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Zeta Function Regularization and its Relationship to Number Theory

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A thesis

presented to

the faculty of the Department of Mathematics & Statistics

East Tennessee State University

In partial fulfillment

of the requirements for the degree

Master of Science in Mathematical Sciences

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by

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May 2021

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ABSTRACT

Zeta Function Regularization and its Relationship to Number Theory

by

Stephen Wang

While the "path integral" formulation of quantum mechanics is both highly intuitive and far reaching, the path integrals themselves often fail to converge in the usual sense. Richard Feynman developed regularization as a solution, such that regularized path integrals could be calculated and analyzed within a strictly physics context. Over the past 50 years, mathematicians and physicists have retroactively introduced schemes for achieving mathematical rigor in the study and application of regularized path integrals. One such scheme was introduced in 2007 by the mathematicians Klaus Kirsten and Paul Loya. In this thesis, we reproduce the Kirsten and Loya approach to zeta function regularization and explore more fully the relationship between operators in physics and classical zeta functions of mathematics. In so doing, we highlight intriguing connections to number theory that arise.

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1 PROBABILITY THEORY IN QUANTUM MECHANICS

While the "path integral" formulation of quantum mechanics is both highly intuitive and far reaching, the path integrals themselves often fail to converge in the usual sense. In this section, we discuss how regularization developed from an extension of the idea of determinant of a finite matrix to the idea of a functional determinant in the infinite-dimensional spaces of operators. We also discuss how Norbert Wiener first established this idea in the context of probability theory and then Richard Feynman strengthened the idea in the context of "the sum over histories" used in quantum mechanics.

1.1 Finite Dimensional Operators

Let $\mathbb{E}^n$ be $\mathbb{R}^n$ with the standard inner product

$$\langle x, y\rangle = x^T y$$

(i.e., $\mathbb{E}^n$ is the matrix model of *Euclidean space).* The *adjoint* A^* of an $n \times n$ complex matrix A is the Hermitian

$$A^H = \overline{A^T}$$

where the bar denotes complex conjugation of each coefficient. If A is self-adjoint (symmetric in the real case), then

$$\int\int \dots \int_{\mathbb{E}^n} e^{\pi\langle Ax,x\rangle} dx_1 dx_2 \dots dx_n = \frac{1}{\sqrt{\det(A)}}.$$

Such multiple integrals occur often in applications, and when they do, they can be evaluated if we can calculate the determinant of A. Moreover, such integrals are often extended into an infinite dimensional context, in which case A is a self-adjoint operator on an infinite dimensional *Hilbert space*, where the concept of a determinant may not make sense. In such cases, it is often possible to use zeta function regularization to *assign* a meaningful value to $\det(A)$ via analytic continuation.

1.2 Quantum Calculations Related to Probability Theory

Quantum mechanics is a stochastic theory (it uses probability densities), so we must take a stochastic approach. In particular, we begin by developing a model of Brownian motion of a random walk as a limit as step size approaches zero. For fixed $\varepsilon > 0$, suppose that at discrete times steps $t = t_i, t_i+\varepsilon, t_i+2\varepsilon, t_i+3\varepsilon, \ldots$ the probability distribution $P_k(x)$ for the position of a random walker at time $t_i + k\varepsilon$ satisfies the Markov chain property

$$P_k(x) = \int_{-\infty}^{\infty} P_{k-1}(u)\, p(x,u)\, du,$$

where $p(x,u)$ is the probability of transition from $u \in \mathbb{R}$ to $x \in \mathbb{R}$. Einstein showed that for Brownian motion, this transition probability is [9]

$$p(x,u) = R(x-u) \qquad \text{where} \qquad R(x) = \frac{1}{\sqrt{2\pi\xi}} e^{-x^2/(2\xi)},$$

where ξ is the "mass" variable. According to Norbert Wiener, if we let $P_0(x)$ be a delta function centered about the initial position x_0, then [9]

$$P_1(x) \;=\; \int_{-\infty}^{\infty} R(x-u)\, \delta(u-x_0)\, du = R(x-x_0).$$

Extending this to more discrete time steps, we get

$$\begin{aligned}
P_2(x) &= \int_{-\infty}^{\infty} R(x-x_1) R(x_1-x_0)\, dx_1 \\
P_3(x) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R(x-x_2) R(x_2-x_1) R(x_1-x_0)\, dx_2 dx_1 \\
&\vdots \qquad\qquad \vdots \\
P_n(x) &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} R(x-x_{n-1}) R(x_{n-1}-x_{n-2})\cdots R(x_1-x_0)\, dx_{n-1}\cdots dx_1.
\end{aligned}$$

If we let x be denoted x_n at the n^{th} iteration, then we can rewrite this as

$$P_n(x_n) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \prod_{k=1}^{n} R(x_k - x_{k-1})\ \ dx_{k-1}.$$

However, notice that

$$\begin{aligned}
\prod_{k=1}^{n} R(x_k - x_{k-1}) &= \left(\frac{1}{\sqrt{2\pi\xi}}\right)^{n-1} \prod_{k=1}^{n} e^{-(x_k-x_{k-1})^2/(2\xi)} \\
&= \left(\frac{1}{\sqrt{2\pi\xi}}\right)^{n-1} \exp\left(\frac{-1}{\xi}\frac{1}{2}\sum_{k=1}^{n}(x_k-x_{k-1})^2\right).
\end{aligned}$$

We subsequently define

$$S_\varepsilon(x_0, x_1, \ldots, x_n) = \frac{1}{2}\sum_{k=1}^{n}(x_k - x_{k-1})^2,$$

so that we have

$$P_n(x) = \frac{1}{(2\pi\xi)^{(n-1)/2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-S_\varepsilon(x_0,x_1,\ldots,x_n)/\xi}\ \prod_{k=1}^{n-1} dx_k.$$

Next, Wiener introduced a profound new idea in in Applied Mathematics in the 1920s. Partition $[t_i, t_f]$: For $t_k = t_i + k\varepsilon$, define a piecewise linear function $q_\varepsilon(t)$ such that

$$q_\varepsilon(t) = \sqrt{\varepsilon}\left(\frac{x_k - x_{k-1}}{t_k - t_{k-1}}(t - t_{k-1}) + x_{k-1}\right) \qquad \text{for} \ \ t_{k-1} \le t \le t_k$$

and choose ε such that $t_n = t_f$. Also, require that $q_\varepsilon(t_0) = x_i$ and $q_\varepsilon(t_n) = x_f$ (Endpoints are free in the piecewise linear definition, so there are no issues here.). It follows that we can rewrite $S_\varepsilon(x_0, \ldots, x_n)$ as

$$S_\varepsilon(x_0, \ldots, x_n) = \frac{1}{2}\int_{t_i}^{t_f} \left(\frac{dq}{dt}\right)^2 dt.$$

Wiener showed using probability theory that as the number of time steps in the partition increased toward infinity, the probability distribution becomes [7, 9]

$$\lim_{n\to\infty} \frac{1}{\sqrt{\varepsilon}} P_n(x) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left(\frac{-1}{\xi}\frac{1}{2}\int_{t_i}^{t_f} \dot{q}^2 dt\right) \prod_{k=1}^{\infty} dx_k. \tag{1}$$

He called this new concept a *path integral.* In some sense, it measures the probability that a random walker starts at x_i at time t_i and ends at x_f at time t_f by integrating over every possible path from x_i to x_f.

1.3 Path Integrals in Brownian Motion

Note that the $P_n(x)$ distributions are defined by convolutions, so convergence is transformed via the Convolution Theorem for the Fourier Transform into proving that an infinite product of Fourier transforms converges [2]:

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left(\frac{-1}{\xi}\frac{1}{2}\int_{t_i}^{t_f} \dot{q}^2 dt\right) \prod_{k=1}^{\infty} dx_k = \mathcal{F}^{-1}\left(\prod_{n=1}^{\infty} \hat{f}_k(\omega)\right).$$

The infinite product is indeed in terms of a determinant of an operator. This is how Wiener proved that equation (1) converges to

$$\frac{1}{\sqrt{2\pi\xi(t_f - t_i)}} \exp\left(\frac{-(x_f - x_i)^2}{2\xi(t_f - t_i)}\right).$$

If we let $t_i = 0$ and write t_f as just t, (thus, x_f as simply x), then we can write this as

$$u(x,t) = \frac{1}{\sqrt{2\pi\xi t}} \exp\left(\frac{-(x-x_i)^2}{2\xi t}\right),$$

and it can be shown that $u(x,t)$ is a solution to a heat equation

$$\frac{\partial u}{\partial t} = \frac{\xi}{2}\frac{\partial^2 u}{\partial x^2}$$

with condition $\int_{-\infty}^{\infty} u(x,t)\,dx = 1$ for all $t > 0$ [12]. Moreover, notice that $S_\varepsilon(x_0, x_1, \ldots, x_n)$ is an inner product of the form $\langle A_n x, x\rangle$ for self-adjoint A_n, so that

$$P_n(x) = \frac{1}{(2\pi\xi)^{(n-1)/2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\langle A_n x, x\rangle} \prod_{k=1}^{n-1} dx_k = \frac{1}{\sqrt{\det(A_n)}}.$$

In probability theory, path integrals converge and determinants of the associated operators are well-defined because expected values of probability measures are assumed to be absolutely convergent [4]. Convergence implies $\det(A_n)$ also converges to $\det(A)$, where A is a self-adjoint linear operator on $L^2(\mathbb{R})$:

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left(\frac{-1}{\xi}\frac{1}{2}\int_{t_i}^{t_f} \dot{q}^2 dt\right) \prod_{k=1}^{\infty} dx_k = \frac{1}{\sqrt{\det(A)}}.$$

Thus, with complete mathematical rigor we can write

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left(\frac{-1}{\xi}\frac{1}{2}\int_{t_i}^{t_f} \dot{q}^2 dt\right) \prod_{k=1}^{\infty} dx_k = \frac{1}{\sqrt{2\pi\xi(t_f - t_i)}} \exp\left(\frac{-(x_f - x_i)^2}{2\xi(t_f - t_i)}\right),$$

which is known as a Wiener path integral (or, equivalently, a Wiener process [9]).

1.4 Feynman Path Integrals

In classical physics, the quantity $\int_{t_i}^{t_f} \frac{1}{\xi}\dot{q}^2 dt$ is the action of a free particle with mass ξ. The "dimensionless" Schrödinger of a free particle [9]

$$i\frac{\partial \psi}{\partial t} = \frac{\partial^2 \psi}{\partial x^2}$$

can be written in the form

$$\frac{\partial \psi}{\partial (it)} = \frac{\partial^2 \psi}{\partial x^2},$$

which is a heat equation in "imaginary time" (called a Wick rotation). If we let $t = i\tau$ in the Wiener path integral, then the result is the Feynman path integral for a free particle

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left(i \frac{1}{2\xi} \int_{t_i}^{t_f} \dot{q}^2 d\tau \right) \prod_{k=1}^{\infty} dx_k.$$

Feynman thus generalized Wiener's path integrals in probability theory to Feynman path integrals

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{iS(x)/\hbar} \prod_{k=1}^{\infty} dx_k,$$

where $S(x) = \int_{t_i}^{t_f} L(x, \dot{x})\, d\tau$ is the classical Lagrangian action from classical physics. Feynman certainly was inspired by Wiener, who himself is one of the greatest mathematicians and scientists of the 20th century (he established the field of *cybernetics*, for example; Wiener also defined and established the deep and rich connection between probability theory and analytic function theory [14]).

However, Feynman went far beyond the probabilistic path integrals of Wiener. For example, if S is very large, then [9]

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{iS(x)/\hbar} \prod_{k=1}^{\infty} dx_k \approx \int_{t_i}^{t_f} L(x, \dot{x})\, dt.$$

Thus, large S implies classical Lagrangian mechanics. Feynman uses this idea to extend the Lagrangian mechanics approach to quantum field theory.

1.5 The Role of Zeta Regularization

But Feynman path integrals do not converge. In probability theory, the integrand is $e^{-S(x)}$, whereas in Feyman's path integral formulation the integrand is $e^{iS(s)/\hbar}$. Similarly, the self-adjoint operators related to Schrödinger equations do not tend to have well-defined determinants. However, as Feynman himself realized, it may still be possible to obtain a value for a determinant via analytic continuation, in which case

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{iS(x)/\hbar} \prod_{k=1}^{\infty} dx_k = \frac{1}{\sqrt{\det(S)}},$$

where $\det(S)$ is a *functional determinant* – that is, a number assigned to an operator via analytic continuation of a zeta function regularization [11].

Zeta function regularization is one method that can be used to find the functional determinant of an infinite-dimensional operator. Kirsten and Loya [5] use both zeta function regularization and comparison of similar determinants for regularization. Osgood, Phillips, and Sarnak showed that both methods of regularization are equivalent [8]. Defining the mathematically rigorous zeta function regularization below, we have that the zeta function of an operator S is defined by

$$\zeta_S(z) = \mathrm{tr}\ S^{-z}\ ,$$

For a self-adjoint operator with positive eigenvalues $\{\sigma_k\}_{k=1}^{\infty}$, the equation above is equivalent to

$$\zeta_S(z) = \sum_{k=1}^{\infty} \frac{1}{\sigma_k^z}\ , \tag{2}$$

which will be useful in Chapters 3 and 4. The functional determinant is defined by

[11]

$$\det S = e^{-\zeta_S'(0)}.$$

2 THE RIEMANN ZETA FUNCTION AND ITS PROPERTIES

In order to understand the zeta functions used to regularize these infinite dimensional operators, we must establish an understanding of the classical Riemann zeta function. In this chapter, we present background and some results about the classical Riemann zeta function.

2.1 Complex Analysis Background

In this section, we state definitions from complex analysis that will be used throughout the rest of the thesis. John Conway's *Functions of One Complex Variable* offers a comprehensive resource for complex analysis background.

Definition 2.1. *A holomorphic function is a complex-valued function of a complex variable that is, at every point of its domain, complex differentiable in a neighborhood of the point.*

A major theorem of complex analysis states that all holomorphic functions are analytic, and vice versa. Hence, the term analytic will be used to describe these functions.

Definition 2.2. *A meromorphic function on an open subset D of the complex plane is a function that is analytic on all of D except for a set of isolated points, which are poles of the function.*

The following theorem is known colloquially as Cauchy's Residue Theorem.

Theorem 2.3. *Let f be a function defined and meromorphic on a simply connected open subset of the complex plane D. If C is a positively oriented simple closed curve* *around D containing poles (σ_k) of f(z) and not intersecting any other poles of f(z), then*

$$\oint_C f(z) = 2\pi i \sum \mathrm{Res}(f, \sigma_k)$$

where $\mathrm{Res}(f, \sigma_k)$ *are the corresponding residues at the poles.*

Analytic continuation is a technique by which the analyticity of a function is conserved while extending the domain of the original function. The technique of analytic continuation, though it can have many forms, provides a unique continuation due to the analytic property of the original function.

2.2 The Riemann Zeta Function $\zeta(z)$

The Riemann zeta function is most commonly associated with the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{1}{n^z}.$$

The series converges for $\mathrm{Re}(z) > 1$, and in this domain we write

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}.$$

However, the Riemann zeta function is not restricted to the domain of convergence of the Dirichlet series. Hence, we have the following definition.

Definition 2.4. [3] *The Riemann zeta function $\zeta(z)$ is the analytic continuation of the Dirichlet series*

$$\sum_{n=1}^{\infty} \frac{1}{n^z}$$

to the meromorphic function defined on the whole complex plane, which has a pole at $z = 1$.

2.3 The Gamma Function $\Gamma(z)$

In order to further study the Riemann zeta function, we must also consider the closely related gamma function.

Definition 2.5. [3] *The gamma function* $\Gamma(z)$ *is defined by*

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt.$$

Using integration by parts, we can see that

$$\Gamma(z+1) = \int_0^\infty t^z e^{-t}dt = -t^z e^{-t}\Big|_0^\infty + z\int_0^\infty t^{z-1}e^{-t}dt = z\Gamma(z). \tag{3}$$

That is, $\Gamma(z+1) = z\Gamma(z)$ for any z such that $\mathrm{Re}(z) > 0$. Straightforward integration shows that $\Gamma(1) = 1$. So if n is a positive integer, then the two properties combined imply that $\Gamma(n) = (n-1)!$. Furthermore, the Γ function also extends to a meromorphic function defined on the complex plane. The function has simple poles at each of the integers less than or equal to 0. All other values of the extension can be found by calculating values of the Γ function in the right half-plane directly from the integral and using the equation $\Gamma(z+1) = z\Gamma(z)$ to extend the function to values in the left-half plane.

2.4 The Digamma Function

The digamma function is the quotient of the derivative of the gamma function and the gamma function. We explore this function further in Chapters 3 and 5.

Definition 2.6. [10] *The digamma function* $\psi(z)$ *is defined by*

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}.$$

Theorem 2.7. *The residues of the poles of the digamma function $\psi(z)$ are given by*

$$\text{Res}(\psi, -n) = -1$$

for $n \in \mathbb{Z}^{\leq 0}$.

Proof. The gamma function obeys equation (3):

$$\Gamma(z+1) = z\Gamma(z).$$

So, taking the derivative with respect to z gives

$$\Gamma'(z+1) = z\Gamma'(z) + \Gamma(z).$$

Dividing by $\Gamma(z+1)$ or the equivalent $z\Gamma(z)$ gives

$$\frac{\Gamma'(z+1)}{\Gamma(z+1)} = \frac{\Gamma'(z)}{\Gamma(z)} + \frac{1}{z},$$

or

$$\psi(z+1) = \psi(z) + \frac{1}{z}. \tag{4}$$

Note that the digamma function has poles at the same locations as the gamma function. So, allowing $z \to 0$ in equation (4), we see that $\psi(1)$ is a constant, while $\frac{1}{z}$ goes to infinity. Therefore, $\psi(z)$ must contain a pole of the form $-\frac{1}{z}$ to cancel out the pole of $\frac{1}{z}$. That is,

$$\text{Res}(\psi, 0) = -1.$$

Referring again to equation (4), we see that because $\psi(z)$ has a pole at $z = 0$, it must also have a pole at every negative integer. Likewise,

$$\text{Res}(\psi, -n) = -1$$

for $n \in \mathbb{N}$. $\square$

2.5 The Functional Equation

We now describe and prove the functional equation of the Riemann zeta function.

Theorem 2.8. [3] $\zeta(z)$ *satisfies the following equation:*

$$\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z) = \pi^{-\frac{1-z}{2}}\Gamma\left(\frac{1-z}{2}\right)\zeta(1-z).$$

The functional equation for the Riemann zeta function can be proved via Poisson summation and theta functions.

Proof. Let $\mathcal{H} = \{x + iy : x, y \in \mathbb{R}, y > 0\}$ denote the complex upper half plane. We define the theta function $\Theta : \mathcal{H} \to \mathbb{C}$ by

$$\Theta(\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 \tau}.$$

Restricting to $\tau = it$ with $t \in \mathbb{R}$, $t > 0$, we define

$$\theta(t) = \Theta(it) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t} = 1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 t}.$$

According to Poisson summation, $\theta(1/t) = t^{1/2}\theta(t)$. For a function $f : \mathbb{R}^+ \to \mathbb{C}$, the Mellin transform of f is the integral

$$g(z) = \int_{t=0}^{\infty} f(t)t^z \frac{dt}{t}.$$

We consider the Mellin transform at $z/2$ of

$$f(t) = \frac{1}{2}(\theta(t) - 1) = \sum_{n=1}^{\infty} e^{-\pi n^2 t}, t > 0.$$

That is,

$$\begin{aligned} g(z/2) &= \int_0^{\infty} f(t)t^{z/2}\frac{dt}{t} \\ &= \int_0^{\infty} \sum_{n=1}^{\infty} e^{-\pi n^2 t} t^{z/2}\frac{dt}{t}. \end{aligned}$$

Since the sum converges for $t > 0$ and the integral converges absolutely for all values of z, we may switch the sums and the integral in the domain of convergence:

$$g(z/2) = \sum_{n=1}^{\infty} \int_0^{\infty} e^{-\pi n^2 t} t^{z/2} \frac{dt}{t}.$$

We rewrite the integrals by making the substitution $u = \pi n^2 t$,

$$g(z/2) = \sum_{n=1}^{\infty} (\pi n^2)^{-z/2} \int_0^{\infty} e^{-u} u^{z/2} \frac{du}{u}.$$

This yields the final form of $g(z/2)$,

$$g(z/2) = \pi^{-z/2} \Gamma(z/2) \zeta(z), \mathrm{Re}(z) > 1,$$

which includes both the gamma function and the Riemann zeta function. We write this equation in another form. Splitting the integral at $t = 1$, we get

$$\begin{aligned} \pi^{-z/2}\Gamma(z/2)\zeta(z) &= \int_0^{\infty} \frac{1}{2} \left(\theta(t) - 1\right) t^{z/2} \frac{dt}{t} \\ &= \frac{1}{2} \int_0^1 \left(\theta(t) - 1\right) t^{z/2} \frac{dt}{t} + \frac{1}{2} \int_1^{\infty} \left(\theta(t) - 1\right) t^{z/2} \frac{dt}{t}. \end{aligned}$$

Multiplying the terms in the left integral together, we can write it as a sum of two integrals

$$\pi^{-z/2}\Gamma(z/2)\zeta(z) = \frac{1}{2} \int_0^1 \theta(t) t^{z/2} \frac{dt}{t} - \frac{1}{2} \int_0^1 t^{z/2-1} dt + \frac{1}{2} \int_1^{\infty} \left(\theta(t) - 1\right) t^{z/2} \frac{dt}{t}.$$

The integral of the monomial can be computed directly:

$$\pi^{-z/2}\Gamma(z/2)\zeta(z) = \frac{1}{2} \int_0^1 \theta(t) t^{z/2} \frac{dt}{t} - \frac{1}{z} + \frac{1}{2} \int_1^{\infty} \left(\theta(t) - 1\right) t^{z/2} \frac{dt}{t}.$$

The first integral can be rewritten by substituting $1/t$ as the variable:

$$\pi^{-z/2}\Gamma(z/2)\zeta(z) = \frac{1}{2} \int_1^{\infty} \theta(1/t) t^{-z/2} \frac{dt}{t} - \frac{1}{z} + \frac{1}{2} \int_1^{\infty} \left(\theta(t) - 1\right) t^{z/2} \frac{dt}{t}.$$

Applying the Poisson summation result, we get

$$\pi^{-z/2}\Gamma(z/2)\zeta(z) = \frac{1}{2}\int_1^\infty \theta(t)t^{\frac{1-z}{2}}\frac{dt}{t} - \frac{1}{z} + \frac{1}{2}\int_1^\infty (\theta(t)-1)\,t^{z/2}\frac{dt}{t}.$$

We add and subtract a copy of the same integral and rearrange to get

$$\begin{aligned}
\pi^{-z/2}\Gamma(z/2)\zeta(z) &= \frac{1}{2}\int_1^\infty \theta(t)t^{\frac{1-z}{2}}\frac{dt}{t} - \frac{1}{2}\int_1^\infty t^{\frac{1-z}{2}}\frac{dt}{t} - \frac{1}{z} + \frac{1}{2}\int_1^\infty t^{\frac{1-z}{2}}\frac{dt}{t} \\
&\quad + \frac{1}{2}\int_1^\infty (\theta(t)-1)\,t^{z/2}\frac{dt}{t} \\
&= \frac{1}{2}\int_1^\infty (\theta(t)-1)\,t^{\frac{1-z}{2}}\frac{dt}{t} - \frac{1}{z} + \frac{1}{2}\int_1^\infty t^{\frac{1-z}{2}}\frac{dt}{t} \\
&\quad + \frac{1}{2}\int_1^\infty (\theta(t)-1)\,t^{z/2}\frac{dt}{t}.
\end{aligned}$$

Computing the integral of the monomial and combining the other two integrals, for $\mathrm{Re}(z) > 1$ we get

$$\begin{aligned}
\pi^{-z/2}\Gamma(z/2)\zeta(z) &= \frac{1}{2}\int_1^\infty (\theta(t)-1)\,t^{\frac{1-z}{2}}\frac{dt}{t} - \frac{1}{z} - \frac{1}{1-z} + \frac{1}{2}\int_1^\infty (\theta(t)-1)\,t^{z/2}\frac{dt}{t} \\
&= \frac{1}{2}\int_1^\infty (\theta(t)-1)\left(t^{\frac{1-z}{2}} + t^{z/2}\right)\frac{dt}{t} - \frac{1}{z} - \frac{1}{1-z}.
\end{aligned}$$

Note that this expression is meromorphic over the whole complex plane with simple poles at $z = 0$ and $z = 1$. The expression is also invariant under the transformation $z \mapsto 1 - z$, and so we have proved the functional equation. □

2.6 Specific Values of the Riemann Zeta Function

An equivalent alternate form of the functional equation is

$$\zeta(z) = 2^z\pi^{z-1}\sin\left(\frac{\pi z}{2}\right)\Gamma(1-z)\,\zeta(1-z).$$

From this equation, we can notice that $\sin\left(\frac{\pi z}{2}\right)$ has zeros at all the even integers. While all the zeros of $\sin\left(\frac{\pi z}{2}\right)$ greater than or equal to 0 are canceled out by poles of

$\Gamma(1-z)$ or the pole at $\zeta(1)$, the zeros of $\sin\left(\frac{\pi z}{2}\right)$ less than 0 are not canceled out by any poles. Hence, the negative even integers are called the trivial zeros of $\zeta(z)$. The famed Riemann Hypothesis states that all other zeros of the Riemann zeta function like within the critical strip, $0 < \text{Re}(z) < 1$, on the line $\text{Re}(z) = \frac{1}{2}$ [3]. The previously mentioned values and other values of the Riemann zeta function, including some that can be found in the domain of convergence of the Dirichlet series, are in Table 1 [13].

Table 1: Specific Values of the Riemann Zeta Function

z	$\zeta(z)$
$\ldots$	$\ldots$
-6	0
-4	0
-2	0
-1	$-\frac{1}{12}$
2	$\frac{\pi^2}{6} = 1.6449340\ldots$
3	$1.2020569\ldots$
4	$\frac{\pi^4}{90} = 1.0823232\ldots$
∞	1

3 MATHEMATICAL AND ANALYTIC TOOLS

We list theorems that will prove useful in Chapters 4 and 5.

3.1 A Classical Theorem

The following theorem is known colloquially as the Fundamental Theorem of Algebra.

Theorem 3.1. *Every non-zero, single-variable, degree n polynomial with complex coefficients has, counted with multiplicity, exactly n complex roots.*

3.2 Theorems Inspired by Kirsten and Loya

Kirsten and Loya [5] describe a process of calculating zeta functions from a function whose zeros are the same as the eigenvalues of some operator in physics. The authors developed the following theorem in order to further explore Kirsten and Loya contour integration:

Theorem 3.2. *Suppose that $f(t)$ is of the form*

$$f(t) = c\prod_{k=1}^{\infty}\left(1 - \frac{t}{\sigma_k}\right)^{a_k}$$

where $a_k \in \mathbb{R}^+$. If $f(t)$ is analytic on a domain containing σ_k, then

$$\operatorname{Res}\left(t^{-z}\frac{f'(t)}{f(t)}, \sigma_k\right) = a_k\sigma_k^{-z}.$$

Proof. Without loss of generality, let us consider $f'(t)$ at $t = \sigma_1$. So,

$$f(t) = c\left(1 - \frac{t}{\sigma_1}\right)^{a_1}\prod_{k=2}^{\infty}\left(1 - \frac{t}{\sigma_k}\right)^{a_k}.$$

Taking the derivative, we get

$$f'(t) = c\left(1-\frac{t}{\sigma_1}\right)^{a_1}\frac{d}{dt}\left(\prod_{k=2}^{\infty}\left(1-\frac{t}{\sigma_k}\right)^{a_k}\right) + \; c\,\frac{a_1}{\sigma_1}\left(1-\frac{t}{\sigma_1}\right)^{a_1-1}\prod_{k=2}^{\infty}\left(1-\frac{t}{\sigma_k}\right)^{a_k}.$$

Therefore,

$$\frac{f'(t)}{f(t)} = \frac{\frac{d}{dt}\left(\prod_{k=2}^{\infty}\left(1-\frac{t}{\sigma_k}\right)^{a_k}\right)}{\prod_{k=2}^{\infty}\left(1-\frac{t}{\sigma_k}\right)^{a_k}} + \frac{\frac{a_1}{\sigma_1}\left(1-\frac{t}{\sigma_1}\right)^{a_1-1}}{\left(1-\frac{t}{\sigma_1}\right)^{a_1}}.$$

The second term simplifies to $\dfrac{a_1}{\sigma_1 - t} = \dfrac{-a_1}{t-\sigma_1}$, while the first term has no $\left(1-\dfrac{t}{\sigma_1}\right)$ terms in it. So,

$$\operatorname{Res}\left(\frac{f'(t)}{f(t)}, \sigma_1\right) = -a_1.$$

Since t^{-z} is analytic at $t = \sigma_1$,

$$\operatorname{Res}\left(t^{-z}\frac{f'(t)}{f(t)}, \sigma_1\right) = -a_1\sigma_1^{-z}.$$

Therefore,

$$\operatorname{Res}\left(t^{-z}\frac{f'(t)}{f(t)}, \sigma_k\right) = -a_k\sigma_k^{-z}$$

for all $k \in \mathbb{N}$. □

The following theorem is central to the work of Kirsten and Loya:

Theorem 3.3. [5] *The zeta function $\zeta_f(z) = \sum_{k=1}^{\infty} a_k\sigma_k^{-z}$ associated with a function f with positive zeros at $\{\sigma_k\}_{k=1}^{\infty}$ can be found via the integral*

$$\zeta_f(z) = \frac{1}{2\pi i}\oint_C t^{-z}\frac{f'(t)}{f(t)}dt$$

for any closed curve C containing the zeros but not intersecting the negative real axis.

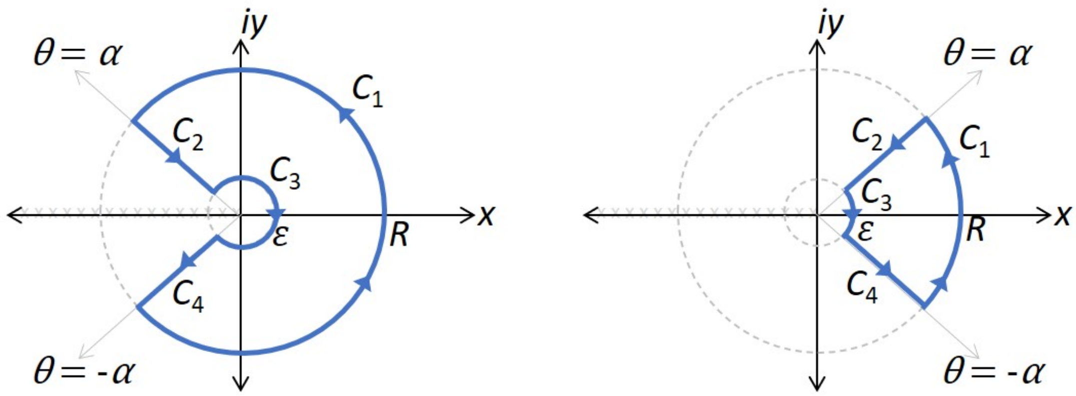

Figure 1: Contours used for the Kirsten and Loya process of zeta function regularization. Since both contours contain the same poles of $\dfrac{f'(z)}{f(z)}$, the contour integrals are equal by Theorem 2.3.

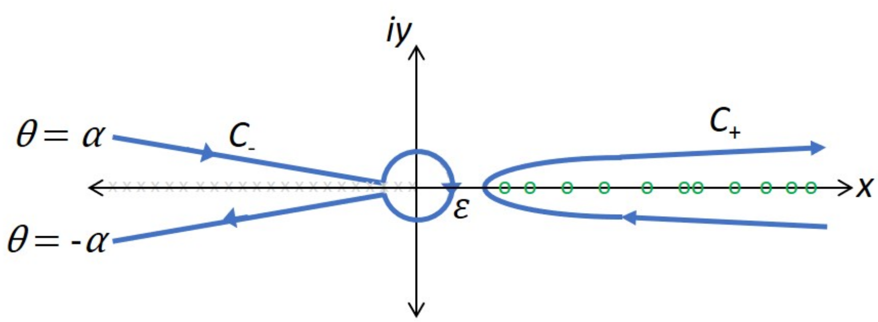

Figure 2: Contours used for the Kirsten and Loya process of zeta function regularization. C_- is the contour such that $\alpha \to \pi^-$, while C_+ is the same contour such that $\alpha \to 0^+$.

Proof. Let us show the method by which Kirsten and Loya establish a zeta function for a known or unknown operator via a contour integral.

We consider the contour as in Figure 1 where $\alpha \to 0^+$. Note, $C_2 \cup C_3 \cup C_4 = C_+$ as in Figure 2. If $f(z)$ has infinitely many zeros $\{\sigma_k\}_{k=1}^{\infty}$ extending along the positive real axis, let us define C as the limiting case of a succession of closed contours containing more zeros of $f(z)$ along the positive real axis. This corresponds nicely with the definition of an infinite series as a limit of the partial sums. Therefore, by Theorems 2.3 and 3.2, we have

$$\begin{aligned} \frac{1}{2\pi i} \oint_C t^{-z} \frac{f'(t)}{f(t)} dt &= \lim_{n \to \infty} \sum_{k=1}^{n} a_k \sigma_k^{-z} \\ &= \sum_{k=1}^{\infty} a_k \sigma_k^{-z}. \end{aligned}$$

It immediately follows that

$$\zeta_f(z) = \sum_{k=1}^{\infty} a_k \sigma_k^{-z}.$$

□

A second theorem also due to Kirsten and Loya is as follows:

Theorem 3.4. [5] *The zeta function $\zeta_f(z) = \sum_{k=1}^{\infty} a_k \sigma_k^{-z}$ associated with a function f with positive zeros at $\{\sigma_k\}_{k=1}^{\infty}$ can be found via the integral*

$$\zeta_f(z) = \frac{\sin(-\pi z)}{\pi} \int_0^{\infty} r^{-z} \frac{f'(-r)}{f(-r)} dr$$

for all z for which the integral exists.

Remark. We reproduce the proof of the theorem for polynomials, but the proof can be extended to a larger class of functions. For polynomials, the integral converges for all $0 < \mathrm{Re}(z) < 1$.

Proof. We consider the contour as in Figure 1 where $\alpha \to \pi^-$. By Theorem 2.3, since C contains the same poles of $\dfrac{f'(t)}{f(t)}$ for constant R and $0 < \alpha < \pi$, therefore the contour integrals are equal for constant R and $0 < \alpha < \pi$. So, the result of Theorem 3.3 still holds:

$$\zeta_f(z) = \frac{1}{2\pi i} \oint_C t^{-z} \frac{f'(t)}{f(t)} dt.$$

We define the contour in a positive orientation as $C = C_1 \cup C_2 \cup C_3 \cup C_4$. Note, $C_2 \cup C_3 \cup C_4 = C_-$ as in Figure 2. As $\alpha \to \pi^-$, C_2 and C_4 approach the negative real axis, where C_2 is oriented toward $x = 0$ and C_4 is oriented toward $x = -R$. C_3 becomes an almost-complete circle of radius $r = \varepsilon$, oriented clockwise, whereas C_1 becomes an almost-complete circle of radius $r = R$, oriented counterclockwise. If we compute the contour integral for C_2 first, then $t = re^{i\pi}$ from $r = R$ to $r = \varepsilon$ and $dt = e^{i\pi} dr$. That is,

$$\int_{C_2} t^{-z} \frac{f'(t)}{f(t)} dt = \int_R^{\varepsilon} (re^{i\pi})^{-z} \frac{f'(re^{i\pi})}{f(re^{i\pi})} e^{i\pi} dr.$$

Note that we can cancel out the $e^{i\pi} = -1$ by flipping the limits on the integral:

$$\int_{C_2} t^{-z} \frac{f'(t)}{f(t)} dt = \int_{\varepsilon}^{R} (re^{i\pi})^{-z} \frac{f'(re^{i\pi})}{f(re^{i\pi})} dr.$$

If we then compute the contour integral for C_4, we get $t = re^{-i\pi}$ from $r = \varepsilon$ to $r = R$ and $dt = e^{-i\pi} dr$. That is,

$$\int_{C_4} t^{-z} \frac{f'(t)}{f(t)} dt = \int_{\varepsilon}^{R} (re^{-i\pi})^{-z} \frac{f'(re^{-i\pi})}{f(re^{-i\pi})} e^{-i\pi} dr.$$

So, adding the two contour integrals together, we get

$$\begin{aligned} \int_{C_2} t^{-z} \frac{f'(t)}{f(t)} dt + \int_{C_4} t^{-z} \frac{f'(t)}{f(t)} dt \quad &= \quad \int_{\varepsilon}^{R} (re^{i\pi})^{-z} \frac{f'(re^{i\pi})}{f(re^{i\pi})} dr \\ &\quad + \int_{\varepsilon}^{R} (re^{-i\pi})^{-z} \frac{f'(re^{-i\pi})}{f(re^{-i\pi})} e^{-i\pi} dr. \end{aligned}$$

We factor out a $e^{-i\pi z}$ from the first integral and a $e^{i\pi z}$ from the second integral. Also, note that in the limiting sense, $re^{i\pi} = re^{-i\pi} = -r$. We also factor out an $e^{-i\pi} = -1$ from the second integral. So,

$$\begin{aligned}\int_{C_2} t^{-z}\frac{f'(t)}{f(t)}dt + \int_{C_4} t^{-z}\frac{f'(t)}{f(t)}dt &= e^{-i\pi z}\int_{\varepsilon}^{R} r^{-z}\frac{f'(-r)}{f(-r)}dr \\ &\quad - e^{i\pi z}\int_{\varepsilon}^{R} r^{-z}\frac{f'(-r)}{f(-r)}dr.\end{aligned}$$

Let us compute the contour integrals for C_1 and C_3. For C_1, $t = Re^{i\theta}$ from $\theta = -\pi$ to $\theta = \pi$ and $dt = iRe^{i\theta}d\theta$. So,

$$\int_{C_1} t^{-z}\frac{f'(t)}{f(t)}dt = \int_{-\pi}^{\pi}(Re^{i\theta})^{-z}\frac{f'(Re^{i\theta})}{f(Re^{i\theta})}iRe^{i\theta}d\theta.$$

Factoring out a R^{-z} and rearranging, we get

$$\int_{C_1} t^{-z}\frac{f'(t)}{f(t)}dt = R^{-z}\int_{-\pi}^{\pi} e^{-i\theta z}\frac{Rf'(Re^{i\theta})}{f(Re^{i\theta})}ie^{i\theta}d\theta.$$

Let us look at whether the integral absolutely converges. That is,

$$\begin{aligned}\left|\int_{C_1} t^{-z}\frac{f'(t)}{f(t)}dt\right| &= \left|R^{-z}\int_{-\pi}^{\pi} e^{-i\theta z}\frac{Rf'(Re^{i\theta})}{f(Re^{i\theta})}ie^{i\theta}d\theta\right| \\ &= \left|R^{-z}\right|\left|\int_{-\pi}^{\pi} e^{-i\theta z}\frac{Rf'(Re^{i\theta})}{f(Re^{i\theta})}ie^{i\theta}d\theta\right|.\end{aligned}$$

We can see that for fixed z, $\left|e^{-i\theta z}\right|$ has a finite supremum for $\theta \in [-\pi, \pi]$. That is, $|e^{-i\theta z}|$ is bounded. If we here restrict our function f to the class of polynomials, then we see that $\left|\frac{Rf'(Re^{i\theta})}{f(Re^{i\theta})}\right|$ is also bounded on $[-\pi, \pi]$. Finally, $\left|ie^{i\theta}\right| = 1$. Therefore,

$$\begin{aligned}\left|\int_{C_1} t^{-z}\frac{f'(t)}{f(t)}dt\right| &\leq \left|R^{-z}\right|\int_{-\pi}^{\pi}\left|e^{-i\theta z}\frac{Rf'(Re^{i\theta})}{f(Re^{i\theta})}ie^{i\theta}\right|d\theta \\ &\leq \left|R^{-z}\right|\int_{-\pi}^{\pi}\left|e^{-i\theta z}\right|\left|\frac{Rf'(Re^{i\theta})}{f(Re^{i\theta})}\right|\left|ie^{i\theta}\right|d\theta \\ &\leq 2\pi M_1\left|R^{-z}\right|,\end{aligned}$$

for some positive constant M_1. Therefore, for $\text{Re}(z) > 0$, as $R \to \infty$,

$$\int_{C_1} t^{-z} \frac{f'(t)}{f(t)} dt \to 0.$$

Note that for f in other classes of function, $\left| \frac{Rf'(Re^{i\theta})}{f(Re^{i\theta})} \right|$ may be bounded for a different region of z. That is, the above integral may converge to 0 for a different region of z.

Computing the contour integral for C_3, we use $t = \varepsilon e^{i\theta}$ from $\theta = -\pi$ to $\theta = \pi$ and $dt = i\varepsilon e^{i\theta} d\theta$. That is,

$$\int_{C_3} t^{-z} \frac{f'(t)}{f(t)} dt = \int_{-\pi}^{\pi} (\varepsilon e^{i\theta})^{-z} \frac{f'(\varepsilon e^{i\theta})}{f(\varepsilon e^{i\theta})} i\varepsilon e^{i\theta} d\theta.$$

Factoring out a ε^{1-z} and rearranging, we get

$$\int_{C_3} t^{-z} \frac{f'(t)}{f(t)} dt = \varepsilon^{1-z} \int_{-\pi}^{\pi} ie^{i\theta(1-z)} \frac{f'(\varepsilon e^{i\theta})}{f(\varepsilon e^{i\theta})} d\theta.$$

It follows that

$$\begin{aligned} \left| \int_{C_3} t^{-z} \frac{f'(t)}{f(t)} dt \right| &= \left| \varepsilon^{1-z} \int_{-\pi}^{\pi} ie^{i\theta(1-z)} \frac{f'(\varepsilon e^{i\theta})}{f(\varepsilon e^{i\theta})} d\theta \right| \\ &= \left| \varepsilon^{1-z} \right| \left| \int_{-\pi}^{\pi} ie^{i\theta(1-z)} \frac{f'(\varepsilon e^{i\theta})}{f(\varepsilon e^{i\theta})} d\theta \right|. \end{aligned}$$

Note that as we let $\varepsilon \to 0^+$, $f(0)$ is a nonzero constant, and $f'(0)$ is also constant. So, $\frac{f'(\varepsilon e^{i\theta})}{f(\varepsilon e^{i\theta})}$ will approach a constant. As before, we can see that for fixed $1-z$, $\left| e^{i\theta(1-z)} \right|$ has a supremum for $x \in [-\pi, \pi]$ and is bounded. Therefore,

$$\begin{aligned} \left| \int_{C_3} t^{-z} \frac{f'(t)}{f(t)} dt \right| &\leq \left| \varepsilon^{1-z} \right| \int_{-\pi}^{\pi} \left| ie^{i\theta(1-z)} \frac{f'(\varepsilon e^{i\theta})}{f(\varepsilon e^{i\theta})} \right| d\theta \\ &\leq \left| \varepsilon^{1-z} \right| \int_{-\pi}^{\pi} |i| \left| e^{i\theta(1-z)} \right| \left| \frac{f'(\varepsilon e^{i\theta})}{f(\varepsilon e^{i\theta})} \right| d\theta \\ &\leq 2\pi M_3 \left| \varepsilon^{1-z} \right|, \end{aligned}$$

for some positive constant M_3. Therefore, for $\mathrm{Re}(z) < 1$, as $\varepsilon \to 0$,

$$\int_{C_3} t^{-z} \frac{f'(t)}{f(t)} dt \to 0.$$

Thus, letting $\varepsilon \to 0^+$ and $R \to \infty$, we have that for $C = C_1 \cup C_2 \cup C_3 \cup C_4$ and $0 < \mathrm{Re}(z) < 1$:

$$\begin{aligned}
\int_C t^{-z} \frac{f'(t)}{f(t)} dt &= \int_{C_2} t^{-z} \frac{f'(t)}{f(t)} dt + \int_{C_4} t^{-z} \frac{f'(t)}{f(t)} dt \\
&= e^{-i\pi z} \int_0^\infty r^{-z} \frac{f'(-r)}{f(-r)} dr \\
&\quad - e^{i\pi z} \int_0^\infty r^{-z} \frac{f'(-r)}{f(-r)} dr.
\end{aligned}$$

That is,

$$\int_C t^{-z} \frac{f'(t)}{f(t)} dt = \left(e^{-i\pi z} - e^{i\pi z}\right) \left(\int_0^\infty r^{-z} \frac{f'(-r)}{f(-r)} dr \right).$$

Note that $e^{-i\pi z} - e^{i\pi z}$ can be written as $2i \sin(-\pi z)$. So,

$$\begin{aligned}
\zeta_f(z) &= \frac{1}{2\pi i} \int_C t^{-z} \frac{f'(t)}{f(t)} dt \\
&= \frac{\sin(-\pi z)}{\pi} \int_0^\infty r^{-z} \frac{f'(-r)}{f(-r)} dr.
\end{aligned}$$

Note that coincidentally enough, the region of convergence of $0 < \mathrm{Re}(z) < 1$ matches the critical strip of the Riemann zeta function. $\square$

3.3 Theorems Applied to the Digamma Function

We apply the theorems of Section 3.2 to the digamma function. These results are original and are related to Ramanujan's master theorem in Chapter 5.

Theorem 3.5. *Let $\zeta(z)$ be the Riemann zeta function. Then,*

$$\zeta(z) = -\frac{1}{2\pi i} \oint_C t^{-z} \frac{\Gamma'(1-t)}{\Gamma(1-t)} dt$$

and

$$\zeta(z) = \frac{\sin(\pi z)}{\pi} \int_0^\infty r^{-z} \frac{\Gamma'(1+r)}{\Gamma(1+r)} dr.$$

Proof. Note that $\psi(1-z)$ is a reflected and translated version of $\psi(z)$ such that the poles of $\psi(1-z)$ are the positive integers instead of the integers less than or equal to zero. Let us now use

$$\frac{f'(z)}{f(z)} = \frac{\Gamma'(1-z)}{\Gamma(1-z)} = \psi(1-z)$$

in the Kirsten and Loya contour integral. By Theorems 2.3 and 2.7, we have that

$$-\frac{1}{2\pi i} \oint_C t^{-z} \frac{\Gamma'(1-t)}{\Gamma(1-t)} dt = -\frac{1}{2\pi i} \oint_C t^{-z} \sum_{n=1}^{\infty} \frac{-1}{t-n} dt.$$

So, by Theorem 3.2,

$$-\frac{1}{2\pi i} \oint_C t^{-z} \frac{\Gamma'(1-t)}{\Gamma(1-t)} dt = \sum_{n=1}^{\infty} n^{-z}.$$

Therefore, by definition of the Riemann zeta function,

$$\zeta(z) = -\frac{1}{2\pi i} \oint_C t^{-z} \frac{\Gamma'(1-t)}{\Gamma(1-t)} dt.$$

Also, by the proof of Theorem 3.4,

$$\begin{aligned} \zeta(z) &= -\frac{\sin(-\pi z)}{\pi} \int_0^\infty r^{-z} \frac{\Gamma'(1+r)}{\Gamma(1+r)} dr \\ &= \frac{\sin(\pi z)}{\pi} \int_0^\infty r^{-z} \frac{\Gamma'(1+r)}{\Gamma(1+r)} dr. \end{aligned}$$

$\square$

4 KIRSTEN LOYA ZETA REGULARIZATION

The approach in the physics community, and thus also the approach by Kirsten and Loya, is to apply the contour integral method to specific operators that occur in particular quantum field theory models. Our research seeks to identify *classes* of functions for which the contour integral form of zeta regularization converges and thus for which Theorem 3.3 also holds, beginning with the polynomials. Thus, we explore Kirsten and Loya's process of calculating a zeta function for many examples and show certain classes of functions for which this process involves convergent contour integrals.

4.1 Class 1: Polynomials

Any polynomial f with positive zeros (σ_k) can be factored according to Theorem 3.1 as

$$f(t) = c\prod_{k=1}^{n}\left(1 - \frac{t}{\sigma_k}\right)^{a_k}$$

for $a_k \in \mathbb{N}$. We can relate this function to its zeta function via the Kirsten and Loya contour integral. That is, by Theorems 3.2 and 3.3,

$$\zeta_f(z) = \sum_{k=1}^{n} a_k\sigma_k^{-z} = \frac{1}{2\pi i}\oint_C t^{-z}\frac{f'(t)}{f(t)}dt$$

for any closed curve C that contains all the zeros of f (and, hence, all the poles of $\frac{f'}{f}$) and does not intersect $t = 0$ and the negative real axis. In the following sections, we look toward classes of functions f with infinitely many zeros, where we must deal with convergence issues for the contour integral.

4.2 Class 2: Powers with a Sum that Absolutely Converges for Infinitely Many Zeros

Suppose that f is of the form

$$f(t) = c \prod_{k=1}^{\infty} \left(1 - \frac{t}{\sigma_k}\right)^{a_k}$$

where $a_k \in \mathbb{R}^+$ such that $\sum_{k=1}^{\infty} a_k$ converges. By Theorems 3.2 and 3.3,

$$\zeta_f(z) = \sum_{k=1}^{\infty} a_k \sigma_k^{-z}.$$

Since $\sigma_k \in \mathbb{R}^+$ and $\sum_{k=1}^{\infty} a_k$ converges, therefore the series for $\zeta_f(z)$ converges for $\mathrm{Re}(z) > 0$. Note that if $\{\sigma_k\}_{k=1}^{\infty} = \{k\}_{k=1}^{\infty}$, then $\zeta_f(z)$ is an L-function, relevant in number theory.

4.2.1 Class 2a: Geometric Powers for Infinitely Many Zeros

Suppose that $\{a_k\}_{k=1}^{\infty}$ is a geometric sequence defined by

$$a_k = a^k$$

with $0 < a < 1$. In this case, we have

$$\zeta_f(z) = \sum_{k=1}^{\infty} a^k \sigma_k^{-z}.$$

Note that this is a special case of $a_k \in \mathbb{R}^+$ such that $\sum_{k=1}^{\infty} a_k$ converges.

4.3 Class 3: Powers with Partial Sums that Satisfy the Wiener-Ikehara Theorem for Infinitely Many Zeros

Note that if $\{\sigma_k\}_{k=1}^{\infty} = \{k\}_{k=1}^{\infty}$, then

$$\zeta_f(z) = \sum_{k=1}^{\infty} a_k k^{-z}$$

is an L-function. Suppose that f is of the form

$$f(t) = c \prod_{k=1}^{\infty} \left(1 - \frac{t}{\sigma_k}\right)^{a_k}$$

where $a_k \in \mathbb{R}^+$ such that

$$\sum_{k \leq X} a_k = \frac{c}{b} X^b.$$

We no longer specify as in Class 2 that $\sum_{k=1}^{\infty} a_k$ converges, and we apply the Wiener-Ikehara Theorem [6]. By Theorems 3.2 and 3.3,

$$\zeta_f(z) = \frac{1}{2\pi i} \oint_C t^{-z} \frac{f'(t)}{f(t)} dt = \sum_{k=1}^{\infty} a_k k^{-z}.$$

Therefore, the Wiener-Ikehara Theorem implies that $\sum_{k=1}^{\infty} a_k k^{-z}$ converges to an analytic function in $\mathrm{Re}(z) \geq b$ with a simple pole of residue c at $z = b$. So, defining $\zeta_f(z)$ as the analytic continuation of the Dirichlet series, $\zeta_f(z)$ has a pole that is both shifted and of a different residue than the Riemann zeta function.

4.4 Class 4: Infinitely Many Regularly Spaced Zeros of Order 1

For an operator with infinitely many eigenvalues $\{\sigma_k\}_{k=1}^{\infty}$ modeled by a function f with infinitely many zeros of order 1 (also $\{\sigma_k\}_{k=1}^{\infty}$), we can relate the zeta function produced by the Kirsten-Loya method to the Riemann zeta function as follows:

Theorem 4.1. *For an entire function*

$$f(t) = c\prod_{k=1}^{\infty}\left(1 - \frac{t}{\sigma_k}\right)$$

with zeros that are defined by $\sigma_k = bk^a$*, where* $b, a \in \mathbb{R}^+$ *with* $a > 1$*, we have*

$$\zeta_f(z) = b^{-z}\zeta(az).$$

We can therefore relate the computed zeta function directly to the classical Riemann zeta function. We prove the result below.

Proof. Let the zeros of f be defined by $\sigma_k = bk^a$, where $b, a \in \mathbb{R}^+$ so that

$$f(t) = c\prod_{k=1}^{\infty}\left(1 - \frac{t}{bk^a}\right).$$

So, by Theorems 3.2 and 3.3,

$$\begin{aligned} \zeta_f(z) &= \frac{1}{2\pi i}\oint_C t^{-z}\frac{f'(t)}{f(t)}dt \\ &= \sum_{k=1}^{\infty}\frac{1}{(bk^a)^z} \\ &= \sum_{k=1}^{\infty}\frac{1}{b^z k^{az}}. \end{aligned}$$

Factoring out the b^{-z}, we have

$$\zeta_f(z) = b^{-z}\sum_{k=1}^{\infty}\frac{1}{k^{az}}.$$

Therefore, we have

$$\zeta_f(z) = b^{-z}\zeta(az).$$

$\square$

Note that b^{-z} is entire, and the Dirichlet series for $\zeta(az)$ converges for $\mathrm{Re}(az) > 1$. So, the integral and series for $\zeta_f(z)$ converge for $\mathrm{Re}(z) > \frac{1}{a}$. However, since the Riemann zeta function is analytically continued to the whole complex plane, $\zeta_f(z)$ also can be computed for all $z \in \mathbb{C}$, with poles at $z = 0$ and $z = \frac{1}{a}$.

4.4.1 Examples

We now apply Theorem 4.1 to the example that Kirsten and Loya showed in [5] ($f_1(t) = \sin(\sqrt{t}L)/\sqrt{t}$) and a couple similar examples:

Table 2: Zeta Functions for Functions with Zeros that Follow Monomials of Positive Exponent

f_n	Expression	Product	$\zeta_{f_n}(z)$
f_1	$\dfrac{\sin(\sqrt{t}L)}{\sqrt{t}}$	$\prod_{k=1}^{\infty}\left(1 - \dfrac{t}{\left(\frac{\pi}{L}\right)^2 k^2}\right)$	$\left(\frac{\pi}{L}\right)^{-2z}\zeta(2z)$
f_2	$\dfrac{1-\cos\sqrt{t}}{t}$	$\dfrac{1}{2}\prod_{k=1}^{\infty}\left(1 - \dfrac{t}{(2\pi)^2 k^2}\right)$	$(2\pi)^{-2z}\zeta(2z)$
f_3	$\dfrac{\sin(\sqrt[4]{t})\sinh(\sqrt[4]{t})}{\sqrt{t}}$	$\prod_{k=1}^{\infty}\left(1 - \dfrac{t}{\pi^4 k^4}\right)$	$\pi^{-4z}\zeta(4z)$

5 CONNECTIONS TO NUMBER THEORY

5.1 Ramanujan's Master Theorem for the Digamma Function

Used often in number theory, Ramanujan's Master Theorem provides a formula for the Mellin transform of complex-valued functions.

Theorem 5.1. [1] *If a complex-valued function $f(x)$ has an expansion of the form*

$$f(x) = \sum_{k=0}^{\infty} \lambda(k)(-x)^k$$

then the Mellin transform of $f(x)$ is given by

$$\int_0^\infty x^{z-1} \left(\lambda(0) - x\,\lambda(1) + x^2\,\lambda(2) - \cdots \right) dx = \frac{\pi}{\sin(\pi z)}\,\lambda(-z).$$

Ramanujan's Master Theorem is a powerful and beautiful theorem but also a theorem that is hard to prove for many classes of functions. Let us prove this theorem for the digamma function ψ, which we were working with in Chapter 3. According to [10], $\psi(x+1)$ has a Taylor series expansion of

$$\psi(x+1) = -\gamma - \sum_{k=1}^{\infty} \zeta(k+1)(-x)^k.$$

The corresponding classical result from Ramanujan's Master Theorem is as follows:

Corollary 5.2.

$$\int_0^\infty x^{z-1}\psi(x+1)dx = -\frac{\pi}{\sin(\pi z)}\,\zeta(1-z)$$

Remark. The proof, obtained by the Kirsten Loya contour integral method, does not rely on Ramanujan's Master Theorem.

Proof. By Theorem 3.5, we have

$$\zeta(z)=\frac{\sin(\pi z)}{\pi}\int_0^\infty x^{-z}\frac{\Gamma'(1+x)}{\Gamma(1+x)}dx.$$

If we substitute $1-z$ for z in the above equation, we get

$$\begin{aligned}\zeta(1-z) &= \frac{\sin(\pi(1-z))}{\pi}\int_0^\infty x^{-(1-z)}\frac{\Gamma'(1+x)}{\Gamma(1+x)}dx\\ &= \frac{\sin(\pi(1-z))}{\pi}\int_0^\infty x^{z-1}\frac{\Gamma'(1+x)}{\Gamma(1+x)}dx.\end{aligned}$$

Note that

$$\frac{\sin(\pi(1-z))}{\pi}=\frac{\sin(\pi z)}{\pi}.$$

So,

$$\begin{aligned}\zeta(1-z) &= \frac{\sin(\pi z)}{\pi}\int_0^\infty x^{-(1-z)}\frac{\Gamma'(1+x)}{\Gamma(1+x)}dx\\ \frac{\pi}{\sin(\pi z)}\zeta(1-z) &= \int_0^\infty x^{z-1}\frac{\Gamma'(1+x)}{\Gamma(1+x)}dx.\end{aligned}$$

That is,

$$\int_0^\infty x^{z-1}\psi(1+x)dx=\frac{\pi}{\sin(\pi z)}\zeta(1-z).$$

□

Correspondingly, similar results for the classes of functions in Chapter 4 can be obtained from Ramanujan's Master Theorem, which is one of our "future directions" for this project.

5.2 Discussion

In Chapter 4, the zeta functions generated from functions of infinitely many zeros are L-functions, which are studied in number theory. In fact, the result of the Wiener-Ikehara Theorem described in Section 4.4 is used to prove the Prime Number Theorem based on the Riemann zeta function [6]. Furthermore, Theorem 4.1 in Section 4.5 relates the generated zeta functions directly to the Riemann zeta function. Given the nature of the analytic tools used in zeta function regularization and the large overlap with the field of analytic number theory shown below, there is more to be discovered than the connection to Ramanujan's Master Theorem and the already discovered results of the Riemann zeta function. Physicists looking to make the process of zeta function regularization simpler and more mathematical can follow the Kirsten Loya process and obtain results by relating their generated zeta functions to the classical Riemann zeta function. The authors recognize analytic number theory and complex analysis as further research areas that connect to the rich mathematical topics explored above.

BIBLIOGRAPHY

[1] Tewodros Amdeberhan, Olivier Espinosa, Ivan Gonzalez, Marshall Harrison, Victor H Moll, and Armin Straub. Ramanujan's master theorem. *The Ramanujan Journal*, 29(1-3):103–120, 2012.

[2] Ronald N. Bracewell. *The Fourier Transform and Its Applications*, volume 31999. McGraw-Hill New York, 1986.

[3] John B. Conway. *Functions of One Complex Variable*. Springer-Verlag, New York Inc., second edition, 1978.

[4] Rick Durrett. *Probability: Theory and Examples*, volume 49. Cambridge University Press, 2019.

[5] Klaus Kirsten and Paul Loya. Calculation of determinants using contour integrals. *American Journal of Physics*, 76(1):60–64, 2008.

[6] Jaap Korevaar. The Wiener–Ikehara theorem by complex analysis. *Proceedings of the American Mathematical Society*, 134(4):1107–1116, 2006.

[7] Roberto Mauri and S Haber. Applications of Wiener's path integral for the diffusion of Brownian particles in shear flows. *SIAM Journal on Applied Mathematics*, 46(1):49–55, 1986.

[8] Brad Osgood, Ralph Phillips, and Peter Sarnak. Extremals of determinants of Laplacians. *Journal of Functional Analysis*, 80(1):148–211, 1988.

[9] Daniel Parrochia. Some remarks on history and pre-history of Feynman path integral. *arXiv preprint arXiv:1907.11168*, 2019.

[10] Andrei D Polyanin and Vladimir E Nazaikinskii. *Handbook of Linear Partial Differential Equations for Engineers and Scientists*. CRC press, 2015.

[11] Nicolas M Robles. *Zeta Function Regularization*. PhD thesis, Imperial College London, 2009.

[12] Walter A Strauss. *Partial Differential Equations: An Introduction*. John Wiley & Sons, second edition, 2007.

[13] Edward Charles Titchmarsh, David Rodney Heath-Brown, Edward Charles Titchmarsh Titchmarsh, et al. *The Theory of the Riemann Zeta-Function*. Oxford University Press, 1986.

[14] Norbert Wiener. *Cybernetics or Control and Communication in the Animal and the Machine*. MIT press, 2019.

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