$$

5. If $g : \mathbb{T} \to \mathbb{C}$ is a continuous function and $\varepsilon > 0$ then by Theorem 2.2 there exists a trigonometric polynomial P with $|P(x) - g(x)| \leq \frac{\varepsilon}{3}$ for every $x \in \mathbb{T}$. By the result of 3^{rd} step, there exists an m_0 such that $|F_m| \leq \frac{\varepsilon}{3}$ for every $m \geq m_0$. However by the 4^{th} step, we have $|F_m(g) - F_m(P)| \leq \frac{2\varepsilon}{3}$ and so $|F_m(g)| \leq |F_m(P)| + |F_m(g) - F_m(P)| \leq \varepsilon$ for all $m \geq m_0$.

It comes after that $F_m(g) \to 0$ as $m \to \infty$ and so the first part is demonstrated. The only left is the last part. In fact, it is a simple problem to solve, for each $\varepsilon > 0$, continuous functions $g_+, g_- : \mathbb{T} \to \mathbb{R}$ such that

(a) $g_+(x) \geq 1 \geq g_-(x)$ for all $x \in [2\pi\alpha, 2\pi\beta]$,

(b) $g_+ \geq 0$ for all $x \in \mathbb{T}$,



(c) $g_-(x) = 0$ for all $x \notin [2\pi\alpha, 2\pi\beta]$,

(d) $(\beta - \alpha) + \varepsilon \geq (2\pi)^{-1} \int_{x \in \mathbb{T}} g_+(x)dx$,

(e) $(2\pi)^{-1} \int_{x \in \mathbb{T}} g_-(x)dx \geq (\beta - \alpha) - \varepsilon$.

($g_+ = f_+ =$ and $g_- = f_-$ are described in the above figure). Using (a), (b) and (c) we get that

$$\sum_{s=1}^{m} g_+(2\pi s\gamma) \geq card\{m \geq s \geq 1 : 2\pi s\gamma \in [2\pi\alpha, 2\pi\beta]\} \geq \sum_{s=1}^{m} g(2\pi s\gamma).$$

However, from part (1) we can find there an $m_0(\varepsilon)$ such that, $m \geq m_0(\varepsilon)$, $|F_m(g_+)|, |F_m(g_-)|) \leq \varepsilon$ and so

$$\frac{1}{2\pi} \int_{x \in \mathbb{T}} g_+(x)dx + \varepsilon \geq m^{-1} card\{m \geq s \geq 1 : 2\pi s\gamma \in [2\pi\alpha, 2\pi\beta]\} \geq \frac{1}{2\pi} \int_{x \in \mathbb{T}} g_-(x)dx - \varepsilon.$$

Then utilizing (d) and (e) we acquire

$$(\beta - \alpha) + 2\varepsilon \geq m^{-1} card\{m \geq s \geq 1 : 2\pi s\gamma \in [2\pi\alpha, 2\pi\beta]\} \geq (\beta - \alpha) - 2\varepsilon,$$

Then because $\varepsilon \geq 0$ was randomly chosen,

$m^{-1} card\{m \geq s \geq 1 : 2\pi s\gamma \in [2\pi\alpha, 2\pi\beta]\} \to (\beta - \alpha)$ as $m \to \infty$. ■

Note that the results of Theorem 2.3 and Theorem [1] 2.4 are trivially false if γ is rational, so the problem of characterising those γ with $\langle s\gamma \rangle$ equidistributed (i.e. with

$m^{-1} card\{m \geq s \geq 1 : \langle s\gamma \rangle \in [\alpha, \beta]\} \to \beta - \alpha,$ whenever $0 \leq \alpha \leq \beta \leq 1$) is solved completely by the condition γ irrational.

2.3 Constructions in the Bolzano Weierstrass Theorem

First, let's start with the sequence $(\sin(n))_{n=1}^{\infty}$**.**

As we know the functionsin(n) is bounded (by $\pm$ 1), then [10] by the Bolzano-Weierstrass Theorem, there exists a convergent subsequence. [2][3][4] To affirm that there exists a subsequence with limit zero, it suffices to demonstrate that, for any strict positive integer i, there are integers x_i and y_i such that [11]

$$|x_i - y_i\pi| < \frac{1}{i}.$$

Note: $\langle z\rangle$ is the fractional part of z, $\langle z\rangle \in (0, 1)$.

It is understandably that at least we can find 2 integers n_i and m_i between any $(i + 1)$ different integers so that

$$|\langle n_i\pi\rangle - \langle m_i\pi\rangle||\langle(n_i - m_i)\pi\rangle| < \frac{1}{i},$$

as we know that $\langle p\pi\rangle \in (0, 1)$ where p is an integer. Then

$$|x_i - y_i\pi| < \frac{1}{i}$$

where $y_i = n_i - m_i$ and $x_i = \lfloor n_i\pi\rfloor - \lfloor m_i\pi\rfloor$.

It is so exciting that the range of the function $\sin(x)$ has many limit points. Now we are going to show this fact. See [2]

Proposition 2.1

For every $\alpha \in [-1, 1]$ there exists a subsequence $(x_i)_i$ of $\mathbb{Z}^+$ so that

$$\lim_{i\to\infty} \sin(x_i) = \alpha.$$

Proof. Actually if β is not an element of the set $\mathbb{Q}$ and $n \in \mathbb{Z}^+$, then $\langle x\beta \rangle$ is dense in $(0,1)$[2, 4] [3]. Then $\{\langle 2x\pi \rangle + i\}$ is dense and uniformly distributed in the interval $(i, i+1)$ for each integer k, which means that the density of the sequence $\{\sin(n)\}_n$ in the closed unit interval. The analogous conclusions hold for general continuous functions with irrational periods.

The next 2 lemmas describe a useful recursive procedure for any given irrational number γ . We get two sequences x_k and y_k of integers such that $y_i\gamma + x_i$ approaches zero as i goes to infinity. [3]

Lemma 2.1

Let $\gamma \in (\mathbb{R}/\mathbb{Q})$ such that $\gamma > 1$, and suppose that $0 < z_1 < z_0$ and $z_0/z_1 = \gamma$.

Then

$$z_{n+2} = z_n - \lfloor z_n/z_{n+1} \rfloor z_{n+1} \tag{i}$$

is well defined for all $n \in \mathbb{Z}^+$ (z_n is never 0), and

$$0 < z_{n+2} < \tfrac{z_n}{2} \quad \text{for all } n. \tag{ii}$$

Proof. By assumption $0 < z_1 < z_0$ and z_0/z_1 is irrational. Assume that $0 < z_{k+1} < z_k$ with z_k/z_{k+1} irrational. By (i), $z_{k+2} = z_k - \lfloor z_k/z_{k+1} \rfloor z_{k+1}$ and $z_{k+2}/z_{k+1} - \lfloor z_k/z_{k+1} \rfloor \in \mathbb{R}/\mathbb{Q}$ with $0 < z_{k+2} < z_{k+1}$.

By induction, $z_{n+1}/z_{n+2} \in \mathbb{R}/\mathbb{Q}$ with $0 < z_{n+2} < z_{n+1}$ for all $n \in \mathbb{Z}^+$. Moreover $z_{n+2} = z_n - \lfloor z_n/z_{n+1} \rfloor z_{n+1} \leq z_n - z_{n+1} < z_n - z_{n+2}$. Thus $z_{n+2} < z_n/2$ for $n \in \mathbb{Z}^+$.

Remarks. The inequality (ii) and $0 < z_{n+2} < z_{n+1}$ implies $\lim_{n\to\infty}(z_n) = 0$ where (z_n) determined by (i). See [4]

Lemma 2.2

For each z_i determined by (i) we can find integers y_i and x_i such that

$z_i = y_i\gamma + x_i$, with $y_i = (-1)^i|y_i|$ and $x_i = (-1)^{i-1}|x_i|$ for $i \in \mathbb{Z}^+/\{1\}$

Proof. By induction, We find the y_i and x_i . Without lost of generality $z_0 = \gamma > 1$ and $z_1 = 1$. Then

$$z_2 = z_0 - \lfloor z_0/z_1 \rfloor z_1 = \gamma - \lfloor \gamma \rfloor = y_2\gamma + x_2,$$

with $y_2 = 1$ and $x_2 = -\lfloor \gamma \rfloor =$ -$|x_2|$. We have

$$z_3 = z_1 - \lfloor z_1/z_2 \rfloor z_2 = 1 - \lfloor z_1/z_2 \rfloor (\gamma - \lfloor \gamma \rfloor)$$

$$= -\lfloor z_1/z_2 \rfloor \gamma + 1 + \lfloor z_1/z_2 \rfloor \lfloor \gamma \rfloor$$

$$z_3 = y_3\gamma + x_3,$$

with $y_3 = -\lfloor z_1/z_2 \rfloor = -|y_3|$ and $x_3 = 1 + \lfloor z_1/z_2 \rfloor \lfloor \gamma \rfloor = |x_3|$.

Now suppose that y_i, x_i, y_{i+1} and x_{i+1} have been found , which means that

$z_i = y_i\gamma + x_i$, $y_i = (-1)^i|y_i|$ and $x_i = (-1)^{i-1}|x_i|$, also

$z_{i+1} = y_{i+1}\gamma + x_{i+1}$, $y_{i+1} =$ (-1)$^{i+1}|y_{i+1}|$ and $x_{i+1} = (-1)^i|x_{i+1}|$.

then

$$z_{i+2} = z_i - \lfloor z_i/z_{i+1} \rfloor z_{i+1} = y_i\gamma + x_i - \lfloor z_i/z_{i+1} \rfloor (y_{i+1}\gamma + x_{i+1})$$

$$= (y_i - \lfloor z_i/z_{i+1} \rfloor y_{i+1})\gamma + x_i - \lfloor z_i/z_{i+1} \rfloor x_{i+1}$$

$$= y_{i+2}\gamma + x_{i+2},$$

where

$$y_{i+2} = y_i - \lfloor z_i/z_{i+1} \rfloor y_{i+1} = (-1)^i |y_i| - (-1)^{i+1} |y_{i+1}| \lfloor z_i/z_{i+1} \rfloor$$

$$= (-1)^{i+2}(|y_i| + |y_{i+1}| \lfloor z_i/z_{i+1} \rfloor)$$

$$= (-1)^{i+2} |y_{i+2}|,$$

and

$$x_{i+2} = x_i - \lfloor z_i/z_{i+1} \rfloor x_{i+1} = (-1)^{i-1} |x_i| - (-1)^i |x_{i+1}| \lfloor z_i/z_{i+1} \rfloor$$

$$= (-1)^{i+1}(|x_i| + |x_{i+1}| \lfloor z_i/z_{i+1} \rfloor)$$

$$= (-1)^{i+1} |x_{i+2}|.$$

■

Now let $f(x) = \sin(x)$, and express our results.

Theorem 2.5

There exists a subsequence $(n_i)_i$ of $\mathbb{Z}^+$ that satisfy

$$\lim_{i \to \infty} \sin(n_i) = 0.$$

Proof. Let $\gamma = \pi$ in Lemma 2.1 and 2.2 and $n_i = |x_i|$ such that (x_i is defined in Lemma 2.2). Then

$$|\sin(n_i)| = |x_i| = |\sin(z_i - y_i \pi)| = |\sin(z_i)|.$$

By Lemma 2.1, $\lim_{i \to \infty} z_i = 0$, the it follows that $\lim_{i \to \infty} \sin(n_i) = 0$, as claimed.

Every element of $[-1, 1]$ is a limit point of $(\sin(n))_n$.

Theorem 2.6

For every $\delta \in [-1, 1]$ there exists a subsequence $(p_k)_k$ of $\mathbb{Z}^+$ that satisfy

$$\lim_{i\to\infty} \sin(n_i) = \delta.$$

Proof. For $\delta = 0$ we get Theorem 2.5. Now let $\gamma = 2\pi$.

(i) : $0 < \delta \leq 1$. We have $A = \arcsin(\delta) \in (0, \frac{pi}{2}]$. Let $\rho_i = A - \lfloor A/z_i \rfloor z_i$, with z_i defined by (i). We suppose $A > z_i$ since z_i goes to zero. Obviously, $0 \leq \rho_i \leq z_i$. By Lemma 2,

$$z_{2i-1} = 2y_{2i-1}\pi + x_{2i-1},$$

with

$$x_{2i-1} = (-1)^{2i}|x_{2i-1}| = |x_{2i-1}| > 0.$$

Put

$$n_i = \lfloor A/z_{2i-1} \rfloor x_{2i-1},$$

and

$$m_i = \lfloor A/z_{2i-1} \rfloor y_{2i-1}.$$

Then $n_i \in \mathbb{N}$ and $A - \rho_{2i-1} = \lfloor A/\rho_{2i-1} \rfloor z_{2i-1} = n_i + 2m_i\pi$, so

$$\sin(n_i) = \sin(A - \rho_{2i-1} - 2m_i\pi) = \sin(A - \rho_{2i-1}).$$

Then since ρ_{2i-1} tends to zero as i goes to infinity

$$\lim_{i\to\infty} \sin(n_i) = \sin(A) = \delta.$$

(ii) : $-1 \leq \delta < 0$. We have $A = \arcsin(\delta) \in [-\frac{\pi}{2}, 0]$. Let

$\rho_i = |A| - \lfloor |A|/z_i \rfloor z_i$, where z_i determined by (i), and suppose $0 < z_i < |A|$. As subsequent from Lemma 2.2 that

$$z_{2i} = 2y_{2i}\pi + x_{2i} = 2y_{2i} + (-1)^{2i-1}|x_{2i}| = 2y_{2i}\pi - |x_{2i}|.$$

If we write

$$n_i' = \lfloor |A|/z_{2i} \rfloor x_{2i} \text{ and } m_i = \lfloor |A|/z_{2i} \rfloor y_{2i},$$

Then $|A| - \rho_{2i} = n_i' + 2m_i\pi$ and $n_i = -n_i' \in \mathbb{N}$. Thus

$$\sin(n_i) = \sin(-n_i') = -\sin(|A| - \rho_{2i} - 2m_i\pi) = -\sin(|A| - \rho_{2i}),$$

and it follows that

$$\lim_{i\to\infty} \sin(n_i) = -\sin(|A|) = \sin(A) = \delta.$$

■

Second, let's see the sequence $(\sqrt{n})_{n=1}^{\infty}$.

Definition: Equidistribution

Let's $(x_n)_n$ be a sequence of elements from the interval $[0,1]$.

Let $[\alpha, \beta] \subset [0,1]$. For each $n \in \mathbb{N}$, we define $u_n(a,b)$ to be number of integers i ($i \in \mathbb{N}$) with $x_i \in [\alpha, \beta]$. Then we say (x_n) is equidistributed in $[0,1]$ if for all a, b: $[\alpha, \beta] \subset [0,1]$

$$\lim_{n\to\infty} \frac{u_n(\alpha, \beta)}{n} = \beta - \alpha.$$

Our question now, is $\langle \sqrt{n} \rangle$ equidistributed?

First, define the "discrepancy" of the sequence $(x_n)_n$ in $(0,1)$ as:

$$D_N = Sup\left\{ \left| \frac{u_N(\alpha, \beta)}{N} - (\beta - \alpha) \right| : 0 \le \alpha \le \beta \le 1 \right\}.$$

The property of equidistribution of $(x_n)_n$ can be verbalized other way, in terms of the discrepancy. Let's define

$$D_N^* = Sup\left\{\left|\frac{s_N(0,\alpha)}{N} - \alpha\right| : 0 \leq \alpha \leq 1\right\}$$

Compare D_N and D_N^*.

We can see that $D_N^* \leq D_N$. Also, let's $\varepsilon > 0$ and $(\alpha, \beta) \subset (0,1)$, then

$$u_N(\alpha,\beta) \leq u_N(0,\beta) - u_N(0,\alpha-\varepsilon)$$

When $\varepsilon \to 0$, we have :

$$D_N^* \leq D_N \leq 2D_N^*$$

We conclude that as $N \to 0$, we have $D_N \to 0$ if only and if $D_N^* \to 0$.

If $D_N \to 0$ then (x_n) is equidistributed in $(0,1)$(by definition).

Then we can use $u_n(0,\lambda)$ instead of $u_n(\alpha,\beta)$. Now suppose [2] $\lambda \in (0,1)$, let estimate the number of integers n such that:

$$\langle\sqrt{n}\rangle \in [0,\lambda).$$

For any n, let $i = [\sqrt{n}]$ be the greatest element of $\mathbb{Z}$ less than or equal to $\sqrt{n}$. We have $0 \leq \langle\sqrt{n}\rangle \leq \lambda \implies$ i $\leq \sqrt{n} \leq i+\lambda$, then, $i^2 \leq n \leq (i+\lambda)^2 = i^2+2i\lambda+\lambda^2$.

So for a given i, there are $1+[2i\lambda+\lambda^2]$ such n. Furthermore for any other i , since $(i+\lambda)^2 < (i+1)^2$, these are disjoint.

Particularly, for any i, the cardinality $u_{i^2}(0,\lambda)$ of $\{i : 0 \leq i \leq i^2$ and $\langle\sqrt{n}\rangle \leq \lambda\}$ is equal to

$$\sum_{j=o}^{i-1}(1+[2j\lambda+\lambda^2]).$$

Therefore for any n and for $i = [\sqrt{n}]$, we have :

$$|u_n(0,\lambda) - n\lambda| = |u_n(0,\lambda) - u_{i^2}(0,\lambda) + u_{i^2}(0,\lambda) - n\lambda|$$

Using the triangle inequality we get

$$\begin{aligned} |u_n(0,\lambda) - n\lambda| &\le |u_n(0,\lambda) - u_{i^2}(0,\lambda)| + |u_{i^2}(0,\lambda) - n\lambda| \\ &\le n - i^2 + \left|\sum_{j=0}^{i-1}(1 + [2j\lambda + \lambda^2]) - n\lambda\right| \\ &< 2i + 1 + \left|\sum_{j=0}^{i-1}(1 + [2j\lambda + 2]) - n\lambda\right|, \end{aligned}$$

and hence $|u_n(0,\lambda) - n\lambda| < 7i + 2 \le 7\sqrt{n} + 2$.

Then

$$\frac{|u_n(0,\lambda) - n\lambda|}{n} \le \frac{7\sqrt{n}+2}{n} = \frac{7\sqrt{n}}{n} + \frac{2}{n}.$$

which means

$$\left|\frac{u_n(0,\lambda)}{n} - \lambda\right| < \frac{7}{\sqrt{n}} + \frac{2}{n}.$$

Thus

$$\lim_{n\to\infty}\left|\frac{u_n(0,\lambda)}{n} - \lambda\right| = 0.$$

In other words

$$\lim_{N\to\infty} D_N^* = 0.$$

Therefore, $\langle\sqrt{n}\rangle$ is equidistributed in $(0,1)$. ■

Fix $\varepsilon > 0$. Suppose we seek a subsequence of $\langle\sqrt{n}\rangle$ that converges to $l \in (0,1)$. Start with $n_1 = 1$ so $\langle\sqrt{n_1}\rangle = 0$. By the equidistribution of $\langle\sqrt{n}\rangle$ we find $n_2 > n_1$ such that $\frac{l}{2} - \varepsilon \le \langle\sqrt{n_2}\rangle \le \frac{l}{2}$. Then find $n_3 > n_2$ such that $\frac{3l}{4} - \varepsilon \le \langle\sqrt{n_3}\rangle \le \frac{3l}{4}$. Continuing in this fashion we see that $\langle\sqrt{n_k}\rangle \to l$.

Now, let check the sequence $(\ln(n))_{n=1}^{\infty}$.

We have $\langle \ln(n) \rangle$ is not equidistributed. To show this we have to use Euler summation formula.

Euler Summation Formula

If $g(t)$ is a complex function with a continuous derivative on the interval $[1, N]$, such that $N \geq 1$ is an integer, then

$$\sum_{n=1}^{N} g(n) = \int_1^N g(t)dt + \frac{1}{2}(g(1) + g(N)) + \int_1^N (t - \frac{1}{2})g^{'}(t)dt.$$

Let $g(t) = e^{2\pi i \ln(t)}$, and divide both sides by N. Then the first term of the RHS is

$$\frac{Ne^{2\pi i \ln(N)} - 1}{N(2\pi i + 1)}$$

and this expression diverge as N goes to the infinity. The second term on the RHS, divided by N, goes to zero as N goes to infinity, as does the third term on the RHS divided by N, as follows from

$$\left| \int_1^N (t - \frac{1}{2})g^{'}(t)dt \right| \leq \pi \int_1^N \frac{dt}{t}.$$

Hence, Weyl's Criterion in page 13 with $x_n = \ln(n)$ and $m = 1$ is not satisfied. ■

We now provide elementary constructions of convergent subsequences of both $\langle \sqrt{n} \rangle$ and both $\langle \ln(n) \rangle$. Such an elementary construction is needed for $\langle \ln(n) \rangle$ since we can not use equidistribution.

We start with $\langle \sqrt{n} \rangle$. Let $n_1 = n^2 + \lfloor ln \rfloor$ for n to be chosen so that

$$n \leq \sqrt{n_1} = \sqrt{n^2 + \lfloor ln \rfloor} \leq n + \frac{l}{2}$$

and so $\langle \sqrt{n_1} \rangle = \frac{l}{2}$, where l is the desired limit. Moreover if n is large enough

$\sqrt{n_1} \sim n + \frac{l}{2}$.

The above is true since $n^2 + \lfloor ln \rfloor \leq \left(n + \frac{l}{2}\right)^2 = n^2 + ln + \frac{l^2}{4}$, we next let n_2 be such that $n_2 = m^2 + \lfloor \frac{3lm}{2} \rfloor$ where $m > n$ is large enough so that $\sqrt{n_2} \sim m + \frac{3l}{4}$ and $\langle \sqrt{n_2} \rangle = \frac{3l}{4}$. Continuing using larger numbers $n_3 = p^2 + \lfloor \frac{14lp}{8} \rfloor$ with $\langle \sqrt{n_3} \rangle \sim p + \frac{7l}{8}$ and $\langle \sqrt{n_3} \rangle = \frac{7l}{8}$, we find a sequence n_k such that $\langle \sqrt{n_k} \rangle \to l$.

Next we turn to $\langle \ln(n) \rangle$. Let l be the desired limit of $\langle \ln(n_k) \rangle$ as $k \to \infty$.

Fix $0 < l < 1$, then choose n_1 large enough so that $\langle \ln(n_1) \rangle < l$. (We can do this since the natural logarithm ln function grows very slowly.)

Now $\ln(n_1 + 1) - \ln(n_1) = \ln(1 + \frac{1}{n_1}) \sim \frac{1}{n_1}$. Thus $\ln(n_1 + 1) \sim \ln(n_1) + \frac{1}{n_1}$ and so $\langle \ln(n_1 + 1) \rangle < \langle \ln(n_1) + \frac{1}{n_1} \rangle < l$. for large n_1. Let $n_2 = n_1 + 1$, we continue making incremental increases until $\langle \ln(n_1 + M) \rangle > l$. We then choose another large integer N so that $\langle \ln(n_1 + M - 1) < \langle \ln(N) \rangle < l$ and let N be the next term in the subsequence. Continuing this process leads to the desired subsequence.

3 HÖLDER INEQUALITY

Hölder's inequality (we restrict this to continuous functions f and g so that f, g are p-integrable for each p) states that :

$$\|fg\|_1 \leq \|f\|_p \|g\|_q \quad \text{for } \frac{1}{p} + \frac{1}{q} = 1, \quad 1 \leq p \leq \infty \ [1]$$

where

$$\|fg\|_1 = \int_a^b |f(x)g(x)|dx, \ \|f\|_p = \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}}, \ \|g\|_q = \left(\int_a^b |g(x)|^q dx\right)^{\frac{1}{q}}$$

and $\|f\|_\infty = Sup|f(x)|$

Definition:

Let $p \geq 1$ be a real number and f be a measurable function. f is said to be p-integrable if and only if:

$$\int |f(x)|^p dx < \infty.$$

3.1 Hölder inequality for beta and gamma integrals

The Beta function is defined for $a, b > 0$ by

$$\beta(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx,$$

while the Gamma function is defined for $a > 0$ by

$$\Gamma(a) = \int_0^\infty x^{a-1}e^{-x}dx.$$

These functions have the following properties:

$\Gamma(n) = (n-1)!$, $\Gamma(z+1) = z\Gamma(z)$ and $\beta(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$. Since both functions are expressed as the integral of the product of two functions, we will examine the efficiency of Hölder's inequality for these integrals.

We start with some easy examples and present several proofs of each. Then we turn to the Beta and Gamma integrals.

3.1.1. *Examples*

$$\text{Example 1.} \qquad \mathbf{A = \int_0^1 x^2(1-x)^3 dx}$$

a. Exact Value: We have that

$$\begin{aligned} A &= \int_0^1 x^2(1-x)^3 dx = \int_0^1 x^{3-1}(1-x)^{4-1} dx \\ &= \frac{\Gamma(3)\Gamma(4)}{\Gamma(7)} \\ &= \frac{2!3!}{6!} = \frac{2\times 6}{720} = \frac{1}{60}. \end{aligned}$$

b. Using expansion of $(1-x)^3$**:**

$$\begin{aligned} A &= \int_0^1 x^2(1-x)^3 dx \\ &= \int_0^1 x^2(1-3x+3x^2-x^3) dx \\ &= \int_0^1 (x^2-3x^3+3x^4-x^5) dx \\ &= \left(\frac{x^3}{3} - 3\frac{x^4}{4} + 3\frac{x^5}{5} - \frac{x^6}{6} \bigg|_0^1 \right) \\ &= \frac{1}{3} - \frac{3}{4} + \frac{3}{5} - \frac{1}{6} \\ &= \frac{20-45+36-10}{60} = \frac{1}{60}. \end{aligned}$$

c. Hölder inequality bounds:

$$A = \int_0^1 x^2(1-x)^3 dx \le \left(\int_0^1 x^{2p} dx\right)^{\frac{1}{p}} \left(\int_0^1 (1-x)^{\frac{3p}{p-1}} dx\right)^{\frac{p-1}{p}}$$

$$\le \left(\frac{x^{2p+1}}{2p+1}\Big|_0^1\right)^{\frac{1}{p}} \left(\frac{(p-1)(x-1)(1-x)^{\frac{3p}{p-1}}}{4p-1}\Big|_0^1\right)^{\frac{p-1}{p}}$$

$$= \left(\frac{1}{2p+1}\right)^{\frac{1}{p}} \left(\frac{p-1}{4p-1}\right)^{\frac{p-1}{p}}.$$

Let

$$F(p) = \left(\frac{1}{2p+1}\right)^{\frac{1}{p}} \left(\frac{p-1}{4p-1}\right)^{\frac{p-1}{p}}.$$

To minimize $F(p)$, we instead minimize $\ln F(p)$. Letting $\varphi(p) = \ln(F(p))$, we have

$$\varphi(p) = \ln\left(\left(\frac{1}{2p+1}\right)^{\frac{1}{p}} \left(\frac{p-1}{4p-1}\right)^{\frac{p-1}{p}}\right)$$

$$= -\frac{1}{p}\ln(2p+1) + \frac{p-1}{p}\ln(\frac{p-1}{4p-1})$$

$$= -\frac{1}{p}\ln(2p+1) + \frac{p-1}{p}\ln(p-1) - \frac{p-1}{p}\ln(4p-1).$$

The derivative of $\varphi(p)$ is

$$\varphi^{'}(p) = \frac{\ln(2p+1)}{p^2} - \frac{2}{p(2p+1)} + \frac{\ln(p-1)}{p} + \frac{1}{p} - \frac{\ln(p-1)(p-1)}{p^2}$$

$$- \frac{\ln(4p-1)}{p} + \frac{p-1}{p^2}\ln(4p-1) - \frac{4(p-1)}{p(4p-1)}$$

$$= \frac{\ln(2p+1)}{p^2} + \frac{\ln(p-1)}{p^2} - \frac{\ln(4p-1)}{p^2} - \frac{2}{p(2p+1)} + \frac{1}{p} - \frac{4(p-1)}{p(4p-1)}.$$

Let's denote the logarithmic terms above by:

$$\varphi_1(p) = \frac{\ln(2p+1)}{p^2} + \frac{\ln(p-1)}{p^2} - \frac{\ln(4p-1)}{p^2} = \frac{1}{p^2}\ln\left(\frac{(2p+1)(p-1)}{4p-1}\right), \quad (1)$$

and the non-logarithmic terms by

$$\varphi_2(p) = -\frac{2}{p(2p+1)} + \frac{1}{p} - \frac{4(p-1)}{p(4p-1)} = \frac{-2p+5}{p(2p+1)(4p-1)}. \quad (2)$$

We notice that

$$\varphi_1\left(\frac{5}{2}\right) = \frac{1}{\left(\frac{5}{2}\right)^2}\ln\left(\frac{(5+1)(\frac{5}{2}-1)}{10-1}\right)$$

$$= \frac{4}{25}\ln\left(\frac{\frac{18}{2}}{9}\right) = \frac{4}{25}\ln(1)$$

$$= 0.$$

Also

$$\varphi_2\left(\frac{5}{2}\right) = -\frac{-5+5}{\frac{5}{2}(5+1)(10-1)} = -2\frac{0}{270}$$

$$= 0.$$

What a pleasant surprise, since this yields

$$\varphi^{'}\left(\frac{5}{2}\right) = \varphi_1\left(\frac{5}{2}\right) + \varphi_2\left(\frac{5}{2}\right) = 0,$$

and therefore the minimum of F occurs at $\frac{5}{2}$ (2.5) and has value

$$F\left(\tfrac{5}{2}\right) = \left(\tfrac{1}{6}\right)^{\frac{2}{5}}\left(\frac{\frac{3}{2}}{9}\right)^{\frac{\frac{5}{2}-1}{\frac{5}{2}}} = \left(\tfrac{1}{6}\right)^{\frac{2}{5}}\left(\tfrac{1}{6}\right)^{\frac{3}{5}} = \left(\tfrac{1}{6}\right)^{1} = \tfrac{1}{6}.$$

Example 2. Next we let $\mathbf{B} = \int_{\mathbf{0}}^{\mathbf{1}} \sqrt{\mathbf{x}}\sqrt{\mathbf{1-x}}\mathbf{dx}$

a. Exact Value:

$$
\begin{aligned}
B &= \int_0^1 \sqrt{x}\sqrt{1-x}dx \\
&= \int_0^1 x^{\frac{3}{2}-1}(1-x)^{\frac{3}{2}-1}dx \\
&= \beta\left(\frac{3}{2}, \frac{3}{2}\right) = \frac{\Gamma(\frac{3}{2})\Gamma(\frac{3}{2})}{\Gamma(3)} \\
&= \frac{\frac{1}{2}\Gamma(\frac{1}{2})\frac{1}{2}\Gamma(\frac{1}{2})}{2} = \frac{1}{8}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right),
\end{aligned}
$$

but since $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, we get

$$B = \frac{\pi}{8}.$$

b. Integration:

$$
\begin{aligned}
B &= \int_0^1 \sqrt{x}\sqrt{1-x}dx \\
&= \int_0^1 \sqrt{x}\sqrt{1-x}.\frac{2\sqrt{x}}{2\sqrt{x}}dx \\
&\text{(Let } u=\sqrt{x}, du=\tfrac{dx}{2\sqrt{x}} \text{ and } x=0,1 \implies u=0,1) \\
&= \int_0^1 2u^2\sqrt{1-u^2}du.
\end{aligned}
$$

Let $u = \sin(\theta), du = \cos(\theta)d\theta$, so that

$$
\begin{aligned}
B &= \int_0^{\frac{\pi}{2}} 2\sin^2(\theta).\cos(\theta).\cos(\theta)d\theta = \int_0^{\frac{\pi}{2}} 2\sin^2(\theta).\cos^2(\theta)d\theta \\
&\text{(We have that } \sin(2\theta) = 2\sin(\theta)\cos(\theta)) \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} 4\sin^2(\theta).\cos^2(\theta)d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} \big(2\sin(\theta).\cos(\theta)\big)^2 d\theta \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^2(2\theta)d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1-\cos(4\theta)}{2}d\theta \\
&= \frac{1}{4}\left(\theta - \frac{\sin(4\theta)}{4}\bigg|_0^{\frac{\pi}{2}}\right) = \frac{1}{4}\left(\frac{\pi}{2} - \frac{\sin(2\pi)}{4} - 0 + \frac{\sin(0)}{4}\right) \\
&= \frac{\pi}{8}.
\end{aligned}
$$

c. Power series:

$$\begin{aligned}
B &= \int_0^1 \sqrt{x}\sqrt{1-x}dx = \int_0^1 x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}dx \\
&= \int_0^1 x^{\frac{1}{2}} \left(1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} \cdots\cdots \right) dx \\
&= \int_0^1 x^{\frac{1}{2}}dx - \frac{1}{2}\int_0^1 x^{\frac{3}{2}}dx - \frac{1}{8}\int_0^1 x^{\frac{5}{2}}dx - \frac{1}{16}\int_0^1 x^{\frac{7}{2}}dx \cdots\cdots \\
&= \left.\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right|_0^1 - \left.\frac{1}{2}\frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right|_0^1 - \left.\frac{1}{8}\frac{x^{\frac{7}{2}}}{\frac{7}{2}}\right|_0^1 - \left.\frac{1}{16}\frac{x^{\frac{9}{2}}}{\frac{9}{2}}\right|_0^1 \cdots\cdots \\
&= \frac{2}{3} - \frac{1}{2}.\frac{2}{5} - \frac{1}{8}.\frac{2}{7} - \frac{1}{16}.\frac{2}{9} \cdots\cdots \\
&= \frac{1}{8}\left(\frac{16}{3} - \frac{8}{5} - \frac{8}{28} - \frac{8}{72} \cdots\cdots \right) \\
&= \frac{1}{8}\left(5.333 - 1.6 - 0.286 - 0.111 - \cdots\cdots\right) \\
&= \frac{1}{8}\left(3.336 - \cdots\cdots\right) \\
&\simeq \frac{3.14}{8} \simeq \frac{\pi}{8}
\end{aligned}$$

d. Hölder:

$$\begin{aligned}
B = \int_0^1 \sqrt{x}\sqrt{1-x}dx &\le \left(\int_0^1 \left(\sqrt{x}\right)^p dx\right)^{\frac{1}{p}} \left(\int_0^1 \left(\sqrt{1-x}\right)^{\frac{p}{p-1}} dx\right)^{\frac{p-1}{p}} \\
&\le \left(\left.\frac{2x^{\frac{p+2}{2}}}{p+2}\right|_0^1\right)^{\frac{1}{p}} \left(\left.\frac{-(2p-1)(1-x)^{\frac{p}{2p-2}+1}}{3p-2}\right|_0^1\right)^{\frac{p-1}{p}} \\
&\le \left(\frac{2}{p+2}\right)^{\frac{1}{p}} \left(\frac{2p-2}{3p-2}\right)^{\frac{p-1}{p}}
\end{aligned}$$

Let

$$G(p) = \left(\frac{2}{p+2}\right)^{\frac{1}{p}} \left(\frac{2p-2}{3p-2}\right)^{\frac{p-1}{p}}$$

To minimize $G(p)$, we let $\psi(p) = \ln(G(p))$ and see that

$$\begin{aligned}\psi(p) &= \ln\left(\left(\frac{2}{p+2}\right)^{\frac{1}{p}}\left(\frac{2p-2}{3p-2}\right)^{\frac{p-1}{p}}\right)\\ &= \frac{1}{p}\ln\left(\frac{2}{p+2}\right) + \frac{p-1}{p}\ln\left(\frac{2p-2}{3p-2}\right).\end{aligned}$$

The derivative of ψ is

$$\begin{aligned}\psi'(p) &= \frac{-\ln\left(\frac{2}{p+2}\right)}{p^2} - \frac{1}{p(p+2)} + \frac{\ln\left(\frac{2p-2}{3p-2}\right)}{p} - \frac{(p-1)\ln\left(\frac{2p-2}{3p-2}\right)}{p^2}\\ &+ \frac{(p-1)(3p-2)\left(\frac{2}{3p-2} - \frac{3(2p-2)}{(3p-2)^2}\right)}{p(2p-2)}\\ &= \frac{-\ln\left(\frac{2}{p+2}\right)}{p^2} - \frac{1}{p(p+2)} + \frac{\ln\left(\frac{p}{2p-2}\right) - 1}{p^2}\\ &= -\frac{1}{p^2}\left(\ln\left(\frac{2}{p+2}\right) - \ln\left(\frac{2p-2}{3p-2}\right)\right) - \frac{1}{p(p+2)} + \frac{1}{p(3p-2)}.\end{aligned}$$

Let's denote the logarithmic terms above by:

$$\psi_1(p) = -\frac{1}{p^2}\left(\ln\left(\frac{2}{p+2}\right) - \ln\left(\frac{2p-2}{3p-2}\right)\right), \tag{3}$$

and the non-logarithmic terms by

$$\psi_2(p) = -\frac{1}{p(p+2)} + \frac{1}{p(3p-2)}. \tag{4}$$

We notice that

$$\psi_1(2) = -\frac{1}{4}\left(\ln\left(\frac{2}{2+2}\right) - \ln\left(\frac{4-2}{6-2}\right)\right) = -\frac{1}{4}\left(\ln\left(\frac{1}{2}\right) - \ln\left(\frac{1}{2}\right)\right) = 0,$$

and

$$\psi_2(2) = -\frac{1}{2\times 4} + \frac{1}{2\times 4} = 0.$$

Then

$$\psi^{'}(2) = \psi_1(2) + \psi_2(2) = 0.$$

The minimum occurs at $p = 2$ and $G(2) = 0.5$ which is greater that $\frac{\pi}{8} \simeq 0.392$. ✓

Example 1.3. Finally we let $C = \int_0^1 x^a(1-x)^b dx$,

which is the object of our primary interest.

Theorem 3.1: Hölder inequality for Beta integrals

For $a > -1$, $b > -1$, and $a + b \neq -1$

$$\int_0^1 x^a(1-x)^b dx \leq \left(\int_0^1 x^{ap} dx\right)^{\frac{1}{p}} \left(\int_0^1 (1-x)^{\frac{bp}{p-1}} dx\right)^{\frac{p-1}{p}},$$

anf the best p that minimize the right hand side is

$$p = \frac{a+b}{a}.$$

Proving the theorem by finding an extreme point is quite complicated, and a formal proof will not be given. Instead we'll show that the stated value "works".

In fact we have

$$\begin{aligned}\int_0^1 x^a(1-x)^b dx &\leq \left(\int_0^1 x^{ap} dx\right)^{\frac{1}{p}} \left(\int_0^1 (1-x)^{\frac{bp}{p-1}} dx\right)^{\frac{p-1}{p}} \\ &\leq \left(\left.\frac{x^{ap+1}}{ap+1}\right|_0^1\right)^{\frac{1}{p}} \left(\left.\frac{(p-1)(x-1)(1-x)^{\frac{bp}{p-1}}}{(b+1)p-1}\right|_0^1\right)^{\frac{p-1}{p}} \\ &\leq \left(\frac{1}{ap+1}\right)^{\frac{1}{p}} \left(\frac{p-1}{(b+1)p-1}\right)^{\frac{p-1}{p}}.\end{aligned}$$

Letting

$$\varphi(p) = \ln\left(\left(\frac{1}{ap+1}\right)^{\frac{1}{p}}\left(\frac{p-1}{(b+1)p-1}\right)^{\frac{p-1}{p}}\right) \tag{5}$$

$$= \frac{1}{p}\ln\left(\frac{1}{ap+1}\right) + \frac{p-1}{p}\ln\left(\frac{p-1}{(b+1)p-1}\right) \tag{6}$$

we have

$$\begin{aligned}\varphi^{'}(p) &= \frac{\ln(ap+1)}{p^2} - \frac{a}{p(ap+1)} + \frac{\ln\left(\frac{p-1}{(b+1)p-1}\right)}{p} - \frac{(p-1)\ln\left(\frac{p-1}{(b+1)p-1}\right)}{p^2} \\ &+ \frac{(b+1)(p-1)\left(\frac{1}{(b+1)p-1} - \frac{(b+1)(p-1)}{((b+1)p-1)^2}\right)}{p} \\ &= \frac{\ln(ap+1)}{p^2} - \frac{a}{p(ap+1)} + \frac{((b+1)p-1)\ln\left(\frac{p-1}{(b+1)p-1}\right) + bp}{p^2((b+1)p-1)}.\end{aligned}$$

Let φ_1 be the sum of the logarithmic terms only;

$$\begin{aligned}\varphi_1(p) &= \frac{\ln(ap+1)}{p^2} + \frac{((b+1)p-1)\ln\left(\frac{p-1}{(b+1)p-1}\right)}{p^2((b+1)p-1)} \\ &= \frac{\ln(ap+1)}{p^2} + \frac{\ln\left(\frac{p-1}{(b+1)p-1}\right)}{p^2}.\end{aligned}$$

Inspired by Example 1.1, we plug in $p = \frac{a+b}{a}$ and get

$$\begin{aligned}
\varphi_1\left(\frac{a+b}{a}\right) &= \frac{a^2}{(a+b)^2}\ln(a+b+1) + \frac{a^2}{(a+b)^2}\ln\left(\frac{\frac{a+b}{a}-1}{(b+1)\frac{a+b}{a}-1}\right) \qquad (7)\\
&= \frac{a^2}{(a+b)^2}\left(\ln(a+b+1) + \ln\left(\frac{\frac{b}{a}}{\frac{ab+b^2+b}{a}}\right)\right)\\
&= \frac{a^2}{(a+b)^2}\left(\ln(a+b+1) + \ln\left(\frac{b}{ab+b^2+b}\right)\right)\\
&= \frac{a^2}{(a+b)^2}\left(\ln(a+b+1) + \ln\left(\frac{1}{a+b+1}\right)\right)\\
&= \frac{a^2}{(a+b)^2}\left(\ln(a+b+1) - \ln(a+b+1)\right)\\
&= \frac{a^2}{(a+b)^2}(0)\\
&\text{so that } \varphi_1\left(\frac{a+b}{a}\right) = 0
\end{aligned}$$

Note that (7) is of the form $-\infty$ if $a+b=-1$, but the case $a=b=-\frac{1}{2}$ is the only one that leads to a zero denominator in the oeriginal expression (6) when $p=2$.

Next letting φ_2 be the sum of the non-logarithmic terms:

$$\varphi_2(p) = -\frac{a}{p(ap+1)} + \frac{b}{p((b+1)p-1)},$$

and plugging in $p = \frac{a+b}{a}$ we get

$$\begin{aligned}
\varphi_2\left(\frac{a+b}{a}\right) &= -\frac{a}{\frac{a+b}{a}(a+b+1)} + \frac{b}{\frac{a+b}{a}\left((b+1)\frac{a+b}{a}-1\right)} \qquad (8)\\
&= -\frac{a^2}{(a+b)(a+b+1)} + \frac{a^2 b}{(a+b)((b+1)(a+b)-a)}\\
&= -\frac{a^2}{(a+b)(a+b+1)} + \frac{a^2 b}{(a+b)(b^2+ab+b)}\\
&= -\frac{a^2}{(a+b)(a+b+1)} + \frac{a^2}{(a+b)(a+b+1)}\\
&\textit{so } \varphi_2\left(\frac{a+b}{a}\right) = 0,
\end{aligned}$$

and thus

$$\varphi'(\frac{a+b}{a}) = \varphi_1\left(\frac{a+b}{a}\right) + \varphi_2\left(\frac{a+b}{a}\right) = 0.$$

Note that (8) gives zero denominators if $a+b=-1$ but the case $a=b=-\frac{1}{2}$ is the only one that gives a zero denominator in (6) at $p=2$.

As a corollary, we see that the Cauchy-Schwartz does best ($p=2$) if $a=b\neq-\frac{1}{2}$.

Example 1.4. Incomplete Gamma integral $\quad D = \int_0^A e^{-x}x^a dx$

Note that

$$\lim_{A\to\infty} D = \Gamma(a+1).$$

We have that

$$\int_0^A e^{-x}x^a dx = \Gamma(a+1) - \Gamma(a+1,A).$$

Note that we have to use an incomplete gamma integral for a specific reason. If we had $A=\infty$ and $a>0$ then the function $g(x)=x^a$ does not belong to L^p for any $p\in[1,\infty]$, and the right side in Hölder's inequality would be infinity for any p, leading to a meaningless optimization question.

By Hölder's inequality for $A<\infty$, however, we get that

$$\begin{aligned} D = \int_0^A e^{-x}x^a dx &\leq \left(\int_0^A e^{-px}dx\right)^{\frac{1}{p}} \left(\int_0^A x^{\frac{ap}{p-1}}dx\right)^{\frac{p-1}{p}} \\ &\leq \left(\left.\frac{e^{-px}}{-p}\right|_0^A\right)^{\frac{1}{p}} \left(\left.\frac{x^{\frac{ap}{p-1}+1}}{\frac{ap}{p-1}+1}\right|_0^A\right)^{\frac{p-1}{p}} \\ &\leq \left(\frac{1}{p} - \frac{e^{-Ap}}{p}\right)^{\frac{1}{p}} \left(\frac{A^{\frac{ap+p-1}{p-1}}}{\frac{ap+p-1}{p-1}}\right)^{\frac{p-1}{p}}. \end{aligned}$$

Letting $\psi(p)$ be logarithm of the right hand side of the above inequality, we see that

$$\begin{aligned}\psi(p) &= \ln\left(\left(\frac{1}{p}-\frac{e^{-Ap}}{p}\right)^{\frac{1}{p}}\left(\frac{A^{\frac{ap+p-1}{p-1}}}{\frac{ap+p-1}{p-1}}\right)^{\frac{p-1}{p}}\right)\\ &= \frac{1}{p}\ln\left(\frac{1-e^{-Ap}}{p}\right)+\frac{p-1}{p}\ln\left(\frac{A^{\frac{ap+p-1}{p-1}}}{\frac{ap+p-1}{p-1}}\right) \\ &= \frac{1}{p}\ln\left(\frac{1-e^{-Ap}}{p}\right)+\frac{(a+1)p-1}{p}\ln(A)-\frac{p-1}{p}\ln\left(\frac{(a+1)p-1}{p-1}\right)\end{aligned}$$

[

the derivative of $\psi(p)$ is

$$\begin{aligned}\psi^{'}(p) &= \frac{\frac{Ae^{-Ap}}{p}-\frac{1-e^{-Ap}}{p^2}}{1-e^{-Ap}}-\frac{\ln\left(\frac{1-e^{-Ap}}{p}\right)}{p^2}+\frac{a+1}{p}\ln(A)-\frac{\ln(A)((a+1)p-1)}{p^2}\\ &-\frac{\ln\left(\frac{(a+1)p-1}{p-1}\right)}{p}+\frac{(p-1)\ln\left(\frac{(a+1)p-1}{p-1}\right)}{p^2}-\frac{(p-1)^2\left(\frac{a+1}{p-1}-\frac{(a+1)p-1}{(p-1)^2}\right)}{p((a+1)p-1)}\\ &= \frac{(e^{Ap}-1)\ln\left(\frac{1-e^{-Ap}}{p}\right)+e^{Ap}-Ap-1}{p^2(e^{Ap}-1)}+\frac{\ln(A)}{p^2}\\ &-\frac{((a+1)p-1)\ln\left(\frac{(a+1)p-1}{p-1}\right)-ap}{p^2((a+1)p-1)}\\ &= \frac{-1}{p^2}\left(\ln\left(\frac{1-e^{-Ap}}{p}\right)+\ln\left(\frac{(a+1)p-1}{p-1}\right)\right)+\frac{1}{p^2}\left(\frac{Ape^{-Ap}-1+e^{-Ap}}{1-e^{-Ap}}\right)\\ &+\frac{1}{p^2}\left(\ln(A)+\frac{ap}{(a+1)p-1}\right)\end{aligned}$$

Separating out the logarithmic and non-logarithmic terms doesn't help since we don't have an educated guess of the best p.

Letting for example, $A = a = 2$, using WolframAlpha we get that the minimum is attained for $p = 2.598$. Notice that the above equation is not easy to solve as in the case of beta integrals. Most importantly, we have numerical evidence to support the fact that for any a, the minimum value of p for the incomplete gamma integral in Hölder's inequality decreases to $p = 1$ as $A \to \infty$, which is what we turn to next.

Theorem 3.2

For any a, denoting the minimum value of p for Hölder's inequality in the incomplete gamma function by $\phi(A, a)$, we have that

$$\lim_{A \to \infty} \phi(A, a) = 1.$$

Proof. We do not work with $\psi'(p)$ but rather with $\psi(p)$ directly. Furthermore, we just prove the theorem for $a = 2$, since the proof is identical for general values of a. For $a = 2$,

$$\begin{aligned}\psi(p) &= \frac{1}{p} \ln\left(\frac{1 - e^{-Ap}}{p}\right) + \frac{(a+1)p - 1}{p} \ln(A) - \frac{p-1}{p} \ln\left(\frac{(a+1)p - 1}{p-1}\right) \\ &= T_1 + T_2 + T_3, \text{ say.}\end{aligned}$$

We will approximate T_1 by $\frac{1}{p} \ln\left(\frac{1}{p}\right)$. Our goal is to show that $T_1 + T_3$ is negative and bounded below by a constant. If we did not approximate T_1 as mentioned, we could assume that $A > 10$, for example, and use the fact that in this case

$$T_1 \geq \frac{1}{p} \ln\left(\frac{1 - e^{-10}}{p}\right),$$

and arrive at different finite upper and lower bounds for $T_1 + T_3$. We simplify and get

$$T_1 + T_3 = \ln p^{-1/p} \left(\frac{3p - 1}{p - 1}\right)^{-(p-1)/p},$$

which WolframAlpha reveals is between $\ln(1/3) \approx -1.09$ and $\ln 1 = 0$. Our goal is to show that for each A, there is an $\epsilon > 0$ so that

$$\psi(p) \geq \psi(1+\epsilon) \quad (p \geq 1+2\epsilon)$$

and that $\epsilon \to 0$ as $A \to \infty$. To prove (letting $G(p) = T_1 + T_3$) that

$$\begin{aligned} \psi(p) &= G(p) + \frac{3p-1}{p} \ln A \\ &\geq G(1+\epsilon) + \frac{3(1+\epsilon)-1}{1+\epsilon} \ln A \end{aligned}$$

we show instead (recalling the maximum and minimum values of $T_1 + T_3$) that

$$-1.09 + \frac{3p-1}{p} \ln A \geq \frac{2+3\epsilon}{1+\epsilon},$$

which simplifies to

$$p \geq \frac{(1+\epsilon)\ln A}{\ln A - (1.09)(1+\epsilon)}.$$

But we have assumed that $p \geq (1+2\epsilon)$, so the question is whether

$$(1+2\epsilon) \geq \frac{(1+\epsilon)\ln A}{\ln A - (1.09)(1+\epsilon)},$$

or

$$\epsilon \ln A \geq (1.09)(1+\epsilon)(1+2\epsilon).$$

Assuming without loss of generality that $\epsilon \leq 1/2$, the above reduces to

$$\epsilon \ln A \geq (1.09) \cdot \frac{3}{2} \cdot 2,$$

i.e.,

$$\epsilon \geq \frac{3.27}{\ln A} = \epsilon_A \to 0 \ (A \to \infty).$$

This proves that ψ is increasing on $[1+2\epsilon, \infty)$ and thus that

$$\min_{x>1} \psi(p) \leq \psi(1+\epsilon)$$

Theorem 4.1: Approximation Theorem

If g is a continuous real function on $[\alpha, \beta]$ then for any $\varepsilon > 0$, we can find a polynomial $E = E_\varepsilon$ on $[\alpha, \beta]$ such that

$$|g(t) - E(t)| < \varepsilon$$

for every t element of $[\alpha, \beta]$. Which means that any continuous function on a closed and bounded interval can be uniformly approximated on that interval by polynomials to any degree of precision.

Restricting to $a = 0, b = 1$ and rephrasing in the context that we seek to investigate, we have [5]

Theoerem 4.2: Weierstrass Approximation Theorem

If g is any continuous function on the interval [0,1], it is always possible, regardless how small ε is, to determine a polynomial

$$E_n(x) = \sum_{k=0}^{n} a_{n-k} x^k$$

of the degree n high enough such that we have

$$|g(x) - E_n(x)| < \varepsilon$$

for all point in the interval under consideration.

We are interested in investigating the degree of the polynomial. But which polynomial? For a given continuous function g on $[0,1]$, the Bernstein polynomial of degree n for g that is determined in terms of the Bernstein basis polynomials

$$B_n(g)(t) = \sum_{k=0}^{n} g\left(\frac{k}{n}\right) B_{k,n}(t) = \sum_{k=0}^{n} g\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-t)^{n-k}. [5], [6]$$

We are specifically interested in the degree of the approximating Bernstein polynomial. First we outline the proof of the fact that the Bernstein polynomials *do* provide a constructive proof of the Weierstrass theorem. The following adequate "absorption" identity on binomial coefficients is involved in what follows:

$$k\binom{n}{k} = n\binom{n-1}{k-1}. \qquad (0)$$

(Letting $n = 5, k = 2$, as example, gives $2\binom{5}{2} = 5\binom{4}{1}$ or $2.10 = 5.4$. The absorption identity is easily proved by writing each side in terms of factorials.) The following equations are key, and have clearly to do with the mean and variance of the binomial distribution:

$$\sum_{k=0}^{n} B_{k,n}(t) = 1. \qquad (1)$$

$$\sum_{k=0}^{n} k.B_{k,n}(t) = nx. \qquad (2)$$

$$\sum_{k=0}^{n} k(k-1).B_{k,n}(t) = n(n-1)t^2. \qquad (3)$$

(1) is true because

$$\sum_{k=0}^{n} B_{k,n}(t) = (t + (1-x))^n = 1.$$

(2) is proved as follows:

$$\sum_{k=0}^{n} k.B_{k,n}(t) = \sum_{k=0}^{n} k\binom{n}{k}x^k(1-t)^{n-k}.$$

The absorption identity (0) gives us

$$= \sum_{k=0}^{n} n\binom{n-1}{k-1}t^k(1-t)^{n-k}.$$

Letting $j = k - 1$ we simplify as

$$= n \times \sum_{j=0}^{n-1} \binom{n-1}{j} x^{j+1}(1-x)^{(n-1)-j}$$

$$= nt \times \sum_{j=0}^{n-1} \binom{n-1}{j} t^j(1-x)^{(n-1)-j}$$

(1) implies that

$$= nt \times B_{j,n-1}(t)$$

$$= nt \times 1 = nt.$$

By the same way we can prove (3) (using absorption twice). Using (1) - (3), we get

$$\sum_{k=0}^{n}(k-nt)^2 B_{k,n}(t) = \sum_{k=0}^{n}\left[k(k-1)-(2nt-1)k+n^2t^2\right] B_{k,n}(t)$$

$$= n(n-1)t^2-(2nt-1)nt+n^2t^2$$

$$= nt(1-t)$$

Since $t \in [0,1]$, we have that $t(1-t) \leq \frac{1}{4}$, and follows

$$\sum_{k=0}^{n}(k-nt)^2 B_{k,n}(t) = nt(1-t) \leq \frac{1}{4}n.$$

$$\sum_{k=0}^{n} n^2\left(\frac{k}{n}-t\right)^2 B_{k,n}(t) \leq \frac{1}{4}n.$$

$$\sum_{k=0}^{n}\left(\frac{k}{n}-t\right)^2 B_{k,n}(t) \leq \frac{1}{4n}.[5][6]$$

For $\varepsilon > 0$, let $\delta = \delta(\varepsilon)$ be the delta that guarantees uniform continuity of g.

Next let $t \in [0,1]$ and $\delta > 0$, and consider the sum of all $B_{k,n}$ with k such that $|\frac{k}{n}-t| \geq \delta$; that is, where $\frac{k}{n}$ is bounded away from t:

$$\sum_{k:|\frac{k}{n}-t|\geq\delta} B_{k,n}(t) \leq \frac{1}{\delta^2}\sum_{k:|\frac{k}{n}-t|\geq\delta}\left(\frac{k}{n}-x\right)^2 B_{k,n}(t) \leq \frac{1}{\delta^2}.\frac{1}{4n}$$

This inequality is true only for k such that $|\frac{k}{n}-t| \geq \delta$. For those k, $\frac{|\frac{k}{n}-t|}{\delta} \geq 1$ and therefore $\frac{\left(\frac{k}{n-t}\right)^2}{\delta^2} \geq 1$, justifying the introduction of $\left(\frac{k}{n-t}\right)^2$ and δ^2 in the inequality. So if $|g(t)| < M$ on [0, 1], then

$$\left|g(t) - \sum_{k:|\frac{k}{n}-t|\geq\delta} g\left(\frac{k}{n}\right) B_{k,n}(t)\right| \leq \sum_{k:|\frac{k}{n}-t|\geq\delta}\left|g(t)-g\left(\frac{k}{n}\right)\right| B_{k,n}(t) < \frac{2M}{4\delta^2 n} = \frac{M}{2\delta^2 n}.(4)$$

(4) holds only for k with $|\frac{k}{n}-t| \geq \delta$. For k with $\frac{k}{n}$ closer to t, the uniform continuity of g can be used as follows.

Let ε be a given and choose δ such that $|g(u) - g(v)| < \frac{\varepsilon}{2}$ when $|u - v| < \delta$ throughout [0, 1]. So for k with $|\frac{k}{n} - t| < \delta$:

$$\left| g(t) - \sum_{|\frac{k}{n} - t| < \delta} g\left(\frac{k}{n}\right) B_{k,n}(t) \right| \leq \sum_{|\frac{k}{n} - t| < \delta} \left| g(t) - g\left(\frac{k}{n}\right) \right| B_{k,n}(t) < \frac{\varepsilon}{2} \times 1 = \frac{\varepsilon}{2}. \qquad (5)$$

The functional arguments are close together in this range, therefore the functional values are; furthermore, the $B_{k,n}(t)$ all sum to 1, so the sum of some of them will be less than that.

The theorem follows by combining (4) and (5). To spell it out, Let $\varepsilon > 0$ be given and choose δ so that (5) holds, as discussed above. Then for that δ, choose n high enough so that $\frac{M}{2\delta^2 n} < \frac{\varepsilon}{2}$ and the left side of (4) is less than $\frac{\varepsilon}{2}$. It follows that all $t \in [0, 1]$:

$$\left| g(t) - \sum_{k=0}^{n} g\left(\frac{k}{n}\right) B_{k,n} \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

■

It is now clear that, given $\varepsilon > 0$ the degree of the approximating polynomial is the n that makes $\frac{M}{2\delta^2 n} < \frac{\varepsilon}{2}$, or simply $\frac{M}{\delta^2 n} < \varepsilon$ where g is bounded by M and $\delta = \delta_\varepsilon$ is the one that occurs in the validation of uniform continuity of g.

4.1 Examples:

We give two types of examples. First we choose very smooth functions for which the n^{th} Taylor polynomial does vastly better. Then we consider two nowhere differentiable continuous functions on [0,1], and find the degree of the approximating Bernstein polynomial.

Example **a.** $\mathbf{f(x) = x}$ (Silly example)

Let $\varepsilon > 0$ be given. Choose $\delta = \varepsilon$ and assume $|x - y| < \delta$

$$|f(x) - f(y)| = |x - y| < \varepsilon$$

Then Weierstrass approximation on [0, 1] gives us:

$\frac{M}{2\delta^2} n < \frac{\varepsilon}{2} \implies \frac{1}{\varepsilon^2 n} < \varepsilon \implies n > \frac{1}{\varepsilon^3}$ for $M = 1$. e.g. $n = 1000$ if $\varepsilon = 0.1$

As we can see the degree n is very high since x is itself a polynomial but the best polynomial is x.

Exanple **b**. $\mathbf{f}(\mathbf{x}) = \ln(\mathbf{1} + \mathbf{x})$, on [0, 1]

Let $x, y \in [0, 1]$, given $\varepsilon > 0$.

$$|f(x) - f(y)| = |\ln(1 + x) - \ln(1 + y)| < \varepsilon$$
$$= \left|\ln\left(\frac{1 + x}{1 + y}\right)\right| < \varepsilon,$$

it follows that $\frac{1 + x}{1 + y} < e^{\varepsilon}$ we know that $1 - u \leq e^{-u}$ then

$\frac{1 + x}{1 + y} < e^{\varepsilon} < 1 + \frac{\varepsilon}{1 - \varepsilon}$. Without loss of generality, assume $x > y$

$$\frac{1 + y + (x - y)}{1 + y} < 1 + \frac{\varepsilon}{1 - \varepsilon} \implies 1 + \frac{x - y}{1 + y} < 1 + \frac{\varepsilon}{1 - \varepsilon}$$

Then

$$\frac{x - y}{2} < \frac{x - y}{1 + y} < \frac{\varepsilon}{1 - \varepsilon} \implies x - y < \frac{2\varepsilon}{1 - \varepsilon}$$

Let $\delta = \frac{2\varepsilon}{1-\varepsilon}$ and $M = \ln(2)$

Then the Weierstrass polynomial degree is given by $\frac{M}{2\delta^2 n} < \frac{\varepsilon}{2}$, i.e,

$$\frac{\ln(2)}{\left(\frac{2\varepsilon}{1-\varepsilon}\right)^2 \varepsilon} < n \implies \frac{\ln(2)(1 - \varepsilon)^2}{4\varepsilon^3} < n$$

$$\text{If} \quad \varepsilon = 0.1 \quad \text{then} \quad n > 140.362.$$

Example **c**. $\mathbf{f(x) = \sin(x)}$

Given $\varepsilon > 0$ and $x, y \in \mathbb{R}$, we want

$$|f(x) - f(y)| < \varepsilon \text{ which implies that } |\sin(x) - \sin(y)| < \varepsilon$$

$$\text{i.e,} \quad \left|2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)\right| < \varepsilon, \quad \text{We know that}$$

$$\left|2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)\right| \le 2\left|\sin\left(\frac{x-y}{2}\right)\right|$$

When $|x-y| < \delta$, also $\left|\frac{x-y}{2}\right| < \delta$ and since $|\sin(x)| \le |x|$, we get

$$2\left|\sin\left(\frac{x-y}{2}\right)\right| \le 2\left|\frac{x-y}{2}\right| \le 2\delta.$$

Choose $\delta = \frac{\varepsilon}{2}$, and $M = 1$ we define the the degree of the polynomial n as $n > \frac{2}{\varepsilon^3}$ and it follows for $\varepsilon = 0.1$ we get $n > 2000$.

Example **d**. $\mathbf{f(x) = e^x}$

Given $\varepsilon > 0$, there exists $\delta > 0$, such that for all $x, y \in [0, 1]$:

$$|x - y| < \delta \text{ implies that } |e^x - e^y| < \varepsilon,$$

$-\delta < x - y < \delta$ gives us $y - \delta < x < y + \delta$.

Since e^x is increasing on [0, 1], we have

$$e^{y-\delta} < e^x < e^{y+\delta} \implies e^{y-\delta} - e^y < e^x - e^y < e^{y+\delta} - e^y$$

$$\implies \frac{e^y(1-e^\delta)}{e^\delta} < e^x - e^y < e^y(e^\delta - 1)$$

As $\delta > 0$, we have that $1 - e^{\delta} < 0$ so that

$$e^y(1-e^{\delta}) < e^x - e^y < e^y(e^{\delta}-1)$$

$$|e^x - e^y| < e^y(e^{\delta}-1) < M(e^{\delta}-1)$$

$$\implies |e^x - e^y| < Me^{\delta}$$

Let $\varepsilon = M(e^{\delta}-1)$, i.e, $\delta = \ln(\frac{\varepsilon}{M}+1)$ then we have

$$\frac{M}{2\delta^2 n} < \frac{\varepsilon}{2} \implies \frac{M}{\varepsilon\left(\ln\left(\frac{\varepsilon}{M}+1\right)\right)^2} < n$$

We can get $M = e$ and $\varepsilon = 0.1$

$$\frac{e}{0.1\left(\ln\left(\frac{0.1}{e+1}\right)\right)^2} = 20826.707 < n.$$

Also we can see that the degree is extremely high for all examples but how high? We compare to the n^{th} Taylor polynomial approximation for example 4.1.d.

We know that Taylor series for e^x on [0, 1] is

$$e^x = 1 + x + \frac{x^2}{2!} + ... + \frac{x^n}{n!} + ...$$

Then if we want the n^{th} Taylor polynomial to be close to the function, we must have

$$\left|e^x - \left(1 + x + \frac{x^2}{2!} + ... + \frac{x^n}{n!}\right)\right| < 0.1$$

$$R_n(x) = \frac{M}{(n+1)!}\underbrace{(x-a)^{n+1}}_{\leq 1}$$

$$\text{M=e and } R_n(x) \leq 0.1$$

$$\frac{e}{(n+1)!} \leq 0.1$$

$$(n+1)! \geq \frac{e}{0.1} \geq 27$$

$$\therefore \quad \text{n=4.}$$

But in general we need $f^n(x)$ to exist on $[a, b]$ for the n^{th} Taylor approximation to be valid. For Weierstrass we just need f to be continuous! Then we do the following, using as examples two of the most non-smooth continuous functions we can think of.

4.2 Weierstrass Polynomials for Nowhere Differentiable Everywhere Continuous Functions

1. Brownian Motion

Standard one dimensional Brownian Motion (also known as the Wiener process) is a stochastic process $\{\mathcal{X}(t) : t \geq 0\}$ satisfying

$$\cdot \mathcal{X}(\mathbf{0}) = \mathbf{0}.$$

$$\cdot \mathcal{X}(\mathbf{t}) \sim \mathcal{N}(\mathbf{0}, \mathbf{t}).$$

$$\cdot \{\mathcal{X}(\mathbf{t}) : \mathbf{t} \geq \mathbf{0}\} \ \text{ has independent increments,}$$

i.e. for $u < v < w$, $\mathcal{X}(v) - \mathcal{X}(u)$ and $\mathcal{X}(w) - \mathcal{X}(v)$ are independent.

It is known [7] that the sample paths of Brownian motion are with probability 1 nowhere differentiable and everywhere continuous. But what of the modulus of continuity? Is it true, e.g., that $|\mathcal{X}(t)| \leq 10\sqrt{t}, \forall t$? This may enable us to establish a modulus of continuity (The "10" above is an arbitrary large number and reflect the fact that most of the mass of a normal variable is within 3 standard deviations of its mean).

For a fixed t, this is true with high probability

$$\mathcal{X}(t) \sim \mathcal{N}(0, t).$$

$$\text{so } \ P\Big(-10\sqrt{t} \leq \mathcal{X}(t) \leq 10\sqrt{t}\Big) = P\Big(\frac{-10\sqrt{t} - 0}{\sqrt{t}} \leq \mathcal{N}(0,1) \leq 10\Big)$$

$$= P(-10 \leq z \leq 10)$$

$$= \int_{-10}^{10} \frac{1}{\sqrt{2\pi}} e^{\frac{-t^2}{2}} dt \approx 1.$$

But we have to prove that $|\mathcal{X}(t) - \mathcal{X}(s)| \le 10\sqrt{t-s}$ with high probability for all t, s. This seems unlikely.

However, Lévy [7] proved that Brownian motion is Hölder continuous with exponent $\alpha < \frac{1}{2}$, i.e, for each s, t

$$|\mathcal{X}(t) - \mathcal{X}(s)| \le K|t-s|^{\alpha} \qquad \left(\alpha < \frac{1}{2}, \text{ and K constant}\right)$$

with probability one. In fact, more is known: We have that

$$\mathcal{B}(t+h) - \mathcal{B}(t) \le \sqrt{2+\eta}\sqrt{h\log(\tfrac{1}{h})},\ \ h \text{ sufficiently small, for all} \quad 0 \le t \le 1-h,$$

which is better than Hölder continuity with $\alpha < \frac{1}{2}$ since when $h \to 0$ we have that $h\ln(\frac{1}{h}) = \frac{\ln\frac{1}{h}}{\frac{1}{h}} \to \frac{\infty}{\infty}$ which is an indeterminate form. So we can apply L'Hôpital's rule and get

$$\lim_{h\to o} h\ln\left(\frac{1}{h}\right) = \lim_{h\to 0}\frac{h(\frac{-1}{h^2})}{\frac{-1}{h^2}} = \lim_{h\to 0} h = 0$$

In any case, if we use $\alpha = 0.49$, for example then we have by the earlier discussion that

$$|\mathcal{X}(t) - \mathcal{X}(s)| \le (1.42)|t-s|^{0.49} < \epsilon$$

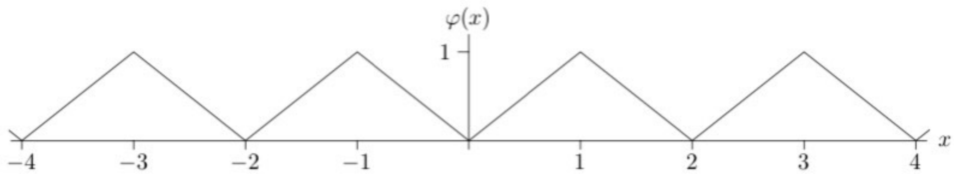
if $|t-s| < (0.7)\epsilon^{2.041} = \delta$, so that the degree of the ϵ-approximating polynomial is $1.42/\epsilon^{5.08}$. If $\epsilon = 0.1$, e.g., then the degree is

$$1.42 \times 10^{5.08} = 170,722$$

.

2. Let's construct the function $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ as $\psi(t) = |t|$ for

$t \in$ [-1, 1] and that $\psi(t+2) = \psi(t)$ for every real number t. By definition is periodic of period 2. ψ is continuous everywhere (see [8]).



Now define

$$g(t) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \psi(4^n t).$$

Since $\psi(t)$ is bounded (± 1), the sequence converges uniformly by the Weierstrass M-test with $M_n = \left(\frac{3}{4}\right)^n$. As ψ is continuous, $g(t)$ is a uniform limit of continuous functions and then is continuous.

Fix any $t \in \mathbb{R}$ and show that g is not differentiable at t by demonstrating a real sequence $(\mu_k)_{k\in\mathbb{N}}$ that converges to 0 such that $\frac{1}{\mu_k}[g(t+\mu_k)-g(t)]$ diverges as $k \to \infty$.

Actually, $\mu_k = \pm\frac{1}{2}4^{-k}$ with the sign chosen so that there is no integer strictly between $4^k t$ and $4^k(t+\mu_k)$.

Now compute the magnitude of the n^{th} term in $\frac{1}{\mu_k}(g(t+\mu_k)-g(t)) = \gamma_{k,n}$.

$$\gamma_{k,n} = \frac{1}{\mu_k}\left(\frac{3}{4}\right)^n \left[\psi(4^k t + 4^k \mu_k) - \psi(4^k t)\right] = \pm 2(3^n)4^{k-n}\left(\psi(4^k t \pm \frac{1}{2}4^{n-k}) - \psi(4^n t)\right).$$

We have 3 cases:

Case 1: $n > k$. In this case $\frac{1}{2}4^{n-k}$ is an even integer. So $\gamma_{k,n} = 0$

since $\psi(4^n t \pm \frac{1}{2}4^{n-t}) = \psi(4^n t)$ because ψ has a period 2.

Case 2: $n = k$. Recall that the sign of μ_k so that there is no

integer strictly between $4^k t$ and

$4^k(t+\mu_k)$. So $(4^k t, \psi(4^k t))$ and $(4^k(t+\mu_k), \psi(4^k t + 4^k \mu_k))$ lie

on the same ramp (i.e. straight line segment) in the graph of ψ,

above. Each of those ramps has slope -1 or +1. So

$|\psi(4^k t + 4^k \mu_k) - \psi(4^k t)| = 4^k |\mu_k| = \frac{1}{2}$ and $|\gamma_{k,n}| = 2(3^k)4^{k-k}\frac{1}{2} =$ 3^k.

Case 3: $n < k$. Since $|\psi(z) - \varphi(t)| \leq |z - t|$ for all $t, z \in \mathbb{R}$, we always have that

$$|\gamma_{k,n}| \leq 2(3^n)4^{k-n}\frac{1}{2}4^{n-k} = 3^n$$

Putting these bounds together

$$\begin{aligned} \left|\frac{1}{\mu_k}\left[g(t+\mu_k) - g(t)\right]\right| = \left|\sum_{n=0}^{\infty}\gamma_{k,n}\right| = \left|\sum_{n=0}^{k}\gamma_{k,n}\right| &\geq |\gamma_{k,k}| - \sum_{n=0}^{k-1}|\gamma_{k,n}| \\ &\geq 3^k - \sum_{n=0}^{k-1}3^n \\ &= 3^k - \frac{1-3^k}{1-3} = \frac{1}{2}(3^k+1) \end{aligned}$$

As $k \to \infty$, this does not converge. Consequently g is not differentiable at t.

So

$$\begin{aligned} |g(t+\mu_k) - g(t)| &= \sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n (\psi(4^n(t+\mu_k)) - \psi(4^n x)) \\ &= \sum_{n=0}^{k}\left(\frac{3}{4}\right)^n (\psi(4^n(t+\mu_k)) - \varphi(4^n t)) \\ &= \sum_{n=0}^{k}\frac{1}{2.4^{k-n}}\left(\frac{3}{4}\right)^n \\ &= \sum_{n=0}^{k}\frac{1}{2}\frac{3^n}{4^k} = \frac{1}{2.4^k}\sum_{n=0}^{k}3^n \\ &\approx \frac{C.3^k}{2.4^k} \end{aligned}$$

$$\therefore \quad |g(t+\mu_k) - g(t)| \leq C\left(\tfrac{3}{4}\right)^k \quad \text{if} \quad |t+\mu_k - t| \leq \tfrac{1}{4^k}$$

Let $\delta = \frac{1}{4^k}$ then $\ln\left(\frac{1}{4^k}\right) = k\ln\left(\frac{1}{4}\right)$ and $\ln\left(\left(\frac{3}{4}\right)^k\right) = k\ln\left(\frac{3}{4}\right)$

$\therefore\ \ln(\delta) = \ln(\varepsilon).\frac{\ln(1/4)}{\ln(3/4)}$, i.e, $\delta = e^{\ln(\varepsilon).\frac{\ln(1/4)}{\ln(3/4)}}$

$\therefore\quad \delta = \varepsilon^{\frac{\ln(1/4)}{\ln(3/4)}} = \varepsilon^{C}$

Thus, for the Weierstrass polynomial, where $M = 1$ and $\varepsilon = 0.1$, the degree of the polynomial using the formula $\frac{M}{\delta^2\varepsilon} < n$ is

$$n > 5.75440...10^{10}.$$

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