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The 2-Domination Number of a Caterpillar

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The 2-Domination Number of a Caterpillar

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by

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ABSTRACT

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A set D of vertices in a graph G is a 2-dominating set of G if every vertex in $V - D$ has at least two neighbors in D. The 2-domination number of a graph G, denoted by $\gamma_2(G)$, is the minimum cardinality of a 2-dominating set of G. In this thesis, we discuss the 2-domination number of a special family of trees, called caterpillars. A caterpillar is a graph denoted by $P_k(x_1, x_2, ..., x_k)$, where x_i is the number of leaves attached to the i^{th} vertex of the path P_k. First, we present the 2-domination number of some classes of caterpillars. Second, we consider several types of complete caterpillars. Finally, we consider classification of caterpillars with respect to their spine length and 2-domination number.

DEDICATION

With so much love, I would like to dedicate this thesis to my late father, Elvis Ephraim Idu, in appreciation for his love, encouragement and support during my undergraduate and to my lovely family for their love and patience when I was out working on my thesis.

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First, I would like to give thanks to God Almighty for His wisdom, knowledge and understanding while I was writing this thesis. I never thought I would see the end of this thesis.

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1 INTRODUCTION

First we need to define some terminology and notation for the purpose of this thesis. Let G be a finite, simple, and undirected graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *order* of G, denoted $|V(G)| = n$, is the number of vertices in G. The *size* of G, denoted $|E(G)| = m$, is the number of edges in G. For any two vertices $x, y \in V(G)$, x and y are adjacent if the edge $xy \in E(G)$. The *open neighborhood* of v in V is the set $N(v) = \{u \in V : uv \in E\}$ and the *closed neighborhood* of $v \in V$ is the set $N[v] = N(v) \cup \{v\}$. The *open neighborhood* of a set $D \subseteq V$ is the set $N(D) = \cup_{v \in D} N(v)$, and the *closed neighborhood* of a set D is the set $N[D] = N(D) \cup D$. The *degree* of v is the cardinality of the open neighborhood of v, or $deg_G(v) = |N(v)|$. A vertex with exactly one neighbor is called a $leaf$ and its neighbor is a *support vertex*. A support vertex with two or more leaf neighbors is called a *strong support vertex* [2]. The independence number of G, denoted $\beta(G)$, is the cardinality of the largest independent set of vertices in G. A *path* P_k is a graph of order k and size $k - 1$ with vertices denoted $v_1, v_2, ..., v_k$ and edges $v_i v_{i+1}$ for $i = 1, 2, ..., k - 1$. A *subgraph* H of a graph G is a graph contained in G, i.e., $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$.

A *dominating set* of a graph G is a nonempty subset D of the vertex set V such that for each $u \in V - D$, there exists a $v \in D$ adjacent to u. Equivalently, a subset D of V is a dominating set if for each $v \in V$, $|N[v] \cap D| \geq 1$ [4]. A dominating set having the smallest cardinality among all dominating sets in a given graph is called a *minimum dominating set*. The cardinality of a minimum dominating set in graph G is called the *domination number* of G and is denoted $\gamma(G)$.

In [9], Fink and Jacobson introduced the concept of k-domination, of which 2-domination is a special case. It was shown in [9] that the dominating property of a minimum dominating set can be destroyed by removing at most two edges or vertices from the graph. In most cases, the removal of only one edge or vertex from a graph will leave some vertices undominated by what had been a minimum dominating set. For example, in Figure 1, a minimum dominating set D of graph G_1 is $D = \{v, c, u, d\}$. Figure 2 shows that the removal of vertex v in G_1 will leave vertices a and b undominated in G_2 by what had been a minimum dominating set in G_1.

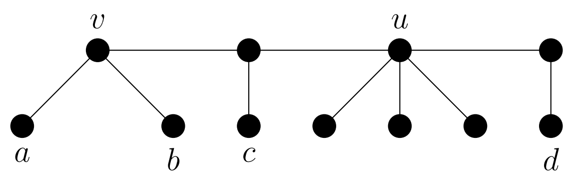


Figure 1: Graph G_1

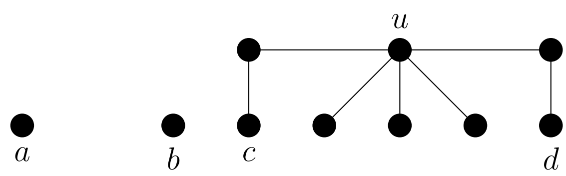


Figure 2: Graph $G_2 = G_1 - v$: removal of vertex v

As a result of this, the 2-domination number was introduced in [9] and mentioned in [3, 12]. For a graph G, if D is a subset of V and $u \in V - D$ is adjacent to at least two members of D, we say that u is 2-*dominated* by D. If every vertex in $V - D$ is 2-dominated by D, then D is called a 2-*dominating set*. Among all 2-dominating sets of the graph G, if D has the smallest cardinality then D is a *minimum* 2-*dominating set* and its cardinality is the 2-*domination number* of G, denoted $\gamma_2(G)$. An example of 2-domination is given in Figure 3. $D = \{a, b, c, d, e, f, g, h, i, j, u, t\}$ is a 2-dominating set of minimum cardinality, so $\gamma_2(G_3) = 12$.

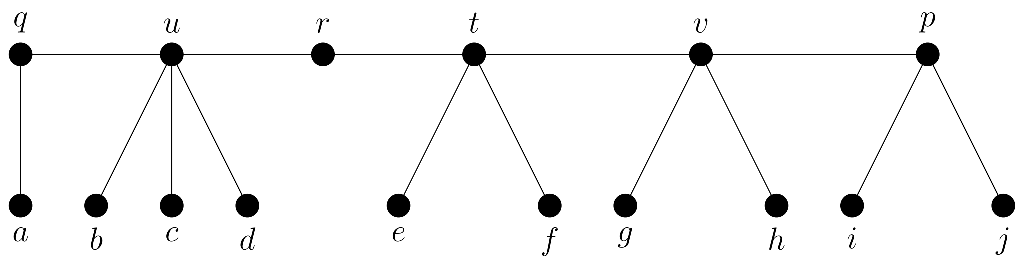


Figure 3: 2-domination of graph G_3

With this in mind, the removal of only one vertex or edge from a graph will leave the vertices still dominated.

A *caterpillar* is a tree with the property that the removal of its leaves and incident edges results in a path, which we call the *spine* of the caterpillar. We say a caterpillar is *complete* if every vertex on the spine of the caterpillar has at least one leaf.

In this thesis, we study the 2-domination of a caterpillar. In Section 2, we will give a literature survey over some related work which is relevant to this thesis. In Section 3, we will discuss 2-domination of paths and by considering several simple cases of caterpillars with the number of leaves attached to the spine of the caterpillar. We will also consider upper bounds for all complete and general caterpillars. It is actually simple to figure out the 2-domination number of some classes of caterpillars, but making a generalization is a little more difficult. Thus we close with an open problem.

2 LITERATURE SURVEY

In this section, we review some background results to this thesis. These results present the motivation and origin of 2-domination. First, we review the basic results regarding this type of domination and then proceed with a simple observation of Fink and Jacobson.

2.1 2-domination

The concept of domination in graphs was known earlier, but Ore [13] was the first to use the term domination, who noticed that for every graph G, the relation $\gamma(G) \leq \beta(G)$ holds. Since the origin of Ore's initial introduction of domination, a large amount of work has been done with dominating sets and the domination number. The reader is referred to [5, 6, 11]. In [9], 2-domination is motivated by the following theorem which says that the dominating property of a minimum dominating set can be destroyed by the removal of only one or two edges (or vertices) from the graph G.

Theorem 2.1. *[9, 11] If D is a minimum dominating set in a non-empty graph G, then at least one vertex in $V - D$ is dominated by at most two members of D.*

As mentioned earlier, given any minimum dominating set D of G, one can remove two edges from G such that D is no longer a dominating set of G. As a result of this process, in [9], a greater degree of assurance is introduced via 2-domination so that the removal of at most two edges or vertices from the graph G will still retain the dominating property.

In [9, 11], a vertex $v \in V - D$ is *k-dominated* if it is dominated by at least k vertices in D, that is $|N(v) \cap D| \geq k$. If every vertex in $V - D$ is k-dominated, then D is called a *k-dominating set*. The minimum cardinality of a k-dominating set is called the *k-domination number* of G, denoted $\gamma_k(G)$.

Proposition 2.2. *[9, 11] A k-dominating set D is minimal if and only if for every vertex $v \in D$, either, (1) $|N(v) \cap D| < k$ or (2) there exists a vertex $u \in V - D$ such that $|N(v) \cap D| = k$ and $u \in N(v)$.*

It was also noted in [9] that every k-dominating set is a dominating set and thus, for every graph G we have $\gamma(G) \leq \gamma_k(G)$ for each $k \geq 1$. Furthermore, if $1 \leq j \leq k$, then every k-dominating set in G is also a j-dominating set and hence $\gamma_j(G) \leq \gamma_k(G)$. In [9], a lower bound for γ_k involving only the number of vertices and the number of edges was obtained. This leads to an extremal result for the 2-domination number of a tree.

Theorem 2.3. *[9, 11] If G has n vertices and m edges, then $\gamma_k(G) \geq n - (m/k)$.*

The subdivision of an edge e with endpoints $\{u, v\}$ yields a graph containing one new vertex w and with an edge set replacing e by two new edges, uw and wv.

Definition 2.4. *[14] A subdivision of a graph $G = (V, E)$ is a graph where each edge is subdivided exactly once in G.*

The path P_n can be subdivided into another path P_m by subdividing each edge of P_n exactly once, that is, $subdiv(P_n) = P_m$ where $m = 2n - 1$ is odd. For example, if $n = 3$, then $m = 2(3) - 1 = 5$. Thus, $T' = P_5$ is the subdivision graph of $T = P_3$, i.e., $subdiv(P_3) = P_5$ as seen in Figure 4.

The edges uw and wv have been subdivided into four edges, ux, xw, wy, and yv.

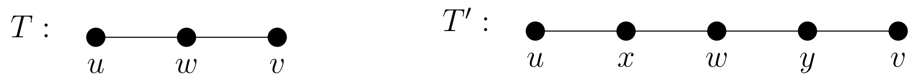


Figure 4: T' is the subdivision of T

A *tree* is an undirected graph in which any two vertices are connected by exactly one simple path. Alternatively, a tree can be defined as a connected graph without a cycle subgraph. A caterpillar is a special type of tree. Theorem 2.3 yields the following bounds on the 2-domination number of a tree T of order n.

Corollary 2.5. *[9, 11] If T is a tree with $n \geq 2$ vertices, then $\gamma_2(T) \geq \frac{n+1}{2}$.*

Corollary 2.6. *[9, 11] If T is a tree with $n \geq 2$ vertices, then $\gamma_2(T) = \frac{n+1}{2}$ if and only if T is a subdivision graph of a tree T'.*

An important theorem involving the number of vertices that are of degree one was shown in [9].

Theorem 2.7. *[7, 8] For a tree T, $\gamma_2(T) \leq \frac{n+\ell}{2}$, where ℓ refers to the number of vertices that are of degree one and n is number vertices of T.*

The vertices of degree one are included in any 2-dominating set D.

3 RESULTS

3.1 2-domination of Paths

The path graph P_n is a tree with two end vertices of degree 1, and the other $n-2$ vertices of degree 2. Examples are given in Figure 5.

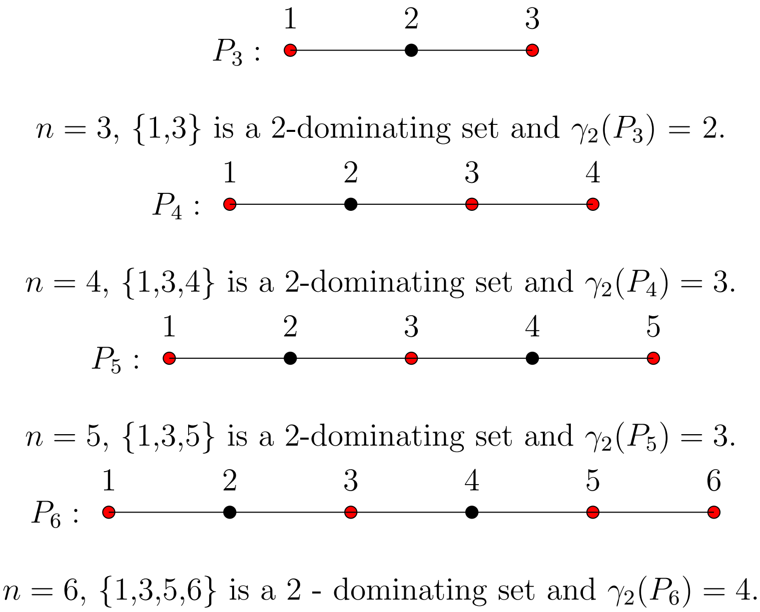


Figure 5: Examples of 2-domination of paths

From the above examples, we have a pattern for odd and even vertices for any path graph, P_n. When n is even, the 2-domination number of a path graph P_n is given by $\gamma_2(P_n) = \frac{n+2}{2}$. When n is odd, the 2-domination number of a path graph P_n is given by $\gamma_2(P_n) = \frac{n+1}{2}$. From the patterns above, we have the following theorem.

Theorem 3.1. *For a path P_n, the 2-domination number is given by* $\gamma_2(P_n) = \left\lceil \frac{n+1}{2} \right\rceil$.

Proof. Let V be the vertex set and D be a minimum 2-dominating set of P_n. For n odd, the path P_n is a subdivision graph of $P_{\frac{n+1}{2}}$. Thus, by Corollary 2.6, $\gamma_2(P_n) = \frac{n+1}{2}$.

For n even, we show that $\gamma_2(P_n) = \frac{n+2}{2}$. First, we show that $\gamma_2(P_n) \geq \frac{n+2}{2}$. By Corollary 2.5 and 2.6, $\gamma_2(P_n) > \frac{n+1}{2}$, which implies that $\gamma_2(P_n) \geq \frac{n+2}{2}$.

Next, we show that $\gamma_2(P_n) \leq \frac{n+2}{2}$. Consider a 2-dominating set of a path P_n, where $D = \{v_1, v_3, ..., v_{n-1}, v_n\}$. Every vertex $u \in V - D - \{v_n\}$ is at least 2-dominated by two vertices of D. Since $v_n \in D$, then $\gamma_2(P_n) \leq |D| = \frac{n}{2} + 1 = \frac{n+2}{2}$. Hence, combining the results, we have $\gamma_2(P_n) = \lceil \frac{n+1}{2} \rceil$. □

3.2 Caterpillar

Recall, a caterpillar is a graph which can be obtained from the path on k vertices by appending x_i pendant vertices to the the i^{th} vertex of the path, P_k. The caterpillar with parameters k, $x_1, ..., x_k$, where $x_1, x_k \neq 0$, will be denoted $P_k(x_1, x_2, ..., x_k)$ as in [1].

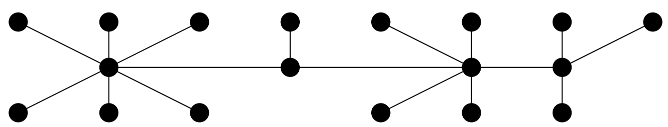


Figure 6: A caterpillar $P_4(6, 1, 4, 3)$

Note, this is a tree with the property that the removal of its leaves and incident edges results in a path P_k called the *spine* of the caterpillar. Let ℓ denote the number of leaves, i.e., $\ell = \sum_{i=1}^{k} x_i$. We say a caterpillar is complete if every vertex on the spine of the caterpillar is adjacent to at least one leaf.

3.3 Preliminary Results

Let $P_k(x_1, x_2, ..., x_k)$ be a caterpillar. We first consider the case of caterpillars where $x_i \neq 1$ for $1 \leq i \leq k$. An example is given in Figure 7.

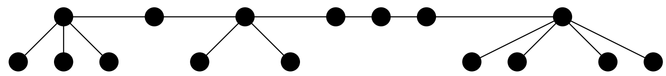


Figure 7: A caterpillar $P_7(3, 0, 2, 0, 0, 0, 4)$

Note, the 2-domination number of the above caterpillar is $\gamma_2(P_7(3, 0, 2, 0, 0, 0, 4)) = \ell + 3$. First, we dissect our caterpillar into complete caterpillars and paths of maximal length. We denote the complete caterpillars with $x_i > 1$ as C_i and the paths as P_{k_i}. From the above example in Figure 7, $C_1 = P_1(3)$, $C_2 = P_1(2)$, $C_3 = P_1(4)$, $k_1 = 1$ and $k_2 = 3$.

Proposition 3.2. *For a caterpillar, $P_{k_i}(x_1, ..., x_k)$, where $x_i \neq 1$ for $1 \leq i \leq k$, then $\gamma_2(P_{k_i}(x_1, ..., x_k)) \leq \ell + \sum_{i=1}^{r} \lceil \frac{k_i+1}{2} \rceil$.*

Proof. Let D be a minimum 2-dominating set. Let k_i be the order of the path P_{k_i}, where P_{k_i} is the set of vertices with $x_i = 0$ on the spine for $i = 1, ..., r$. Since the leaves are in D, we only need to 2-dominate each of the paths P_{k_i}. Thus, D is the union of the leaves and a 2-dominating set for each path P_{k_i}. Hence, $|D| \leq \ell + \sum_{i=1}^{r} \lceil \frac{k_i+1}{2} \rceil$. □

Thus, from Figure 7, we have

$$\begin{aligned}\gamma_2(P_7(3, 0, 2, 0, 0, 0, 4)) &= (3 + 2 + 4) + \left\lceil \frac{k_1 + 1}{2} \right\rceil + \left\lceil \frac{k_2 + 1}{2} \right\rceil \\ &= \ell + \left\lceil \frac{1 + 1}{2} \right\rceil + \left\lceil \frac{3 + 1}{2} \right\rceil \\ &= \ell + 3.\end{aligned}$$

This is an example where equality holds, but below is another example where equality fails.

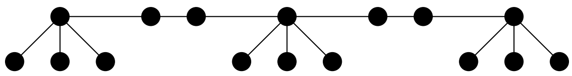


Figure 8: A caterpillar $P_7(3,0,0,3,0,0,3)$.

From Figure 8, $C_1 = P_1(3)$, $C_2 = P_1(3)$, $C_3 = P_1(3)$, $k_1 = 2$ and $k_2 = 2$. Thus,

$$\begin{aligned}\gamma_2(P_7(3,0,0,3,0,0,3)) &< (3+3+3) + \left\lceil \frac{k_1+1}{2} \right\rceil + \left\lceil \frac{k_2+1}{2} \right\rceil \\ &< \ell + \left\lceil \frac{2+1}{2} \right\rceil + \left\lceil \frac{2+1}{2} \right\rceil \\ &< \ell + 4.\end{aligned}$$

3.4 Complete Caterpillars

Recall, a caterpillar, $P_k(x_1, x_2, ..., x_k)$, is complete if the $x_i > 0$ for $1 \leq i \leq k$. Note, since each leaf has degree one, each leaf must be in a 2-dominating set. First, let us consider the case of complete caterpillars with $x_i > 1$ for $1 \leq i \leq k$. An example of a complete caterpillar with $x_i > 1$ for $1 \leq i \leq k$ is given in Figure 8.

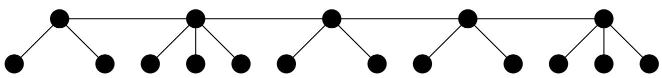


Figure 9: A complete caterpillar $P_5(2,3,2,2,3)$.

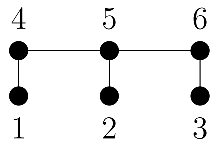
Proposition 3.3. *For a complete caterpillar, $P_k(x_1, x_2, ..., x_k)$, where $x_i > 1$ for $1 \leq i \leq k$, then $\gamma_2(P_k(x_1, x_2, ..., x_k)) = \ell$.*

Proof. Let D be a minimum 2-dominating set of $P_k(x_1, x_2, ..., x_k)$, where $x_k > 1$ for $1 \leq i \leq k$.

First, we show that $|D| \leq \ell$. Suppose D is the set of leaves. Then each vertex on the spine, say v_i, has $x_i > 1$ neighbors in D. So, D is a 2-dominating set. Thus $|D| \leq \ell$. Next we show that $|D| \geq \ell$. Suppose to the contrary that $|D| < \ell$. Then, at least one leaf $v \notin D$. That is, v has only one neighbor. So, D is not a 2-dominating set. Thus, $|D| \geq \ell$. Hence, $|D| = \ell$ and we are done. □

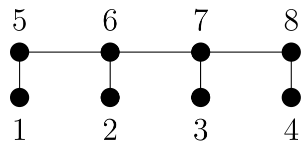
The next class of caterpillars that we consider will be complete caterpillars where $x_i = 1$ for $1 \leq i \leq k$, that is, having only one leaf attached to each vertex on the spine of the caterpillar. Examples are given in Figure 9.

$P_3(1, 1, 1)$:



$D = \{1, 2, 3, 5\}$, $\gamma_2(P_3(1, 1, 1)) = 4$.

$P_4(1, 1, 1, 1)$:



$D = \{1, 2, 3, 4, 6, 8\}$, $\gamma_2(P_4(1, 1, 1, 1)) = 6$.

Figure 10: Complete caterpillars where $x_i = 1$.

Observe that the 2-domination number of the above caterpillars is the sum of the number of leaves and the domination number of path.

We need the following lemma to prove our result.

Lemma 3.4. *[10] For $k \geq 3$, $\gamma(P_k) = \lceil \frac{k}{3} \rceil$.*

Now, we are ready to prove Proposition 3.5.

Proposition 3.5. *For a complete caterpillar,* $P_k(x_1, x_2, ..., x_k)$*, where* $x_i = 1$ *for* $1 \leq i \leq k$*, then* $\gamma_2(P_k(x_1, x_2, ..., x_k)) = \ell + \lceil \frac{k}{3} \rceil$.

Proof. Let D be a minimum 2-dominating set. Since the leaves are of degree one, each leaf must be in D. The set of leaves dominate each vertex on the path exactly once. Thus, a minimum dominating set of the path P_k unioned with the set of leaves ℓ is a minimum 2-dominating set of the caterpillar. Hence, $|D| = \ell + \lceil \frac{k}{3} \rceil$ and we are done. □

Theorem 3.6. *For a caterpillar,* $P_k(x_1, x_2, ..., x_k)$*, where* $x_i \geq 1$ *for* $1 \leq i \leq k$*, we have* $\gamma_2(P_k(x_1, x_2, ..., x_k)) \leq \ell + \gamma(P_k) = \ell + \lceil \frac{k}{3} \rceil$.

Proof. Let V be the vertex set and D be a minimum 2-dominating set of $P_k(x_1, x_2, ..., x_k)$, where $x_i \geq 1$ for all positive integers i. Since the leaves are of degree one, then each leaf must be in D. Since $x_i \geq 1$ for all i, a 1-dominating set of the spine P_k is sufficient to 2-dominate the caterpillar. □

Our next result of interest gives both a descriptive and a constructive family of caterpillars denoted $\mathcal{T}$.

Let $\mathcal{T}$ be a family of caterpillars $P_{k_i}(x_1, ..., x_k)$ satisfying the following conditions:

- $x_1 = x_k = 1$ and
- if $x_i > 1$, then $x_{i+1} = 1$, for $i = 2, ..., k-1$.

Complete caterpillars satisfying the above conditions for the family of caterpillars, $\mathcal{T}$, can be written in the form $C_1 - v_1 - C_2 - v_2 - C_3 - v_3 - ... - v_{r-1} - C_r$, where

$C_j = P_{k_j}(1, 1, ..., 1)$, v_j is the vertex on the spine with more than one leaf and $|C_j|$ is the length of P_{k_j} with $x_i = 1$ for $1 \leq i \leq k_j$. An example of a caterpillar satisfying the given conditions is given in Figure 10.

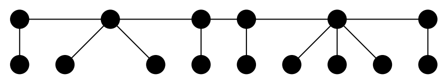


Figure 11: A complete caterpillar $P_6(1, 2, 1, 1, 3, 1)$.

In Figure 11, the complete caterpillar $P_6(1, 2, 1, 1, 3, 1)$ is of the form $C_1 - v_1 - C_2 - v_2 - C_3$ where $|C_1| = 1$, $|C_2| = 2$, $|C_3| = 1$, and $r = 3$. Note that $x_1 = x_k = 1$ and every $x_i > 1$ is followed by $x_{i+1} = 1$. The 2-domination number, $\gamma_2 = \ell + 2 = \ell + \lceil \frac{k}{3} \rceil$. Note, there are some cases when equality fails as well, i.e., $\gamma_2(T) < \ell + \lceil \frac{k}{3} \rceil$. What matters is the how much the vertex v_j with $x_j > 1$ for $1 \leq j \leq r$ is "helping" both sides. An example where equality fails is given in Figure 12.

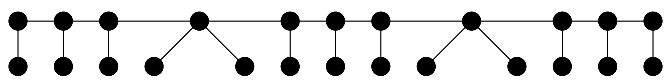


Figure 12: A complete caterpillar $P_{11}(1, 1, 1, 2, 1, 1, 1, 2, 1, 1, 1)$.

In Figure 12, $\gamma_2 = \ell + 3 < \ell + \lceil \frac{k}{3} \rceil = \ell + 4$. We can dissect the complete caterpillar into $C_j = P_{k_j}(1, 1, ..., 1)$ and v_j which is the vertex on the spine with $x_j > 1$ for $1 \leq j \leq k$. We have the following upper bounds since equality fails in some families of caterpillars $\mathcal{T}$.

Proposition 3.7. *For a complete caterpillar in $\mathcal{T}$,*

$$\gamma_2 \leq \ell + \sum_{j=1}^{r} \left\lceil \frac{k_j}{3} \right\rceil. \tag{1}$$

Proof. Let D be a minimum 2-dominating set and k_j be the length of the spine with $x_i = 1$ for $1 \leq k_j \leq r$, $1 \leq j \leq r$. Since all leaves ℓ are of degree one, the leaves are in D, so $|D| \geq \ell = \sum_{i=1}^{r} x_i$. Next, we dominate the spine of each of the sub-caterpillars k_j with $x_i = 1$ and thus, $\gamma_2 = |D| \leq \ell + \sum_{j=1}^{r} \left\lceil \frac{k_j}{3} \right\rceil$. □

Proposition 3.8. *For a complete caterpillar in $\mathcal{T}$,*

$$\gamma_2 \leq \ell + \sum_{j=1}^{r-1} |v_j| + \sum_{j=2}^{r-1} \left\lceil \frac{k_j - 2}{3} \right\rceil + \left\lceil \frac{k_1 - 1}{3} \right\rceil + \left\lceil \frac{k_r - 1}{3} \right\rceil. \quad (2)$$

Proof. Let D be a minimum 2-dominating. Let k_j be the length of the spine with $x_i = 1$ and v_j the vertex with $x_j > 1$ for $1 \leq i \leq r$. First, all the leaves are in D and, which 2-dominates each of the interior vertices v_j with $x_j > 1$ for $1 \leq i \leq r$. Thus, we are dominating a path of length smaller than k_1 and k_r by 1 and by 2 in the middle. Hence, $|D| \leq \ell + \sum_{j=1}^{r-1} |v_j| + \sum_{j=2}^{r-1} \left\lceil \frac{k_j - 2}{3} \right\rceil + \lceil \frac{k_1 - 1}{3} \rceil + \lceil \frac{k_r - 1}{3} \rceil$. □

We consider some cases to show that sometimes (1) is better than (2) and vice versa. We have the following:

Case 1: $\gamma_2 = \textit{Upper Bound}\ (1) = \textit{Upper Bound}\ (2)$.

Let us consider the caterpillar $P_7(1, 2, 1, 1, 1, 3, 1)$ where $\gamma_2 = \ell + 3$ as seen in Figure 13.

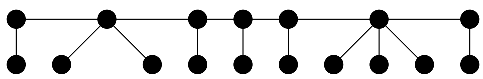


Figure 13: A complete caterpillar $P_7(1, 2, 1, 1, 1, 3, 1)$.

For Upper Bound (1):

$$
\begin{aligned}
\gamma_2(P_7(1,2,1,1,1,3,1)) &\leq \ell + \sum_{i=1}^{r} \left\lceil \frac{k_j}{3} \right\rceil \\
&= \ell + \left\lceil \frac{k_1}{3} \right\rceil + \left\lceil \frac{k_2}{3} \right\rceil + \left\lceil \frac{k_3}{3} \right\rceil \\
&= \ell + \left\lceil \frac{1}{3} \right\rceil + \left\lceil \frac{3}{3} \right\rceil + \left\lceil \frac{1}{3} \right\rceil \\
&= \ell + 1 + 1 + 1 \\
&= \ell + 3.
\end{aligned}
$$

For Upper Bound (2):

$$
\begin{aligned}
\gamma_2 &\leq \ell + \sum_{j=1}^{r-1} |v_j| + \sum_{j=2}^{r-1} \left\lceil \frac{k_j - 2}{3} \right\rceil + \left\lceil \frac{k_1 - 1}{3} \right\rceil + \left\lceil \frac{k_r - 1}{3} \right\rceil \\
&= \ell + \sum_{j=1}^{2} |v_j| + \sum_{j=2}^{2} \left\lceil \frac{k_j - 2}{3} \right\rceil + \left\lceil \frac{k_1 - 1}{3} \right\rceil + \left\lceil \frac{k_3 - 1}{3} \right\rceil \\
&= \ell + 2(1) + \left\lceil \frac{3-2}{3} \right\rceil + \left\lceil \frac{1-1}{3} \right\rceil + \left\lceil \frac{1-1}{3} \right\rceil \\
&= \ell + 2 + 1 + 0 + 0 \\
&= \ell + 3.
\end{aligned}
$$

Case 2: $\gamma_2 < \textit{Upper Bound}\ (2) < \textit{Upper Bound}\ (1)$.

Let us consider the caterpillar $P_{19}(1,1,1,1,2,1,1,1,1,2,1,1,1,1,2,1,1,1,1)$ where $\gamma_2 = \ell + 6$ as seen in Figure 14.

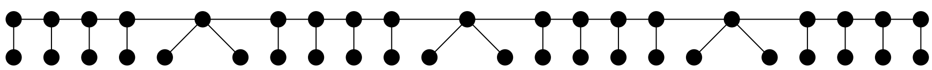


Figure 14: A caterpillar $P_{19}(1,1,1,1,2,1,1,1,1,2,1,1,1,1,2,1,1,1,1)$.

For Upper Bound (1):

$$
\begin{aligned}
\gamma_2 &\leq \ell + \sum_{j=1}^{r} \left\lceil \frac{k_j}{3} \right\rceil \\
&= \ell + \left\lceil \frac{k_1}{3} \right\rceil + \left\lceil \frac{k_2}{3} \right\rceil + \left\lceil \frac{k_3}{3} \right\rceil + \left\lceil \frac{k_4}{3} \right\rceil \\
&= \ell + \left\lceil \frac{4}{3} \right\rceil + \left\lceil \frac{4}{3} \right\rceil + \left\lceil \frac{4}{3} \right\rceil + \left\lceil \frac{4}{3} \right\rceil \\
&= \ell + 2 + 2 + 2 + 2 \\
&= \ell + 8.
\end{aligned}
$$

For Upper Bound (2):

$$
\begin{aligned}
\gamma_2 &\leq \ell + \sum_{j=1}^{r-1} |v_j| + \sum_{j=2}^{r-1} \left\lceil \frac{k_j - 2}{3} \right\rceil + \left\lceil \frac{k_1 - 1}{3} \right\rceil + \left\lceil \frac{k_r - 1}{3} \right\rceil \\
&= \ell + \sum_{j=1}^{3} |v_j| + \sum_{j=2}^{3} \left\lceil \frac{k_j - 2}{3} \right\rceil + \left\lceil \frac{k_1 - 1}{3} \right\rceil + \left\lceil \frac{k_3 - 1}{3} \right\rceil \\
&= \ell + 3(1) + \left\lceil \frac{4-2}{3} \right\rceil + \left\lceil \frac{4-2}{3} \right\rceil + \left\lceil \frac{4-1}{3} \right\rceil + \left\lceil \frac{4-1}{3} \right\rceil \\
&= \ell + 3 + 1 + 1 + 1 + 1 \\
&= \ell + 7.
\end{aligned}
$$

Case 3: $\gamma_2 < \textit{Upper Bound}\ (1) < \textit{Upper Bound}\ (2)$.

Let us consider the caterpillar $P_{15}(1,1,3,1,3,1,1,1,3,1,3,1,2,1,1)$ where $\gamma_2 = \ell + 5$ as seen in Figure 15.

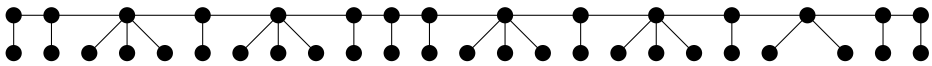


Figure 15: A complete caterpillar $P_{15}(1,1,3,1,3,1,1,1,3,1,3,1,2,1,1)$.

For Upper Bound (1):

$$\begin{aligned}
\gamma_2 &\leq \ell + \sum_{j=1}^{r} \left\lceil \frac{k_j}{3} \right\rceil \\
&= \ell + \left\lceil \frac{k_1}{3} \right\rceil + \left\lceil \frac{k_2}{3} \right\rceil + \left\lceil \frac{k_3}{3} \right\rceil + \left\lceil \frac{k_4}{3} \right\rceil + \left\lceil \frac{k_5}{3} \right\rceil + \left\lceil \frac{k_6}{3} \right\rceil \\
&= \ell + \left\lceil \frac{2}{3} \right\rceil + \left\lceil \frac{1}{3} \right\rceil + \left\lceil \frac{3}{3} \right\rceil + \left\lceil \frac{1}{3} \right\rceil + \left\lceil \frac{1}{3} \right\rceil + \left\lceil \frac{2}{3} \right\rceil \\
&= \ell + 1 + 1 + 1 + 1 + 1 + 1 \\
&= \ell + 6.
\end{aligned}$$

For Upper Bound (2):

$$\begin{aligned}
\gamma_2 &\leq \ell + \sum_{j=1}^{r-1} |v_j| + \sum_{j=2}^{r-1} \left\lceil \frac{k_j - 2}{3} \right\rceil + \left\lceil \frac{k_1 - 1}{3} \right\rceil + \left\lceil \frac{k_r - 1}{3} \right\rceil \\
&= \ell + \sum_{j=1}^{5} |v_j| + \sum_{j=2}^{5} \left\lceil \frac{k_j - 2}{3} \right\rceil + \left\lceil \frac{k_1 - 1}{3} \right\rceil + \left\lceil \frac{k_6 - 1}{3} \right\rceil \\
&= \ell + 5(1) + \left\lceil \frac{1-2}{3} \right\rceil + \left\lceil \frac{3-2}{3} \right\rceil + \left\lceil \frac{1-2}{3} \right\rceil + \left\lceil \frac{1-2}{3} \right\rceil \\
&+ \left\lceil \frac{2-1}{3} \right\rceil + \left\lceil \frac{2-1}{3} \right\rceil \\
&= \ell + 5 + \left\lceil \frac{-1}{3} \right\rceil + \left\lceil \frac{1}{3} \right\rceil + \left\lceil \frac{-1}{3} \right\rceil + \left\lceil \frac{-1}{3} \right\rceil + \left\lceil \frac{1}{3} \right\rceil + \left\lceil \frac{1}{3} \right\rceil \\
&= \ell + 5 + 0 + 1 + 0 + 0 + 1 + 1 \\
&= \ell + 8.
\end{aligned}$$

Observation: In Case 1, we found that the Upper Bound (1) equals Upper Bound (2) and both equal $\gamma_2 = \ell + \lceil \frac{k}{3} \rceil$.

In Case 2, the Upper Bound (2) is better than the Upper Bound (1). This is an example where equality fails. That is $\gamma_2 < \ell + \lceil \frac{k}{3} \rceil$.

In Case 3, the Upper Bound (1) is better than the Upper Bound (2). This is an example where equality fails. Hence, $\gamma_2 < \ell + \lceil \frac{k}{3} \rceil$.

From the cases above, we found out that in general neither of the upper bounds are better than the other. Thus, both serve as a useful upper bound.

3.5 Upper Bounds for all Complete Caterpillars

In this section, we construct a congruence class of caterpillars in $\mathcal{T}$ for which we conjecture that equality holds in Theorem 3.6. That is $\gamma_2 = \ell + \lceil \frac{k}{3} \rceil$.

We let $C_j = P_{k_j}(1, 1, ..., 1)$. Let us consider the following congruence classes for $\mathcal{T}$:

Class 1: Let $|C_j| \equiv 0 (\text{mod } 3)$ for $1 \leq j \leq r$.

For $P_7(1, 1, 1, x_4, 1, 1, 1)$, $\gamma_2(P_7(1, 1, 1, x_4, 1, 1, 1)) = \ell + 2 \leq \ell + \lceil \frac{7}{3} \rceil$. So, equality fails with $x_4 > 1$.

Class 2: Let $|C_j| \equiv 1 (\text{mod } 3)$ for $1 \leq j \leq r$.

For $P_3(1, x_2, 1)$, $\gamma_2(P_3(1, x_2, 1)) = \ell + 1 = \ell + \lceil \frac{3}{3} \rceil$.

For $P_5(1, x_2, 1, x_4, 1)$, $\gamma_2(P_5(1, x_2, 1, x_4, 1)) = \ell + 2 = \ell + \lceil \frac{5}{3} \rceil$.

For $P_7(1, x_2, 1, x_4, 1, x_6, 1)$, $\gamma_2(P_7(1, x_2, 1, x_4, 1, x_6, 1)) = \ell + 2 < \ell + \lceil \frac{7}{3} \rceil$. So, equality holds when $k = 3, 5$, but fails when $k = 7$ where $x_i > 1$ for $i = 2, 4, 6$.

Class 3: Let $|C_j| \equiv 2 (\text{mod } 3)$ for $1 \leq j \leq r$.

For $P_5(1, 1, x_3, 1, 1)$, $\gamma_2(P_5(1, 1, x_3, 1, 1)) = \ell + 2 = \ell + \lceil \frac{5}{3} \rceil$.

For $P_8(1, 1, x_3, 1, 1, x_6, 1, 1)$,

$\gamma_2(P_8(1, 1, x_3, 1, 1, x_6, 1, 1)) = \ell + 3 = \ell + \lceil \frac{8}{3} \rceil$.

For $P_{11}(1, 1, x_3, 1, 1, x_6, 1, 1, x_9, 1, 1)$,

$\gamma_2(P_{11}(1, 1, x_3, 1, 1, x_6, 1, 1, x_9, 1, 1)) = \ell + 4 = \ell + \lceil \frac{11}{3} \rceil$.

For $P_{11}(1,1,1,1,1,x_6,1,1,1,1,1)$,

$\gamma_2(P_{11}(1,1,1,1,1,x_6,1,1,1,1,1)) = \ell + 4 = \ell + \lceil \frac{11}{3} \rceil$.

Thus, equality holds in the cases above.

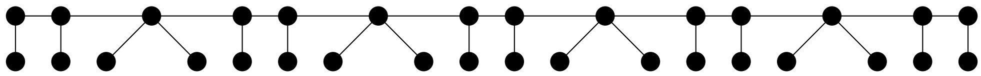


Figure 16: A complete caterpillar $P_{14}(1,1,2,1,1,2,1,1,2,1,1,2,1,1)$.

In Figure 16, $\gamma_2(P_{14}(1,1,x_3,1,1,x_6,1,1,x_9,1,1,x_{12},1,1)) = \ell + 5 = \ell + \lceil \frac{14}{3} \rceil$, where $x_i > 1$ for $i = 3,6,9,12$.

We also have other classes of $\mathcal{T}$ where equality fails when $C_j = P_{k_j}(1,1,...,1)$ for $1 \le j \le k$.

Class 4: Let $|C_j| \not\equiv |C_{j+1}| (\text{mod } 3)$ for $1 \le j \le r$.

For $P_4(1,x_2,1,1) \equiv P_4(1,1,x_3,1)$, $\gamma_2(P_4(1,x_2,1,1)) = \ell + 2 = \ell + \lceil \frac{4}{3} \rceil$.

For $P_6(1,1,1,x_4,1,1) \equiv P_6(1,1,x_3,1,1,1)$,

$\gamma_2(P_6(1,1,1,x_4,1,1)) = \ell + 2 = \ell + \lceil \frac{6}{3} \rceil$.

For $P_{13}(1,x_2,1,1,x_5,1,1,1,x_9,1,1,1,1)$,

$\gamma_2(P_{13}(1,x_2,1,1,x_5,1,1,1,x_9,1,1,1,1)) = \ell + 5 = \ell + \lceil \frac{13}{3} \rceil$.

For $1 \le j \le r$, equality holds when $|C_j| \not\equiv |C_{j+1}| (\text{mod } 3)$ in the examples above but fails for $P_{26}(1,2,1,1,2,1,2,1,1,2,1,2,1,1,2,1,2,1,1,2,1,2,1,1,2,1)$.

We calculate

$$\begin{aligned}\gamma_2(P_{26}(1,2,1,1,2,1,2,1,1,2,1,2,1,1,2,1,2,1,1,2,1,2,1,1,2,1)) &= \ell + 8 \\ &< \ell + \left\lceil \frac{26}{3} \right\rceil \\ &= \ell + 9.\end{aligned}$$

Let $\mathcal{T}_{]}$ be the family of caterpillars in $\mathcal{T}$ satisfying the additional condition:

- $|C_j| \equiv 2 (\text{mod } 3)$ for $1 \leq j \leq r$.

Conjecture 3.9. *Let T be a caterpillar, $P_k(x_1, x_2, ..., x_k)$, with ℓ leaves. If $T \in \mathcal{T}_c$ then* $\gamma_2(T) = \ell + \lceil \frac{k}{3} \rceil$.

3.6 Upper Bounds for all General Caterpillars

In this section, we obtain an upper bound for the 2-domination number of general caterpillars. We begin by dissecting a caterpillar C into subgraphs of the following types

- P_k
- $P_k(1, 1, 1, 1, ..., 1)$
- $P_k(x_1, ..., x_k)$, $x_i > 1$.

Denote the first type subgraph by P_{k_j}, the second type by C_ℓ, and the third type by CC_m. An example is given in Figure 17.

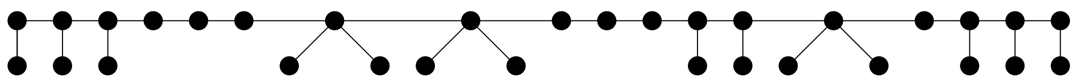


Figure 17: The caterpillar $P_{18}(1, 1, 1, 0, 0, 0, 2, 2, 0, 0, 0, 1, 1, 2, 0, 1, 1, 1)$.

In Figure 17, the above caterpillar can be dissected into $C_1 = P_3(1, 1, 1)$, $C_2 = P_2(1, 1)$, $C_3 = P_3(1, 1, 1)$, $P_{k_1} = P_3$, $P_{k_2} = P_3$, $P_{k_3} = P_1$, $CC_1 = P_2(2, 2)$, and

$CC_2 = P_1(2)$. Thus,

$$\begin{aligned}\gamma_2(P_{18}(1,1,1,0,0,0,2,2,0,0,0,1,1,2,0,1,1,1)) &= \ell + 8 \\ &\leq \ell + \sum_{j=1}^{3}\gamma(C_j) + \sum_{j=1}^{3}\gamma_2(P_{k_j}).\end{aligned}$$

Theorem 3.10. *For a complete caterpillar, $P_k(x_1, x_2, ..., x_k)$, dissected as above we have, $\gamma_2 \leq \ell + \sum_{j=1}^{r}\gamma(C_j) + \sum_{j=1}^{r}\gamma_2(P_{k_j})$.*

Proof. Let D be a minimum 2-dominating set. First, all the leaves are in D. Next, we dissect the caterpillar into $C_j's$, $P_{k_j}'s$ and $CC_j's$. Then, dominate each of the spines of C_j and 2-dominate each of the paths P_{k_j}. Hence, $|D| \leq \ell + \sum_j \gamma(C_i) + \sum_j \gamma_2(P_{k_i})$. □

3.7 Caterpillars of Small Length

Our aim in this section is to determine the 2-domination number of caterpillars with small length. Let us consider a caterpillar, $P_k(x_1, ..., x_k)$ for $1 < k \leq 5$. The following tables below give the 2-domination number of the caterpillars with a spine of small length. In the tables below we consider the caterpillars up to isomorphism and order them lexicographically in x_i. Also, we always choose the isomorphism class so that $x_1 \geq x_k$. Furthermore, when we don't specify the value of x_i, we have $x_i > 1$ for $1 \leq i \leq k$.

$P_2(x_1, x_2)$	γ_2
(1,1)	$\ell + 1$
(x_1,1)	$\ell + 1$
(x_1,x_2)	ℓ

Table 1: Caterpillars of length 2

$P_3(x_1, x_2, x_3)$	γ_2
(1,0,1)	$\ell + 1$
(1,1,1)	$\ell + 1$
(1,x_2,1)	$\ell + 1$
(x_1,0,1)	$\ell + 1$
(x_1,1,1)	$\ell + 1$
(x_1,1,x_3)	$\ell + 1$
(x_1,x_2,1)	$\ell + 1$
(x_1,x_2,x_3)	ℓ

Table 2: Caterpillars of length 3

$P_4(x_1, x_2, x_3, x_4)$	γ_2
(1,0,0,1)	$\ell + 2$
(1,0,1,1)	$\ell + 2$
(1,0,x_3,1)	$\ell + 2$
(1,1,1,1)	$\ell + 2$
(1,1,x_3,1)	$\ell + 2$
(1,x_2,x_3,1)	$\ell + 2$
(x_1,0,0,1)	$\ell + 2$
(x_1,0,0,x_4)	$\ell + 2$
(x_1,0,1,1)	$\ell + 2$
(x_1,0,1,x_4)	$\ell + 1$

$P_4(x_1, x_2, x_3, x_4)$	γ_2
(x_1,0,x_3,x_4)	$\ell + 1$
(x_1,1,0,1)	$\ell + 1$
(x_1,1,1,1)	$\ell + 1$
(x_1,1,1,x_4)	$\ell + 1$
(x_1,1,x_3,1)	$\ell + 1$
(x_1,x_2,0,1)	$\ell + 1$
(x_1,x_2,1,1)	$\ell + 1$
(x_1,x_2,1,x_4)	$\ell + 1$
(x_1,x_2,x_3,1)	$\ell + 1$
(x_1,x_2,x_3,x_4)	ℓ

Table 3: Caterpillars of length 4

$P_5(x_1, x_2, x_3, x_4, x_5)$	γ_2	$P_5(x_1, x_2, x_3, x_4, x_5)$	γ_2
(1,0,0,0,1)	$\ell+2$	(x_1,0,1,1,1)	$\ell+2$
(1,0,0,1,1)	$\ell+2$	(x_1,0,1,x_4,1)	$\ell+2$
(1,0,0,x_4,1)	$\ell+2$	(x_1,0,x_3,1,1)	$\ell+2$
(1,0,1,0,1)	$\ell+2$	(x_1,0,x_3,0,1)	$\ell+2$
(1,0,1,1,1)	$\ell+2$	(x_1,1,0,0,1)	$\ell+2$
(1,0,x_3,1,1)	$\ell+2$	(x_1,1,1,1,1)	$\ell+2$
(1,0,x_3,x_4,1)	$\ell+2$	(x_1,1,1,1,x_5)	$\ell+1$
(1,1,0,1,1)	$\ell+2$	(x_1,1,x_3,0,1)	$\ell+2$
(1,1,0,x_4,1)	$\ell+2$	(x_1,1,x_3,1,x_5)	$\ell+1$
(1,1,1,1,1)	$\ell+2$	(x_1,x_2,0,0,1)	$\ell+2$
(1,1,1,x_4,1)	$\ell+2$	(x_1,x_2,0,1,1)	$\ell+2$
(1,1,x_3,x_4,1)	$\ell+2$	(x_1,x_2,0,x_4,x_5)	$\ell+1$
(1,x_2,0,x_4,1)	$\ell+2$	(x_1,x_2,1,1,1)	$\ell+1$
(1,x_2,1,x_4,1)	$\ell+2$	(x_1,x_2,1,x_4,x_5)	$\ell+1$
(1,x_2,x_3,x_4,1)	$\ell+2$	(x_1,x_2,x_3,0,1)	$\ell+1$
(x_1,0,0,0,1)	$\ell+2$	(x_1,x_2,x_3,0,x_5)	$\ell+1$
(x_1,0,0,0,x_5)	$\ell+2$	(x_1,x_2,x_3,1,1)	$\ell+1$
(x_1,0,0,x_4,1)	$\ell+2$	(x_1,x_2,x_3,1,x_5)	$\ell+1$
(x_1,0,0,x_4,x_5)	$\ell+2$	(x_1,x_2,x_3,x_4,1)	$\ell+1$
(x_1,0,1,0,x_5)	$\ell+2$	(x_1,x_2,x_3,x_4,x_5)	ℓ

Table 4: Caterpillars of length 5

4 CONCLUDING REMARKS

After many cases were discussed and many different caterpillars were examined, we obtained a general upper bound for the 2-domination number of caterpillars, $\gamma_2(P_k(x_1, ..., x_k)) \leq \ell + \lceil \frac{k}{3} \rceil$. We discussed different cases of caterpillars with varying number of leaves on the spine of caterpillars.

In Section 3.5, we conjectured that which equality holds for the family of caterpillars $\mathcal{T}_c$.

4.1 Open Problems

In this thesis, we have that for any caterpillar, $\gamma_2 \leq \ell + \sum_{j=1}^{r} \gamma(C_j) + \sum_{j=1}^{r} \gamma_2(P_{k_j})$.

- Can we characterize the caterpillar for which the bound is sharp?
- Can we determine the 2-domination number of caterpillars?

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