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Properties of Small Ordered Graphs Whose Vertices are Weighted by Their Degree

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ABSTRACT

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Graphs can effectively model biomolecules, computer systems, and other applications. A weighted graph is a graph in which values or labels are assigned to the edges of the graph. However, in this thesis, we assign values to the vertices of the graph rather than the edges and we modify several standard graphical measures to incorporate these vertex weights. In particular, we designate the degree of each vertex as its weight. Previous research has not investigated weighting vertices by degree. We find the vertex weighted domination number in connected graphs, beginning with trees, and we define how weighted vertices can affect eccentricity, independence number, and connectivity.

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TABLE OF CONTENTS

LIST OF FIGURES

1 INTRODUCTION

A *graph G* consists of a finite set of vertices together with a set of edges that connect those vertices. If vertices u and v are connected by an edge, then it is denoted as edge uv. The vertices u and v are therefore *adjacent*. A vertex u and edge uv are called *incident*. A graph is said to be a weighted graph if the edges are assigned weights. There is a large volume of work on weighted graphs in the literature. However, there is not a comparable amount of research dedicated to graphs whose vertices are weighted. In fact, a literature search for weighted graphs in mathematical journals results almost exclusively in edge weighted graphs [6, 10, 14, 16]. In previous work by Knisley et al. [7, 8, 9] small ordered graphs were used to model biomolecules and the vertices were weighted by the molecular mass of the corresponding atoms. In this thesis, we designate the degree of each vertex as its weight. In particular, we studied small-ordered graphs, such as trees, whose vertices are weighted by their degree.

A review of available publications on vertex weighting shows that vertex weighting has been typically limited to various algorithms for assigning vertex weights [1, 11, 13, 14, 16]. For instance, Southey and Henning [14] developed a scheme of weighting vertices by summing the values of the incident edges based on an edge weighting function on dominating sets. Our goal is to assign weights to the vertices and use these weights to define graphical invariants that incorporate vertex weights. We find this an area of open investigation and have defined several invariants using degree weights. We most fully examine domination by degree weights and thus start this paper with weighted domination.

A graph is connected if for any pair of vertices in the graph there is a path of consecutive edges between them. That is, for every vertex v in graph G, we can find a path of edges connecting vertices $vu_1, u_1u_2, ..., u_{i-1}u_i$ that leads directly to any other given vertex u_i in G. We limit our research to connected graphs.

A graph *invariant* is a number derived from the structure of the graph. Some such invariants are *order*, n, which is the number of vertices in a graph, and *size*, m, which is the number of edges of the graph. We use other invariants in this thesis, such as domination number, maximum degree, radius, diameter, connectivity, and others that will be formally defined as they are introduced. Observe that we often find the cardinality of a minimum (maximum) vertex set under a specified constraint such as the minimum dominating set of vertices. To incorporate the weights of the vertices into the definition of a graphical invariant we find the minimum (maximum) **weight** vertex set under that specified constraint.

The *degree* of a vertex is the number of vertices adjacent to that vertex. These standard graphical definitions, and others in this thesis, have been compiled using texts by Chartrand et al. [2], Haynes [5] et al. and Harary [4]. We now modify these definitions by including the weight of the vertices.

Definition 1.1 *A graph whose vertices are assigned weights is called a vertex-weighted graph. A D-graph is a vertex-weighted graph, one whose vertices are assigned the weight equal to its degree.*

The ability to weight vertices in graphs for consideration in determining an invariant should be useful in a variety of task projects, biomedical applications, chemical molecular modeling, game theory, travel planning, city planning, communication

networks, and data organization. Some of our observations and propositions are applicable to all graphs, but for some we will limit our discussions to graphs of order $n \leq 6$.

2 DOMINATION IN D-GRAPHS

Domination in graphs has received much attention in the last few decades. A vertex v in a graph G is said to *dominate* itself and all of the vertices adjacent to it. A set S is called a *dominating set* of G if every vertex in the graph is dominated by at least one vertex in set S. The minimum cardinality of all the dominating sets of G is called the *domination number*, denoted $\gamma(G)$. A dominating set of minimum cardinality is called a γ-set. Ore's Theorem [12] provides an upper bound for the domination number using order.

Theorem 2.1 *[12] If a graph G of order n has no isolated vertices, then $\gamma(G) \leq \frac{n}{2}$.*

The maximum degree of a graph, denoted $\Delta(G)$, is the largest degree of all the vertices in the graph. Using maximum degree, a lower bound for the domination number is provided by Waliker et al. [15].

Theorem 2.2 *[15] For any graph G of order n, $\left\lceil \frac{n}{1+\Delta(G)} \right\rceil \leq \gamma(G)$.*

So a functional set of bounds for the domination number on a connected graph G is $\left\lceil \frac{n}{1+\Delta(G)} \right\rceil \leq \gamma(G) \leq \frac{n}{2}$. From Haynes et al. [5], we know that one vertex can dominate at most itself and the maximum degree of G, $\Delta(G)$, other vertices.

For a vertex weighted graph, instead of the minimum cardinality, we find the minimum weight by summing the degree weight of each vertex in a dominating set.

Definition 2.3 *Let the weighted domination number of a D-graph, denoted $\gamma_w(G)$, be the minimum sum of the weights of the vertices among all possible dominating sets. A dominating set of vertices with minimum weight will be denoted by γ_w-set.*
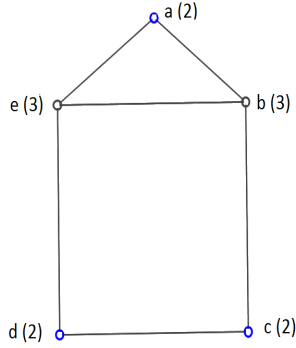
Figure 1: The house graph as a D-graph, with vertex weights in parentheses

As an example of a D-graph, recall G, the house graph, shown in Figure 1. Let G have vertices $\{a, b, c, d, e\}$ as shown in Figure 1. It has three vertices of degree 2, namely, $\{a, c, d\}$. It has two vertices of degree 3, namely, $\{b, e\}$. The dominating vertex sets of G are $\{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{e, d\}, \{e, c\}$, and $\{b, e\}$ and thus $\gamma(G) = 2$. The weighted domination number of G can be found by checking the sum of the degrees of these sets. They are 4, 4, 5, 5, 5, 5, and 6 respectively. Thus $\gamma_w(G) = 4$, the minimum weight of these dominating sets, which can be found using either $\{a, c\}$ or $\{a, d\}$.

If a graph G has a γ-set that is also a γ_w-set, then we say that there exists a γ_w-set such that γ_w-set $= \gamma$-set. The house graph in Figure 1 has a γ_w-set $= \gamma$-set, namely $\{a, c\}$ or $\{a, d\}$. On the other hand, the graph in Figure 2 does not. Its γ-sets are $\{a, d\}, \{b, d\}$, and $\{c, d\}$. The weighted domination for these sets are 7, 6, and 6 respectively. We can, however, find a smaller $\gamma_w(G)$ by choosing as our γ_w-set as $\{a, e, f\}$ or $\{c, f, e\}$. Thus $\gamma_w(G) = 5$.
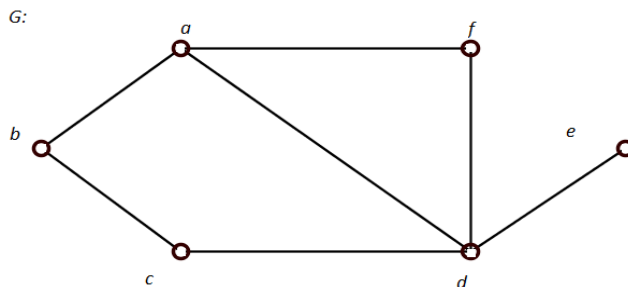
Figure 2: The graph above does not have a γ_w-set equal to its γ-set

2.1   Domination in D-Trees

We first investigate domination in D-trees. In a graph G, a *cycle* is a path of edges connecting vertices $vu_1, u_1u_2, ..., u_iv$ that leads directly back to the original vertex. A *tree* is a connected graph without cycles. It is interesting to note that trees always have size $m = n - 1$ for order n and that the addition of a single edge in a tree creates a cycle which changes the graph from a tree. The deletion of a single edge disconnects the graph which also changes it to a graph that is not a tree.

We began with the tree with the highest degree, $n - 1$, known as a star. Figure 3 depicts a star. A *star* is a tree with n vertices, $n - 1$ of which are degree 1. The designation for the star is $K_{1,n-1}$, that is, a tree with one central vertex and $n - 1$ leaves with a *leaf* being a vertex of degree 1. Our first observation was that the cardinality of γ_w-set of a star is the same cardinality of the γ-set, which is also the
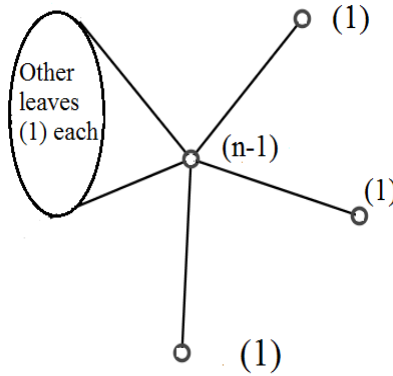
Figure 3: The star graph with weights in parentheses used in Proposition 1

domination number of that star.

Proposition 2.4 *For any star $K_{1,n-1}$, there exists γ_w-set, such that γ_w-set $= \gamma$-set, and $\gamma_w(K_{1,n-1}) = n - 1$.*

Proof. Let $K_{1,n-1}$ be a star graph with order n. Then $\gamma(K_{1,n-1}) = 1$ because a star is dominated by its central vertex. The degree of the central vertex is $n - 1$. The number of leaves (of degree one) is $n - 1$ and thus $\gamma_w(K_{1,n-1}) = n - 1$ can be achieved by either using the central vertex of degree $n - 1$ or by using the $n - 1$ vertices of degree one. These are the only two possible degree weighted dominating sets and both are equal and thus minimum. Therefore $\gamma_w(K_{1,n-1}) = n - 1$. Because the degree weighted domination number of the star can be formed by a single vertex, then there exists γ_w-set $= \gamma$-set. ■

We now consider paths where $\Delta(G) = 2$, the lowest maximum degree possible if $n \geq 3$. The *path*, is a tree of order n and size $n - 1$ whose vertices can be labeled

12

$v_1, v_2, ..., v_n$ where all v_i are distinct. Paths of order $n = 1$ are single vertices and those of order $n = 2$ have maximum degree one. The maximum degree of any path of order $n \geq 3$ is $\Delta(P_n) = 2$. The bounds on $\gamma(P_n)$ for any path of order n can be deduced from Cockayne et al. [3].

Theorem 2.5 *[3] If a connected graph G of order n is claw-free and net-free, then $\gamma(G) \leq \lceil \frac{n}{3} \rceil$.*

Since a path is both claw-free and net-free, $\gamma(P_n) = \lceil \frac{n}{3} \rceil$. To find the domination number and γ-set of a path, $\gamma(P_n)$, we use the simple process of starting with the second vertex in the path, which will dominate the end vertex, itself, and the third vertex, and choosing every third vertex after that. If n is divisible by 3, then (P_n) can be partitioned into subsets of three vertices each with the center vertex dominating the vertices on either side. If n is not divisible by 3, then an additional end vertex will need to be in the dominating set. There are other means for determining $\gamma(P_n)$ but this produces a simple way to determine a weighted dominating set for the path. For weighting purposes, it may be that using one or both end vertices to determine a minimum degree weighted domination number will provide a lower bound on $\gamma_w(G)$ because the end vertices have degree 1 rather than 2. We define in what circumstances this occurs.

Note, for $n = 3$, we obtain $\gamma(P_3) = 1$ by choosing the center vertex and $\gamma_w(P_3) = 2$ whether we use the two end vertices or the center vertex. For $n = 4$, $\gamma(P_4) = 2$ and $\gamma_w(P_4) = 2$, obtained by using the end vertices of the path. From observation, we have discovered the following equations for determining the degree weighted domination

number for paths of $n > 4$ using $\gamma(P_n) = \lceil \frac{n}{3} \rceil$:

$$\gamma_w(P_n) = \begin{cases} 2\gamma(P_n) & = 2(\frac{n}{3}) & \text{if} \quad n \equiv 0 \text{ mod } 3, \text{ or} \\ 2\gamma(P_n) - 2 & = 2\lceil \frac{n}{3} \rceil - 2 & \text{if} \quad n \equiv 1 \text{ mod } 3, \text{ or} \\ 2\gamma(P_n) - 1 & = 2\lceil \frac{n}{3} \rceil - 1 & \text{if} \quad n \equiv 2 \text{ mod } 3. \end{cases}$$

Proposition 2.6 *For any D-path of order $n > 4$, the weighted domination number is given by:*

$$\gamma_w(P_n) = \begin{cases} 2(\frac{n}{3}) & \text{if} \quad n \equiv 0 \text{ mod } 3, \text{ or} \\ 2\lceil \frac{n}{3} \rceil - 2 & \text{if} \quad n \equiv 1 \text{ mod } 3, \text{ or} \\ 2\lceil \frac{n}{3} \rceil - 1 & \text{if} \quad n \equiv 2 \text{ mod } 3. \end{cases}$$

Proof. We have three cases to determine $\gamma_w(P_n)$. Let P_n be a D-graph that is a path with n vertices where $n = 3k, 3k + 1$, or $3k + 2$, as needed for each case in the proposition. Figure 4 illustrates these cases.

Case 1
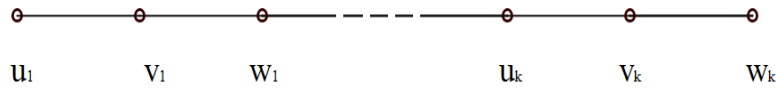
Case 2
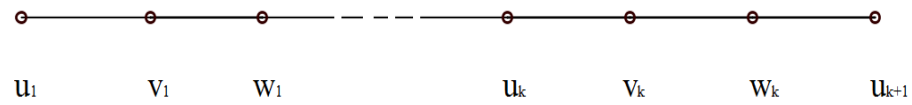
Case 3
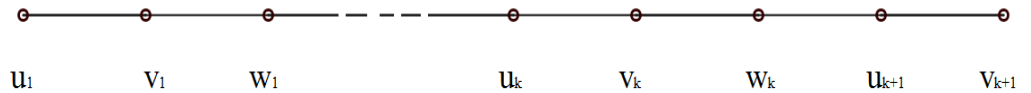
Figure 4: The three cases in Proposition 5

Case 1. If $n \equiv 0 \text{ mod } 3$, then $n = 3k$. The vertex set of $P_n = P_{3k}$ can be partitioned into subsets of three vertices each with the center vertex dominating the

14

vertices on either side. That is, for each 3-subset of vertices of P_n, of the form $\{u_i, v_i, w_i\}$, v_i dominates u_i and w_i and thus $\gamma(P_n) = \frac{n}{3}$. For the degree weighted domination number, $\gamma_w(P_n)$, we have the choice to use the same set of vertices as $\gamma(P_n)$. Because the degree of each of these vertices is 2, we have a result of $\gamma_w(P_n) = 2(\frac{n}{3})$. Alternately, we can attempt to get a smaller weight by using one or both of the end vertices of degree one. If we start with the end vertex u_1, then that vertex only dominates itself and v_1. But we have dropped our weight by one. Each $\frac{n}{3}$ subset after u_1 is now $\{w_i, u_{i+1}, v_{i+1} : i = 1, 2, ..., k\}$. Now the final vertex chosen for the dominating set is u_k, two vertices from the end and the end vertex w_k must be added to the dominating set, thus bringing our weight number back up by one. So in Case 1, changing the weighted domination set to include an end vertex means we will need to include the other and our net weight is the same. Our weighted domination number cannot be made less than $2(\frac{n}{3})$.

Case 2. If $n \equiv 1 \mod 3$, then we have $n = 3k + 1$. But because the end vertices are not used to dominate P_{3k}, we need to add a new vertex, u_{k+1}, into our dominating set for P_{3k+1}, thus increasing $\gamma(P_{3k+1})$ by one over $\gamma(P_{3k})$. Therefore, to determine the weighting of P_{3k+1}, we can again choose to use the same vertex set as $\gamma(P_{3k+1})$, which is one more that $\gamma(P_{3k})$. This gives a weight of $2\lceil \frac{n}{3} \rceil - 1$ because the end vertex, u_{k+1}, has degree one. We can improve the $\gamma_w(G)$ number by using the other end vertex. As in Case 1, we will choose the end vertex u_1, which will dominate itself and v_1 and the $\frac{n}{3}$ subsets will again be $\{w_i, u_{i+1}, v_{i+1}\}$, decreasing the weight by one. But this time we have the end vertex u_{k+1}, which was included in the dominating set. So our weighting number decreases by one. Taking advantage of the lower degree of

the end vertices results in a $\gamma_w(P_{3k+1}) = 2\gamma(P_{3k+1}) - 2 = 2\lceil\frac{n}{3}\rceil - 2$. So we have a smaller weighted domination number.

Case 3. If $n \equiv 2 \mod 3$, then we now have $n = 3k + 2$. Note that P_{3k+2} adds one more vertex over P_{3k+1}, say v_{k+1}. However, no additional dominating vertices are necessary because vertex u_{k+1} of P_{3k+1} was already in the dominating set and can dominate the added vertex v_{k+1} of P_{3k+2}. Adding this vertex does not change the dominating number. Thus $\gamma(P_{3k+2}) = \gamma(P_{3k+1})$. For weighting, we can again accept the same set of dominating vertices. Therefore, $\gamma_w(P_{3k+2}) = 2\gamma(P_{3k+2}) - 1 = 2\lceil\frac{n}{3}\rceil - 1$ because the new end vertex increases the degree of u_{k+1} in P_{3k+1} to degree 2. We attempt to improve the number. Choose the end vertex u_1, which only dominates itself and v_1, and the $\frac{n}{3}$ subsets will again be (w_i, u_{i+1}, v_{i+1}), decreasing the weight by one to $2\lceil\frac{n}{3}\rceil - 2$. Now this time, the last dominating vertex is u_{k+1}. This is the second to last vertex and we cannot use the lower weight of the end vertex v_{k+1}, raising the number back to $2\lceil\frac{n}{3}\rceil - 1$. Thus, we have the same result. Here, we can use one but not both end vertices and different weighted dominating set does not produce a lower weighted domination number in the D-path.

We can take advantage of the lower degree of the end vertices only in Case 2 to reduce the weighted domination number. All three cases are proved and the equations for determining the weighted vertex domination number of a D-path is as observed. ∎

2.2 Additional Graphs Where γ_w-set Equals γ-set

Graphs with a maximum degree of $n-1$ have the property that there exists a γ_w-

set $= \gamma$-set. Recall that the *open neighborhood* of v is the set $N_G(v) = \{u \in V | uv \in E\}$ and the *closed neighborhood* of v is $N_G[v] = \{v\} \cup N_G(v)$.

Proposition 2.7 *For all connected D-graphs, if the maximum degree, $\Delta(G) = n-1$, then there exists γ_w-set $= \gamma$-set.*

Proof. Let G be a D-graph and $\Delta(G) = n - 1$. All graphs where $\Delta(G) = n - 1$ are dominated by a single vertex of maximum degree, that is, the dominating vertex. Thus $\gamma(G) = 1$. We want to show that $\gamma_w(G) = n - 1 = \Delta(G)$ and that there exists a single vertex where the weighted domination number is minimum and thus there exists γ_w-set $= \gamma$-set. Obviously, this is true if G is a complete graph as all vertices have $\Delta(G) = n - 1$ and we have already proven it for stars in Proposition 3.

Let v be a vertex of degree $n - 1$. We claim that there is no dominating set of vertices whose weight is less than $n - 1$. Suppose, to the contrary, there is such a set. Then no vertex of degree $n - 1$ will be in this γ_w-set. We build a weighted dominating set S of a smaller weight excluding v. Let the remaining vertices of G be u_i for $i = \{1, ... n - 1\}$. Let u_1 be in the dominating set and have degree $n - k$ for some $k \in \{2, 3, ..., n - 2\}$. If $k = n - 1$ then we would produce a star and we have already discussed stars in Proposition 3. If $k = 1$, then we have another vertex of maximum degree that can exist but cannot be in the dominating set as noted above. Because u_1 has degree $n - k$ the $\gamma_w(N[u_1]) = n - k$ and the cardinality of the open neighborhood of u_1 is $n - k$.

Let the vertices u_i that are not adjacent to u_1 be B. Thus, the cardinality of B is the order of G minus one for the vertex v, minus the cardinality of the open

G: v deg (n -1)

B

b_j

|B| = k - 1

u_1

deg (n-k)
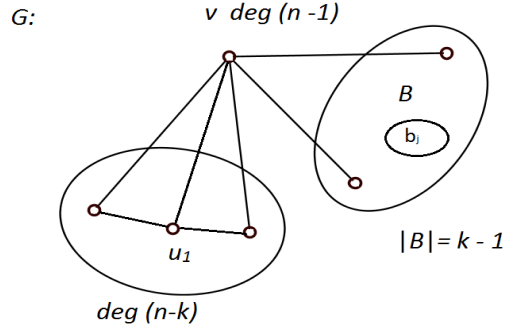
Figure 5: Building S, a weighted dominating set consisting of u_1 and the set of b_j vertices

neighborhood of u_1, or

$$|B| = n - 1 - (n - k) = k - 1.$$

Let some set of vertices $b_j \in B$, for an indexing set j, dominate B. We take u_1 and the set of b_j vertices to dominate G without using v. Each b_j can dominate at most $\deg(b_j) - 1$ new vertices and adds $\deg(b_j)$ to the weight of S. Thus, all vertices b_j that are added to the dominating set S will add more weight than they dominate. So $\gamma_w(B) \geq |B| = k - 1$. Hence,

$$\gamma_w(G) \geq n - k + |B| = n - k + k - 1 = n - 1.$$

Therefore, the single dominating vertex of maximum degree also has the minimum weighted domination number and there exists γ_w-set $= \gamma$-set. ∎

18

2.3 Small Connected Graphs Where There Exists γ_w-set That Equals γ-set and Where No Such Set Exists

Using Frank Harary's [4] tables of small graphs, we looked further at the connected graphs that are not trees. Graphs of order 1 and 2 are trees. There is one graph of order $n = 3$ that is connected but not a tree, and only four graphs of order $n = 4$ in this category. For $n = 5$, there are 18 connected graphs that are not trees. This gives a total of 23 graphs that are connected but are not trees for $n \leq 5$. We found that for these 23 graphs, there exists γ_w-set $= \gamma$-set. But when we look at $n = 6$, of the possible 103 graphs that are not trees or disconnected, we have nine that do not have this property, and the weighted domination number for these D-graphs is only minimum when more vertices than the γ-set are strategically chosen. All of these graphs have size $m = 6$ or $m = 7$. For $n = 6$, all graphs of size $m \geq \frac{4n}{3}$ have the property that γ_w-set $= \gamma$-set. None of these graphs have $\Delta(G) = n - 1$. But where we have shown that when a graph has $\Delta(G) = n - 1$, the γ_w-set $= \gamma$-set, the converse is not true.

The graphs shown in Figure 6 of order $n = 6$ have a smaller weighted domination number if more than $\gamma(G)$ vertices are chosen. A review of their characteristics may lead to an explanation of why. We investigated size, maximum degree, diameter, radius, independence number, girth, circumference, and domination number of both the line graphs and complements.

Some of these terms have not yet been defined in this paper. We provide those definitions now. The *distance* between two vertices in a graph is the length of the shortest path between the vertices, if such a path exists. The *eccentricity*, $\epsilon(v)$, of
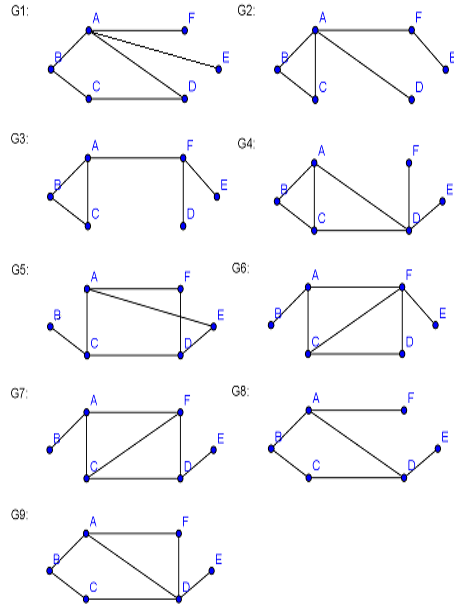
Figure 6: These D-graphs have a smaller $\gamma_w(G_i)$ if more than $\gamma(G_i)$ vertices are chosen

a vertex v is the greatest distance between v and any other vertex. *Diameter* is the maximum eccentricity of any vertex in the graph and *radius* is the minimum eccentricity of any vertex in the graph. A set U of vertices in a graph G is *independent* if no two vertices in U are adjacent. The *vertex independence number* of G is the maximum number of vertices in an independent set U. The independence number is denoted either $\alpha(G)$ or $\beta_0(G)$, depending on the author. We use the second designation in accordance with Haynes et al. [5]. *Girth* is the smallest cycle in a graph and *circumference* is the largest cycle. The *complement* of graph G, denoted $\overline{G}$, is a graph with the same vertices as G but for every non-edge in G, there is an edge in $\overline{G}$. The *line graph*, $L(G)$ of graph G, is a graph whose vertices correspond with the edges (lines) of G, so that two vertices of $L(G)$ are adjacent if and only if the corresponding

edges of G are adjacent.

First, note those invariants that are the same for all of the nine graphs: $\gamma(G_i) = 2$, $rad(G_i) = 2$, $\gamma L(G_i) = 2$, $\gamma(\overline{G_i}) = 2$, and $\gamma_w(\overline{G_i}) = 5$. Table 1 summarizes the remaining graphical invariants.

Table 1: Summary of invariants examined

Graph	Size	$\Delta(G_i)$	Diam	$\beta_0(G_i)$	Girth	Cir	$\gamma_w(G_i)$	$\gamma_w L(G_i)$
(G_1)	6	4	3	4	4	4	4	5
(G_2)	6	4	3	3	3	3	4	5
(G_3)	6	4	3	3	3	3	4	5
(G_4)	7	4	3	3	3	4	4	5
(G_5)	7	3	3	3	4	4	5	5
(G_6)	7	4	3	3	3	4	4	5
(G_7)	7	3	4	4	3	4	5	6
(G_8)	6	3	3	3	4	4	4	5
(G_9)	7	4	3	4	3	4	5	6

None of these measures are unique to these nine graphs for $n = 6$. For example, we can find graphs in the other 94 that have radius 2 or $\gamma(G) = 2$. Therefore, this remains an open problem for further research.

3   DISTANCE MEASURES FOR D-GRAPHS

Recall the definitions for distance, eccentricity, diameter, and radius in the previous section. Each of these invariants can be vertex weighted.

Definition 3.1 *Weighted distance is the sum of the vertex weights, in a D-graph, of the minimum weighted path between two vertices.*

Definition 3.2 *The weighted eccentricity of a vertex v of a D-graph, denoted $\epsilon_w(v)$, is the greatest weighted distance between v and any other vertex in the graph.*

Definition 3.3 *Let the weighted diameter of a D-graph G, denoted $diam_w(G)$, be the maximum weighted eccentricity of any vertex in the graph. Let the weighted radius of a D-graph G, denoted $rad_w(G)$, be the minimum weighted eccentricity of any vertex in the graph.*
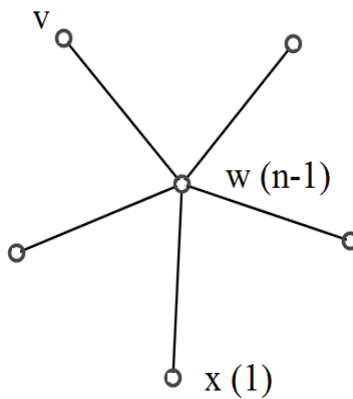

Figure 7: Eccentricity in the star graph with weights in parentheses

To visualize these definitions, consider the star graph in Figure 7. Let v be a vertex in the outer ring of vertices, w be the central vertex, and x any outer vertex other than v. Then the distance from v to x is two and $\epsilon(v) = 2$. The diameter of the star is obviously two. Using weighted diameter, from v we have the degree of w, $n-1$, plus the degree of x, 1. Thus $diam_w(G) = n-1+1 = n$. The radius of the star is the minimum eccentricity that is w to any v or 1. The $rad_w(G)$ is also 1 because w to v means we only count the weight of the vertex v.
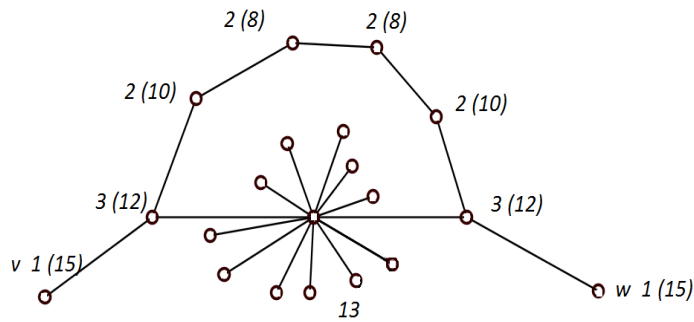

Figure 8: A D-graph to illustrate weighted distance, with eccentricity (ϵ) noted

23

For another example, consider the graph in Figure 8. This graph shows degree weights and eccentricity (ϵ) for the displayed vertices. Note that the shortest distance between v and w has a weight of $3 + 13 + 3 + 1 = 20$. The longer distance provides a weighted distance of only 15 as noted. Therefore, the shortest distance is larger in weight and the longer distance over the top of the cycle is the least weighted. Thus, we use the lower weight of the longer distance to calculate the weighted diameter and radius. In this graph, $diam_w(G) = 15$ and the $rad_w(G) = 8$.

A suggested usage for this convention of vertex degree weighting in distance measurement is traffic movement and GPS instructions. Think of a higher weighted vertex as a busy intersection, a long stoplight, a school zone, or a crowded business area. Weighted vertex modeling can be used in GPS programming to assist drivers to avoid these types of traffic delays.

Remember the definition of independence number from Section 2.3. We have elected to vertex weight independence number as follows.

Definition 4.1 *Let the weighted independence number of a D-graph be the maximum weight of the vertices in an independent set U, denoted $\beta_{0w}(G)$.*
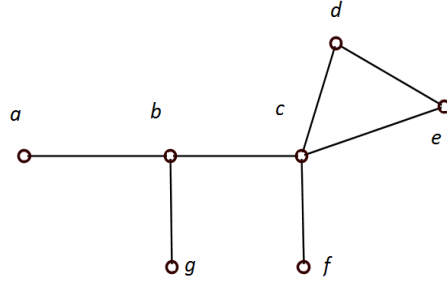

Figure 9: A D-graph to illustrate weighted independence number, $\beta_{0w}(G)$

To understand weighted independence, refer to Figure 9. The independent sets for this graph are $U(G) = \{\{a, e, f.g\}, \{a, d, f, g\}, \{a, d, f\}, \{a, c, g\}, \{a, d, g\}, \{a, e, f\}, \{a, e, g\}, \{a, c\},$ $\{a, d\}, \{a, e\} \{a, f\}, \{a, g\}, \{b, d\}, \{b, e\}, \{b, f\}, \{b, d, f\}, \{b, e, f\}, \{c, g\}, \{d, f, g\}, \{d, b\}, \{d, f\},$ $\{d, g\}, \{e, b\}, \{e, f\}, \{e, g\}, \{f, g\}, \{a, f, g\}\}$. Thus, $\beta_0(G) = 4$ for the set $U = \{a, d, f, g\}$ or $\{a, e, f, g\}$. Therefore, $\{a, d, f, g\}$ and $\{a, e, f.g\}$ are of maximum cardinality of all the independent sets, making $\beta_0 = 4$. To find the weighted independence number, find the weights of all the 27 independent sets and choose the maximum, per the definition above. These weights are 5, 5, 4, 6, 4, 4, 4, 5, 3, 3, 2, 2, 5, 5, 4, 6, 6, 5, 4, 5, 3, 3, 3, 3, 5, 2, and 3, respectively. Therefore, $\beta_{0w}(G) = 6$, the maximum weight

25

of the vertices in an independent set of G. The sets that are of maximum weight are $\{a, c, g\}, \{b, e, f\}$, and $\{b, d, f\}$. So β_0-set is not the same as β_{0w}-set. Here is another area of open research.

5   CONNECTIVITY FOR D-GRAPHS

A *vertex cut* set of a connected graph G is a set of vertices whose removal renders G disconnected or trivial. The vertex *connectivity* $\kappa(G)$ is the size of a minimum vertex cut. A complete graph requires $n-1$ vertices in the vertex cut set and is thus a trivial case. A graph is called *k-connected* if $\kappa(G) \geq k$. And $\kappa(G) \leq \delta(G)$, the minimum degree of the graph, because deleting all neighbors of a vertex of minimum degree will disconnect that vertex from the rest of the graph.

Definition 5.1 *Let the weighted vertex cut of a D-graph G be the minimum weight of a vertex cut of G, denoted $\kappa_w(G)$.*

To understand weighted vertex cut, we use the same Figure 9 as we used to demonstrate independence number. Consider the graph vertex cut sets $\{b\}$ and $\{c\}$. The removal of either of these vertices will disconnect the graph. The weight of b is 3 and the weight of c is 4. Therefore, using b as the cut vertex of minimum weight, the $\kappa_w(G) = 3$.

6 MAIN RESULTS AND FUTURE PROBLEMS

In this thesis, we introduced D-graphs, a graph whose vertices are weighted by their respective degree. We have provided equations for finding the weighted domination number, $\gamma_w(G)$, for all paths, and shown that the cardinality of this set of vertices for all stars, paths, and connected graphs where $\Delta(G) = n-1$ can be made equal to the γ-set. We have defined degree weighted vertex invariants for distance, independence, and connectivity.

We have researched D-graphs with $n \leq 6$ that have the property γ_w-set $= \gamma$-set and those that do not. A problem for further investigation is to provide characterizations of these graphs, which can be expanded to other families of graphs and D-graphs of order $n \geq 7$. We have identified new areas of research in weighted domination and weighted independence number. Other graphical invariants can also be modified to include vertex weights.

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