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Italian Domination in Complementary Prisms

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A thesis

presented to

the faculty of the Department of Mathematics

East Tennessee State University

In partial fulfillment

of the requirements for the degree

Master of Science in Mathematical Sciences

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by

Haley Desiree Russell

May 2018

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Keywords: graph theory, complementary prism, Roman {2}-domination, Italian domination

ABSTRACT

Italian Domination in Complementary Prisms

by

Haley Russell

Let $G$ be any graph and let $\overline{G}$ be its complement. The complementary prism of $G$ is formed from the disjoint union of a graph $G$ and its complement $\overline{G}$ by adding the edges of a perfect matching between the corresponding vertices of $G$ and $\overline{G}$. An Italian dominating function on a graph $G$ is a function such that $f : V \to \{0, 1, 2\}$ and for each vertex $v \in V$ for which $f(v) = 0$, it holds that $\sum_{u \in N(v)} f(u) \geq 2$. The weight of an Italian dominating function is the value $f(V) = \sum_{u \in V(G)} f(u)$. The minimum weight of all such functions on $G$ is called the Italian domination number. In this thesis we will study Italian domination in complementary prisms. First we will present an error found in one of the references [5]. Then we will define the small values of the Italian domination in complementary prisms, find the value of the Italian domination number in specific families of graphs complementary prisms, and conclude with future problems.

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# DEDICATION

With love I would like to dedicate this thesis to my adoring husband, who made the coffee more often than not, to my wonderful family, for always being supportive, and last but not least to my crazy friends, who always seem to know when I need to be bullied or babied. Without you all I'd be a bigger mess than I already am. Love you guys.

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First, I would like to thank Dr. Teresa Haynes for not only her help and guidance while I was writing this thesis, but for also being an amazing person. During this whole process, no matter how wrong I was, she was always kind, patient, and uplifting. Second, I would like to thank Dr. Anant Godbole without whom I would not have had this amazing opportunity. One small conversation with him set me on this path. Finally, I would like to thank Dr. Robert Beeler for introducing me to graph theory and teaching me there is always more to a subject than what is taught in class. I would also like to thank Dr. Beeler for warning me about the snakes (hopefully mine doesn't end up being too big or poisonous). There are really no words to describe the impact each of you have had on me during in my time here at ETSU. So I will simply finish by saying, thank you all from the bottom of my heart.

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# 1 INTRODUCTION

In this thesis we study Italian domination in the family of graphs known as complementary prisms. Before progressing into our discussion we first need to define some terminology and notation. Let $G = (V, E)$ be a simple undirected graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *size* of $G$, $|E(G)| = m$, is the number of edges in $G$. Similarly, the *order* of $G$, $|V(G)| = n$, is the number of vertices in $G$. For any two vertices $a, b \in V(G)$, $a$ and $b$ are *adjacent* if the edge $ab \in E(G)$. The *open neighborhood* of a vertex $v \in V$, denoted $N(v)$, consists of the vertices adjacent to $v$, and its *closed neighborhood*, denoted $N[v]$, is the open neighborhood of vertex $v$ together with the vertex $v$. That is, $N(v) = \{x \mid vx \in E\}$ and $N[v] = N(v) \cup \{v\}$. A vertex $v \in S$ has a *private neighbor* with respect to the set $S$ if there is a $w \in N(v) \cap (V - S)$ for which $N(w) \cap S = \{v\}$. The *degree* of $v$ is the cardinality of the open neighborhood of $v$, or $deg_G(v) = |N(v)|$. The *maximum degree* of a graph $G$ is denoted $\Delta(G) = \max\{deg_G(v) | v \in V(G)\}$ and the *minimum degree* of graph $G$ is denoted $\delta(G) = \min\{deg_G(v) | v \in V(G)\}$. A vertex of degree zero is known as an *isolate*, or an *isolated vertex.* A *leaf* of a graph is vertex of degree one, and the vertex adjacent to a leaf is called a *support vertex.* We will let $mH$ denote the union of $m$ disjoint copies of $H$. For example, $mK_2$ is the graph formed from $m$ copies of $K_2$. The *complement* of a graph G, denoted $\overline{G}$, is a graph such that $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{xy \mid xy \notin V(G)\}$.

Complementary products were first introduced in [17] as a generalization of Cartesian products of graphs. In this thesis, our main focus will be on complementary prisms which are a sub-family of complementary products. A *complementary prism* of a graph $G$, denoted $G\overline{G}$, is the disjoint union of $G$ and $\overline{G}$ formed by adding a perfect matching between corresponding vertices of $G$ and $\overline{G}$. For each $v \in V(G)$ let the corresponding vertex in $\overline{G}$ be denoted as $\overline{v}$. In other words, the graph $G\overline{G}$ is formed from $G \cup \overline{G}$ by adding the edge $v\overline{v}$ for all $v \in V(G)$. Several different parameters have been studied in complementary prisms, see [8, 18, 20, 21]. It is important to observe that complementary prisms are a generalization of several well known graphs, such as the Petersen graph which is the complementary prism $C_5\overline{C_5}$, as seen in Figure 1, and the corona $K_n \circ K_1$ which is the complementary prism $K_n\overline{K}_n$. An example $K_5\overline{K}_5$ is seen in Figure 1.

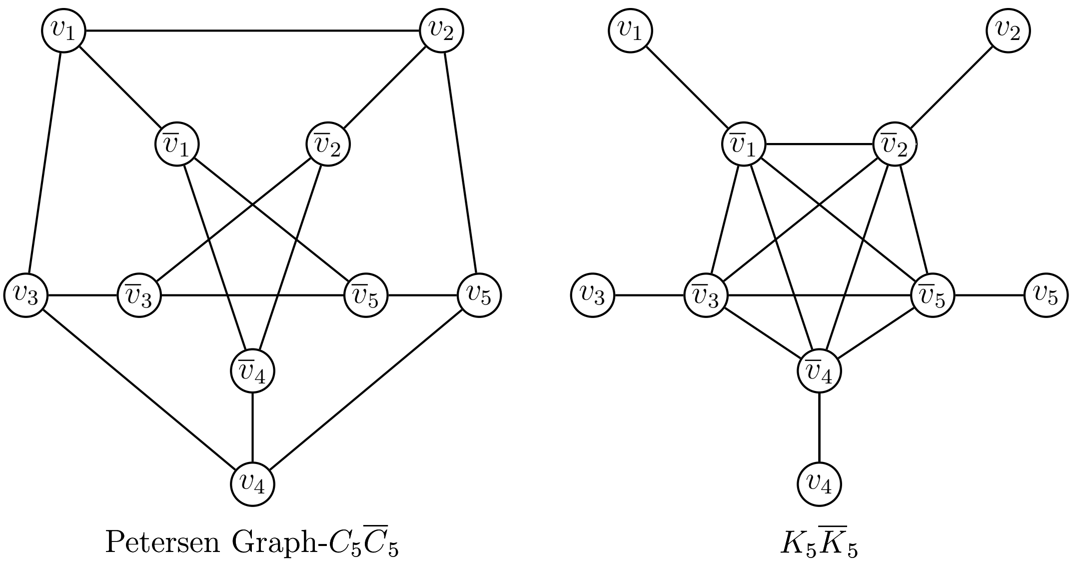


Figure 1: Complementary prisms: Petersen graph and $K_5\overline{K}_5$

A *dominating set* of $G$ is a subset $S$ of $V$ such that every vertex in $V - S$ is adjacent to at least one vertex in $S$. That is, $N[S] = V$. The minimum cardinality of the dominating sets of $G$ is the *domination number*, denoted $\gamma(G)$. A subset $S \subseteq V$ is a *2-dominating set* if every vertex of $V - S$ is adjacent to at least two vertices in $S$. The minimum cardinality amongst all 2-dominating sets is the *2-domination number*, denoted $\gamma_2(G)$. The 2-domination number was first introduced in [12] and is generalized as $n$-domination, see also [3, 22].

Another, equivalent, definition of a dominating set is a subset $S$ of $V$ is a dominating set only if $|N[v] \cap S| \geq 1$, for each $v \in V$. And just as we had previously, the minimum cardinality amongst these dominating sets of $G$ the domination number. A *double dominating set* is a subset $S$ of $V$ such that $|N[v] \cap S| \geq 2$, for each $v \in V$. The minimum cardinality of a double dominating set of $G$ is called the *double domination number*, denoted $\gamma_{\times 2}(G)$. Double domination was introduced in [16] and is a case of $k$-tuple domination, see also [7, 9, 15]. It is important to observe that the double domination number is not defined for graphs that have an isolated vertex.

A *Roman dominating function*, or RDF, on $G$ is a function such that $f : V \to \{0, 1, 2\}$ and every $v \in V$ for which $f(v) = 0$ is adjacent to at least one vertex $u$ for which $f(u) = 2$. For any Roman dominating function $f$ of $G$, and $i \in \{0, 1, 2\}$, let $V_i = \{v \in V \mid f(v) = i\}$. Since this partition determines $f$, we write $f = (V_0, V_1, V_2)$. The weight of a Roman dominating function is the value $f(V) = \sum_{u \in V(G)} f(u)$, equivalently $f(V) = |V_1| + 2|V_2|$. The minimum weight of a Roman dominating function is the *Roman domination number* of $G$, denoted $\gamma_R(G)$. Roman domination was motivated by Stewart in [27] and a Roman dominating function was first formally defined in [6]. Roman domination has been studied in numerous papers, see [1, 2, 4, 10, 11, 13, 14, 23, 24, 25, 26, 29].

The final parameter we define is a variation of both 2-domination and Roman domination. An *Italian dominating function*, or IDF, on a graph $G$ is a function such that $f : V \to \{0, 1, 2\}$ and for each vertex $v \in V$ for which $f(v) = 0$, it holds that $\sum_{u \in N(v)} f(u) \geq 2$. Similar to Roman domination, an Italian dominating function $f$ of $G$ can also be partitioned into three sets such that $f = (V_0, V_1, V_2)$. The weight of an Italian dominating function is the value $f(V) = \sum_{u \in V(G)} f(u)$, equivalently $f(V) = |V_1| + 2|V_2|$. The minimum weight of all such Italian dominating functions on $G$ is called the *Italian dominating number*, denoted $\gamma_I(G)$. Italian domination was first introduced in [5] under the name Roman $\{2\}$-domination, in 2016. The concept has also been explored in [19] and [28]. An example of Italian domination is given in Figure 2, for two graphs where the assignment of a $\gamma_I(G)$-function of the graph is on the vertices.


Figure 2: Examples of Italian domination

It follows directly from the above definitions that the Italian domination number of any graph $G$ is bounded by

$$\gamma(G) \leq \gamma_I(G) \leq \gamma_R(G) \leq 2\gamma(G).$$

This string of inequalities was given in [5, 28]. Also following directly from the above definitions the Italian domination number of an isolate free graph is bounded by

$$\gamma(G) \le \gamma_I(G) \le \gamma_2(G) \le \gamma_{\times 2}(G).$$

This string of inequalities was partially studied in [5] and given in its entirety in [28]. These two strings of inequalities that bound the Italian domination number are of particular interest because the Roman domination number and the double domination number of a graph are generally incomparable parameters; this was the motivation behind [28].

To further our discussion of Italian domination, we need to define some related terminology given in [28]. A graph is defined to be an *I1 graph* if every minimum weight Italian dominating function exclusively uses the set $\{0, 1\}$. Analogously, a graph is defined to be an *I2 graph* if every minimum weight Italian dominating function is exclusively the set $\{0, 2\}$. Finally a graph is an *I1a graph* if the range of some minimum weight Italian dominating function is the set $\{0, 1\}$. A graph is called a *Roman graph* if $\gamma_R(G) = 2\gamma(G)$. Similarly, a graph is defined to be an *Italian graph* if $\gamma_I(G) = 2\gamma(G)$.

It is clear from the definitions that if $G$ is Italian, then it is also Roman. It is not necessarily true that if a graph is Roman, then it is Italian. Consider the graph $P_5$ as an example. We have that $\gamma(P_5) = 2$, $\gamma_R(P_5) = 4$, and $\gamma_I(P_5) = 3$, making $P_5$ a graph that is Roman but not Italian. Notice any graph that is I1 cannot be an Italian or Roman graph. Also every I2 graph is an Italian graph and therefore a Roman graph as well. Not every Italian graph, however, is necessarily an I2 graph. Consider the graph $P_2$ for example. It is clear that $\gamma(P_2) = 1$ and $\gamma_I(P_2) = 2$ making

$P_2$ an Italian graph. Observe that one possible Italian dominating function of $P_2$ assigns both vertices a 1. Thus, implying that, $P_2$ is an Italian graph that is not an I2 graph.

As stated previously, in this thesis we study Italian domination of complementary prisms. First, in Section 2, we will conduct a literature survey over all known, relevant work to this thesis. In Section 3 we will correct an error found in [5], the small values of Italian domination in complementary prisms are defined and discussed, the Italian domination number is found for specific classes of graph's complementary prisms, and general bounds are observed for the Italian domination number in complementary prisms. Then we conclude with future problems.

# 2 LITERATURE SURVEY

This section will present some background results to this research. These results were the catalyst and motivation behind this thesis. First we present all known relevant results regarding the Italian dominating number, and then we present the results from a similar paper to this thesis.

## 2.1 Italian Domination

The following theorems and results are not an exhaustive list of results regarding the Italian domination number, rather, it is a list of all known relevant results. For a more complete list of results regarding the Italian dominating number the reader is referred to [5, 19, 28]. It is important to note that Italian domination was first introduced in [5] as Roman {2}-domination and was denoted $\gamma_{\{R2\}}(G)$. For clarity all results from [5] have been alerted to reflect our notation for Italian domination number. First we have several bounds on the Italian domination number.

**Proposition 2.1.** *[5] For every graph $G$, $\gamma(G) \leq \gamma_I(G) \leq \gamma_R(G)$.*

**Observation 2.2.** *[5] For a graph $G$, $\gamma(G) < \gamma_I(G) < \gamma_R(G)$ is possible, even for paths.*

**Theorem 2.3.** *[5, 28] For every graph $G$, $\gamma_I(G) \leq 2\gamma(G)$.*

**Proposition 2.4.** *[5] For every graph $G$, $\gamma_I(G) \leq \gamma_2(G)$.*

The next result observes a case when the bound in Proposition 2.4 is sharp.

**Corollary 2.5.** *[5] For every graph $G$ with $\Delta(G) \leq 2$, $\gamma_I(G) = \gamma_2(G)$.*

Using Corollary 2.5 the Italian domination number for two important families of graphs are given below.

**Corollary 2.6.** *[5, 28] For the classes of paths $P_n$ and cycles $C_n$, $\gamma_I(P_n) = \lceil \frac{n+1}{2} \rceil$ and $\gamma_I(C_n) = \lceil \frac{n}{2} \rceil$.*

The following proposition characterizes when the bound in Proposition 2.4 is sharp.

**Proposition 2.7.** *[28] For all $G$, $\gamma_I(G) = \gamma_2(G)$ if and only if $G$ is I1a.*

These next results give upper bounds on the Italian domination number.

**Theorem 2.8.** *[28] For all connected graphs $G$ with $n \geq 3$ vertices, $\gamma_I(G) \leq \frac{3n}{4}$.*

**Theorem 2.9.** *[28] Let $G$ be a graph with $n \geq 3$ vertices and $\delta \geq 2$. Then $\gamma_I(G) \leq \frac{2n}{3}$.*

**Theorem 2.10.** *[28] Let $G$ be a graph with $n$ vertices and minimum degree $\delta \geq 3$. Then $\gamma_I(G) \leq \frac{n}{2}$.*

Let a *circulant graph* $C_n(x_1, x_2, x_3)$ be a graph with $n$ vertices $v_0, v_1, ..., v_{n-1}$ with $v_i$ adjacent to each vertex $v_{i \pm x_j}$, $j \in \{1, 2, 3\}$, where addition is done mod $n$. The following is a conjecture regarding circulant graphs.

**Conjecture 2.11.** *[28] The Italian domination number $\gamma_I(G) = \frac{n}{2}$ for $G = C_{4k}(1, 2k, 4k-1)$ and for $G = C_{8k}(1, 4k, 8k-1)$.*

A *regular graph* is a graph in which every vertex has the same degree. If every vertex of graph $G$ has degree $r$, then we say $G$ is a *r-regular graph.* The next two results are about 4-regular graphs.

**Proposition 2.12.** *[28] There exist infinitely many bipartite 4-regular graphs $G$ such that $G$ has $n$ vertices, $\gamma_2(G) = \frac{n}{2}$, and $\gamma_I(G) < \gamma_2(G)$.*

**Proposition 2.13.** *[28] Let $G$ be a 4-regular, diameter two graph with $n = 10$ vertices. Then $\gamma(G) \leq 3$ and $\gamma_I(G) \leq 5$.*

The final results in this subsection are fundamental results regarding the Italian domination number of graphs that are trees.

**Corollary 2.14.** *[28] For the path $P_n$ with $n$ odd and $n \neq 3$ ia an 1i graph and these are the only paths that are I1.*

**Proposition 2.15.** *[28] The paths $P_2, P_3, P_6$ are the only Italian paths and no paths are I2.*

**Theorem 2.16.** *[28] Let $T$ be a tree with $n > 1$ vertices. Then $\gamma(T) < \gamma_I(T)$.*

**Observation 2.17.** *[19] If $T'$ is a subtree of a tree $T$, then $\gamma(T') \leq \gamma(T)$ and $\gamma_I(T') \leq \gamma_I(T)$.*

## 2.2 Complementary Prisms

Now results from a similar paper to this thesis, that is about Roman domination in complementary prisms, is presented, see [1]. The first result characterizes the complementary prisms having $\gamma_R(G\overline{G}) \in \{2, 3, 4\}$.

**Theorem 2.18.** *[1] Let $G$ be a graph of order $n$. Then*

1. *$\gamma_R(G\overline{G}) = 2$ if and only if $G = K_1$.*
2. *$\gamma_R(G\overline{G}) = 3$ if and only if $G = K_2$ or $\overline{G} = K_2$.*

*3.* $\gamma_R(G\overline{G}) = 4$ *if and only if* $\gamma_R(G) = 3$ *and* $G$ *has an isolated vertex or* $\gamma_R(\overline{G}) = 3$ *and* $G$ *has an isolated vertex*

The next lower bound follows directly from Theorem 2.18.

**Corollary 2.19.** *[1] If* $G$ *and its complement* $\overline{G}$ *are isolate-free graphs, then* $\gamma_R(G\overline{G}) \geq 5$.

Next the Roman domination number of the complementary prism when $G = K_n$ is given.

**Proposition 2.20.** *[1] If* $G = K_n$*, then* $\gamma_R(G\overline{G}) = n + 1$.

The following two results give lower bounds on the Roman domination number for complementary prisms.

**Theorem 2.21.** *[1] For any graph* $G$ *of order* $n$*,* $\gamma_R(G\overline{G}) \geq max\{\gamma_R(G), \gamma_R(\overline{G})\} + 1$ *with equality if and only if* $G$ *or* $\overline{G}$ *has an isolated vertex.*

**Corollary 2.22.** *[1] If neither graph* $G$ *nor its complement* $\overline{G}$ *has an isolated vertex, then* $\gamma_R(G\overline{G}) \geq max\{\gamma_R(G), \gamma_R(\overline{G})\} + 2$.

The following shows that the bound given in Corollary 2.22 is sharp.

**Theorem 2.23.** *[1] If neither graph* $G$ *nor its complement* $\overline{G}$ *has an isolated vertex and one of them has a vertex of degree one, then* $\gamma_R(G\overline{G}) = max\{\gamma_R(G), \gamma_R(\overline{G})\} + 2$.

Theorem 2.23 gives the Roman domination number for the complementary prisms of paths.

**Corollary 2.24.** *[1] For paths* $G = P_n$ *where* $n \geq 3$*,* $\gamma_R(G\overline{G}) = \lceil \frac{2n}{3} \rceil + 2$.

Next we have an upper bound for the Roman domination number of complementary prisms.

**Observation 2.25.** *[1] For any graph $G$, $\gamma_R(G\overline{G}) \leq \gamma_R(G) + \gamma_R(\overline{G})$.*

Lastly we are given a lower bound and an upper bound for the Roman domination number in complementary prisms when the diam(G) $\geq 3$ such that neither $G$ nor $\overline{G}$ has an isolated vertex.

**Theorem 2.26.** *[1] Let $G$ be a graph with diam(G) $\geq 3$ such that neither $G$ nor $\overline{G}$ has an isolated vertex. Then $\gamma_R(G) + 2 \leq \gamma_R(G\overline{G}) \leq \gamma_R(G) + 4$.*

# 3 RESULTS

In this section we present an error that we found in [5] and correct it by defining a new type of private neighbor. Then we present key results regarding the Italian domination number in complementary prisms.

## 3.1 Error

Below is the statement made in [5] that is incorrect.

**Statement 3.1.** *[5] For every graph $G$, there exists a $\gamma_I$-function $f = (V_0, V_1, V_2)$ such that either $V_2 = \emptyset$ or every vertex of $V_2$ has at least three private neighbors in $V_0$ with respect to the set $V_1 \cup V_2$.*

A counterexample to Statement 3.1 is the star graph $K_{1,3}$ with one of its edges subdivided. Figure 3 gives an $\gamma_I$-function of this graph. Let us assume Statement 3.1 holds. In the Italian domination function given in Figure 3 the only vertex in $V_2$ is $v_3$ and it does not have three private neighbors in $V_0$ with respect to the set $V_1 \cup V_2$, by Statement 3.1, this implies there exists a $\gamma_I$-function such that $V_2 = \emptyset$. But this

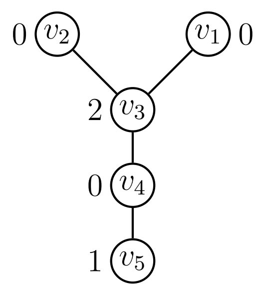


Figure 3: Counterexample to Statement 3.1

means the graph in Figure 3 can be Italian dominated by assigning three vertices a 1; however, this is clearly impossible. Thus, as stated previously, Statement 3.1 is false.

We would like to alter Statement 3.1 to make it true. To do so we will define a new type of private neighbor. Let $f = (V_0, V_1, V_2)$ be a $\gamma_I$-function and $S = V_1 \cup V_2$. We say that a vertex $w \in V_0$ is an *Italian private neighbor* of $v \in S$ with respect to $S$ if $\sum_{u \in N(w)-\{v\}} f(u) < 2$. Implementing this new definition we have the following.

**Proposition 3.2.** *For every graph $G$, there exists a $\gamma_{\{R2\}}$-function $f = (V_0, V_1, V_2)$ such that either $V_2 = \emptyset$ or every vertex of $V_2$ has at least three Italian private neighbors in $V_0$ with respect to the set $V_1 \cup V_2$.*

*Proof.* Amongst all $\gamma_I$-functions, let $f = (V_0, V_1, V_2)$ be one with $|V_2|$ as small as possible. If $V_2 = \emptyset$, then we are finished. Thus, assume $V_2 \neq \emptyset$. If a vertex $a \in V_2$ has no Italian private neighbors in $V_0$ with respect to $V_1 \cup V_2$, then we can reduce the weight of $f$ by assigning the value of 1 to $a$ instead of 2, a contradiction. Hence, every vertex in $V_2$ must have at least one Italian private neighbor in $V_0$.

Assume that a vertex $x \in V_2$ has at most two Italian private neighbors in $V_0$. Suppose $x$ has only one Italian private neighbor $b \in V_0$. Then either $\sum_{y \in N(b)-\{x\}} f(y) = 1$ or $\sum_{y \in N(b)-\{x\}} f(y) = 0$. If $\sum_{y \in N(b)-\{x\}} f(y) = 1$, then we can reduce the weight of $f$ by assigning the value of 1 to $x$ instead of 2, a contradiction. Hence, $\sum_{y \in N(b)-\{x\}} f(y) = 0$. Now we let $f' = (V_0 - \{b\}, V_1 \cup \{b, x\}, V_2 - \{x\})$. Clearly, $f'$ is a is a $\gamma_I(G)$-function with fewer vertices assigned a 2 than under $f$, a contradiction to our choice of $f$. Thus, $x$ has two Italian private neighbors in $V_0$, say $u$ and $v$.

We have the following possibilities $\sum_{y \in N(u)-\{x\}} f(y) = 1$ and $\sum_{y \in N(v)-\{x\}} f(y) = 1$, $\sum_{y \in N(u)-\{x\}} f(y) = 0$ and $\sum_{y \in N(v)-\{x\}} f(y) = 0$, or, without loss of generality,

$\sum_{y\in N(u)-\{x\}} f(y) = 1$ and $\sum_{y\in N(v)-\{x\}} f(y) = 0$. If $\sum_{y\in N(u)-\{x\}} f(y) = 1$ and $\sum_{y\in N(v)-\{x\}} f(y) = 1$, then we can reduce the weight of $f$ by assigning the value 1 to $x$ instead of 2, a contradiction. Let us assume that $\sum_{y\in N(u)-\{x\}} f(u) = 0$ and $\sum_{y\in N(v)-\{x\}} f(u) = 0$. Define a function $f^1 = (\{x\}\cup V_0-\{u,v\}, V_1\cup\{u,v\}, V_2-\{x\})$. Clearly, $f^1$ is a $\gamma_I(G)$-function with fewer vertices assigned a 2 under $f^1$ than under $f$, contradicting our choice of $f$. Hence, it must be the case, without loss of generality, that $\sum_{y\in N(u)-\{x\}} f(y) = 1$ and $\sum_{y\in N(v)-\{x\}} f(y) = 0$. Define a function $f^2 = (V_0 - \{v\}, V_1 \cup \{v,x\}, V_2 - \{x\})$. Clearly, $f^2$ is a $\gamma_I(G)$-function with fewer vertices assigned a 2 under $f^2$ than under $f$, again contradicting our choice of $f$. □

## 3.2 Small Values

We begin with an observation from the definition of Italian domination and complementary prisms.

**Observation 3.3.** *For any graph $G$, $\gamma_I(G\overline{G}) \geq 2$.*

Here we will consider complementary prisms having small Italian domination numbers, specifically the complementary prisms having $\gamma_I(G\overline{G}) \in \{2,3,4\}$. First we characterize when $\gamma_I(G\overline{G}) = 2$ and $\gamma_I(G\overline{G}) = 3$.

**Theorem 3.4.** *For any graph $G$, $\gamma_I(G\overline{G})$=2 if and only if $G = K_1$.*

*Proof.* If $G = K_1$, then $G\overline{G} = K_2$ and $\gamma_I(G\overline{G}) = 2$.

Assume $\gamma_I(G\overline{G}) = 2$. It follows that any $\gamma_I$-function of weight 2 allows for only two possible labelings, **(1)** $f(v) = 2$ and $f(u) = 0$ for every $u \in V(G\overline{G}) - \{v\}$ and **(2)** $f(v) = 1$, $f(u) = 1$, and $f(w) = 0$ for all $w \in V(G\overline{G}) - \{u,v\}$. For the former, without loss of generality, let $f(v) = 2$ and $v \in V(G)$. Thus, the only vertices Italian dominated by $f$ are in $N[v]$. Since $N[v] \cap V(\overline{G}) = \{\overline{v}\}$, it follows that $|V(\overline{G})|$=1. Hence, $G = K_1$. See Figure 4.

Next assume that $f(v) = 1$, $f(u) = 1$, and $f(w) = 0$ for all $w \in V(G\overline{G}) - \{u, v\}$. If $u$ and $v$ are both in $G$ (respectively, $\overline{G}$), then the vertices of $\overline{G}$ (respectively, $G$) are not Italian dominated. Hence, we may assume, without loss of generality, that $v \in V(G)$ and $u \in V(\overline{G})$. If $u = \overline{v}$, then it follows that $G = K_1$. Hence, assume that $u \neq \overline{v}$, say $u = \overline{w}$ where $w \neq v$. Thus, in order to Italian dominate $\overline{v}$, it follows that $\overline{v} \in N(\overline{w})$. But then $w$ is not adjacent to $v$, so $w$ is not Italian dominated, a contradiction. □

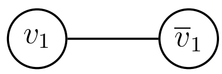


Figure 4: Complementary prism where $G = K_1$

**Lemma 3.5.** *[5, 28] For the classes of paths $P_n$ and cycles $C_n$, $\gamma_I(P_n) = \lceil \frac{n+1}{2} \rceil$ and $\gamma_I(C_n) = \lceil \frac{n}{2} \rceil$.*

**Theorem 3.6.** *For any graph $G$, $\gamma_I(G\overline{G})$=3 if and only if $G = K_2$.*

*Proof.* If $G = K_2$ or $\overline{G} = K_2$, then $G\overline{G}$ is the path $P_4$ and by Lemma 3.5, we have that $\gamma_I(P_4) = \lceil \frac{5}{2} \rceil = 3$.

Assume $\gamma_I(G\overline{G}) = 3$. Let $f$ be a $\gamma_I$-function of $G\overline{G}$. By Theorem 3.4, G has at least two vertices. We consider the two possible labelings **(1)** one vertex is assigned weight 2 and one vertex is assigned a weight 1 and **(2)** three vertices are each assigned a weight 1.

**Case 1.** $f(v) = 2$ and $f(u) = 1$.

We may assume that $v \in V(G)$. If $u$ is also in $G$, then its corresponding vertex $\overline{u} \in \overline{G}$ is not Italian dominated. Thus, $u \in V(\overline{G})$. This implies that $u$ and $\overline{v}$ are the only vertices in $\overline{G}$, so $\overline{G}$ has order two. Furthermore, $G = K_2$ or $\overline{G} = K_2$. See Figure 5.

**Case 2.** $f(v) = f(u) = f(w) = 1$.

If all vertices assigned 1 under $f$ are contained in $G$ (respectively, $\overline{G}$), then the vertices of $\overline{G}$ (respectively, G) are not Italian dominated. Thus, we may assume without loss of generality, that $f(u) = f(v) = 1$ and $f(\overline{w}) = 1$ where $u, v \in V(G)$ and $\overline{w} \in V(\overline{G})$. If $\overline{w} \notin \{\overline{u}, \overline{v}\}$, then $\overline{w}$ is adjacent to both $\overline{u}$ and $\overline{v}$ to Italian dominate them. But then $w$ is not adjacent to $u$ nor $v$ and hence is not Italian dominated, a contradiction. Thus, $\overline{w} \in \{\overline{u}, \overline{v}\}$, say $\overline{w} = \overline{u}$. Now every vertex in $\overline{V} - \{\overline{u}\}$ is adjacent to $\overline{u}$ and at least one of $u$ and $v$, implying that $\overline{G}$ has order 2. Thus, $G \in \{K_2, \overline{K}_2\}$. See Figure 5. □

**Corollary 3.7.** *For any graph $G \neq K_1$ and $G \neq K_2$, $\gamma_I(G\overline{G}) \geq 4$.*

Next, we will there are several types of graphs whose complementary prisms have $\gamma_I(G\overline{G}) = 4$.

**Theorem 3.8.** *If $\gamma_I(G) = 3$ and $G$ has an isolated vertex, then $\gamma_I(G\overline{G}) = 4$.*

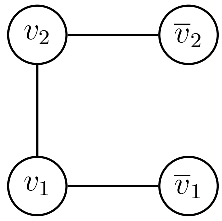


Figure 5: Complementary prism where $G = K_2$

*Proof.* Assume that $\gamma_I(G) = 3$ and $G$ has an isolated vertex, say $v$. By Corollary 3.7, we have $\gamma_I(G\overline{G}) \geq 4$. Note that by assigning a 2 to $\overline{v}$, $\overline{v}$ Italian dominates $V(\overline{G}) \cup \{v\}$. Since $v$ is an isolate of $G$ and $\gamma_I(G) = 3$, it follows that $\gamma_I(G - v) = 2$. Thus, assigning 2 to $\overline{v}$ and a 0 to each vertex in $V(\overline{G}) \cup \{v\}$and a $\gamma_I$-function of $G - \{v\}$ yields an Italian dominating function of $G\overline{G}$. Hence, $\gamma_I(G\overline{G}) \leq 4$, and so, $\gamma_I(G\overline{G}) = 4$. $\square$

**Corollary 3.9.** *If $G$ is a star with order $n \geq 3$, then $\gamma_I(G\overline{G}) = 4$.*

*Proof.* Let $G$ be a star with order $n \geq 3$. Since $G$ is a star, the support vertex $v \in V(G)$ is an isolated vertex in $\overline{G}$ and the leaf vertices in $G$ will form a complete graph on $n - 1$ vertices in $\overline{G}$. It is now clear that, $\overline{G}$ has an isolated vertex and $\gamma_I(\overline{G}) = 3$. It follows directly from Theorem 3.8 that $\gamma_I(G\overline{G}) = 4$. $\square$

Even though there is an infinite family of graphs for which Theorem 3.8 holds. The converse of Theorem 3.8 does not always hold. Notice if $G = C_4$, then $\gamma_I(G) = 2$, $\overline{G} = 2K_2$, and $\gamma_I(\overline{G}) = 4$. We have the following.

**Theorem 3.10.** *If $G = C_4$, then $\gamma_I(G\overline{G}) = 4$.*

*Proof.* Consider $G\overline{G}$ where $G = C_4$. Let the vertices of $G$ and $\overline{G}$ be labeled as shown in Figure 6. Consider the Hamiltonian path $v_1, \overline{v}_1, \overline{v}_3, v_3, v_2, \overline{v}_2, \overline{v}_4, v_4$. Beginning at

$v_1$ label this vertex 1, and proceed by labeling every other vertex with a 1. This produces the assignments shown in Figure 7. Thus, $\gamma_I(G\overline{G}) \leq 4$. Equality follows from Corollary 3.7. Hence, $\gamma_I(G\overline{G}) = 4$. □

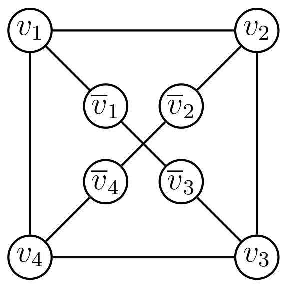


Figure 6: Complementary prism where $G = C_4$

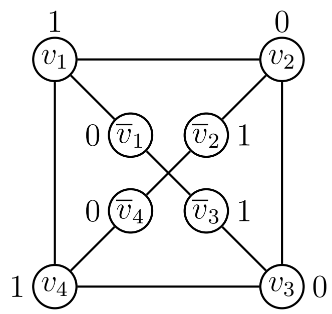


Figure 7: Italian domination of complementary prism where $G = C_4$

For the following lemma we need to define a family of graphs $\mathcal{F}$, from [9] we have that, $\mathcal{F} = \{G \,|\, G \in \{P_2, P_3\}\} \cup \{G \,|\, G$ is a graph with an induced $P_4$ such that every vertex in $G - P_4$ is adjacent to the support vertices of $P_4$ and not adjacent to the leaves of $P_4\}$.

**Lemma 3.11.** *[9] Let $G$ be a graph. Then $\gamma_{\times 2}(G\overline{G}) = 4$ if and only if $G \in \mathcal{F}$.*

Notice that $P_2 = K_2$, and by Theorem 3.6, we have $\gamma_I(P_2\overline{P}_2) = 3$. Also notice that $\gamma_I(\overline{P}_3) = 3$ and $\overline{P}_3$ has an isolated vertex. Then by Theorem 3.8, we already know that, $\gamma_I(P_3\overline{P}_3) = 4$. We define a family of graphs $\mathcal{G} = \{G \,|\, G$ is a graph with an induced $P_4$ such that every vertex in $G - P_4$ is adjacent to the support vertices of $P_4$ and not adjacent to the leaves of $P_4\}$ such that we have the following.

**Theorem 3.12.** *If $G \in \mathcal{G}$, then $\gamma_I(G\overline{G}) = 4$.*

*Proof.* Recall that $\gamma_I(G) \leq \gamma_{\times 2}(G)$ when $G$ has no isolated vertices, and by definition $G\overline{G}$ will be isolate free. Notice that $\mathcal{G} \subset \mathcal{F}$. Thus, by Corollary 3.7 together with Lemma 3.11, we have that, $4 \leq \gamma_I(G\overline{G}) \leq \gamma_{\times 2}(G\overline{G}) = 4$. Ergo, we have the desired result that $\gamma_I(G\overline{G}) = 4$. □

Once again, we have that the converse of Theorem 3.11 is not necessarily true. For example, by Theorem 3.9, $\gamma_I(C_4\overline{C}_4) = 4$ but $C_4 \notin \mathcal{G}$.

### 3.3 Specific Families

Notice when $G = K_1, K_2, K_3$, by Theorems 3.4, 3.6, 3.8 (respectively), we have that $\gamma_I(G\overline{G}) = 2, 3, 4$ (respectively). This prompts us to determine what the Italian domination number of a complementary prism when $G = K_n$ (see Figure 8) and inspires the following.

**Theorem 3.13.** *If $G = K_n$ or $\overline{G} = K_n$, then $\gamma_I(G\overline{G}) = n + 1$.*

*Proof.* Without loss of generality, let $G = K_n$, where $n \geq 1$. Then $\overline{G}$ is the empty graph on $n$ vertices. Now consider $G\overline{G}$. So, for every $\overline{v} \in V(\overline{G})$, its open neighborhood will only consist of its corresponding vertex in $G$. Notice that a function assigning a 1 to each $\overline{v} \in V(\overline{G})$, a 1 to any vertex $v \in V(G)$, and a 0 to all other vertices is indeed an Italian dominating function of $G\overline{G}$. Hence, $\gamma_I(G\overline{G}) \leq n+1$.

Now, we must show $\gamma_I(G\overline{G}) \geq n+1$. Let $f$ be a $\gamma_I$-function of $G$. It is clear that for every vertex in $V(\overline{G})$, either $f(\overline{v}) \geq 1$ or $f(v) = 2$. This implies that $\gamma_I(G\overline{G}) \geq n$. Note that if $f$ has total weight $n$, then each vertex in $V(\overline{G})$ is assigned a 1 under $f$, and every vertex in $V(G)$ is assigned a 0. But then the vertices in $G$ are not Italian dominated by $f$. Hence, $\gamma_I(G\overline{G}) \geq n+1$ and so $\gamma_I(G\overline{G}) = n+1$ as desired. $\square$

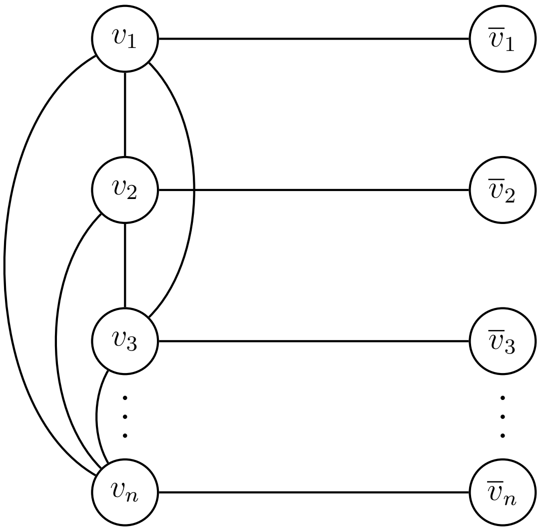


Figure 8: Complementary prism where $G = K_n$

For the complementary prism of any graph we observe the following lower bound.

**Observation 3.14.** *For any graph $G$, $\gamma_I(G\overline{G}) \geq max\ \{\gamma_I(G), \gamma_I(\overline{G})\}$.*

Next we show that the bound of Observation 3.14 is sharp for complementary prisms with $G = mK_2$, where $m \geq 2$.

**Theorem 3.15.** *If $G = mK_2$ with order $n$, where $m \geq 2$, then $\gamma_I(G) = n$ and $\gamma_I(G\overline{G}) = n$.*

*Proof.* Let $n$ be the order of $G$. By Theorem 3.4, $\gamma_I(K_2) = 2$. Thus if $G = mK_2$, where $m \geq 2$, then $\gamma_I(G) = m\gamma_I(K_2) = n$.

First let $m = 2$. Notice $\overline{G} = C_4$. And by Theorem 3.10, $\gamma_I(G\overline{G}) = 4$ and we have the desired result. Hence we may assume $m \geq 3$ which means $n = 2m \geq 2(3) = 6$. Label the vertices of $G$ as shown in Figure 9. Notice the function that assigns a 1 to each $v_k$ with odd $k \leq n-1$, a 1 to each $\overline{v_l}$ with even $l \leq n$, and a 0 otherwise is indeed an Italian dominating function of $G\overline{G}$. Thus, $\gamma_I(G\overline{G}) \leq \frac{n}{2} + \frac{n}{2} = n$

Finally, we must show $\gamma_I(G\overline{G}) \geq n$. First note that $\overline{G}$ has $n$ vertices and since the degree of every $v \in V(G)$ is 1, we have the degree of every vertex in V($\overline{G}$) to be $n-1-1 = n-2$. Clearly if you assign any two non-adjacent vertices in $\overline{G}$ a 1 and assign a 0 to all other vertices you have $\gamma_I$-function for $\overline{G}$ and $\gamma_I(\overline{G}) = 2$. By Observation 3.14, $\gamma_I(G\overline{G}) \geq \max\ \{\gamma_I(G), \gamma_I(\overline{G})\} = \gamma_I(G) = n$. Ergo, $\gamma_I(G\overline{G}) = n$. $\square$

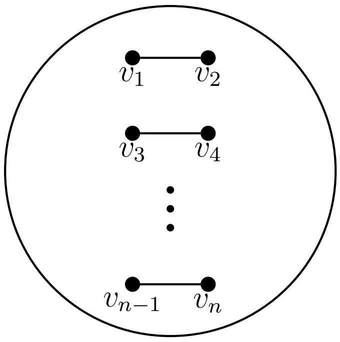


Figure 9: Vertex labeling of $G = mK_2$

For the complementary prism of any graph, we observe the following upper bound on the Italian domination number.

**Observation 3.16.** *For any graph $G$, $\gamma_I(G\overline{G}) \leq \gamma_I(G) + \gamma_I(\overline{G})$.*

Now we show that the bound of Observation 3.16 is sharp for complementary prisms with $G = K_{m,l}$, where $m, l \geq 5$.

**Theorem 3.17.** *If $G = K_{m,l}$, where $m, l \geq 5$, then $\gamma_I(G\overline{G}) = 8$.*

*Proof.* Let G=$K_{m,l}$, where $m, l \geq 5$ and let the bipartite set containing $m$ number of vertices be called $M$ and the bipartite set containing $l$ number of vertices be $L$. Then $\overline{G}$ is two disjoint complete graphs $K_m$ and $K_l$. Note that the function assigning a 1 to any two vertices in each of $V(M)$, $V(L)$ ,$V(K_m)$, $V(K_l)$, and a 0 to all other vertices is indeed an Italian dominating function of $G\overline{G}$. Thus, $\gamma_I(G\overline{G}) \leq 8$.

Assume $\gamma_I(G\overline{G}) \leq 7$. Let $f$ be a $\gamma_I(G\overline{G})$-function that has a weight of 7. By the Pigeonhole Principle, $f$ will assign at least one of $M$, $L$, $K_m$, $K_l$ a weight of 0 or 1. First assume, without loss of generality, that $f$ assigns $K_m$ a weight of 0. Then the vertices in $M$ must Italian dominate their corresponding vertices in $K_m$. Hence, $f$ assigns each vertex in $M$ a 2 giving $M$ a weight of at least 10, a contradiction. Thus, the function $f$ must assign a weight of at least 1 to both $K_m$ and $K_l$. Now assume, without loss of generality, that $f$ assigns $K_m$ a weight of 1. Let $f$ assign any one vertex, say $\overline{v}$, in $K_m$ a 1 and a 0 to all other vertices in $K_m$. To Italian dominate the vertices assigned a 0 in $K_m$, we must assign their corresponding vertices in $M$ at least a 1, implying that the weight of $M$ is at least four by our choice of $m$. Since $f$ assigns a weight at least 1 to $K_l$, at most one other vertex in $G\overline{G}$ can be assigned a 1. Notice that $v$ has not been Italian dominated. To Italian dominate $v$, $f$ either assigns $v$ a 1 or a vertex in $L$ a 1. If $f$ assigns $v$ a 1, then $l-1$ vertices in $K_l$ are not Italian dominated, a contradiction. So, $f$ assigns a vertex in $L$ a 1. Then at least $l-2$ in $K_l$ are not Italian dominated, a contradiction. Therefore, $f$ must assign both

$K_m$ and $K_l$ at least a weight of 2.

Now assume, without loss of generality, that $M$ is assigned a weight of 0. Then the vertices in $L$ must either Italian dominate themselves or the vertices of $K_l$ must Italian dominate them.

For each $v \in L$, if $v$ Italian dominates itself, then a weight of at least one is added, and if $\overline{v} \in V(K_l)$ Italian dominates $v$, then a weight of at least 2 is added. Hence, $\gamma_(G\overline{G}) \geq 4 + 5$, a contradiction. Thus, $M$ and $L$ must both be assigned at least a weight of 1. Assume, without loss of generality, that $M$ has a weight of 1. Let $f$ assign any vertex in $M$ a 1, say $v$, and all other vertices in $M$ a 0. Since $f$ assigns a weight of at least 1 to $L$ and a weight of at least 2 to both $K_m$ and $K_l$, exactly one other vertex can be assigned 1 under $f$. Notice that $v$ and at most two other vertices of $M$ are Italian dominated. If $L$ is assigned only a weight of 1, then each of the $m-1$ vertices assigned a 0 in $M$ must have their corresponding vertices assigned a weight of at least 1 in $K_m$. This means at least 4 vertices in $K_m$ are assigned a 1 by choice of $m$, a contradiction. Thus $L$ must be assigned a weight of 2. But then at least $l-4$ vertices in $L$ assigned 0 under $f$ are not Italian dominated, a contradiction. Therefore, $\gamma_I(G\overline{G}) = 8$ as desired. □

Notice that Observation 3.16 together with Lemma 3.5 gives the following bounds for paths $\gamma_I(P_n\overline{P}_n) \leq \lceil \frac{n+1}{2} \rceil + 3$ and $\gamma_I(C_n\overline{C}_n) \leq \lceil \frac{n}{2} \rceil + 3$ for cycles. Our next two results show that we can do better.

**Theorem 3.18.** *If $G$=$P_n$, where $n \geq 4$, then $\gamma_I(G\overline{G}) \leq \lceil \frac{n+3}{2} \rceil$.*

*Proof.* Let $n$ be the order of $G$ and let $G = P_n$, where $n \geq 4$. Label the vertices as shown in Figure 10, where $\overline{v}_3$ and $\overline{v}_{n-2}$ are adjacent only if they are not adjacent in

$G$. Notice that a function assigning a 1 to $\overline{v}_1$, a 1 to $\overline{v}_n$, and an Italian domination of the path $P_{n-2}$ with weight $\lceil\frac{(n-2)+1}{2}\rceil$ to the path $v_2, ...v_{n-1}$ in which $v_2$ and $v_{n-1}$ are assigned a 1, and a 0 otherwise is an Italian dominating function of $G\overline{G}$. Thus, $\gamma_I(G\overline{G}) \leq \lceil\frac{n-1}{2}\rceil + 2 = \lceil\frac{n-1}{2} + \frac{4}{2}\rceil = \lceil\frac{n-1+4}{2}\rceil = \lceil\frac{n+3}{2}\rceil$. □

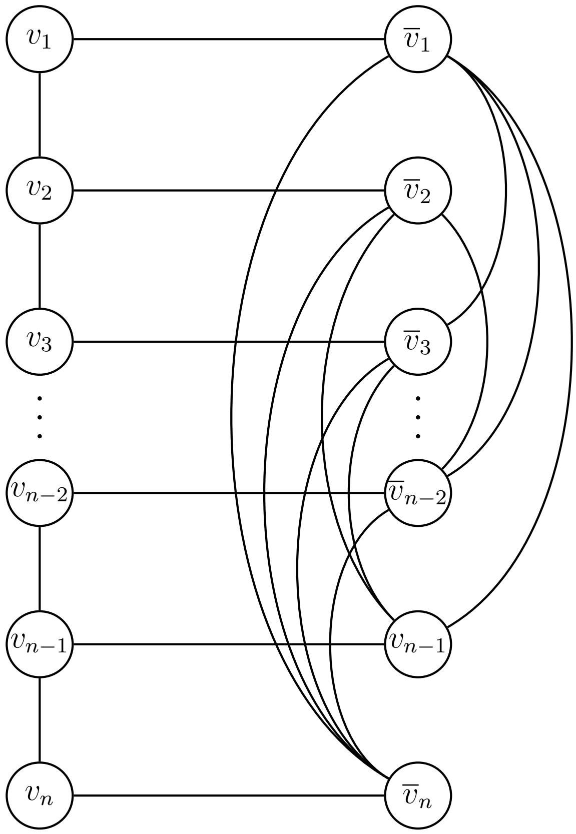


Figure 10: Complementary prism where G=$P_n$

**Theorem 3.19.** *If G=$C_n$, where $n \geq 4$, then $\gamma_I(G\overline{G}) \leq \lceil\frac{n+3}{2}\rceil$.*

*Proof.* Let $n$ be the order of $G$ and let $G = C_n$, where $n \geq 4$. Label the vertices as shown in Figure 11, where $\overline{v}_3$ and $\overline{v}_{n-2}$ are adjacent only if they are not adjacent in $G$. Notice that a function assigning a 1 to $\overline{v}_1$, a 1 to $\overline{v}_n$, and an Italian domination of the path $P_{n-2}$ with weight $\lceil\frac{(n-2)+1}{2}\rceil$ to the path $v_2, ...v_{n-1}$ in which $v_2$ and $v_{n-1}$ are assigned a 1, and a 0 otherwise is a $\gamma_I$-function of $G\overline{G}$. Thus,

$\gamma_I(G\overline{G}) \leq \lceil\frac{n-1}{2}\rceil + 2 = \lceil\frac{n-1}{2} + \frac{4}{2}\rceil = \lceil\frac{n-1+4}{2}\rceil = \lceil\frac{n+3}{2}\rceil$. □

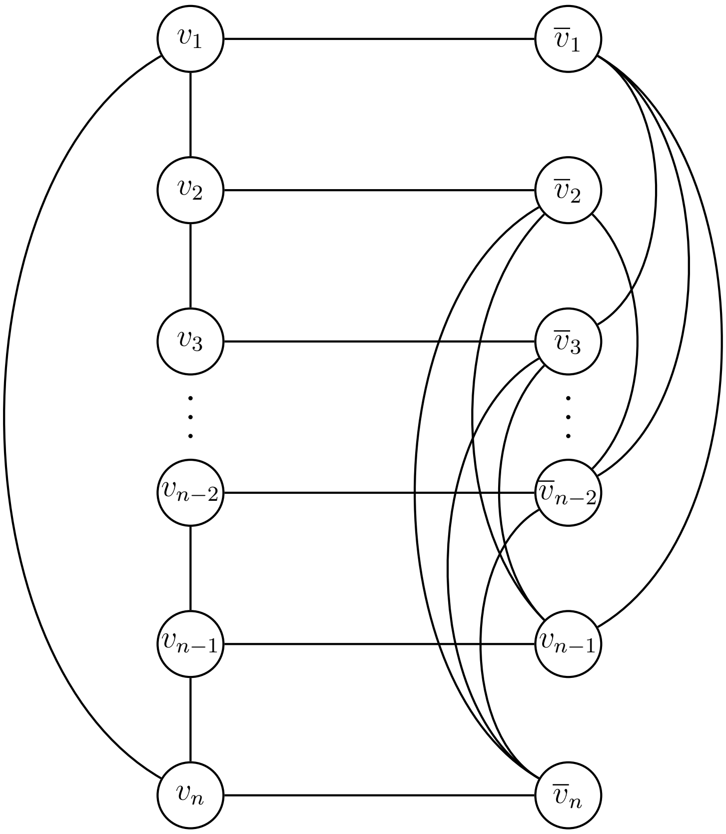


Figure 11: Complementary prism where G=$C_n$

## 3.4 Concluding Remarks

The investigation of domination parameters of complementary prisms is still on going. We conclude this thesis by mentioning some open questions and problems suggested by this research.

1. Characterize the graphs $G$ with $\gamma_I(G\overline{G}) = 4$.

2. Characterize the graphs attaining the bounds in Observation 3.14 and Observation 3.16.

3. Characterize the I2/ I1 graphs.

4. It was observed in the Introduction that a graph that is I2 is Italian, but not every Italian graph is I2. Characterize Italian graphs.

5. Given that a graph $G$ is I1, I1a, or I2 what can be said about relating $\gamma_I(G)$ and the following domination parameters $\gamma_2(G)$, $\gamma_R(G)$, and $\gamma_{\times 2}(G)$?

6. Since complementary prisms by definition have no isolated vertices, an interesting problem is to consider the relationship between $\gamma_{\times 2}(G\overline{G})$ and $\gamma_R(G\overline{G})$.

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