8-2019

Roman Domination Cover Rubbling

Nicholas Carney
East Tennessee State University

Roman Domination Cover Rubbling

A thesis

presented to

the faculty of the Department of Mathematics

East Tennessee State University

In partial fulfillment

of the requirements for the degree

Master of Science in Mathematical Sciences

---

by

Nicholas Carney

August 2019

Rodney Keaton, Ph.D., Chair

Robert A. Beeler, Ph.D.

Teresa Haynes, Ph.D.

Keywords: graph theory, Roman domination, rubbling

ABSTRACT

Roman Domination Cover Rubbling

by

Nicholas Carney

In this thesis, we introduce Roman domination cover rubbling as an extension of domination cover rubbling. We define a parameter on a graph G called the *Roman domination cover rubbling number*, denoted $\rho_R(G)$, as the smallest number of pebbles, so that from any initial configuration of those pebbles on G, it is possible to obtain a configuration which is Roman dominating after some sequence of pebbling and rubbling moves. We begin by characterizing graphs G having small $\rho_R(G)$ value. Among other things, we also obtain the Roman domination cover rubbling number for paths and give an upper bound for the Roman domination cover rubbling number of a tree.

ACKNOWLEDGMENTS

I would like to thank Dr. Haynes for her great sequence on graph theory which has served as great preparation for my thesis. I would like to thank Dr. Beeler for his involvement in our meetings. I would like to thank Dr. Keaton for his help in revising my manuscript, his time during our weekly meetings, and his candid responses to the reality of research mathematics. I would like to thank both my mother and brother for being understanding during the preparation of my manuscript and throughout the research process.

TABLE OF CONTENTS

LIST OF FIGURES

1 INTRODUCTION

In 1989, Lagarias and Saks posed a question to Fan Chung where they first define a pebbling move. They asked whether there exists a sequence of pebbling moves which can reach any vertex in an n-dimensional cube for each initial configuration of 2^n pebbles. Chung solved this problem in [6]. Hurlbert provides a survey of graph pebbling in addition to two new probabilistic results for graph pebbling in [11]. Asplund, Hurlbert, and Kenter consider graph pebbling on binary graph constructs in [1]. Bunde et. al give a linear time algorithm for solvability of pebbling distributions on trees in [4].

In 2009, Belford and Sieben introduce graph rubbling in [3]. Katona and Papp extend this concept to optimal rubbling in [12]. Katona and Sieben give bounds for rubbling and optimal rubbling in [13].

Crull et al. introduce the concept of cover pebbling in [8]. Gardner et al. extend this concept to what they domination cover pebbling in [9]. Lourdusamy and Mathivanan consider the domination cover pebbling number for graphs which are squares of paths in [14]. Watson and Yerger provide results for the domination cover pebbling number of more general graphs in [15].

Beeler, Haynes, and Keaton introduce domination cover rubbling in [2]. Among other things, they give characterizations of graphs with small domination cover rubbling number, a formula for the domination cover rubbling number for cycles, and bounds on the domination cover rubbling number for trees.

Roman domination was first explored by Cockayne, et al. in [7]. Henning and Hedetniemi further explore this topic in [10]. As of February 28, 2019, there are 270 papers on MathSciNet on the topic of pebbling and its variants. As of February 28, 2019, there are four papers on MathSciNet on the topic of rubbling and its variants. The oldest of these papers was published in 2009.

2 DEFINITIONS

In this section, we define notation and concepts which we use throughout the thesis. Furthermore, we include an exposition which attempts to familiarize the reader with the topics of this paper and put forth some useful examples and counter-examples in order to help inform the reader's intuition. Unless otherwise specified, all definitions come from Chartrand, Lesniak, and Zhang in [5].

A *graph* G is a collection of vertices V with a collection of edges E between these vertices, written as $G = (V, E)$. The set of vertices for a graph G will be denoted $V(G)$. The set of edges for a graph G will be denoted $E(G)$. An edge between two vertices $x, y \in V(G)$ is denoted by the concatenation $xy \in E(G)$. The graphs considered in this paper are known as *simple graphs*, i.e. they do not allow for multiple edges from one vertex to another or edges from one vertex to itself. Furthermore, simple graphs do not impose a direction on edges. Thus, if $xy \in E(G)$, we have that $yx \in E(G)$ and $xy = yx$. If $xy \in E(G)$, then we say that vertices x and y are *adjacent*, the edge xy is *incident to* both x and y, and both vertices x and y are incident to xy. The *open neighbourhood* of a vertex v is $N(v) = \{u \in V | uv \in E(G)\}$. The *degree* of a vertex v is the cardinality of its open neighborhood, written $\deg(v) = |N(v)|$. The *closed neighborhood* of a vertex v is $N[v] = \{u \in V | uv \in E(G)\} \cup \{v\} = N(v) \cup \{v\}$. A vertex $x \in V(G)$ is a *universal vertex* if and only if, for every $y \in V(G) \setminus \{x\}$, $xy \in E(G)$. A vertex x is a *leaf vertex* or simply a *leaf* if $|N(x)| = 1$.

Some graphs of particular interest in this paper are paths, complete graphs, complete k-partite graphs, and tree graphs. A *path on n vertices*, denoted P_n, is a collection of n vertices, say $V(P_n) = \{x_1, x_2, \ldots, x_n\}$, such that for $i \in \{1, 2, \ldots, n-1\}$,

$x_ix_{i+1} \in E(P_n)$. A *complete graph on n vertices*, denoted K_n, is a collection of n vertices such that, for all $x, y \in V(K_n)$, $x \neq y$, we have that $xy \in E(K_n)$. A *k-partite graph* G is a graph for which $V(G)$ can be partitioned into k subsets $V_1, V_2, \ldots, V_k$ such that every edge of G joins vertices in two different partite sets. A *complete k-partite graph* G is a k-partite graph with the property that two vetices are adjacent in G if and only if the vertices belong to different partite sets. If $|V_i| = n_i$ for $1 \leq i \leq k$, then G is denoted by $K_{n_1,n_2,\ldots,n_k}$. A *cycle graph on n vertices*, also known as an *n-cycle* or simply a *cycle*, denoted C_n, is a graph such that $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(C_n) = \{v_1v_n\} \cup \left(\cup_{i=1}^{n-1}\{v_iv_{i+1}\}\right)$. A graph H is a subgraph of a graph G if both $V(H) \subset V(G)$ and $E(H) \subset E(G)$. For a nonempty subset S of $V(G)$ where G is a graph, the *subgraph $G[S]$ of G induced by set S* has S as its vertex set and two vertices u and v are adjacent in $G[S]$ if and only if u and v are adjacent in G. A subgraph H of a graph G is called an *induced subgraph* if there is a nonempty subset S of $V(G)$ such that $H = G[S]$. A *tree graph* or simply a *tree* is a connected graph which does not contain a cycle as a subgraph. A *double star graph* is a tree graph containing exactly two vertices which are not leaves and we denote double stars by $S_{a,b}$ where $a, b \in \mathbb{N}$ and a and b represent the number of leaves which are adjacent to each of the non-leaf vertices, following the convention used in [2].

One particular process which produces a graph given two other graphs is one called the *Cartesian product* of two graphs. Let G_1 and G_2 be two graphs. The *Cartesian product* G of G_1 and G_2, denoted in this paper as $G = G_1 \Box G_2$, has vertex set $V(G) = V(G_1) \times V(G_2)$, where two distinct vertices (u, v) and (x, y) of $G_1 \Box G_2$ are adjacent if and only if either $u = x$ and $vy \in E(G_2)$ or $v = y$ and $ux \in E(G_1)$.

A vertex v in a graph G is said to *dominate* itself and each of its nehbors, that is, v dominates the vertices in its closed neighborhood $N[v]$. A set S of vertices of G is a *dominating set* of G if every vertex of G is dominatd by at least one vertex of S. The minimum cardinality among all dominating sets of G is called the *domination number* of G and is denoted by $\gamma(G)$.

Cockayne, et al. in [7] define a *Roman dominating function* on a graph $G = (V, E)$ as a function $f : V \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$.

The value of a Roman dominating function f at a vertex x will be called the *Roman weight at vertex x*. We typically say that a vertex with Roman weight 0 is *Roman dominated* (or *dominated* when the context is clear as it will be in this paper) by a vertex with Roman weight 2 if it is adjacent to a vertex with Roman weight 2 under a Roman dominating function. Similarly, we say that vertices with Roman weight 2 and Roman weight 1 dominate themselves. We define the *Roman weight* of a Roman dominating function to be $f(V) = \sum_{u \in V} f(u)$. We define the *Roman domination number*, denoted $\gamma_R(G)$, of G to be the minimum Roman weight of any Roman dominating function on graph G. We define a γ_R-function on a graph G to be any Roman dominating function attaining a Roman weight of $\gamma_R(G)$.

Two graphs G and H are *isomorphic* if there exists a bijective function $\phi : V(G) \to V(H)$ such that two vertices u and v are adjacent in G if and only if $\phi(u)$ and $\phi(v)$ are adjacent in H. The function ϕ is called an *isomorphism* from G to H. If G and H are isomorphic, we write $G \cong H$. An *automorphism* of a graph G is an isomorphism from G to itself. Our use of automorphisms will be restricted to determining the role a vertex plays in the graph.

A configuration of pebbles on a graph G is a function $h : V(G) \to \mathbb{N}$. It is helpful to note that Roman dominating functions are a subclass of configurations of pebbles. In discussion, we will use the term *pebble weight of a vertex* to mean the value assigned to a particular vertex under a configuration of pebbles. We define the term *pebble weight of a configuration* or *pebble weight of a function* to mean the sum of all pebble weights of vertices in the graph under the configuration in question. We will sometimes describe these as simply *pebble weight* and its use will be clear from context. Of interest in this paper are two functions, which we will call *pebbling moves* and *rubbling moves*. A pebbling move, denoted by $p(x \to y)$, takes a configuration of pebbles h_1 on G to a new configuration of pebbles h_2 on G such that $h_2(x) = h_1(x)-2$, $h_2(y) = h_1(y) + 1$, and $h_2(v) = h_1(v)$ where $v \in V(G) \setminus \{x, y\}$ and $x, y \in V(G)$ are both distinct and adjacent. Note that this definition is consistent with [6], [9], and the wider mathematical literature on pebbling. A rubbling move, denoted $r(x, y \to z)$, takes a configuration of pebbles h_1 on G to a new configuration of pebbles h_2 on G such that $h_2(x) = h_1(x)-1$, $h_2(y) = h_1(y)-1$, $h_2(z) = h_1(z)+1$, and $h_2(v) = h_1(v)$ where $x, y \in V(G)$ are both distinct and adjacent, $z \in N[x] \cap N[y]$, and $v \in V(G) \setminus \{x, y, z\}$. It is helpful to note that an application of a single rubbling or pebbling move always results in a configuration with pebble weight one less than the initial configuration. We define the *truncation* $\overline{f}$ of a configuration f of pebbles on a graph G to be the function $\overline{f} : V(G) \to \{0, 1, 2\}$ where $\overline{f}(x) = 2$ for all $x \in V(G)$ with $f(x) \geq 2$ and $\overline{f}(z) = f(z)$ for all other $z \in V(G)$. We will say that a configuration f of pebbles on a graph G is Roman dominating on a graph G if its truncation $\overline{f}$ is a Roman dominating function on G.

We define a parameter on a graph G called the *Roman domination cover rubbling number*, denoted $\rho_R(G)$, as the smallest number of pebbles, so that from any initial configuration of those pebbles on G, it is possible to obtain a configuration which is Roman dominating after some sequence of pebbling and rubbling moves. It is important to note that, if the graph G is not connected, $\rho_R(G)$ is left undefined. Why this is so can be seen by considering an initial configuration of pebbles on G which places all pebbles on the same component. Since there exists no path from vertices x to y if x and y are in different components, there does not exist a sequence of pebbling and rubbling moves which will result in a configuration of pebbles on G which is Roman dominating on G. A reasonable value for $\rho_R(G)$ in such a case is ∞. However, if we define $\rho_R(G)$ in such a way, we must then consider extended real-valued configurations and functions. For this reason, we will consider only connected graphs in the remainder of this paper.

We will say that two Roman dominating functions or two configurations of pebbles f and g on the same graph G are *the same up to isomorphism* if and only if there exists a graph automorphism $\phi : V(G) \to V(G)$ such that $f(v) = (g \circ \phi)(v)$ for all $v \in V(G)$. If no such graph automorphism exists, then we say that f and g are *nonisomorphic.*

Figure 1 depicts a Roman dominating function for a spider graph with six legs. Note that this is just one of many Roman dominating functions on the same graph. Given a Roman dominating function it is quite easy to find others. Simply adding Roman weight to vertices will yield another Roman dominating function for the graph.

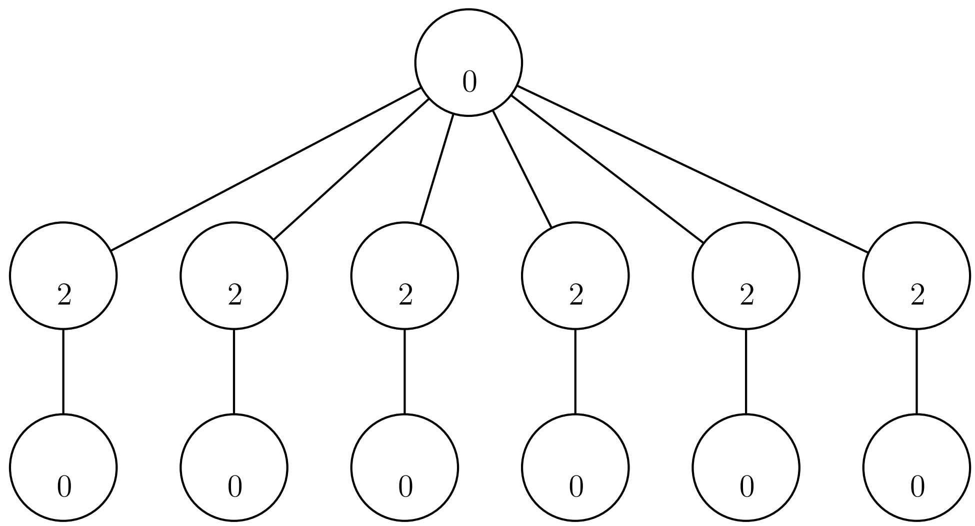

Figure 1: A Roman dominating function for a spider graph

Of course, such a process will necessarily never produce a Roman dominating function of minimum Roman weight for the graph. It is clear from the definition of Roman dominating function that assigning a Roman weight of one or two to every vertex in a graph is necessarily a Roman dominating function for the graph.

While finding Roman dominating functions is not too much trouble, finding Roman dominating functions with minimum Roman weight is quite a difficult problem. Figure 2 depicts a Roman dominating function of minimum Roman weight for a spider graph. For an overview of fundamental properties of Roman dominating functions, Roman dominating sets, Roman domination numbers, and how these quantities relate to dominating functions, dominating sets, and domination numbers, see [7]. As previously described, we can add Roman weight to this Roman dominating function in any fashion we like and what results will be a Roman dominating function. As can be seen from this example, there are graphs which have only one Roman dominating

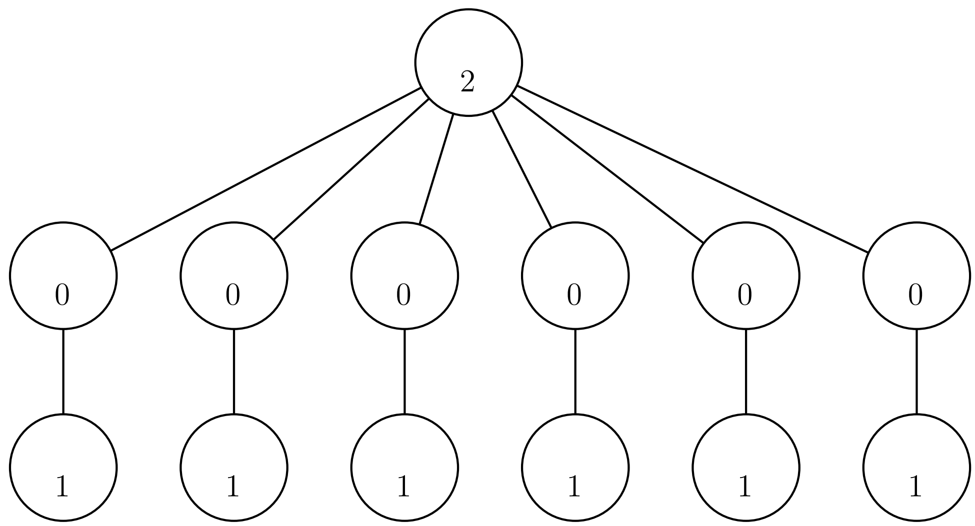

Figure 2: A γ_R-function for a spider graph

function of minimum weight up to isomorphism. However, there are graphs which have at least two nonisomorphic Roman dominating functions of minimum Roman weight. Furthermore, for certain families of graphs, the γ_R-functions are generalizable to all graphs in the family. For instance, consider the family of star graphs $K_{1,n}$ where $n \in \mathbb{N}$. Call the universal vertex x. Then a function which assigns a Roman weight of 2 to x and a Roman weight of 0 to all other vertices is a Roman dominating function for all $n \in \mathbb{N}$. Note that for $n = 1$, we also have that the function which assigns a Roman weight of one to both vertices is a γ_R-function and it is nonisomorphic to the aforementioned γ_R-function.

It is not always clear what Roman dominating function one should pebble or rubble to from an initial configuration of pebbles. At first glance, we may think that we always pebble or rubble to a γ_R-function. This is most definitely not the case. Consider the result of a sequence of pebbling and rubbling moves which began with

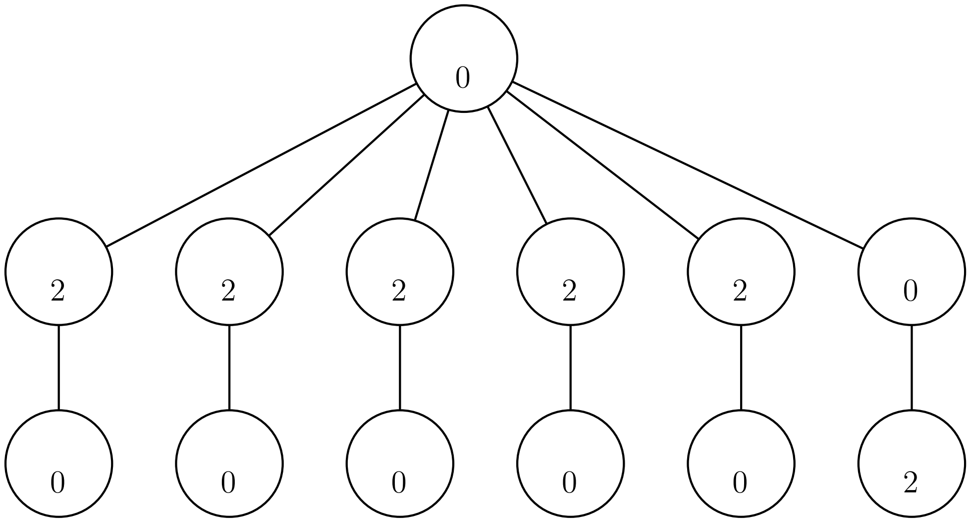

Figure 3: An end Roman dominating configuration for a spider graph

a configuration of 82 pebbles on a leaf vertex of a spider graph G shown in Figure 3. This end configuration is not a γ_R-function as $\gamma_R(G) = 8$, but it is indeed a Roman dominating function.

3 CHARACTERIZATIONS OF CERTAIN ROMAN DOMINATION COVER RUBBLING NUMBERS

In this section, we provide characterizations of graphs G with $1 \leq \rho_R(G) \leq 5$. For $\rho_R(G) = 5$ in particular, we show the nonexistence of such a graph. We also show that the Roman domination number does not imply a particular Roman domination cover rubbling number in general.

Theorem 3.1. *A graph G has $\rho_R(G) = 1$ if and only if $|V(G)| = 1$.*

Proof. ($\Longrightarrow$) Since under Roman domination vertices with Roman weight one may only Roman dominate themselves, we have $\rho_R(G) = 1$ implies $|V(G)| = 1$.

($\Longleftarrow$) A Roman dominating function of a graph with one vertex is the Roman dominating function which assigns a Roman weight of one to the only vertex. Consider an initial configuration of zero pebbles on G. It is clear that there does not exist a sequence of pebbling and rubbling moves which will result in a Roman dominating function. Note that there is only one possible initial configuration of one pebble on G and such a configuration immediately yields a Roman dominating function. Hence $\rho_R(G) = 1$. □

Theorem 3.2. *A graph G has $\rho_R(G) = 2$ if and only if $G \cong P_2$.*

Proof. ($\Longrightarrow$) By Theorem 3.1, $|V(G)| > 1$ since $\rho_R(G) \neq 1$. By definition of $\rho_R(G)$ we may place these two pebbles on different vertices say x and y and still rubble into a Roman dominating set of G.

Observe that if x and y are not adjacent, then the configuration $f(x) = 1$, $f(y) = 1$, and $f(z) = 0$ for all $z \in V(G) \setminus \{x, y\}$ is not Roman dominating. However, any pebbling or rubbling move from this configuration will result in a configuration, say g, having pebble weight one. It is clear that configuration g and all others derived from it cannot be Roman dominating. Thus x and y must be adjacent. Now suppose that there is a vertex z which is distinct from x and y. Since any configuration which results from f after a sequence of pebbling and rubbling moves will have pebble weight one, there exists no such configuration which is Roman dominating. Thus $|V(G)| = 2$. Therefore G is isomorphic to P_2.

($\Longleftarrow$) Let $x \neq y \in V(G)$. By Theorem 3.1, $\rho_R(G) > 1$. Placing two pebbles on x or y Roman dominates the graph. Placing one pebble on x and one pebble on y Roman dominates the graph. Thus $\rho_R(G) \leq 2$. Therefore $\rho_R(G) = 2$. □

Theorem 3.3. *A graph G has $\rho_R(G) = 3$ if and only if G is the complete graph K_n with $n \geq 3$.*

Proof. ($\Longrightarrow$) Theorem 3.1 and Theorem 3.2 combine to give that $|V(G)| \geq 3$. If we place all three pebbles on $x \in V(G)$, then we may only perform a pebbling move $p(x \to y)$ for $y \in N(x)$ which results in only two pebbles on G each of which are on distinct vertices. Since $|V(G)| \geq 3$, this cannot be a Roman dominating function. Thus x is a universal vertex. Therefore, G is the complete graph K_n with $n = |V(G)| \geq 3$.

($\Longleftarrow$) By Theorem 3.1 and Theorem 3.2, $\rho_R(G) > 2$. Placing two or more pebbles on the same vertex automatically yields a Roman dominating function since every vertex in a complete graph is a universal vertex. Consider distinct vertices $x, y, z \in V(G)$. Suppose an initial configuration of one pebble on each of these vertices is not a Roman dominating function for G. Then the sequence $r(x, y \to z)$

will yield a Roman dominating function as z will have two pebbles on it and z is universal. Thus $\rho_R(G) \leq 3$. Hence $\rho_R(G) = 3$. $\square$

Theorem 3.4. *A graph G has $\rho_R(G) = 4$ if and only if G has a universal vertex and $G \ncong K_n$.*

Proof. ($\Longrightarrow$) Note that $G \ncong K_n$ by Theorem 3.3 since $\rho_R(G) = 4$. Consider an initial configuration of all four pebbles on the same vertex, say $x \in V(G)$. If this is a Roman dominating function for G, then G has a universal vertex. Suppose this is not a Roman dominating function for G. Since G is connected, there exists $y \in V(G) \setminus \{x\}$ such that $xy \in E(G)$. Then the sequence $p(x \to y), p(x \to y)$ yields a configuration satisfying $f(y) = 2$ and $f(a) = 0$ for $a \in V(G) \setminus \{y\}$. Note that any further pebbling from this configuration will yield a configuration with only one pebble on G and so cannot be a Roman dominating function of G. Hence if this is a Roman dominating function for G, then G has a universal vertex y. Suppose this is not a Roman dominating function for G. Then there must be some $z \in V(G) \setminus \{x, y\}$ such that $xz \in E(G)$. Then the sequence $p(x \to y), p(x \to z)$ yields a configuration satisfying $f(y) = 1, f(z) = 1$ and $f(a) = 0$ for $a \in V(G) \setminus \{y, z\}$. However, this cannot be a Roman dominating function for G since vertices with Roman weight one can only dominate themselves under Roman domination and $|V(G)| \geq 3$. Any further rubbling from this configuration would leave only one pebble on G and so cannot be a Roman dominating function for G by the same reasoning. Therefore G must have a universal vertex and $G \ncong K_n$.

($\Longleftarrow$) Note that $\rho_R(G) > 3$ by Theorem 3.1, Theorem 3.2, and Theorem 3.3. Let w be a universal vertex of G. Note that placing two or more pebbles on w will immediately produce a Roman dominating configuration on G regardless of how the other pebbles are distributed. Consider any initial configuration f of pebbles which does not place two or more pebbles on w. If w has one pebble on it under this initial configuration, then we must have $\sum_{x\in N(w)} f(x) \geq 2$. This will allow us to pebble or rubble to w once to get two pebbles on w and this will be a Roman dominating configuration on G. If w has no pebbles on it under this initial configuration, then we must have $\sum_{x\in N(w)} f(x) \geq 4$. Thus just four pebbles distributed in any way yields a Roman dominating configuration on G. Hence $\rho_R(G) \leq 4$. Therefore $\rho_R(G) = 4$. □

Corollary 3.5. *It is possible for two graphs having equal Roman domination numbers to have different Roman dominating cover rubbling numbers.*

Proof. Let $G = P_2$ and let $H = K_{1,n}$ where $n \geq 2$. As both G and H have a universal vertex, it is easy to establish that $\gamma_R(G) = 2$ and $\gamma_R(H) = 2$. Thus G and H have the same Roman domination number. However, $\rho_R(G) = 2$ by Theorem 3.2 and $\rho_R(H) = 4$ by Theorem 3.4. □

Theorem 3.6. *There exists no graph with $\rho_R(G) = 5$.*

Proof. Suppose not, i.e., suppose that G is a graph with $\rho_R(G) = 5$. Theorem 3.1, Theorem 3.2, Theorem 3.3, and Theorem 3.4 have characterized all graphs H such that $|V(H)| \leq 3$. Thus $|V(G)| > 3$. Theorem 3.3 and Theorem 3.4 imply that G has no universal vertex. Consider an initial configuration where all five pebbles are placed on a single vertex, say $x \in V(G)$. Since G has no universal vertex, it is clear that this initial configuration is not Roman dominating. Since $|V(G)| > 3$, there exists $y \in V(G) \setminus \{x\}$ such that $xy \in E(G)$.

Suppose there exists a $z \in V(G)$ such that $z \neq y$ and $z \neq x$ and $zx \in E(G)$. The sequence $p(x \to y), p(x \to z)$ yields a configuration which cannot be Roman dominating since $|V(G)| > 3$ and vertices with Roman weight one can only dominate themselves under Roman domination. Both the sequence $p(x \to y)$ and the sequence $p(x \to z)$ yield a configuration which cannot be Roman dominating since G has no universal vertices. Both the sequence $p(x \to y), p(x \to y)$ and the sequence $p(x \to z), p(x \to z)$ yields a configuration which cannot be Roman dominating since G has no universal vertices. Note that any further pebbling or rubbling from the sequences $p(x \to y), p(x \to y)$ and $p(x \to z), p(x \to z)$ will leave two or fewer pebbles on G and thus yield a configuration which cannot be Roman dominating as G has no universal vertex. Therefore there exists no sequence of pebbling and rubbling beginning with all five pebbles on a single vertex which yields a Roman dominating configuration of pebbles on G. Hence $\rho_R(G) > 5$. This is a contradiction. Therefore $\rho_R(G) \neq 5$. $\square$

4 STACKING

In this section, we discuss the issue of stacking on graphs. By definition of the Roman domination cover rubbling number, we must consider all possible configurations of pebbles on a graph. A family of such configurations of pebbles which we must consider is the collection of configurations of pebbles which place all pebbles on a single vertex. Configurations in this family are what we will refer to as stacking configurations. The stacking problem refers to the concept that one must only consider stacking configurations to determine the graph parameter in question, i.e., all configurations which are not stacking configurations will require no more pebbles than a stacking configuration.

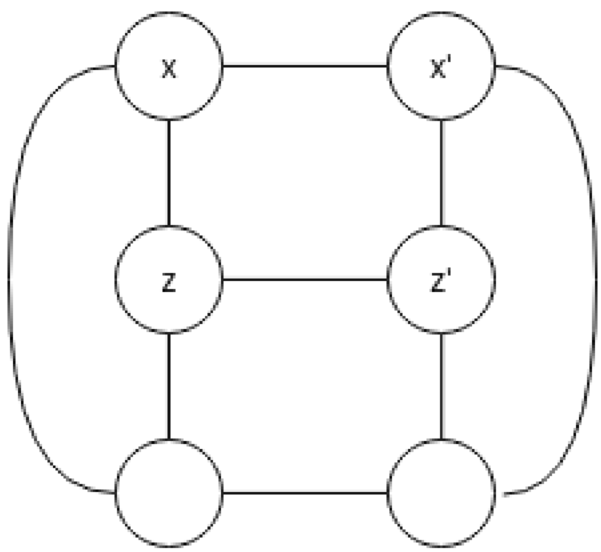

Figure 4: The prism graph $K_3 \square P_2$ requires more than stacking in determining its Roman domination cover rubbling number

It is conjectured in [2] that for tree graphs it is sufficient to only consider initial configurations which stack all pebbles on a single leaf vertex. No counterexample to an analogous formulation of this conjecture for Roman domination cover rubbling has been found. One may initially think that stacking all pebbles on a single vertex will always require the most pebbles in order to reach a Roman dominating function. Such a result is proven in the context of cover pebbling in [9] and is conjectured to hold for cover rubbling in [3]. Stacking is conjectured in the setting of domination cover rubbling on trees and is shown not to hold for general graphs in [2].

Here we show it does not hold in our setting either. Consider a prism graph of the form $K_n \square P_2$ where $n \geq 3$. Choose any vertex and call it x. Call the vertex to which vertex x is adjacent via the edge produced by the P_2 vertex x'. Stacking all pebbles on vertex x will require six pebbles placed on it so that the sequence $p(x \to x'), p(x \to x')$ yields a configuration f on $K_n \square P_2$ such that $f(x) = 2, f(x') = 2$, and $f(v) = 0$ for all other vertices v. As each of the K_n subgraphs of $K_n \square P_2$ contain a vertex which has been assigned two or more pebbles, this is a Roman dominating function. Now, let z be a vertex which is distinct from both x and x' and $xz \in E(K_n \square P_2)$. Define the vertex z' analogously to x'. Consider an initial configuration on $K_n \square P_2$ where three pebbles are placed on x and three pebbles on z. Since $n \geq 3$, $|V(K_n \square P_2)| \geq 6$. Any pebbling or rubbling move will remove one pebble. Thus any sequence of pebbling or rubbling moves from this initial configuration will leave only five or fewer pebbles on the graph. Thus the only Roman dominating function we can hope to achieve is one in which a vertex in each of the K_n subgraphs attains two or more pebbles. Now the sequence $p(x \to x')$ leaves only one pebble on the K_n subgraph containing x'. Thus this is not a Roman dominating function. Furthermore, x now has only one pebble on it. The sequence $p(x \to x'), p(z \to z')$ yields a configuration such that $f(x) = 1, f(z) = 1, f(x') = 1, f(z') = 1$, and $f(v) = 0$ for all other vertices. This is not a Roman dominating function. Any further rubbling or pebbling will leave three or fewer pebbles on $K_n \square P_2$. These cannot be a Roman dominating function. Therefore there is a configuration which is not stacking which requires more pebbles to reach a Roman dominating function than a configuration which is stacking. Therefore it is insufficient to only consider initial configurations which stack all pebbles on a single vertex when determining the parameter $\rho_R(G)$ for arbitrary graphs.

5 PATHS

In this section, we begin by stating a lemma analogous to Lemma 5 in [2] which says that the most number of pebbles exhausted in order to reach a particular configuration after a sequence of pebbling and rubbling moves on paths and cycles is given by utilizing only pebbling moves. We then prove two technical lemmas which are used in the proof of the closed formula for the Roman domination cover rubbling number for paths. The closed formula for paths improves upon the result given by Gardner, et al. in [9] by eliminating an adjustment parameter.

Lemma 5.1. *Let G be either a path or a cycle. Suppose that a Roman dominating configuration is reachable via a sequence of pebbling and rubbling moves from some initial configuration of pebbles on G. Then a Roman dominating configuration is reachable from this same initial configuration using only pebbling moves.*

Proof. The proof is exactly the same as given in [2]. $\square$

Lemma 5.2. *For all $n \in \mathbb{N}$,*

$$(n \mod 3) + \sum_{k=0}^{\lfloor \frac{n}{3} \rfloor - 1} 2^{n-1-3k} = \sum_{k=0}^{\lfloor \frac{n-1}{3} \rfloor} 2^{n-1-3k}$$

Proof. If $n \equiv 0 \mod 3$, then $n = 3x$ for some $x \in \mathbb{N}$. Thus

$$\begin{aligned}
(n \mod 3) + \sum_{k=0}^{\lfloor \frac{n}{3} \rfloor - 1} 2^{n-1-3k} &= 0 + \sum_{k=0}^{\lfloor \frac{3x}{3} \rfloor - 1} 2^{3x-1-3k} \\
&= \sum_{k=0}^{x-1} 2^{3x-1-3k} \\
&= \sum_{k=0}^{\lfloor \frac{3x-1}{3} \rfloor} 2^{3x-1-3k} \\
&= \sum_{k=0}^{\lfloor \frac{n-1}{3} \rfloor} 2^{n-1-3k}.
\end{aligned}$$

If $n \equiv 1 \mod 3$, then $n = 3x + 1$ for some $x \in \mathbb{N}$. Thus

$$\begin{aligned}
(n \mod 3) + \sum_{k=0}^{\lfloor \frac{n}{3} \rfloor - 1} 2^{n-1-3k} &= 1 + \sum_{k=0}^{x-1} 2^{3(x-k)} \\
&= 2^0 + \sum_{k=0}^{x-1} 2^{3(x-k)} \\
&= \sum_{k=0}^{x} 2^{3(x-k)} \\
&= \sum_{k=0}^{\lfloor \frac{n-1}{3} \rfloor} 2^{n-1-3k}.
\end{aligned}$$

If $n \equiv 2 \mod 3$, then $n = 3x + 2$ for some $x \in \mathbb{N}$. Thus

$$\begin{aligned}
(n \mod 3) + \sum_{k=0}^{\lfloor \frac{n}{3} \rfloor - 1} 2^{n-1-3k} &= 2 + \sum_{k=0}^{\lfloor \frac{3x+2}{3} \rfloor - 1} 2^{(3x+2)-1-3k} \\
&= 2 + \sum_{k=0}^{x-1} 2^{3x+1-3k} \\
&= 2^1 + \sum_{k=0}^{x-1} 2^{3x+1-3k} \\
&= \sum_{k=0}^{x} 2^{3x+1-3k} \\
&= \sum_{k=0}^{\lfloor \frac{3x}{3} \rfloor} 2^{3x+1-3k} \\
&= \sum_{k=0}^{\lfloor \frac{(3x+2)-1}{3} \rfloor} 2^{(3x+2)-1-3k} \\
&= \sum_{k=0}^{\lfloor \frac{n-1}{3} \rfloor} 2^{n-1-3k}.
\end{aligned}$$

Thus for all $n \in \mathbb{N}$, we have that

$$(n \mod 3) + \sum_{k=0}^{\lfloor \frac{n}{3} \rfloor - 1} 2^{n-1-3k} = \sum_{k=0}^{\lfloor \frac{n-1}{3} \rfloor} 2^{n-1-3k}$$

and the result holds. $\square$

Lemma 5.3. *If* $4 < a \leq \sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} 2^{n-3k} - 2$, *then* $2^n - a + \lfloor \frac{a-4}{8} \rfloor \geq 0$.

Proof. Note that

$$
\begin{aligned}
2^n - a + \lfloor \frac{a-4}{8} \rfloor &\geq 2^n - a + \frac{a-11}{8} \\
&= 2^n - \frac{7}{8}a - \frac{11}{8} \\
&\geq 2^n - \frac{7}{8}\sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} 2^{n-3k} - \frac{11}{8} + \frac{14}{8} \\
&= 2^n - \frac{7}{8}\sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} 2^{n-3k} + \frac{3}{8} \\
&= 2^n - \frac{7}{8}\left(2^n \sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} (2^{-3})^k\right) + \frac{3}{8} \\
&= 2^n - 7 \cdot 2^{n-3} \frac{1-(2^{-3})^{\lfloor \frac{n}{3} \rfloor + 1}}{1-2^{-3}} + \frac{3}{8} \\
&= 2^n - 7 \cdot \frac{2^{n-3} - 2^n (2^{-3})^{\lfloor \frac{n}{3} \rfloor + 2}}{1-2^{-3}} + \frac{3}{8} \\
&= 2^n - 7 \cdot \frac{2^n - 2^{n+3} (2^{-3})^{\lfloor \frac{n}{3} \rfloor + 2}}{7} + \frac{3}{8} \\
&= 2^n - 2^n + 2^{n+3}(2^{-3})^{\lfloor \frac{n}{3} \rfloor + 2} + \frac{3}{8} \\
&= (2^{n+3})(2^{-3\lfloor \frac{n}{3} \rfloor - 6}) + \frac{3}{8} \\
&= 2^{n-3\lfloor \frac{n}{3} \rfloor - 3} + \frac{3}{8}.
\end{aligned}
$$

If $n \equiv 0 \mod 3$, then

$$2^{n-3\lfloor \frac{n}{3} \rfloor - 3} + \frac{3}{8} = 2^{n-3\frac{n}{3}-3} + \frac{3}{8} = 2^{n-n-3} + \frac{3}{8} = 2^{-3} + \frac{3}{8} = \frac{1}{8} + \frac{3}{8} = \frac{1}{2} \geq 0.$$

If $n \equiv 1 \mod 3$, then

$$2^{n-3\lfloor \frac{n}{3} \rfloor - 3} + \frac{3}{8} = 2^{n-3\frac{n-1}{3}-3} + \frac{3}{8} = 2^{n-(n-1)-3} + \frac{3}{8} = 2^{-2} + \frac{3}{8} = \frac{1}{4} + \frac{3}{8} = \frac{5}{8} \geq 0.$$

If $n \equiv 2 \mod 3$, then

$$2^{n-3\lfloor \frac{n}{3} \rfloor -3} + \frac{3}{8} = 2^{n-3\frac{n-2}{3}-3} + \frac{3}{8} = 2^{n-(n-2)-3} + \frac{3}{8} = 2^{-1} + \frac{3}{8} = \frac{1}{2} + \frac{3}{8} = \frac{7}{8} \geq 0.$$

Therefore

$$2^n - a + \lfloor \frac{a-4}{8} \rfloor \geq 0$$

for $n \geq 0$ and $4 < a \leq \sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} 2^{n-3k} - 2$. □

Theorem 5.4. *For $G \cong P_n$, $\rho_R(G) = \sum_{k=0}^{\lfloor \frac{n-1}{3} \rfloor} 2^{n-1-3k}$.*

Proof. Consider a path of arbitrary length n, P_n. Let the vertices of this path be labeled $V(P_n) = \{v_1, v_2, \cdots, v_n\}$ where $E(P_n) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n\}$. Hence vertices v_1 and v_n are both leaf vertices. Now consider an initial configuration which places all pebbles on a single leaf vertex, say v_1. In order for vertex v_n to be Roman dominated, we require that at least 2^{n-1} pebbles be placed initially on v_1. Note that this will allow us to pebble from v_1 to v_{n-1} leaving 2 pebbles on v_{n-1}. Under Roman domination this will also Roman dominate vertices v_{n-2} and v_{n-1}. In order for vertex v_{n-3} to be Roman dominated, we require that at least 2^{n-4}, pebbles be placed initially on v_1. Similarly, this will also Roman dominate vertices v_{n-6} and v_{n-5}. Repeating this process, we see that we must have $\sum_{k=0}^{\lfloor \frac{n}{3} \rfloor -1} 2^{n-1-3k}$ pebbles initially on v_1 to Roman dominate vertices $v_n, v_{n-1}, \ldots, v_{n-1-3\lfloor \frac{n}{3} \rfloor}$ where we take the convention that if the upper index on the sum is less than the lower index, we consider it an empty sum and assign a value of zero to it. Then if $n \equiv 0 \mod 3$, this configuration is Roman dominating. Such a configuration has at least $\sum_{k=0}^{\lfloor \frac{n}{3} \rfloor -1} 2^{n-1-3k}$ pebbles on v_1. If $n \equiv 1 \mod 3$, then we require at least one more pebble to be placed on v_1 to have a Roman dominating configuration. Such a configuration has $1 + \sum_{k=0}^{\lfloor \frac{n}{3} \rfloor -1} 2^{n-1-3k}$ pebbles on v_1.

If $n \equiv 2 \mod 3$, then we require two more pebbles to be placed on v_1 to have a Roman dominating configuration. Such a configuration has $2 + \sum_{k=0}^{\lfloor \frac{n}{3} \rfloor - 1} 2^{n-1-3k}$ pebbles on v_1. In general, stacking all pebbles on v_1 will require $(n \mod 3) + \sum_{k=0}^{\lfloor \frac{n}{3} \rfloor - 1} 2^{n-1-3k}$ pebbles. Lemma 5.2 proves that $(n \mod 3) + \sum_{k=0}^{\lfloor \frac{n}{3} \rfloor - 1} 2^{n-1-3k} = \sum_{k=0}^{\lfloor \frac{n-1}{3} \rfloor} 2^{n-1-3k}$ for all $n \in \mathbb{N}$. Thus $\rho_R(P_n) \geq \sum_{k=0}^{\lfloor \frac{n-1}{3} \rfloor} 2^{n-1-3k}$.

Now consider an arbitrary initial configuration f of $\sum_{k=0}^{\lfloor \frac{n-1}{3} \rfloor} 2^{n-1-3k}$ pebbles on P_n. Let A and B be subgraphs of P_n such that $A = P_n[v_1, v_2, v_3]$ and $B = P_n[v_4, v_5, \ldots, v_n]$, i.e., the vertex sets of A and B partition the vertex set of P_n. Note, both A and B are necessarily paths.

Observe that when $n = 1$, $\sum_{k=0}^{\lfloor \frac{n-1}{3} \rfloor} 2^{n-1-3k} = \sum_{k=0}^{\lfloor \frac{1-1}{3} \rfloor} 2^{1-1-3k} = \sum_{k=0}^{\lfloor \frac{0}{3} \rfloor} 2^{0-3k} = \sum_{k=0}^{0} 2^{-3k} = 2^0 = 1$, when $n = 2$, $\sum_{k=0}^{\lfloor \frac{n-1}{3} \rfloor} 2^{n-1-3k} = \sum_{k=0}^{\lfloor \frac{2-1}{3} \rfloor} 2^{2-1-3k} = \sum_{k=0}^{\lfloor \frac{1}{3} \rfloor} 2^{1-3k} = \sum_{k=0}^{0} 2^{1-3k} = 2^1 = 2$, and when $n = 3$, $\sum_{k=0}^{\lfloor \frac{n-1}{3} \rfloor} 2^{n-1-3k} = \sum_{k=0}^{\lfloor \frac{3-1}{3} \rfloor} 2^{3-1-3k} = \sum_{k=0}^{\lfloor \frac{2}{3} \rfloor} 2^{2-3k} = \sum_{k=0}^{0} 2^{2-3k} = 2^2 = 4$. As $|V(P_1)| = 1$, Theorem 3.1 yields that $\rho_R(P_1) = 1$. Theorem 3.2 yields that $\rho_R(P_2) = 2$. As P_3 has a universal vertex, but is not a complete graph, Theorem 3.4 yields that $\rho_R(P_3) = 4$. Thus for $n \in \{1, 2, 3\}$, the formula $\rho_R(G) = \sum_{k=0}^{\lfloor \frac{n-1}{3} \rfloor} 2^{n-1-3k}$ agrees with the previous theorems. We will use these as our base cases for induction.

We now begin the inductive argument. Suppose that $\rho_R(P_n) = \sum_{k=0}^{\lfloor \frac{n-1}{3} \rfloor} 2^{n-1-3k}$ holds for some $n \geq 4$. Let f be an initial configuration of $\sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} 2^{n-3k}$ pebbles on P_{n+1}. Define the subgraphs A and B in a manner analogous to that above. Set $a = \sum_{v \in V(A)} f(v)$.

Case 1: $a = 0$, i.e., all pebbles are initially placed on B.

By the above observation and the fact that $\rho_R(P_3) = 4$, we will exhaust at most

$4(2^{(n+1)-3}) = 2^n$ pebbles to get four pebbles on v_3 in A. These four pebbles will be enough to yield a Roman dominating configuration for A by Theorem 3.4. Then we have $\left(\sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} 2^{n-3k}\right) - 2^n = \sum_{k=1}^{\lfloor \frac{n}{3} \rfloor} 2^{n-3k}$ pebbles remaining on B. Note that $\sum_{k=1}^{\lfloor \frac{n}{3} \rfloor} 2^{n-3k} = \sum_{k=0}^{\lfloor \frac{n}{3} \rfloor - 1} 2^{n-3(k+1)} = \sum_{k=0}^{\lfloor \frac{n}{3} \rfloor - 1} 2^{n-3-3k} = \sum_{k=0}^{\lfloor \frac{(n-2)-1}{3} \rfloor} 2^{(n-2)-1-3k}$. Thus we have enough pebbles on B by the inductive hypothesis to yield a Roman dominating configuration after some sequence of rubbling and pebbling moves.

Case 2: $0 < a < \left(\sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} 2^{n-3k}\right) - 1$.

If $0 < a \leq 4$, then at most $(4-a)2^{n-2}$ pebbles must be exhausted from B to get four pebbles on A so that A can be Roman dominated. Hence there are $\left(\sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} 2^{n-3k}\right) - a - (4-a)2^{n-2}$ pebbles on B which are not used to Roman dominate A and thus are free to be used to Roman dominate B. By the inductive hypothesis, B will require that $\sum_{k=0}^{\lfloor \frac{n}{3} \rfloor - 1} 2^{n-3(k+1)}$ pebbles be distributed on it. Note that

$$\begin{aligned}
&\left(\sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} 2^{n-3k}\right) - a - (4-a)2^{n-2} - \sum_{k=0}^{\lfloor \frac{n}{3} \rfloor - 1} 2^{n-3(k+1)} \\
&= \left(\sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} 2^{n-3k}\right) - a - (4-a)2^{n-2} - \sum_{k=1}^{\lfloor \frac{n}{3} \rfloor} 2^{n-3k} \\
&= 2^n - a - (4-a)2^{n-2} \\
&= a(2^{n-2} - 1).
\end{aligned}$$

As a is nonnegative and $2^{n-2} - 1 \geq 0$ for $n \geq 2$, $a(2^{n-2} - 1) \geq 0$ for $n \geq 2$. Thus there are enough pebbles on P_{n+1} to reach a Roman dominating truncation of P_{n+1}.

If $4 < a < \left(\sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} 2^{n-3k}\right) - 1$, then at least $\lfloor (a-4)/2^3 \rfloor$ pebbles can be moved onto B from A. Note that four pebbles will remain on A. Hence we can get at least

$\left(\sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} 2^{n-3k}\right) - a + \lfloor (a-4)/2^3 \rfloor$ pebbles on B. Note that

$$\left(\sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} 2^{n-3k}\right) - a + \lfloor (a-4)/2^3 \rfloor - \left(\sum_{k=0}^{\lfloor \frac{n}{3} \rfloor - 1} 2^{n-3(k+1)}\right)$$
$$= 2^n - a + \lfloor (a-4)/2^3 \rfloor.$$

By Lemma 5.3, the previous quantity is nonnegative for $n \geq 4$ and $4 < a \leq \sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} 2^{n-3k} - 2$. Thus we have enough unutilized pebbles to move to B and eventually yield a Roman dominating configuration for $4 < a \leq \sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} 2^{n-3k} - 2$.

Case 3: $a = \left(\sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} 2^{n-3k}\right) - 1$.

If $G \cong P_{n+1}$ where $n+1 \geq 6$, by symmetry and relabeling where A is the subgraph of P_{n+1} induced by the vertices $\{v_{n-1}, v_n, v_{n+1}\}$ and B is the subgraph of P_{n+1} induced by the vertices $\{v_1, v_2, \ldots, v_{n-2}\}$, A now has only one pebble on it. This has already been taken care of previously in Case 2.

Observe that the relabeling argument does not apply when $n + 1 = 4, 5$, as in these cases the new A subgraph which results from the relabeling will share at least one vertex with the original A subgraph. This sharing of the vertices leaves us unable to determine how many pebbles will be on the new A subgraph. Thus we consider those paths of length four and five separately below.

If $G \cong P_4$ and $f(v_1) = 0$, then, by symmetry and relabeling where A is the subgraph of P_4 induced by the vertices $\{v_2, v_3, v_4\}$ and B is the subgraph of P_4 induced by the vertex v_1, A now has all $\sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} 2^{n-3k}$ pebbles on it and this is taken care of in Case 4. If $G \cong P_4$ and $f(v_1) = 1$, then both leaves already have one pebble each on them. Consider the P_2 subgraph of P_4 induced by the vertices $\{v_2, v_3\}$. This subgraph will have $\left(\sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} 2^{n-3k}\right) - 2 = \left(\sum_{k=0}^{\lfloor \frac{4}{3} \rfloor} 2^{4-3k}\right) - 2 = \left(\sum_{k=0}^{1} 2^{4-3k}\right) - 2 = 9 - 2 = 7$

pebbles on it. As $\rho_R(P_2) = 2$ by Theorem 3.2 and $7 \geq 2$, we have a Roman dominating configuration. If $G \cong P_4$ and $f(v_1) \geq 2$, then, by symmetry and relabeling where A is the subgraph of P_4 induced by the vertices $\{v_2, v_3, v_4\}$ and B is the subgraph of P_4 induced by the vertex v_1, A now has $\left(\sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} 2^{n-3k}\right) - f(v_1) \leq \left(\sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} 2^{n-3k}\right) - 2$ pebbles on it and this has been previously taken care of in Case 2.

If $G \cong P_5$ and $\sum_{i=1}^{2} f(v_i) = 0$, by symmetry and relabeling where A is the subgraph of P_5 induced by the vertices $\{v_3, v_4, v_5\}$ and B is the subgraph of P_5 induced by the vertices $\{v_1, v_2\}$, A now has all $\sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} 2^{n-3k}$ pebbles on it and this is taken care of in Case 4. If $G \cong P_5$ and $\sum_{i=1}^{2} f(v_i) = 1$, then execute the sequence $p(v_3 \to v_2)$. This will yield a configuration satisfying $\sum_{i=3}^{5} f(v_i) = \left(\sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} 2^{n-3k}\right) - 1 - 2 = \left(\sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} 2^{n-3k}\right) - 3 = \left(\sum_{k=0}^{\lfloor \frac{5}{3} \rfloor} 2^{5-3k}\right) - 3 = \left(\sum_{k=0}^{1} 2^{5-3k}\right) - 3 = 2^5 + 2^2 - 3 = 2^5 + 1 = 33$ and either $f(v_1) = 1, f(v_2) = 1$ or $f(v_1) = 0, f(v_2) = 2$. In either case we have that both vertices v_1 and v_2 are Roman dominated. Since the subgraph of P_5 induced by the vertices $\{v_3, v_4, v_5\}$ is isomorphic to a P_3 and thus has $\rho_R(P_3) = 4$ by Theorem 3.4 and $33 \geq 4$, the subgraph of P_5 induced by the vertices $\{v_3, v_4, v_5\}$ is also Roman dominated. Therefore we have a Roman dominating configuration. If $G \cong P_5$ and $\sum_{i=1}^{2} f(v_i) \geq 2$, by symmetry and relabeling where A is the subgraph of P_5 induced by the vertices $\{v_3, v_4, v_5\}$ and B is the subgraph of P_5 induced by the vertices $\{v_1, v_2\}$, A has at most $\left(\sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} 2^{n-3k}\right) - 1 - 2 + 1 = \left(\sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} 2^{n-3k}\right) - 2$ pebbles on it. This has already been taken care of previously in Case 2.

Case 4: $a = \sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} 2^{n-3k}$, i.e., all pebbles are initially placed on A.

Define A' to be the subgraph of P_{n+1} induced by the vertices $\{v_{n-1}, v_n, v_{n+1}\}$. Note that A' is a P_3 and thus will require at least four pebbles able to be be moved onto A' to yield a Roman dominating configuration.

If $n+1=4$, then A and A' share two vertices, namely v_2 and v_3. Furthermore, nine pebbles are initially distributed on P_{n+1}. If four or more pebbles are placed on $v_2 \cup v_3$, A' has four pebbles on it and so we have a Roman dominating configuration of P_{n+1}. Suppose there are $b \in \{0,1,2,3\}$ pebbles initially placed on A'. Then $9-b$ pebbles are placed on $v_1 \in V(A)$. Observe that if $b=0$, then the sequence $p(v_1 \to v_2), p(v_1 \to v_2), p(v_1 \to v_2), p(v_1 \to v_2), p(v_2 \to v_3), p(v_2 \to v_3)$ yields a configuration g satisfying $g(v_1)=1, g(v_2)=0, g(v_3)=2, g(v_4)=0$. Thus we have a Roman dominating configuration. If $b \neq 0$, then we need to pebble fewer times from v_1 to get four pebbles on A'. Thus it is clear that we will have a Roman dominating configuration for any value of b. Therefore if $n+1=4$, $\rho_R(P_{n+1}) \leq \sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} 2^{n-3k}$.

If $n+1=5$, then A and A' share one vertex, namely v_3. Furthermore, 18 pebbles are initially distributed on P_{n+1}. If four or more pebbles are placed on v_3, A' has four pebbles on it and so we have a Roman dominating configuration of P_{n+1}. Suppose there are $b \in \{0,1,2,3\}$ pebbles initially placed on A'. Then $18-b$ pebbles are placed on $v_1 \cup v_2$. Observe that if $b=0$, then we will require at most $4(2^2)=16$ pebbles to pebble or rubble four onto A'. In fact, we will require exactly 16 pebbles if and only if all pebbles are stacked on v_1. In such a case, two pebbles will be left on v_1 and that will be sufficient to Roman dominate $v_1 \cup v_2$, while the pebbles which were moved to A' are sufficient to yield a Roman dominating dominating configuration for A'. In any other configuration, we will have more than two pebbles on A not utilized to move four pebbles to A'. Furthermore, either v_1 or v_2 will have at least two pebbles on it which will Roman dominate v_1 and v_2. Thus from any initial configuration of 18 pebbles on P_{n+1} there exists a sequence of pebbling and rubbling moves which yields a Roman dominating configuration. If $b \neq 0$, then we need to pebble fewer times from v_1 to get four pebbles on A'. Thus it is clear that we will have a Roman dominating configuration for any value of b. Therefore if $n+1=5$, $\rho_R(P_{n+1}) \leq \sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} 2^{n-3k}$.

For $n+1 \geq 6$, A and A' share no vertices. Hence we may without loss of generality switch the roles of A and B. Thus there exists a sequence of pebbling and rubbling moves which will yield a Roman dominating configuration by Case 1.

The four cases above establish $\sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} 2^{n-3k}$ as an upper bound for $\rho_R(P_{n+1})$ for arbitrary n. As the upper and lower bounds are the same, we must have that $\rho_R(P_n) = \sum_{k=0}^{\lfloor \frac{n-1}{3} \rfloor} 2^{n-1-3k}$ for all $n \in \mathbb{N}$. $\square$

6 TREES

In this section, we give an upper bound for the Roman domination cover rubbling number for arbitrary, non-trivial trees. We then give an exact formula for the Roman domination cover rubbling number for double star graphs.

Theorem 6.1. *If T is a non-trivial tree with diam$(T) = d$ and domination number γ, then $\rho_R(T) \leq 2^d\gamma - 2^d + 4$*

Proof. The proof of Theorem 6.1 follows exactly from the proof of Theorem 10 given in [2] with the only change being that we need to move 2 pebbles to each vertex of the dominating set and thus their result is multiplied by 2. □

Theorem 6.2. *For the double star graph $S_{a,b}$ with $a, b \geq 1$,*

$$\rho_R(S_{a,b}) = \begin{cases} 9 & \text{if } a = b = 1 \\ 12 & \text{otherwise} \end{cases}.$$

Proof. Observe that if $a = b = 1$, we have that $S_{a,b} \cong S_{1,1} \cong P_4$. Hence $\rho_R(S_{a,b}) = \rho_R(S_{1,1}) = \rho_R(P_4) = 9$ by Theorem 5.4.

Now let at least one of a or b not equal 1. Suppose without loss of generality that $a \neq 1$. Let v be a vertex in $V(S_{a,b})$ such that $\deg(v) = a + 1$. Let w be a vertex in $V(S_{a,b})$ such that $\deg(w) = b + 1$ and $w \neq v$. Let x be a leaf vertex adjacent to v. Consider an initial configuration which stacks 11 pebbles on x. Observe that pebbling 1 pebble to any leaf adjacent to v will exhaust 4 pebbles. Pebbling 2 pebbles to v will exhaust 4 pebbles. Under Roman domination, vertices with Roman weight 2 dominate themselves and their open neighborhoods and vertices with Roman weight

1 only dominate themselves. Thus we will never pebble to a leaf of v. Similarly, pebbling 1 pebble to any leaf adjacent to w will exhaust 8 pebbles. Pebbling 2 pebbles to w will exhaust 8 pebbles. Thus we will never pebble to a leaf of w. Hence our target Roman dominating function g will be one satisfying $g(v) = g(w) = 2$ and $g(n) = 0$ for all $n \in V(S_{a,b}) \setminus \{v, w\}$. As established previously, pebbling 2 pebbles to both v and w will require $4+8 = 12$ pebbles. Therefore an initial configuration which stacks 11 pebbles on a leaf vertex cannot be Roman dominating. Thus $\rho_R(S_{a,b}) \geq 12$.

Observe that $S_{a,b}$ is also tree. Note $\text{diam}(S_{a,b}) = 3$ and $\gamma(S_{a.b}) = 2$. Thus by Theorem 6.1, $\rho_R(S_{a.b}) \leq 2^3(2) - 2^3 + 4 = 16 - 8 + 4 = 12$.

As the upper and lower bounds for $\rho_R(S_{a,b})$ are both equal to 12, we have that $\rho_R(S_{a,b}) = 12$.

$\square$

REFERENCES

[1] J. Asplund, G. Hurlbert, and F. Kenter. “Pebbling on graph products and other binary graph constructions”. In: *Australas. J. Combin.* 71 (2018), pp. 246–260.

[2] R. A. Beeler, T. W. Haynes, and R. Keaton. “Domination cover rubbling”. In: *Discrete Appl. Math.* 260 (2019), pp. 75–85.

[3] C. Belford and N. Sieben. “Rubbling and optimal rubbling of graphs”. In: *Discrete Math.* 309.10 (2009), pp. 3436–3446.

[4] D. P. Bunde, E. W. Chambers, D. Cranston, K. Milans, and D. B. West. “Pebbling and optimal pebbling in graphs”. eng. In: *J. Graph Theory* 57.3 (2008), pp. 215–238.

[5] G. Chartrand, L. Lesniak, and P. Zhang. *Graphs & Digraphs*. Textbooks in mathematics. CRC Press, Taylor & Francis Group, 2016.

[6] F. R. K. Chung. “Pebbling in hypercubes”. In: *SIAM J. Discrete Math.* 2.4 (1989), pp. 467–472.

[7] E. J. Cockayne, P. A. Dreyer Jr., S. M. Hedetniemi, and S. T. Hedetniemi. “Roman domination in graphs”. In: *Discrete Math.* 278.1-3 (2004), pp. 11–22.

[8] B. Crull, T. Cundiff, P. Feltman, G. Hurlbert, L. Pudwell, Z. Szaniszlo, and Z. Tuza. “The cover pebbling number of graphs”. In: *Discrete Math.* 296 (2005), pp. 15–23.

[9] J. J. Gardner, A. P. Godbole, A. M. Teguia, A. Z. Vuong, N. G. Watson, and C. R. Yerger. “Domination cover pebbling: graph families”. In: *J. Combin. Math. Combin. Comput.* 64 (2008), pp. 255–271.

[10] M. A. Henning and S. T. Hedetniemi. “Defending the Roman Empire–A new strategy”. In: *Discrete Math.* 266.1 (2003). The 18th British Combinatorial Conference, pp. 239 –251.

[11] G. H. Hurlbert. “A survey of graph pebbling”. In: *Proceedings of the Thirtieth Southeastern International Conference on Combinatorics, Graph Theory, and Computing (Boca Raton, FL, 1999)*. Vol. 139. 1999, pp. 41–64.

[12] G. Y. Katona and L. F. Papp. “The optimal rubbling number of ladders, prisms and Möbius-ladders”. In: *Discrete Appl. Math.* 209 (2016), pp. 227–246.

[13] G. Y. Katona and N. Sieben. “Bounds on the rubbling and optimal rubbling numbers of graphs”. In: *Graphs Combin.* 29.3 (2013), pp. 535–551.

[14] A. Lourdusamy and S. T. Mathivanan. “The domination cover pebbling number of the square of a path”. In: *J. Prime Res. Math.* 7 (2011), pp. 1–8.

[15] N. G. Watson and C. R. Yerger. “Domination cover pebbling: structural results”. In: *J. Combin. Math. Combin. Comput.* 72 (2010), pp. 181–196.

VITA

NICHOLAS CARNEY

Education:	M.S. Mathematical Sciences, East Tennessee State University
	Johnson City, Tennessee 2019
	B.S. Mathematics, University of Tennessee
	Knoxville, Tennessee 2015
Professional Experience:	Taught Precalculus 1 (MATH 1710)
	East Tennessee State University, Spring 2019
	Taught Precalculus 2 (MATH 1720)
	East Tennessee State University, Fall 2018
	Tutored Math Courses Including Integral Calculus,
	Linear Algebra, and Differential Equations
	East Tennessee State University, Fall 2017 - Summer 2019