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Bipartitions Based on Degree Constraints

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ABSTRACT

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For a graph $G = (V, E)$, we consider a bipartition $\{V_1, V_2\}$ of the vertex set V by placing constraints on the vertices as follows. For every vertex $v \in V_i$, we place a constraint on the number of neighbors v has in V_i and a constraint on the number of neighbors it has in V_{3-i}. Using three values, namely 0 (no neighbors are allowed), ≥ 1 (at least one neighbor is required), and X (any number of neighbors are allowed) for each of the four constraints, results in 27 distinct types of bipartitions. The goal is to characterize graphs having each of these 27 types. We give characterizations for 21 out of the 27. Three other characterizations appear in the literature. The remaining three prove to be quite difficult. For these, we develop properties and give characterization of special families.

DEDICATION

This thesis is dedicated to my mom Claribell Obregon and to my dad Isidro Delgado.

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1 INTRODUCTION

1.1 Introduction to Graph Theory

Let G be a graph with vertex set $V = V(G)$, edge set $E = E(G)$, and *order* $n = |V|$. Let $\overline{G}$ denote the complement of G. The *open neighborhood* of a vertex $v \in V$ is the set $N_G(v) = \{u \in V \,|\, uv \in E\}$ of vertices adjacent to v, and its *closed neighborhood* is $N_G[v] = N_G(v) \cup \{v\}$. The *open neighborhood of a set* $S \subseteq V$ is $N_G(S) = \bigcup_{v \in S} N_G(v)$, while the *closed neighborhood of a set* $S \subseteq V$ is the set $N_G[S] = \bigcup_{v \in S} N_G[v]$. The *degree* of a vertex v is $deg_G(v) = |N_G(v)|$. An *S-external private neighbor* of a vertex $v \in S$ is a vertex $u \in V \setminus S$ that is adjacent to v but to no other vertex of S. The set of all S-external private neighbors of $v \in S$ is called the *S-external private neighbor set* of v and is denoted $\mathrm{epn}(v, S)$. If G is clear from the context, then we will use $N(v)$, $N[v]$, $N[S]$, $N(S)$, and $deg(v)$ in place of $N_G(v)$, $N_G[v]$, $N_G[S]$, $N_G(S)$, and $deg_G(v)$, respectively.

The minimum and maximum degrees of a vertex in a graph G are denoted $\delta(G)$ and $\Delta(G)$, respectively. A vertex of degree one is called a *leaf*. In a connected graph G, the *distance* $d_G(u, v)$ between vertices u and v is the length of a shortest u-v path in G. For a vertex v, its *eccentricity* $e(v)$ is the distance between v and a vertex farthest from v in G. The maximum eccentricity among the vertices of G is its *diameter*, denoted by $diam(G)$. If $e(v) = diam(G)$, then v is called a *diametrical vertex*. The subgraph of G induced by a set of vertices S is denoted by $G[S]$. If A and B are two subsets of V, we define $[A, B]$ to be the set of edges with one vertex in A and the other in B. For terminology not defined here, the reader is referred to the book [3].

A set $S \subseteq V$ is a *dominating set* of a graph G if $N[S] = V$, that is, every vertex in $V \setminus S$ is adjacent to at least one vertex in S. The minimum cardinality of a dominating set in a graph G is called the *domination number* and is denoted $\gamma(G)$. A dominating set of G with minimum cardinality is called a *$\gamma(G)$-set*. A set S is *independent* if no two vertices in S are adjacent. A set that is both independent and dominating is called an *independent dominating set*, abbreviated *ID-set*. A set $S \subseteq V$ is a *total dominating set* of a graph G, abbreviated *TD-set*, if $N(S) = V$, that is, every $v \in V$ is adjacent to at least one vertex in S. The *total domination number* $\gamma_t(G)$ is the minimum cardinality of a TD-set of G, and a TD-set of G with minimum cardinality is called a *$\gamma_t(G)$-set*. Further, if S is a dominating set and the induced subgraph $G[V \setminus S]$ has no isolated vertices, then S is called a *restrained dominating* set of G. A set $S \subseteq V$ is a *total restrained dominating* set of G if S is both a total dominating set and a restrained dominating set of G. For more details on domination parameters, the reader is referred to the books [7, 8, 11].

A tree T is called a *rooted tree* if it has a vertex r labeled as the *root*, where for each vertex $v \neq r$ of T, the *parent* of v is the neighbor of v on the unique r-v path, while a *child* of v is any other neighbor of v. We let $C(v)$ denote the set of children of v.

1.2 Statement of the Problem

Motivated by [9], we study graphs whose vertex set can be partitioned into two sets by placing degree constraints on the vertices. To allow for different constraints on each set, we denote a bipartition as an ordered pair. We define a (d_1, d_2, d_3, d_4)-

bipartition of the vertex set V into two sets S and $V \setminus S$ to be an ordered pair $\pi = (S, V \setminus S)$ that satisfies the following conditions:

1. $u \in S$ has d_1 neighbors in S and d_2 neighbors in $V \setminus S$ and
2. $v \in V \setminus S$ has d_3 neighbors in S and d_4 neighbors in $V \setminus S$.

We consider three possible values for the number of prescribed neighbors d_i, namely, 0 (no neighbors are allowed), ≥ 1 (at least one neighbor is required), and X (any number of neighbors are allowed). Allowing for symmetry, this results in 27 distinct types of bipartitions, which are described in Table 1. For the 4-tuple (d_1, d_2, d_3, d_4) and associated bipartition $(S, V \setminus S)$, we note that if $d_1 = 0$ (respectively $d_4 = 0$), then S (respectively $V \setminus S$) must be an independent set. If $d_1 \geq 1$ (respectively $d_4 \geq 1$), then $G[S]$ (respectively $G[V \setminus S]$) is isolate-free. Also, note that $d_2 = 0$ if and only if $d_3 = 0$; that is, $[S, V \setminus S] = \emptyset$. Further, if $d_2 \geq 1$ (respectively, if $d_3 \geq 1$), then $V \setminus S$ (respectively, S) is a dominating set. We note that several of these bipartitions result in one or both sets being a type of dominating set as can be seen in Table 1.

Our goal is to characterize the graphs having a (d_1, d_2, d_3, d_4)-bipartition for each of the 27 types.

In Section 2, we discuss the work of Heggernes and Telle [9], which motivated our problem. They were the first to use degree constraints to form partitions of the vertex set of a graph. Based on [9], Dr. Hedetniemi proposed the topic of study addressed in the present thesis during his talk entitled "Towards a Theory of Graph Bipartitions or Graph Bipartition Theory", given on May 25, 2013 at *The 26th Cumberland Conference on Combinatorics, Graph Theory and Computing*. In this Section, we also recall two important results on the topic of bipartitions.

For three of the 27 bipartitions, characterizations are known. In Section 3, we list these results and give characterizations for 21 additional types.

The problem of the remaining three characterizations proves to be quite difficult. For these three bipartitions, we develop properties and give characterization for special families in Section 4.

In Section 5 we conclude with possible avenues for future study on the subject of bipartitions.

Table 1: Types of Bipartitions

(d_1, d_2, d_3, d_4)	**Description of the bipartition**
1. (0,0,0,0)	S and $V \setminus S$ are independent sets, $[S, V \setminus S] = \emptyset$
2. (0,0,0,≥ 1)	S is an independent set, $G[V \setminus S]$ is isolate-free, $[S, V \setminus S] = \emptyset$
3. (0,0,0,X)	S is an independent set, $[S, V \setminus S] = \emptyset$
4. (0,≥ 1,≥ 1,0)	S and $V \setminus S$ are independent dominating sets
5. (0,≥ 1,≥ 1,≥ 1)	S is an independent (restrained) dominating set, $V \setminus S$ is a total dominating set
6. (0,≥ 1,≥ 1,X)	S is an independent dominating set, $V \setminus S$ is a dominating set
7. (0,≥ 1,X,0)	S is an independent set, $V \setminus S$ is an independent dominating set
8. (0,≥ 1,X,≥ 1)	S is an independent set, $V \setminus S$ is a total dominating set
9. (0,≥ 1,X,X)	S is an independent set, $V \setminus S$ is a dominating set
10. (0,X,≥ 1,≥ 1)	S is a independent restrained dominating set
11. (0,X,≥ 1,X)	S is an independent dominating set
12. (0,X,X,0)	S and $V \setminus S$ are independent sets
13. (0,X,X,≥ 1)	S is an independent set, $G[V \setminus S]$ is isolate-free
14. (0,X,X,X)	S is an independent set
15. (≥ 1,0,0,≥ 1)	$G[S]$ and $G[V \setminus S]$ are isolate-free, $[S, V \setminus S] = \emptyset$
16. (≥ 1,0,0,X)	$G[S]$ is isolate-free, $[S, V \setminus S] = \emptyset$
17. (≥ 1,≥ 1,≥ 1,≥ 1)	S and $V \setminus S$ are total (restrained) dominating sets
18. (≥ 1,≥ 1,≥ 1,X)	S is a total dominating set, $V \setminus S$ is a (restrained) dominating set
19. (≥ 1,≥ 1,X,≥ 1)	$V \setminus S$ is a total restrained dominating set
20. (≥ 1,≥ 1,X,X)	$V \setminus S$ is a restrained dominating set
21. (≥ 1,X,≥ 1,X)	S is a total dominating set
22. (≥ 1,X,X,≥ 1)	$G[S]$ and $G[V \setminus S]$ are isolate-free
23. (≥ 1,X,X,X)	$G[S]$ is isolate-free
24. (X,0,0,X)	$[S, V \setminus S] = \emptyset$
25. (X,≥ 1,≥ 1,X)	S and $V \setminus S$ are dominating sets
26. (X,≥ 1,X,X)	$V \setminus S$ is a dominating set
27. (X,X,X,X)	Any bipartition

2 BACKGROUND

2.1 The Work of Heggernes and Telle

In 1998 Heggernes and Telle [9] were the first to use degree constraints to form partitions of the vertex set of a graph into generalized dominating sets. For a graph $G = (V, E)$, they defined a non-empty set $S \subset V$ to be a (σ, ρ)-set, where σ, ρ are sets of non-negative integers, if:

i) for every $u \in S$, $| N(u) \bigcap S | \in \sigma$, and

ii) for $v \in V \setminus S$, $| N(v) \bigcap S | \in \rho$.

The authors limited σ and ρ to be one of :$\{0\}, \{0, 1\}, \{1\}, \mathbb{N}, \mathbb{N}^*$, they listed 13 types of (σ, ρ)-sets. A (k, σ, ρ)-partition problem asks to determine if there exist a partition of the set of vertices of a graph into k subsets such that each of these subsets is a (σ, ρ)-set. They gave a framework within which the computational complexity of many graph partition problems could be determined as the parameters σ, ρ and k vary.

2.2 Two Important Results on the Topic of Bipartitions

The following classic 1962 result by Ore shows that for any graph without isolated vertices, its vertex set can be partitioned into two dominating sets.

Theorem 2.1 (Ore [13]) *In any graph $G = (V, E)$ having no isolated vertices, the complement $V \setminus S$ of any minimal dominating set S is a dominating set.*

Let C_n denote the cycle on n vertices. Henning and Southey [10] established the following useful result for graphs whose vertex set could be partitioned into a dominating set and a total dominating set.

Theorem 2.2 (Henning and Southey [10]) *If G is a graph with $\delta(G) \geq 2$ and G has no C_5 component, then the vertices of G can be partitioned into a dominating set and a total dominating set.*

3 TWENTY-FOUR CHARACTERIZATIONS

In this section we present 24 characterizations. We omit proofs for the three known results, as well as several whose characterizations are straightforward.

We let K_n denote the complete graph on n vertices. In all of the following results we assume that a graph G has order $n \geq 2$. Our first result is obvious.

Proposition 3.1 *A graph G of order n has a Type 1 $(0,0,0,0)$-bipartition if and only if $G = \overline{K}_n$.*

Proposition 3.2 *A graph G has a Type 2 $(0,0,0,\geq 1)$-bipartition if and only if G has at least one isolated vertex and at least one edge.*

Proof. Assume that G has at least one isolated vertex and at least one edge. Let S be the set of all isolated vertices of G. Then clearly, S is an independent set, $N(v) \subseteq V \setminus S$, and $|N(v)| \geq 1$ for all $v \in V \setminus S$. Hence, $(S, V \setminus S)$ is a $(0,0,0,\geq 1)$-partition.

Assume G has a Type 2 $(0,0,0,\geq 1)$-bipartition, say $(S, V \setminus S)$. Then each vertex in S is an isolated vertex in G, and since $V \setminus S \neq \emptyset$ and $d_4 \geq 1$, G has at least one edge. $\square$

Proposition 3.3 *A graph G has a Type 3 $(0,0,0,X)$-bipartition if and only if G is a non-trivial graph with at least one isolated vertex.*

Proof. Assume that G has at least one isolated vertex and order $n \geq 2$. Let S be a subset of V containing at least one but fewer than n isolated vertices. Then $(S, V \setminus S)$ is a $(0,0,0,X)$-bipartition.

Conversely, assume that G has a $(0,0,0,X)$-bipartition, say $(S, V \setminus S)$. Clearly, $n \geq 2$, and every vertex in S is an isolated vertex in G. $\square$

A graph G has a Type 4 $(0, \geq 1, \geq 1, 0)$-bipartition if and only if the set of vertices of G can be partitioned into two independent dominating sets.

Proposition 3.4 *A graph G has a Type 4 $(0, \geq 1, \geq 1, 0)$-bipartition if and only if G is a non-trivial bipartite graph with no isolated vertices.*

The next type of bipartition in this sequence is Type 5 $(0, \geq 1, \geq 1, \geq 1)$. No characterization of this type of bipartition, into an independent dominating set and a total dominating set, is known. We will discuss this bipartition in Section 4.

Problem 3.5 *Characterize the graphs having a Type 5 $(0, \geq 1, \geq 1, \geq 1)$-bipartition.*

Observe that a graph G has a Type 6 $(0, \geq 1, \geq 1, X)$-bipartition $(S, V \setminus S)$ if and only if S is an independent dominating set and $V \setminus S$ is a dominating set. Since every maximal independent set is a minimal dominating set, our next result is an immediate consequence of the theorem of Ore [13].

Proposition 3.6 *A graph G has a Type 6 $(0, \geq 1, \geq 1, X)$-bipartition if and only if G is a non-trivial graph with no isolated vertices.*

Recall that a graph G has a Type 7 $(0, \geq 1, X, 0)$-bipartition if and only if G can be partitioned into an independent set and an independent dominating set.

Proposition 3.7 *A graph G has a Type 7 $(0, \geq 1, X, 0)$-bipartition if and only if G is a bipartite graph with at least one edge.*

Proof. Assume that G is a bipartite graph with at least one edge, and let I be the set of isolated vertices of G. Since G is bipartite, let S and $S^{'}$ be partite sets of $V \setminus I$. Clearly, $S \neq \emptyset$, $S^{'} \neq \emptyset$, and $(S, S^{'} \cup I)$ is a $(0, \geq 1, X, 0)$-bipartition of G.

Now, assume G has a $(0, \geq 1, X, 0)$-bipartition, say $(S, V \setminus S)$. Then both S and $V \setminus S$ are independent sets, so G is bipartite. Since $d_2 \geq 1$, there is at least one edge in G. $\square$

Note that a Type 8 $(0, \geq 1, X, \geq 1)$-bipartition of a graph G is equivalent to partitioning the vertex set of G into an independent set and a total dominating set.

Proposition 3.8 *A graph G has a Type 8 $(0, \geq 1, X, \geq 1)$-bipartition if and only if $\delta(G) \geq 1$ and $\Delta(G) \geq 2$.*

Proof. Assume that G is a graph with $\delta(G) \geq 1$ and $\Delta(G) \geq 2$. Let D be a total dominating set of G. Note that since $\Delta(G) \geq 2$, $G \neq mK_2$. Thus,we can choose D such that $|D| < n$. Let S be a maximum independent set of $G\,[V \setminus D]$. Consider $(S, V \setminus S)$. Note that no vertex in S has a neighbor in S, and since $\delta(G) \geq 1$, every vertex in S has at least one neighbor in $V \setminus S$. Since D is a total dominating set of G and $D \subseteq V \setminus S$, the induced subgraph $G[V \setminus S]$ has no isolated vertices. Thus, $(S, V \setminus S)$ is a $(0, \geq 1, X, \geq 1)$-bipartition.

Assume G has a $(0, \geq 1, X, \geq 1)$-bipartition, say $(S, V \setminus S)$. Since $V \setminus S$ is total dominating set, every vertex in G has a neighbor in $V \setminus S$, so $\delta(G) \geq 1$. Since $S \neq \emptyset$, there exists a vertex $u \in V \setminus S$, such that $|N(u) \cap S| \geqslant 1$ and $|N(u) \cap (V \setminus S)| \geqslant 1$, implying that $\Delta(G) \geq 2$. $\square$

The graphs which have a Type 9 $(0, \geq 1, X, X)$-bipartition are precisely the graphs whose vertex set can be partitioned into an independent set and a dominating set.

Proposition 3.9 *A graph G has a Type 9 $(0, \geq 1, X, X)$-bipartition if and only if G has at least one edge.*

Proof. Assume that G has at least one edge uv. Let $S = \{u\}$. Then clearly S is an independent set, and since u has a neighbor in $V \setminus S$, it follows that $(S, V \setminus S)$ is a $(0, \geq 1, X, X)$-bipartition.

Now, assume G has a $(0, \geq 1, X, X)$-bipartition, $d_2 \geq 1$ implies G has an edge. $\Box$

The next type of bipartition in this sequence is Type 10 $(0, X, \geq 1, \geq 1)$. No characterization of this type of bipartition $(S, V \setminus S)$, where S is an independent restrained dominating set, is known. We will discuss this bipartition in Section 4.

Problem 3.10 *Characterize the graphs having a Type 10 $(0, X, \geq 1, \geq 1)$-bipartition.*

A Type 11 $(0, X, \geq 1, X)$ bipartition is simply a bipartition $(S, V \setminus S)$ in which S is an independent dominating set.

Proposition 3.11 *A graph G of order n has a Type 11 $(0, X, \geq 1, X)$-bipartition if and only if G has an independent dominating set with fewer than n vertices, that is, $G \neq \overline{K}_n$.*

A Type 12 $(0, X, X, 0)$-bipartition is simply the definition of a non-trivial bipartite graph.

Proposition 3.12 *A graph G has a Type 12 $(0, X, X, 0)$-bipartition if and only if G is a non-trivial bipartite graph.*

A graph G has a Type 13 $(0, X, X, \geq 1)$-bipartition if and only if G can be partitioned into an independent set and a set whose induced subgraph has no isolated vertices.

Proposition 3.13 *A graph G of order n has a Type 13 $(0, X, X, \geq 1)$-bipartition if and only if $G \notin \{mK_2, \overline{K}_n\}$.*

Proof. Assume that $G \notin \{mK_2, \overline{K}_n\}$ is a graph of order n. If G has isolated vertices, then let S be the set of isolated vertices of G. Since $G \neq \overline{K}_n$, $V \setminus S \neq \emptyset$ and every vertex in $V \setminus S$ has a neighbor in $V \setminus S$. If G has no isolated vertices, let $S = \{u\}$, where u is a vertex with no leaf neighbors. We know there is such a vertex since $G \neq mK_2$. Since u has no leaf neighbors, every vertex in $V \setminus S$ has a neighbor in $V \setminus S$. In both cases, $(S, V \setminus S)$ is a $(0, X, X, \geq 1)$-bipartition.

Now, assume that G has a $(0, X, X, \geq 1)$-bipartition, say $(S, V \setminus S)$. Since $V \setminus S \neq \emptyset$ and $G[V \setminus S]$ has no isolated vertices, $G \neq \overline{K}_n$. If G has isolated vertices, then $G \neq mK_2$. Thus assume G is isolate-free, and let u be a vertex in S. Since $d_1 = 0$, then $N(u) \subseteq V \setminus S$. Let $v \in V \setminus S$ be a neighbor of u. Since $d_4 \geq 1$, v has at least one neighbor in $V \setminus S$. It follows that $G \neq mK_2$. $\square$

Since the only requirement for a graph G to have a Type 14 $(0, X, X, X)$-bipartition is that G has a proper subset of independent vertices, we have the following trivial characterization.

Proposition 3.14 *A graph G has a Type 14 $(0, X, X, X)$-bipartition if and only if G is a non-trivial graph.*

A graph G has a Type 15 $(\geq 1, 0, 0, \geq 1)$-bipartition if and only if its vertex set can be partitioned into two sets, S and $V \setminus S$, such that every vertex in S (respectively, $V \setminus S$) has a neighbor in S (respectively, $V \setminus S$) and $[S, V \setminus S] = \emptyset$. Our next result follows directly.

Proposition 3.15 *A graph G has a Type 15 $(\geq 1, 0, 0, \geq 1)$-bipartition if and only if G has no isolated vertices and at least two components.*

We can loosen the requirements of Proposition 15 slightly for a Type 16 $(\geq 1, 0, 0, X)$-bipartition.

Proposition 3.16 *A graph G has a Type 16 $(\geq 1, 0, 0, X)$-bipartition if and only if G has at least two components, at least one of which is non-trivial.*

Proof. Assume that G has at least two components, one of which is non-trivial. Let S be the set of vertices in a non-trivial component. Then $(S, V \setminus S)$ is a $(\geq 1, 0, 0, X)$-bipartition.

Assume that G has a $(\geq 1, 0, 0, X)$-bipartition, say $(S, V \setminus S)$. Since $d_2 = 0$ and $d_3 = 0$, G is disconnected. Since $d_1 \geq 1$, $G[S]$ has no isolates. Thus, G has at least one non-trivial component. $\square$

The next type in this sequence is the Type 17 ($\geq$ 1,$\geq$ 1,$\geq$ 1,$\geq$ 1)-bipartition. No characterization of this type of bipartition, into two total dominating sets, or equivalently, into two restrained dominating sets, is known. We will discuss this bipartition in Section 4.

Problem 3.17 *Characterize the graphs having a Type 17 ($\geq$ 1,$\geq$ 1,$\geq$ 1, $\geq$ 1)-bipartition.*

Graphs having a Type 18 $(\geq 1, \geq 1, \geq 1, X)$-bipartition $(S, V \setminus S)$ are precisely the graphs whose set of vertices can be partitioned into a dominating set $(V \setminus S)$ and a total dominating set S. These graphs were characterized by Henning and Southey in [12] through the construction of a family of graphs denoted $\mathcal{G}$.

Theorem 3.18 (Henning and Southey [12]) *A graph* G *has a Type 18* $(\geq 1, \geq 1, X, \geq 1)$*-bipartition if and only if* $G \in \mathcal{G}$*.*

A graph G has a Type 19 $(\geq 1, \geq 1, X, \geq 1)$-bipartition if and only if the vertices of G can be partitioned into two sets S and $V \setminus S$, where $G[S]$ has no isolated vertices and $V \setminus S$ is a total dominating set, that is, $V \setminus S$ is a total restrained dominating set.

Proposition 3.19 *A graph* G *has a Type 19* $(\geq 1, \geq 1, X, \geq 1)$*-bipartition if and only if there exist adjacent vertices* u *and* v *such that* $deg(u) \geq 2$*,* $deg(v) \geq 2$*, and* $G \setminus \{u, v\}$ *has no isolated vertices.*

Proof. Assume that such pair of adjacent vertices, say u, v, exists. Let $S = \{u, v\}$. Since $deg(u) \geq 2$ and $deg(v) \geq 2$, each of u and v has at least one neighbor in $V \setminus S$. Further, since $G \setminus \{u, v\}$ has no isolated vertices, every vertex in $V \setminus S$ has a neighbor in $V \setminus S$. Hence, $(S, V \setminus S)$ is a $(\geq 1, \geq 1, X, \geq 1)$-bipartition.

Assume that G has a $(\geq 1, \geq 1, X, \geq 1)$-bipartition, say $(S, V \setminus S)$. Since $d_1 \geq 1$, $G[S]$ has no isolated vertices, and since $d_2 \geq 1$, every vertex in S has a neighbor in $V \setminus S$. Thus, each vertex in S has degree at least two. Choose u and v to be any adjacent vertices in S. In particular, $deg(u) \geq 2$ and $deg(v) \geq 2$. Moreover, since $d_2 \geq 1$ and $d_4 \geq 1$, every vertex in V has a neighbor in $V \setminus S$, so $G \setminus \{u, v\}$ has no isolated vertices. $\square$

A graph G has a Type 20 $(\geq 1, \geq 1, X, X)$-bipartition if and only if G has a restrained dominating set. The following characterization of these graphs is given by Domke et al. in [4]:

Proposition 3.20 (Domke [4]) *A graph G has a Type 20 $(\geq 1, \geq 1, X, X)$-bipartition if and only if G is not a disjoint union of stars and isolated vertices, or equivalently, G has two adjacent vertices u and v such that $deg(u) \geq 2$ and $deg(v) \geq 2$.*

A graph G having a Type 21 $(\geq 1, X, \geq 1, X)$-bipartition is equivalent to G having a proper total dominating set. Hence, the characterization follows directly from the definition of total domination and the fact that $G = mK_2$ is the only graph requiring all of its vertices in a TD-set.

Proposition 3.21 *A graph G has a Type 21 $(\geq 1, X, \geq 1, X)$-bipartition if and only if $G \neq mK_2$ has no isolated vertices.*

A graph G has a Type 22 $(\geq 1, X, X, \geq 1)$-bipartition if and only if the vertices of G can be partitioned into two sets S and $V \setminus S$ such that $G[V \setminus S]$ and $G[S]$ have no isolated vertices.

Proposition 3.22 *A graph G has a Type 22 $(\geq 1, X, X, \geq 1)$-bipartition if and only if G has no isolated vertices and at least two independent edges.*

Proof. Assume that G has no isolated vertices and at least two independent edges, say uv and xy. Let S be the set consisting of u and v, and any vertex whose open neighborhood is contained in $\{u, v\}$. Clearly, every vertex in S has a neighbor in S. Since uv and xy are not adjacent edges, $N(x)$ (respectively, $N(y)$) is not contained in $\{u, v\}$. It follows that neither x nor y is in S. In particular, $V \setminus S \neq \emptyset$. Since G has no isolated vertices, by our choice of S, every vertex in $V \setminus S$ has a neighbor in $V \setminus S$. Thus, $(S, V \setminus S)$ is a $(\geq 1, X, X, \geq 1)$-bipartition.

Assume that G has a $(\geq 1, X, X, \geq 1)$-bipartition. Since $d_1 \geq 1$ and $d_4 \geq 1$, every vertex in S has a neighbor in S and every vertex in $V \setminus S$ has a neighbor in $V \setminus S$. Since $S \neq \emptyset$ and $V \setminus S \neq \emptyset$, G has no isolated vertices and two independent edges. $\square$

We omit the proofs of the next two straightforward results.

Proposition 3.23 *A graph G of order n has a Type 23 $(\geq 1, X, X, X)$-bipartition if and only if G has at least one edge and $n \geq 3$.*

Proposition 3.24 *A graph G has a Type 24 $(X, 0, 0, X)$-bipartition if and only if G has at least two components.*

As previously mentioned, Ore's Theorem [13] states that in a graph having no isolated vertices, the complement $V \setminus S$ of any minimal dominating set S is a dominating set. In other words, Ore's Theorem implies that the vertex set of any graph with no isolated vertices can be partitioned into two dominating sets. Hence, our next result follows directly.

Proposition 3.25 *A graph G has a Type 25 $(X, \geq 1, \geq 1, X)$-bipartition if and only if G has no isolated vertices.*

Our final two characterizations are trivial.

Proposition 3.26 *A graph G has a Type 26 $(X, \geq 1, X, X)$-bipartition if and only if G has at least one edge.*

Proposition 3.27 *A graph G has a Type 27 (X, X, X, X)-bipartition if and only if G is a non-trivial graph.*

The following table summarizes the necessary and sufficient conditions for a graph to have a bipartition of each of the 27 types.

Table 2: Necessary and Sufficient Conditions

$Type\, i\, (d_1, d_2, d_3, d_4)$	G **has a Type** i**-bipartition if and only if**
1. (0,0,0,0)	$G = \overline{K}_n$
2. (0,0,0,$\geq$ 1)	G has at least one isolated vertex and at least one edge.
3. (0,0,0,X)	G has at least one isolated vertex.
4. (0,$\geq$ 1,$\geq$ 1,0)	G is a non-trivial bipartite graph with no isolated vertices.
5. (0,$\geq$ 1,$\geq$ 1,$\geq$ 1)	**Unknown**
6. (0,$\geq$ 1,$\geq$ 1,X)	G is a non-trivial graph with no isolated vertices.
7. (0,$\geq$ 1,X,0)	G is a bipartite graph with at least one edge.
8. (0,$\geq$ 1,X,$\geq$ 1)	$\delta(G) \geq 1$ and $\Delta(G) \geq 2$.
9. (0,$\geq$ 1,X,X)	G has at least one edge.
10. (0,X,$\geq$ 1,$\geq$ 1)	**Unknown**
11. (0,X,$\geq$ 1,X)	G has an independent dominating set with fewer than n vertices, that is, $G \neq \overline{K}_n$
12. (0,X,X,0)	G is a non-trivial bipartite graph.
13. (0,X,X,$\geq$ 1)	$G \notin \{mK_2, \overline{K}_n\}$.
14. (0,X,X,X)	G is a non-trivial graph.
15. ($\geq$ 1,0,0,$\geq$ 1)	G has no isolated vertices and at least two components.
16. ($\geq$ 1,0,0,X)	G has at least two components, at least one of which is non-trivial.
17. ($\geq$ 1,$\geq$ 1,$\geq$ 1,$\geq$ 1)	**Unknown**
18. ($\geq$ 1,$\geq$ 1,$\geq$ 1,X)	$G \in \mathcal{G}$
19. ($\geq$ 1,$\geq$ 1,X,$\geq$ 1)	G has two adjacent vertices u and v such that $deg(u) \geq 2$, $deg(v) \geq 2$, and $G \setminus \{u, v\}$ has no isolated vertices.
20. ($\geq$ 1,$\geq$ 1,X,X)	G is not a disjoint union of stars and isolated vertices.
21. ($\geq$ 1,X,$\geq$ 1,X)	G has no isolated vertices and $G \neq mK_2$.
22. ($\geq$ 1,X,X,$\geq$ 1)	G has no isolated vertices and at least two independent edges.
23. ($\geq$ 1,X,X,X)	G has at least one edge and $n \geq 3$.
24. (X,0,0,X)	G has at least two components.
25. (X,$\geq$ 1,$\geq$ 1,X)	G has no isolated vertices.
26. (X,$\geq$ 1,X,X)	G has at least one edge.
27. (X,X,X,X)	G is a non-trivial graph.

4 THE REMAINING THREE PROBLEMS

In this section, we address the open problems of characterizing the graphs having the remaining three types of bipartitions:

1. Type 5 $(0, \geq 1, \geq 1, \geq 1)$-bipartition.

   This bipartition is equivalent to partitioning the vertices of G into an independent dominating set and a total dominating set.

2. Type 10 $(0, X, \geq 1, \geq 1)$-bipartition.

   This bipartition is equivalent to partitioning the vertices of G into an independent dominating set and a set whose induced subgraph has no isolated vertices.

3. Type 17 $(\geq 1, \geq 1, \geq 1, \geq 1)$-bipartition.

   This bipartition is equivalent to partitioning the vertices of G into two total dominating sets.

We first note that characterizing graphs having Type 5 bipartition and characterizing graphs having Type 10 bipartition are equivalent problems if the graphs have no isolated vertices. In other words, a $(0, \geq 1, \geq 1, \geq 1)$-bipartition is only possible for a graph with no isolated vertices; and if a graph has no isolated vertices, then a $(0, X, \geq 1, \geq 1)$-bipartition is precisely a $(0, \geq 1, \geq 1, \geq 1)$-bipartition. Furthermore, for a graph G with a set I of isolated vertices and at least one edge, if $(S, V \setminus S)$ is a $(0, \geq 1, \geq 1, \geq 1)$-bipartition of $G - I$, then $(S \cup I, V \setminus (S \cup I))$ is a $(0, X, \geq 1, \geq 1)$-bipartition of G. Hence, for all practical purposes, we have only two distinct open problems remaining, namely, to characterize the graphs whose vertex set can be partitioned into: (1) an independent dominating set and a total dominating set, and (2) into two total dominating sets.

4.1 IDTD-Graphs

In this subsection, we study graphs whose vertex set can be partitioned into an independent dominating set and a total dominating set, that is, graphs having a Type 5 $(0, \geq 1, \geq 1, \geq 1)$-bipartition. We refer to the partition of a graph into an independent dominating set and a total dominating set as a *IDTD-partition*, and we refer to the associated decision problem as the *IDTD-problem*. If G has a IDTD-partition, we say that G is a *IDTD*-graph. As far as we know, the IDTD-graphs have not appeared in the literature.

Given the difficulty of solving the IDTD-problem in general, in the present work we give several sufficient conditions to guarantee that a graph is a IDTD-graph. We also characterize the trees that are IDTD-graphs, following closely the work done by Henning and Southey [12].

4.1.1 Sufficient Conditions

We start with the following observation:

Observation 4.1 *Every IDTD-graph has order at least 3. Trivially, the only IDTD-graph of order 3 is the complete graph K_3.*

Theorem 4.2 *If $\gamma(\overline{G}) \geq 3$, then G is a IDTD-graph.*

Proof. Let I be a maximal independent set of G, clearly $|I| < n$ or else $\gamma(\overline{G}) = 1$. Then I is an independent dominating set of G. Assume $V \setminus I$ is not a total dominating set, that is, there is a vertex $x \in V$ such that x does not have a neighbor in $V \setminus I$. We have the following two possible cases:

Case 1. $x \in I$. Thus, x is an isolated vertex in G, and so, $\{x\}$ dominates $\overline{G}$. Implying that $\gamma(\overline{G}) = 1$. But this is a contradiction since $\gamma(\overline{G}) \geq 3$.

Case 2. $x \in V \setminus I$. Thus x is an isolated vertex in $V \setminus I$, and so, $\{x\}$ dominates $V \setminus I$ in $\overline{G}$. Since any vertex $u \in I$ dominates I in $\overline{G}$, we have that $\{x, u\}$ dominates $\overline{G}$, contradicting that $\gamma(\overline{G}) \geq 3$. □

Theorem 4.3 *If G is a claw-free graph with $\delta(G) \geq 3$, then G is a IDTD-graph.*

Proof. Let G be a claw-free graph with $\delta(G) \geq 3$. Let I be a maximal independent set of G. Then I is an independent dominating set of G. Since $\delta(G) \geq 3$, every vertex in I has a neighbor in $V \setminus I$, so $V \setminus I$ is a dominating set of G. To see that $V \setminus I$ is a total dominating set of G, note that if any vertex of $V \setminus I$ has three neighbors in I, a claw is formed, a contradiction. Hence, every vertex in $V \setminus I$ has at most two neighbors in I, implying that every vertex in $V \setminus I$ has at least one neighbor in $V \setminus I$. In other words, $V \setminus I$ is a TD-set of G. Thus, G is a IDTD-graph. □

Observation 4.4 *If the set of vertices of a graph can be partitioned into an independent dominating set and a non-trivial connected dominating set, then the graph is a IDTD-graph.*

Theorem 4.5 *If $\gamma(\overline{G}) \geq 4$, then the set of vertices of G can be partitioned in a independent dominating set and a non-trivial connected dominating set.*

Proof. Suppose $\gamma(\overline{G}) \geq 4$. Let I be a maximal independent set of G. Then I is an independent dominating set of G. Clearly, $I \neq V$, or else $\gamma(\overline{G}) = 1$. Also, notice $|V \setminus I| > 1$. Suppose $V \setminus I$ is not connected. Let A be the vertex set of a component of $V \setminus I$, then $V \setminus (I \cup A) \neq \emptyset$. Let $a \in A$, then a dominates $V \setminus (I \cup A)$ in $\overline{G}$. Let $u \in I$, then u dominates I in $\overline{G}$. Let $x \in V \setminus (I \cup A)$, then x dominates A in $\overline{G}$. Thus, $\{a, u, x\}$ dominates $\overline{G}$, contradicting that $\gamma(\overline{G}) \geq 4$.
Suppose $V \setminus I$ is not a dominating set, that is, there is a vertex $x \in I$ such that x does not have a neighbor in $V \setminus I$. Then x is an isolated vertex in G, and so dominates $\overline{G}$, again a contradiction since $\gamma(\overline{G}) \geq 4$. □

4.1.2 Trees

Now we proceed to characterize the trees which are IDTD-graphs. We do this in a constructive way, following closely the construction done by Henning and Southey [12] to characterize those trees that can be partition into a total dominating set and a dominating set.

Definition 4.6 *Define a labeling of a graph $G = (V, E)$ as a partition $L = (A, B)$ of V. A vertex v is said to be labeled A (labeled B) if $v \in A$ ($v \in B$).*

Definition 4.7 *By labeled-P_4, we mean a path P_4 with the two central vertices labeled A and the two leaves labeled B.*

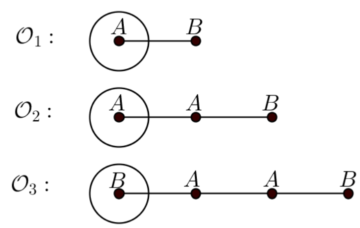

Figure 1: The Three Operations

Definition 4.8 ***The Graph Family $\mathcal{T}$***

Let $\mathcal{T}$ be the minimum family of labeled trees that:

(i) *contains a labeled-P_4 and*

(ii) *is closed under the three operations $\mathcal{O}_1$, $\mathcal{O}_2$, and $\mathcal{O}_3$ listed below:*

- *Operation $\mathcal{O}_1$: Assume the vertex v is labeled A. Add a vertex u labeled B and the edge vu.*
- *Operation $\mathcal{O}_2$: Assume the vertex v is labeled A. Add a path uw, where u is labeled A and w is labeled B and the edge vu.*
- *Operation $\mathcal{O}_3$: Assume the vertex v is labeled B. Add a path uwx, where u is labeled A, w is labeled A and x is labeled B, and the edge vu.*

We illustrate these operations in Figure 1.

Observation 4.9 *Let $(T, L) \in \mathcal{T}$ for some labeling $L = (A, B)$. Then the following properties hold:*

(a) Every vertex labeled A is adjacent to a vertex labeled A and to a vertex labeled B.

(b) Every vertex labeled B is adjacent to a vertex labeled A.

(c) Every neighbor of a vertex labeled B is a vertex labeled A.

(d) The set A is a TD-set of T and the set B is an ID-set of T.

(e) Every leaf of T is labeled B and every support vertex is labeled A.

Observation 4.10 *Let T be a rooted IDTD-tree, and let (D, I) be a partition of its vertex set V into a TD-set D, and an ID-set I. Then, the following properties hold:*

(a) Every leaf belongs to I, while every support vertex belongs to D.

(b) If every child of a vertex is a leaf, then its parent belongs to D.

(c) Any neighbor of a vertex in I must be in D.

A vertex is called a *strong support vertex* if it is adjacent to at least two leaves.

Theorem 4.11 *The IDTD-trees are precisely those trees T such that $(T, L) \in \mathcal{T}$, for some labeling L.*

Proof. Suppose first that $(T, L) \in \mathcal{T}$ for some labeling $L = (A, B)$. By Observation 4.9(d), (A, B) is a partition of V into a TD-set A and an ID-set B.

Now, to prove the necessity we proceed by induction on the order $n \geqslant 4$ of an IDTD-tree T. Since no star $K_{1,n-1}$ is a IDTD-tree, we have that and $diam(T) \geq 3$.

For the base case, if $n = 4$, then $T = P_4$, and $(T, L) \in \mathcal{T}$ where L is the labeling of labeled-P_4.

For the inductive hypothesis, let $n \geq 5$ and assume that for every IDTD-tree T of order less than n there exist a labeling L such that $(T, L) \in \mathcal{T}$.

For the inductive step, let T be an IDTD-tree of order n. Let (D, I) be the partition of V into a TD-set D and an ID-set I. We want to find a labeling such that

$(T, L) \in \mathcal{T}$.

We root the tree T at a diametrical vertex r. Necessarily, r is a leaf. Let u be a vertex at maximum distance from r. Necessarily, u is a leaf. Let v be the parent of u, let w be the parent of v and let x be the parent of w (possibly, $x = r$). Since u is at maximum distance from r, every child of v is be a leaf. Then by Observation 4.10 (a), $C(v) \subseteq I$ (in particular, $u \in I$) and $v \in D$. Also, by Observation 4.10 (b) $w \in D$.

We study the following two possible cases: **1.** T has at least one strong support vertex or **2.** T does not have any strong support vertices.

Case 1. Let z be a strong support vertex of T. Let z_1 and z_2 be two leaf-neighbors of z. Then by Observation 4.10 (a), $z_1, z_2 \in I$ and $z \in D$.
Consider the tree $T \setminus \{z_1\}$. Notice that $(D, I \setminus \{z_1\})$ is a partition of the set of vertices of $T \setminus \{z_1\}$ into a TD-set and ID-set. Applying the inductive hypothesis to $T \setminus \{z_1\}$, there exist a labeling $L = (A, B)$ such that $(T \setminus \{z_1\}, L) \in \mathcal{T}$. By Observation 4.9 (d), we also know A is a TD-set and B is an ID-set of $T \setminus \{z_1\}$.
By Observation 4.10 (a), $z \in A$. Thus, we can restore the tree T by applying Operation $\mathcal{O}_1$ to $T \setminus \{z_1\}$ (at vertex z). Thus, $(T, S) \in \mathcal{T}$ for some labeling S, namely, S is the labeling L for $T \setminus \{z_1\}$ and z_1 is labeled B.

Case 2. Assume that T has no strong support vertices. In particular, $deg(v) = 2$. We study the following two possible cases: **(a)** $deg(w) \geq 3$ or **(b)** $deg(w) = 2$.

Case 2(a). Suppose $deg(w) \geq 3$. Let $v' \in C(w) \setminus \{v\}$.
Two possible cases:

• **Case 2(a.1).** There is at least one child of w besides v with degree greater than one. Suppose v' is such vertex.

By our choice of the vertex u, every child of v' is a leaf. Since T has no strong vertex, $deg(v') = 2$. Let u' be the child of v'. By Observation 4.10 (a), $\{u, u'\} \in I$ and $\{v, v'\} \in D$. Also, by Observation 4.10 (b), $w \in D$.

Consider the tree $T \setminus \{u', v'\}$. Then $(D \setminus \{v'\}, I \setminus \{u'\})$ is a partition of the set of vertices of $T \setminus \{u', v'\}$ into a TD-set and an ID-set. Applying the inductive hypothesis to $T \setminus \{u', v'\}$, there exist a labeling $L = (A, B)$ such that $(T \setminus \{u', v'\}, L) \in \mathcal{T}$. By Observation 4.9 (d), we know that A is a TD-set and B is an ID-set of $T \setminus \{u', v'\}$. By Observation 4.10 (a) and (b), $\{u\} \in B$ and $\{v, w\} \subseteq A$. Thus, we can restore T by applying Operation $\mathcal{O}_2$ to $T \setminus \{u', v'\}$ (at vertex w). Thus, $(T, S) \in \mathcal{T}$ for some labeling S, namely, S is the labeling L for $T \setminus \{u^{'}, v^{'}\}$ along with $v^{'}$ labeled A and $u^{'}$ labeled B.

- **Case 2(a.2).** Every child of w, different from v, is a leaf.

Thus since T does not have any strong support vertices, $deg(w) = 3$ and $C(w) = \{v, v'\}$, where v' is a leaf. By Observation 4.10 (a) and (b), $\{u, v'\} \subseteq I$ and $\{v, w\} \subseteq D$.

There are two possibilities, either $x \in D$ or $x \in I$:

If $x \in D$, then we know that w is adjacent to a vertex in D besides v. Therefore, when considering the tree $T \setminus \{u, v\}$, we have that $(D \setminus \{v\}, I \setminus \{u\})$ is a partition of the set of vertices of $T \setminus \{u, v\}$ into a TD-set and an ID-set. Applying the inductive hypothesis to $T \setminus \{u, v\}$, there exist a labeling $L = (A, B)$ such that $(T \setminus \{u, v\}, L) \in \mathcal{T}$. By Observation 4.9 (d), we know that A is a TD-set and B is an ID-set of $T \setminus \{u, v\}$. By Observation 4.10 (a), $v' \in B$ and $w \in A$. Thus, we can restore T by applying Operation $\mathcal{O}_2$ to $T \setminus \{u, v\}$ (at vertex w). Thus, $(T, S) \in \mathcal{T}$ for some labeling S, namely, S is the labeling L for $T \setminus \{u, v\}$ along with u labeled B and v labeled A.

If $x \in I$, then $(D, I \setminus \{v'\})$ is a partition of the set of vertices of $T \setminus \{v'\}$ into TD-set and ID-set, respectively. Applying the inductive hypothesis to $T \setminus \{v'\}$, there exist a labeling $L = (A, B)$ such that $(T \setminus \{v'\}, L) \in \mathcal{T}$. By Observation 4.9 (d), we know that A is a TD-set and B is an ID-set of $T \setminus \{v'\}$.

By Observation 4.10 (a) and (b), $u \in B$ and $\{v, w\} \subseteq A$. Thus, we can restore T by applying Operation $\mathcal{O}_1$ to $T \setminus \{v'\}$ (at vertex w). Hence, $(T, S) \in \mathcal{T}$ for some labeling S, namely, S is the labeling L for $T \setminus \{v^{'}\}$ and $v^{'}$ labeled B.

Case 2(b). Suppose $deg(w) = 2$. Since $n \geq 5$, the vertex x is not the root r of T (we have that $deg(u) = 1, deg(v) = deg(w) = 2$, thus a fifth vertex would have to be a neighbor of x).

Let y be the parent of x. By Observation 4.10 (a) and (b), $u \in I$ and $v, w \in D$. We must have that $x \in I$ (since $N(w) = \{v, x\}$ and w must have a neighbor in I). Hence, by Observation 4.10 (a), x is not a support vertex, that is, no child of x is a leaf.

We consider the following two possible cases:

- **Case 2(b.1).** $deg(x) \geq 3$.

Let $w' \in C(x) \setminus \{w\}$. Since no child of x is a leaf, $deg(w') \geq 2$.

By our choice of vertex u, each child of the vertex w' is either a support vertex or is a leaf.

If w' is not the parent of any support vertex, all of its children must be leaves. By Observation 4.10 (a) and (b), this implies that $w', x \in D$, contradicting the fact that

$x \in I$. Hence, w' must be parent of a support vertex v'. Let u' be a child of v'.

An identical argument as shown with the vertex w, shows that we may assume $deg(w') = deg(v') = 2$. Hence, by Observation 4.10 (a) and (b), $u' \in I$ and $v', w' \in D$ Thus, x is adjacent to a vertex in D different from w.

Consider $T \setminus \{u, v, w\}$. Then $(D \setminus \{v, w\}, I \setminus \{u\})$ is a partition into TD-set and an ID-set, respectively. Applying the inductive hypothesis to $T \setminus \{u, v, w\}$, there exist a labeling $L = (A, B)$ such that $(T \setminus \{u, v, w\}, L) \in \mathcal{T}$. By Observation 4.9 (d), we know that A is a TD-set and B is an ID-set of $T \setminus \{u, v, w\}$.

By Observation 4.10 (a) and (b), $\{u', x\} \subseteq B$ and $\{v', w'\} \subseteq A$. Thus, we can restore T by applying Operation $\mathcal{O}_3$ to $T \setminus \{u, v, w\}$ (at vertex x). Hence, $(T, S) \in \mathcal{T}$ for some labeling S, namely, S is the labeling L for $T \setminus \{u, v, w\}$ along with u labeled B and $\{v, w\} \subseteq A$.

• **Case 2(b.2).** $deg(x) = 2$. Recall $u, x \in I$ and $v, w \in D$. By Observation 4.10 (c), $y \in D$.

Consider $T \setminus \{u, v, w\}$. Then $(D \setminus \{v, w\}, I \setminus \{u\})$ is a partition into a TD-set and an ID-set, respectively. Applying the inductive hypothesis to $T \setminus \{u, v, w\}$, there exists a labeling $L = (A, B)$ such that $(T \setminus \{u, v, w\}, L) \in \mathcal{T}$.

By Observation 4.9, $x \in B$ (since x is a leaf of $T \setminus \{u, v, w\}$). Thus, we can restore T by applying Operation $\mathcal{O}_3$ to $T \setminus \{u, v, w\}$ (at vertex x). Hence, $(T, S) \in \mathcal{T}$ for some labeling S, namely, S is the labeling L for $T \setminus \{u, v, w\}$ along with u labeled B and $\{v, w\} \subseteq A$. $\square$

4.2 TDTD-Graphs

In this subsection, we study graphs whose vertex set can be partitioned into two total dominating sets, i.e., graphs having a Type 17 $(\geq 1, \geq 1, \geq 1, \geq 1)$-bipartition. We refer to the partition of a graph into two disjoint total dominating sets as a *TDTD-partition*, and we refer to the associated decision problem as the *TDTD-problem*. If G has a TDTD-partition, we say that G is a *TDTD*-graph. In [9], Heggernes and Telle showed that the TDTD-problem is NP-complete even if G is bipartite. Zelinka [14, 15] showed that the minimum degree of a graph is not sufficient to guarantee that a graph has a TDTD-partition. This result was later improved by Calkin and Dankelmann [2] and Feige et al. [6], who showed that if the maximum degree of a graph is not too large relative to the minimum degree, then sufficiently large minimum degree is sufficient to guarantee a TDTD-partition. Broere et al. [1] and Dorfling et al. [5] studied the problem of determining the minimum number of edges necessary to add to a graph in order to ensure that it has a TDTD-partition.

Given the difficulty of solving the TDTD-problem in general, in the present work we give several sufficient conditions to guarantee that a graph is a TDTD-graph. We also show that the Cartesian product of any two isolate-free graphs is a TDTD-graph. Our main result shows that with the exception of the cycle C_5, every self-complementary graph with minimum degree at least two is a TDTD-graph.

4.2.1 Preliminary Results

Throughout the subsection, we make use of the following known properties. Let $\mathrm{diam}(G)$ denote the diameter of G.

Observation 4.12 *Let G be a graph with* $\mathrm{diam}(G) = 2$*, and let v be a vertex of G. Then*

1. *$N(v)$ is a dominating set of G, implying that $\gamma(G) \leq \delta(G)$;*

2. *$N[v]$ is a total dominating set of G, implying that $\gamma_t(G) \leq \delta(G) + 1$.*

Since any two vertices at distance at least three apart in G dominate $\overline{G}$, we have the following useful observation.

Observation 4.13 *If G is a graph with* $\mathrm{diam}(G) \geq 3$*, then $\gamma_t(\overline{G}) = 2$.*

Note that total domination is defined only for graphs without isolated vertices and that $\gamma_t(G) \geq 2$ when defined. Hence, the following conditions are necessary for a graph to be a TDTD-graph:

Observation 4.14 *If G is a TDTD-graph of order n, then $n \geq 4$ and $\delta(G) \geq 2$.*

It follows from Observation 4.14 that no tree is a TDTD-graph. Also, note that not all graphs with minimum degree 2 are TDTD-graphs. For example, the cycles C_5 and C_6 are not TDTD-graphs. The following result from Broere et al. characterizes the cycles that are TDTD-graphs.

Proposition 4.15 [1] *A cycle C_k is a TDTD-graph if and only if $k \equiv 0 (mod\ 4)$.*

This leads to the following corollary.

Corollary 4.16 *If G is a Hamiltonian graph with order $n \equiv 0(mod\ 4)$, then G is a TDTD-graph.*

Next we consider Cartesian products. For graphs G and H, the Cartesian product $G \Box H$ is the graph with vertex set $V(G) \times V(H)$ where two vertices (u, v) and (x, y) are adjacent if and only if either $u = x$ and $vy \in E(H)$ or $v = y$ and $ux \in E(G)$. For each vertex $x \in H$, we denote the copy of G in $G\Box H$ corresponding to x as G_x.

Proposition 4.17 *If G and H are graphs without isolated vertices, then $G\Box H$ is a TDTD-graph.*

Proof. Since $\delta(G) \geq 1$, by Theorem 2.1, G has a partition $P_G = \{A, B\}$, where each of A and B is a dominating set. For each vertex $x \in H$, we denote the P_G partition of G_x as $\{A_x, B_x\}$. We claim that $\{P = \bigcup_{x\in H} A_x, \bigcup_{x\in H} B_x\}$ is a TDTD-partition of $G\Box H$. To see this, first notice that since P_G is a partition of the vertices of G into two dominating sets, P is a partition of the vertices of $G\Box H$ into two dominating sets. Also, since there are no isolated vertices in H, it follows that every vertex in $\bigcup_{x\in H} A_x$ (respectively, $\bigcup_{x\in H} B_x$) has a neighbor in $\bigcup_{x\in H} A_x$ (respectively, $\bigcup_{x\in H} B_x$). Thus, $G\Box H$ is a TDTD-graph. $\Box$

Proposition 4.18 *Let G and H be graphs such that $\delta(H) = 0$. Then $G\Box H$ is a TDTD-graph if and only if G is a TDTD-graph.*

Proof. Suppose G is a TDTD-graph, let $P_G = \{A, B\}$ be a partition of the vertices of G into two TD-sets. For each $x \in H$, we partition the vertices of G_x using P_G. The result is partition of the vertices of $G\Box H$ into two TD-sets, so $G\Box H$ is a TDTD-graph.

Assume that $G\Box H$ is a TDTD-graph, and let P be a partition of the vertices of $G\Box H$ into two TD-sets. Since H has an isolated vertex, say x, G_x is a component of $G\Box H$ that is isomorphic to G. Thus, the restriction of P onto G_x is a partition of the vertices of G_x into two TD-sets, implying that $G_x = G$ is a TDTD-graph, as desired. $\square$

The next result highlights the difficulty of finding a characterization of TDTD-graphs.

Theorem 4.19 *There exists no forbidden subgraph characterization of TDTD-graphs.*

Proof. Let H be an arbitrary graph, and let I be the set of isolated vertices in H. Since $H - I$ has no isolated vertices, Theorem 2.1 implies that the vertices of $H - I$ can be partitioned into two dominating sets, say A and B. Let I_A and I_B denote the set of isolated vertices in $H[A]$ and $H[B]$, respectively. Construct a graph G as follows. For each vertex $v \in I \cup I_A \cup I_B$, add a new path $P_3 = v_1v_2v_3$ and edges vv_1 and vv_3. Clearly, H is an induced subgraph of G.

For each $S \in \{I, I_A, I_B\}$, let $X_S = \bigcup_{v \in S}\{v_1\}$ and $Y_S = \bigcup_{v \in S}\{v_2, v_3\}$. We now show that G is a TDTD-graph by giving a partition of the vertices of G into two total dominating sets, A' and B'. Let $A' = A \cup I \cup X_A \cup X_I \cup Y_B$ and $B' = B \cup X_B \cup Y_A \cup Y_I$. Note that $P = \{A', B'\}$ is a partition of the vertices of G. Further, every vertex in G has a neighbor in A' and a neighbor in B'. Hence, each of A' and B' is a TD-set of G, and so P is a partition of the vertices of G into two TD-sets. Thus, G is a TDTD-graph. $\square$

4.2.2 Sufficient Conditions

Now we present several sufficient conditions for a graph to be a TDTD-graph.

Proposition 4.20 *If G is a graph with $\delta(G) \geq \lceil \frac{n}{2} \rceil + 1$, then G is a TDTD-graph.*

Proof. We partition the vertices of G into sets S and $V \setminus S$, where S is any subset of V such that $|S| = \lceil \frac{n}{2} \rceil$. Then, $|V \setminus S| = \lfloor \frac{n}{2} \rfloor$. Since $\delta(G) \geq \lceil \frac{n}{2} \rceil + 1$, every vertex in V has at least one neighbor in S and at least one neighbor in $V \setminus S$. Thus, $\{S, V \setminus S\}$ is a partition of the vertices of G into two TD-sets, and so, G is a TDTD-graph. $\square$

Proposition 4.21 *If G is a graph with $\gamma_t(G) \leq \delta(G) - 1$, then G is a TDTD-graph.*

Proof. Let G be a graph for which $\gamma_t(G) \leq \delta(G) - 1$, and let S be a $\gamma_t(G)$-set. Since $|S| \leq \delta(G) - 1$, we have $N(v) \not\subseteq S$ for all $v \in V$. In particular, every vertex in V has a neighbor in $V \setminus S$. Thus, both S and $V \setminus S$ are TD-sets, and so G is a TDTD-graph. $\square$

Proposition 4.22 *If G is a graph with $\delta(G) \geq 2$, $\Delta(G) = n - 1$ and there exist a vertex v for which all of its neighbors have degree at least 3, then G is a TDTD-graph.*

Proof. Let $u \in V$ be a vertex with $deg(u) = n - 1$ and let v be a vertex such that every one of its neighbors has degree greater or equal than 3. We partition V into $S = \{u, v\}$ and $V \setminus S$. Since every vertex different that u is adjacent to u, we conclude S is a total dominating set. Since $\delta(G) \geq 2$, we know that both u and v have at least one neighbor in $V \setminus S$, that is $V \setminus S$ is a dominating set. Let $x \in V \setminus S$. If $x \in N(v)$, then $deg(x) \geq 3$, thus, x has at least one neighbor different from u and v, which is going to be an element of $V \setminus S$. If $x \notin N(v)$, we know x has at least one more neighbor besides u, since $\delta(G) \geq 2$. But this neighbor can not be v. Thus x has at least one neighbor in $V \setminus S$. Therefore, $V \setminus S$ is also a total dominating set. $\square$

For the remainder of this subsection 4.2.2, we consider the domination number of the complement of G.

Proposition 4.23 *If G is a graph with $\delta(G) \geq 2$ and $\gamma(\overline{G}) \geq 4$, then G is a TDTD-graph.*

Proof. Let G be a graph with $\delta(G) \geq 2$ such that $\gamma(\overline{G}) \geq 4$. Clearly, G has order $n \geq 4$. If G is a complete graph, then we partition the vertices of G into two total dominating sets S and $V \setminus S$, where S is any set of cardinality two. Hence, we may assume that G is not a complete graph, and so $\text{diam}(G) \neq 1$. Since $\gamma(\overline{G}) \geq 4$, Observation 4.13 implies that $\text{diam}(G) = 2$.

Let v be any minimum degree vertex of G. Since G is not complete, $V \setminus N[v] \neq \emptyset$. Since $\text{diam}(G) = 2$, every vertex of $V \setminus N[v]$ is adjacent to at least one vertex of $N(v)$ in G. We claim that every vertex in $V \setminus N[v]$ is adjacent to at least two vertices of $N(v)$. Suppose, to the contrary, that there exists a vertex $x \in V \setminus N[v]$ such that $N(v) \cap N(x) = \{y\}$. But then $\{v, x, y\}$ is a dominating set of $\overline{G}$ and $\gamma(\overline{G}) \leq 3$, a contradiction. Hence, every vertex in $V \setminus N[v]$ has at least two neighbors in $N(v)$.

Let $\mathcal{I}$ denote the set of vertices that are isolates in $G[V \setminus N[v]]$. Since $|N(v)| = \delta(G)$, it follows that every vertex in $\mathcal{I}$ is adjacent to every vertex of $N(v)$. Also, note that if any vertex, say x, in $N(v)$ has no neighbors in $V \setminus N[v]$, then since $deg(v) = \delta(G)$, it follows that $N[x] = N[v]$. Let u be a vertex in $N(v)$ such that u has a neighbor in $V \setminus N[v]$.

First assume that $\mathcal{I} = \emptyset$. Let $A = N[v] \setminus \{u\}$ and $B = V \setminus A$. We note $\delta(G) \geq 2$ implies that $G[A]$ contains no isolated vertices, and since every vertex of $V \setminus N[v]$ has at least two neighbors in $N(v)$, $N(v) \setminus \{u\}$ dominates $V \setminus N[v]$. Thus, A is a TD-set of G. Since $\mathcal{I} = \emptyset$ and $N(u) \cap V \setminus N[v] \neq \emptyset$, it follows that $G[B]$ has no isolated vertices. The vertex v is dominated by $u \in B$, and every vertex in $N(v)$ that has a neighbor in $V \setminus N[v]$ is dominated by B. If a vertex $x \in N(v) \setminus \{u\}$ has no neighbors in $V \setminus N[v]$, then as previously noted, $N[x] = N[v]$, that is, x is adjacent to $u \in B$. Thus, B is TD-set of G, and so, G is a TDTD-graph.

Secondly, assume that $\mathcal{I} \neq \emptyset$. Let $A = (N(v) \setminus \{u\}) \cup \mathcal{I}$, and let $B = V \setminus A$. Note that $v \in B$. Since every vertex of $\mathcal{I}$ is adjacent to every vertex of $N(v)$, it follows that $G[A]$ contains no isolated vertices. To see that A is a dominating set of G, notice that u has a neighbor in I, every vertex of $V \setminus N[v]$ has a neighbor in $N(v) \setminus \{u\}$, and $\delta(G) \geq 2$ implies that v has a neighbor in $N(v) \setminus \{u\}$. Thus, A is a TD-set of G.

Next, consider the set B. Since u dominates $\mathcal{I}$ and every vertex in $N(v) \setminus \{u\}$ is adjacent to v, we have that B is a dominating set of G. Since u and v are adjacent and every vertex $(V \setminus N[v]) \setminus \mathcal{I}$ has a neighbor in B, B is a TD-set of G, and so G is a TDTD-graph. $\square$

In the case when $\gamma(\overline{G}) = 3$, we have the following sufficient condition for G to be a TDTD-graph.

Proposition 4.24 *If G is a graph with $\gamma(\overline{G}) = 3$ and $\gamma_t(G) \neq \delta(G)$, then G is a TDTD-graph.*

Proof. Let G be a graph for which $\gamma(\overline{G}) = 3$ and $\gamma_t(G) \neq \delta(G)$. Since $\gamma_t(G) \neq \delta(G)$, $G \neq K_3$. Since $G \neq K_3$ and $\gamma(\overline{G}) = 3$, it follows that $n \geq 4$ and $G \neq K_n$. Thus, $\operatorname{diam}(G) \geq 2$. Since $\gamma(\overline{G}) = 3 \leq \gamma_t(\overline{G})$, by Observation 4.13, we have that $\operatorname{diam}(G) = 2$. By Observation 4.12, $\gamma_t(G) \leq \delta(G) + 1$. If $\gamma_t(G) \leq \delta(G) - 1$, then, by Proposition 4.21, G is a TDTD-graph. Suppose that $\gamma_t(G) = \delta(G) + 1$. In this case, $N[v]$ is a $\gamma_t(G)$-set for any minimum degree vertex v. Since $N(v)$ dominates G, but cannot be a TD-set of G, it follows that $G[N(v)]$ has an isolated vertex, say w. But

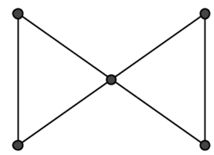

Figure 2: The Bowtie

then $\{w, v\}$ is a dominating set of $\overline{G}$ implying that $\gamma(\overline{G}) \leq 2$, a contradiction. Hence, we conclude that G is a TDTD-graph. $\square$

We note that there exist graphs G for which $\gamma(\overline{G}) = 3$ and $\gamma_t(G) = \delta(G)$ that are not TDTD-graphs. Two small examples are the complete graph K_3 and the bowtie (illustrated in Figure 2). In both cases, $\gamma(\overline{G}) = 3$ and $\gamma_t(G) = 2 = \delta(G)$, but G is not a TDTD-graph.

4.2.3 Self-Complementary Graphs

In this subsection 4.2.3, we prove our main result, namely, that with the exception of the cycle C_5, every self-complementary graph with minimum degree at least 2 is a TDTD-graph. We first prove a lemma which provides an upper bound on the total domination number of self-complementary graphs.

Lemma 4.25 *If G is a self-complementary graph with $\delta(G) \geq 3$, then $\gamma_t(G) \leq 1 + \left\lceil \frac{\delta(G)}{2} \right\rceil$.*

Proof. Let G be a self-complementary graph, and let v be a vertex of maximum degree. Since G is self-complementary, $\Delta(G) = n - 1 - \delta(G)$. Let $A = N_G[v]$ and

$B = V(G) \setminus A$. Clearly, $|B| = \delta(G)$. Let $B = \{b_1, \ldots, b_{\delta(G)}\}$. For $1 \le i < j \le \delta(G)$, if there exist vertices b_i and b_j that have no common neighbor in $N(v)$, then $\{v, b_i, b_j\}$ is a TD-set of $\overline{G}$, and so $\gamma_t(G) \le 3 \le 1 + \left\lceil \frac{\delta(G)}{2} \right\rceil$. Hence, we may assume that every two vertices in B have a common neighbor in A.

Let $k = \left\lfloor \frac{\delta(G)}{2} \right\rfloor$, and let $a_i \in A$ be a common neighbor of b_{2i} and b_{2i-1} for $1 \le i \le k$. If $\delta(G)$ is even, let $a_{k+1} = a_k$. If $\delta(G)$ is odd, let $a_{k+1} \in A$ be a common neighbor of $b_{\delta(G)}$ and b_1. The set $\{v\} \cup \bigcup_{i=1}^{k+1} \{a_i\}$ is a TD-set of G with cardinality at most $1 + \left\lceil \frac{\delta(G)}{2} \right\rceil$. Hence, $\gamma_t(G) \le 1 + \left\lceil \frac{\delta(G)}{2} \right\rceil$. $\square$

Proposition 4.21 and Lemma 4.25 give the following corollary.

Corollary 4.26 *If G is a self-complementary graph with $\delta(G) \ge 4$, then G is a TDTD-graph.*

Proof. By Lemma 4.25, $\gamma_t(G) \le 1 + \left\lceil \frac{\delta(G)}{2} \right\rceil$. Since $\delta(G) \ge 4$, it follows that $1 + \left\lceil \frac{\delta(G)}{2} \right\rceil < \delta(G)$. Hence, by Proposition 4.21, G is a TDTD-graph. $\square$

We will make use of the following well-known properties of self-complementary graphs. The interested reader is referred to [3] for further information.

Observation 4.27 *If G is a non-trivial self-complementary graph of order n, then* $\mathrm{diam}(G) \in \{2, 3\}$ *and n is congruent to 0 or 1 modulo 4.*

For vertices u and v, we let $d_G(u, v)$ denote the distance between u and v. We now give our main result.

Theorem 4.28 *If $G \ne C_5$ is a self-complementary graph with $\delta(G) \ge 2$, then G is a TDTD-graph.*

Proof. Let $G \neq C_5$ be a self-complementary graph of order n with $\delta(G) \geq 2$. If $\delta(G) \geq 4$, then by Corollary 4.26, the result holds. Hence, we may assume that $2 \leq \delta(G) \leq 3$.

By Observation 4.27, n is congruent to 0 or 1 modulo 4 and $2 \leq \mathrm{diam}(G) \leq 3$. Noting that C_5 is the only self-complementary graph with $\delta(G) \geq 2$ and $n \leq 5$, we may assume that $n \geq 8$.

First assume that $\mathrm{diam}(G) = 3$. By Observation 4.13, we have that $\gamma_t(\overline{G}) = 2$, and since G is self-complementary, $\gamma_t(G) = 2$. If $\delta(G) = 3$, then by Proposition 4.21, G is a TDTD-graph. Therefore, assume that $\delta(G) = 2$, and let $S = \{a, b\}$ be a $\gamma_t(G)$-set. Since $\delta(G) = 2$, $V \setminus S$ dominates G. If there does not exist a vertex $v \in V \setminus S$ such that $N(v) = S$, then $\{S, V \setminus S\}$ is a partition of V into two TD-sets of G, and so, G is a TDTD-graph. Hence, we may assume that there exists a vertex $v \in V \setminus S$ such that $N(v) = \{a, b\}$. Since $\mathrm{diam}(G) = 3$ and no vertex is distance three from any of a, b, and v in G, there exist two vertices $x, y \in V \setminus N_G[v]$ such that $d_G(x, y) = 3$. In particular, x and y have no common neighbor in S in G, and so $\{x, y\}$ is a $\gamma_t(\overline{G})$-set, and $V \setminus \{x, y\}$ dominates $\overline{G}$. To see that $V \setminus \{x, y\}$ is a TD-set of $\overline{G}$, note that $v \in V \setminus \{x, y\}$ and v is adjacent to every vertex in $V \setminus \{a, b\}$ in $\overline{G}$. Since $n \geq 8$, v has a neighbor in $V \setminus \{x, y, a, b\}$ in $\overline{G}$. Further, in $\overline{G}$, each of a and b has exactly one neighbor in $\{x, y\}$, and since $\delta(\overline{G}) = 2$, each of a and b has a neighbor in $V \setminus \{x, y\}$ in $\overline{G}$. Thus, $\{\{x, y\}, V \setminus \{x, y\}\}$ is a partition of V into two TD-sets of $\overline{G}$. Since G is self-complementary, we conclude that G is a TDTD-graph, if $\mathrm{diam}(G) = 3$.

Henceforth, we may assume that $\mathrm{diam}(G) = 2$. By Observation 4.13, this implies that $\gamma_t(G) \geq 3$. By Observation 4.12, it follows that $\gamma_t(G) \leq \delta(G) + 1$. Moreover, since $\delta(G) \in \{2,3\}$ and $\gamma_t(G) \geq 3$, we have that $\gamma_t(G) \in \{3,4\}$.

We consider two cases:

Case 1. $\delta(G) = 2$. Since $3 \leq \gamma_t(G) \leq \delta(G)+1$, we have that $\gamma_t(G) = 3$. We first show that there exist two non-adjacent vertices, one having degree two and the other one having degree $n-3$. Let $v \in V$ such that $deg_G(v) = 2$. Then in $\overline{G}$, the vertex v has degree $n-3$, and since G is a self-complementary graph, there exists a vertex x of degree $n-3 \geq 5$ in G. If x is not adjacent to v, we have the desired two vertices. Thus, assume $x \in N_G(v)$. But then in $\overline{G}$, $deg_{\overline{G}}(x) = 2$ and $deg_{\overline{G}}(v) = n-3$ and x is not adjacent to v. Since G is self-complementary, we may assume that two such vertices exist in G.

Let v and x be non-adjacent vertices such that $deg_G(v) = 2$ and $deg_G(x) = n-3$. Let $N_G(v) = \{a,b\}$. Since $\mathrm{diam}(G) = 2$, $\{a,b\}$ is a dominating set of G. But since $\gamma_t(G) = 3$, $\{a,b\}$ is not a TD-set of G, implying that a is not adjacent to b. We consider two possibilities, namely, x has two neighbors in $\{a,b\}$ or x has exactly one neighbor in $\{a,b\}$.

If x is adjacent to both a and b, then x dominates V except for v and some vertex, say $c \in V \setminus N[v]$. Since c is dominated by $\{a,b\}$, without loss of generality, assume that c is adjacent to a. In this case, $\{x,a\}$ is a TD-set of G, contradicting that $\gamma_t(G) = 3$.

Thus, x has exactly one neighbor in $\{a,b\}$, say a, without loss of generality. Since $deg_G(x) = n-3$, $N_G(x) = V \setminus \{b,v\}$. If $deg_G(a) > 2$, consider the sets $S = \{x,a,v\}$ and $V \setminus S$. Note that $G[S]$ has no isolates. Since x dominates $V \setminus \{b,v\}$ and b is adjacent to v, S is a TD-set for G. Now, consider $V \setminus S$. Since $n \geq 8$, $|V \setminus S| \geq 5$.

The vertex v is adjacent to b and both x and a have a neighbor in $V \setminus (S \cup \{b\})$, so $V \setminus S$ is a dominating set of G. We show that $V \setminus S$ is a TD-set, that is, $G[V \setminus S]$ has no isolates. Since $N(v) \cap (V \setminus S) = \{b\}$ and b is not adjacent to a or x, no vertex of degree 3 in $V \setminus S$ is adjacent to all the vertices of S. The only other possibility for $G[V \setminus S]$ to have an isolate is that a vertex $z \in V \setminus S$ such that $N(z) = \{x, a\}$. However, in $\overline{G}$, z is adjacent to every vertex except x and a. Since x and a are not adjacent in $\overline{G}$ and $\text{diam}(\overline{G}) = 2$, there exists a vertex y that is adjacent to both x and a. But then $\{z, y\}$ is a TD-set of $\overline{G}$, contradicting that $\gamma_t(G) = \gamma_t(\overline{G}) = 3$. Thus, $V \setminus S$ is a TD-set of G, implying that G is a TDTD-graph.

Next, assume that $deg_G(a) = 2$, that is, $N_G(a) = \{v, x\}$. Since $\{a, b\}$ dominates G, $N_G(b) = V \setminus \{a, x\}$. Thus, $deg_G(b) = n - 3 \geq 5$. Let $b' \in N_G(b) \setminus \{v\}$. It follows that $S' = \{a, x, b'\}$ is a $\gamma_t(G)$-set. Consider $V \setminus S'$. Note that x and b have a common neighbor, say y, in $V \setminus \{a, b, v, x, b'\}$. Now a is adjacent to v, b' is adjacent to b, and x is adjacent to y in $V \setminus S'$, and so, $V \setminus S'$ is a dominating set of G. Further, $V \setminus S' \subseteq N_G[b]$, so there are no isolates in $G[V \setminus S']$. Hence, $V \setminus S'$ is a TD-set of G, and so G is a TDTD-graph.

Case 2. $\delta(G) = 3$. Recall that $\gamma_t(G) \in \{3, 4\}$. Since no single vertex dominates G and $\text{diam}(G) = 2$, by Observation 4.12, we have that $2 \leq \gamma(G) \leq 3$. If $\gamma(G) = 3$ and $\gamma_t(G) = 4$, then since G is self-complementary, Proposition 4.24 implies that G is a TDTD-graph. Note that if $\gamma(G) = 2$, then since $\gamma_t(G) \geq 3$, the vertices in any $\gamma(G)$-set are not adjacent. Since $\text{diam}(G) = 2$, they have a common neighbor, implying that $\gamma_t(G) \leq 3$. Hence, the possibilities are $\gamma(G) \in \{2, 3\}$ and $\gamma_t(G) = 3$.

Case 2(a). $\gamma(G) = 2$. Let $\{a, b\}$ be any $\gamma(G)$-set. Since diam$(G) = 2$, there exists a vertex c such that c is adjacent to a and b. Let $S = \{a, b, c\}$. Clearly, S is a $\gamma_t(G)$-set. Since $\delta(G) = 3$, $V \setminus S$ is a dominating set of G. If $G[V \setminus S]$ has no isolated vertices, that is, if no minimum degree 3 vertex of $V \setminus S$ is adjacent to all three vertices of S, then S and $V \setminus S$ are TD-sets of G and G is a TDTD-graph. Hence, we may assume there exists a vertex $v \in V \setminus S$ such that $N(v) = S$. Note that since $\{a, b\}$ is a $\gamma(G)$-set, $\{a, v, b\}$ is a $\gamma_t(G)$-set. If $deg_G(c) \geq 4$, then since $\delta(G) = 3$, it follows $\{a, v, b\}$ and $V \setminus \{a, v, b\}$ partition V into two TD-sets, and G is a TDTD-graph as desired.

Thus, we may assume that $deg_G(v) = deg_G(c) = 3$. In $\overline{G}$, v and c are not adjacent, and v (respectively, c) dominates $V \setminus \{c, a, b\}$ (respectively, $V \setminus \{v, a, b\}$). Note that a and b are adjacent in $\overline{G}$ and since diam$(\overline{G})$ = diam$(G) = 2$, a has a neighbor, say x, in $N_{\overline{G}}(v) \cap N_{\overline{G}}(c)$. Note that $x \notin \{a, b, c, v\}$. Then $D = \{a, x, c\}$ is a $\gamma_t(\overline{G})$-set. Note that a is adjacent to b, x is adjacent to v and since $n \geq 8$, c has at least one neighbor in $V \setminus D$. Hence, $V \setminus D$ is a dominating set of $\overline{G}$. If there does not exist some vertex $w \in V \setminus D$ such that $N_{\overline{G}}(w) = D$, then $V \setminus D$ is a TD-set of $\overline{G}$, implying that $\overline{G}$, and hence G, is a TDTD-graph. Thus, assume that such a vertex w exists. Clearly, $w \neq b$ since w is adjacent to c and a. Hence, $w \notin \{a, b, c\}$. Then $deg_G(w) = n - 4 \geq 4$ and $N_G(w) = V \setminus \{a, x, c\}$. In particular, $w \in N_G(v)$, a contradiction since $N_G(v) = \{a, b, c\}$ and $w \notin \{a, b, c\}$. We conclude that in this case G is a TDTD-graph.

Case 2(b). $\gamma(G) = 3$. Let $S = \{a, b, c\}$ be a $\gamma_t(G)$-set such that, without loss of generality, $ab, cb \in E$. Further, let $S_x = \text{epn}(x, S)$ for each $x \in S$. By the minimality of S, $S_a \neq \emptyset$ and $S_c \neq \emptyset$. If $S_b = \emptyset$, then $\{a, c\}$ is a dominating set of G, contradicting that $\gamma(G) = 3$. Hence, $S_b \neq \emptyset$. Let $a' \in S_a$, $b' \in S_b$, and $c' \in S_c$. Thus, $V \setminus S$ is a dominating set of G. If $G[V \setminus S]$ has no isolated vertices, then $V \setminus S$ is a TD-set of G and G is a TDTD-graph as desired. Hence, we may assume that $u \in V \setminus S$ is an isolate in $G[V \setminus S]$. Since $\delta(G) = 3$, it follows that $N_G(u) = S$.

We have just shown:

Fact 1. If G is not a TDTD-graph, every $\gamma_t(G)$-set is contained in the open neighborhood of a vertex with degree 3.

Next we show that:

Fact 2. For every vertex v of minimum degree 3, $N(v)$ is a $\gamma_t(G)$-set.

Let v be a vertex of minimum degree 3. Since $\text{diam}(G) = 2$, $N(v)$ is a dominating set. If $G[N(v)]$ has an isolated vertex, say w, then $\{v, w\}$ is a dominating set of $\overline{G}$, contradicting that $\gamma(G) = \gamma(\overline{G}) = 3$. Hence, $N(v)$ is a $\gamma_t(G)$-set. Thus, Fact 2 holds.

Returning to our $\gamma_t(G)$-set S, we next show that each vertex in S has degree at least 4, implying that no vertex of minimum degree 3 is in any $\gamma_t(G)$-set. Clearly, $deg_G(b) \geq 4$. If $deg_G(a) = 3$, then by Fact 2, $N(a) = \{u, b, a'\}$ is a $\gamma_t(G)$-set. But a' has no neighbor in $N(a)$, a contradiction. Hence, $deg_G(a) \geq 4$, and similarly, $deg_G(c) \geq 4$.

Next we show that u is the only vertex in G with degree 3. Suppose to the contrary that $deg_G(v) = 3$ for some vertex $v \neq u$. Since every vertex in S has degree at least 4, it follows that $v \in V \setminus N[u]$. Then v has a least one neighbor in S.

First assume that $N(v) = S = \{a, b, c\}$. Then $X = \{u, a', b'\}$ is a $\gamma_t(\overline{G})$-set. Since G is self-complementary, by Fact 1, every $\gamma_t(\overline{G})$-set, in particular, X, is contained in the open neighborhood of a vertex of degree 3 in $\overline{G}$. Let $x \in V$ such that $N_{\overline{G}}(x) = X$. Note that $x \neq v$ since $deg_{\overline{G}}(v) = n - 4 \geq 4$, and $x \notin \{a, b, c\}$ since none of a, b, and c is adjacent to u in $\overline{G}$. Also, note that $N_G(x) = V \setminus \{u, a', b'\}$, implying that x is adjacent to v in G, a contradiction. Thus, v is adjacent to at most two of a, b, and c in G.

Again consider $\overline{G}$, and recall that each of $\{u, a', b'\}$, $\{u, a', c'\}$, and $\{u, b', c'\}$ is a $\gamma_t(\overline{G})$-set. Thus, there exist vertices x_1, x_2, and x_3, such that $N_{\overline{G}}(x_1) = \{u, a', b'\}$, $N_{\overline{G}}(x_2) = \{u, a', c'\}$, and $N_{\overline{G}}(x_3) = \{u, b', c'\}$. Note that $x_i \notin \{a, b, c\}$ for $1 \leq i \leq 3$, since $N_{\overline{G}}(u) \cap \{a, b, c\} = \emptyset$. If $v \notin \{a', b', c'\}$, then in $\overline{G}$, v is not adjacent to x_i, for $1 \leq i \leq 3$, and v is not adjacent to at least one of a, b, and c in $\overline{G}$. But this implies that $deg_G(v) \geq 4$, contradicting that $deg_G(v) = 3$.

Hence, we may assume that $v \in \{a', b', c'\}$. Note that this means that besides u, the only vertices that can possibly have degree 3 in G are a', b', or c'. Clearly, $X = \{x_2, a, c\}$ is a TD-set of G. By Fact 1, there exists a vertex x such that $N_G(x) = X$. Since $x_2 \notin \{a, b, c\}$, it follows that $x \neq u$, implying that $x \in \{a', b', c'\}$. But no vertex of $\{a', b', c'\}$ is adjacent to both a and c, a contradiction.

Thus, u is the only vertex of degree 3 in G, and Fact 1 implies that S is the unique $\gamma_t(G)$-set. But again, each of $\{u, a', b'\}$, $\{u, a', c'\}$, and $\{u, b', c'\}$ is a $\gamma_t(\overline{G})$-set, contradicting that G and $\overline{G}$ are self-complementary.

Hence, G is a TDTD-graph. $\square$

5 CONCLUDING REMARKS

We have listed characterizations for 24 out of the 27 types of bipartitions, for which 21 are new additions to the literature. It was also shown that two out of the remaining three bipartitions are equivalent problems when considering graphs with no isolated vertices, which leaves only two types of bipartitions with no characterization for general graphs. For these two types, we provided characterizations of special families. Characterizations of graphs having Type 5 and Type 17 bipartitions for general graphs remain open problems.

In this thesis we extended the study of bipartitions of graphs, initiated by Heggernes and Telle, whereby the three degree conditions were 0 neighbors, ≥ 1 neighbor, and X (no number of neighbors specified). It would be equally interesting to study bipartitions in which the number of neighbors is required either to be $= 1$, which results in either *efficient* or *perfect* dominating sets, or ≤ 1, which results in what are called *nearly perfect* sets.

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