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Neighborhood-Restricted Achromatic Colorings of Graphs

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Neighborhood-Restricted Achromatic Colorings of Graphs

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by

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ABSTRACT

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A (closed) neighborhood-restricted [≤ 2]-coloring of a graph G is an assignment of colors to the vertices of G such that no more than two colors are assigned in any closed neighborhood. In other words, for every vertex v in G, the vertex v and its neighbors are in at most two different color classes. The [≤ 2]-*achromatic number* is defined as the maximum number of colors in any [≤ 2]-coloring of G. We study the [≤ 2]-achromatic number. In particular, we improve a known upper bound and characterize the extremal graphs for some other known bounds.

DEDICATION

I would like to dedicate my thesis to the two most important people in my life, Rebecca Lynn and James Dustin Jr.

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1 INTRODUCTION

1.1 Introduction to Graph Theory

A *graph* $G = (V, E)$ consists of a finite vertex set, $V(G)$, and a finite edge set, $E(G)$. The *order* of a graph G, denoted $n(G)$, is the number of vertices in G, and the *size* of a graph G, denoted $m(G)$, is the number of edges in G; that is, $n(G) = |V(G)|$ and $m(G) = |E(G)|$. If G is clear from the context, we generally use V, E, m, and n. Two vertices u and v are *adjacent* if there is an edge in E, denoted $uv \in E$, connecting u and v. We say that the vertices $u, v \in V$ are *incident* with edge uv. Further, we consider only *simple graphs* where the edges of G do not have a direction component and there are no instances of multiple edges connecting the same two vertices u and v. The *complement* of G, denoted $\overline{G}$, is the graph with $V(G) = V(\overline{G})$ where two vertices are adjacent if and only if they are not adjacent in G. Thus, $E\left(\overline{G}\right) = \overline{E\left(G\right)}$. A Nordhaus-Gaddum type result is a result wherein there is an upper bound on the sum or product of a parameter on G and $\overline{G}$. For any $v \in V$, we denote the graph formed by removing v and all of its incident edges by $G - v$.

For two vertices $u, v \in V$, a *u-v walk* W is a sequence of vertices in G, beginning with u and ending with v, such that the consecutive vertices in W are adjacent in G. A *path* is a walk in which no vertex is repeated. The *distance d(u,v)* between two vertices $u, v \in V$ is the minimum of the lengths of all u-v paths in G. The maximum distance from v to the other vertices of G is called the *eccentricity* of v, $e(v)$; that is, $e(v) = max\{d(u, v)|u \in V\}$. The *diameter* of G, *diam(G)*, is the maximum eccentricity among all the vertices of G. A graph that has a u-v path for all $u, v \in V$ is a *connected graph.*

For a vertex $v \in V$, the set $N(v) = \{u \in V \mid uv \in E\}$ is called the *open neighborhood* of v where $N(v)$ is the set of all vertices adjacent to v in G. Each vertex $u \in N(v)$ is called a *neighbor* of v. The *closed neighborhood* of a vertex v, $N[v]$, is the set of all vertices adjacent to v and v itself. That is, $N[v] = N(v) \cup \{v\}$. The *open neighborhood of a set* $S \subseteq V$ is $N(S) = \bigcup_{v \in S} N(v)$, and the *closed neighborhood of a set* $S \subseteq V$ is $N[S] = \bigcup_{v \in S} N[v]$. The *degree* in G of a vertex v is $deg_G(v) = |N(v)|$; if G is clear from the context then we use $deg(v)$. A vertex v with $deg(v) = 1$ is called a *leaf.* The neighbor of a leaf is called a *support vertex;* a support vertex with more than one leaf neighbor is called a *strong support vertex.*

A *path* P_n is a graph with $V = \{v_1, v_2, ..., v_n\}$ and $E = \{v_i v_{i+1} \mid i = 1, 2, ..., n-1\}$. A *cycle* C_n of order $n \geq 3$ is a graph with $V = \{v_1, v_2, ..., v_n\}$ and $E = \{v_i v_{i+1 \bmod n} \mid i = 1, 2, ..., n\}$. A graph in which every two distinct vertices are adjacent is called a *complete graph* K_n. A connected graph that contains no cycles is a *tree* T. A *star* $S_{1,n-1}$ is a tree with exactly one support vertex and $n-1$ leaves, that is, a star $S_{1,n-1}$ is a tree with diameter 2. A *double star* $S_{r,s}$ is a tree with diameter 3, that is, $S_{r,s}$ has two support vertices $u, v \in V$ such that $uv \in E$ and u has r leaf neighbors while v has s leaf neighbors. The *corona* $G \circ K_1$, denoted *cor*(G), is formed from a graph G by attaching a new vertex v' adjacent to v for each $v \in V(G)$.

A set $S \subseteq V$ is a *dominating set* of G if every vertex $v \in V$ is adjacent to a vertex in S. The minimum cardinality of all possible dominating sets of G is called the *domination number* $\gamma(G)$ of G. A set $S \subseteq V$ is a 2-*packing set* of a graph G if for every $u, v \in S$, $d(u, v) \geq 3$. The 2-*packing number*, $\rho(G)$, is the maximum cardinality of all such 2-packing sets. A dominating set with cardinality $\gamma(G)$ is called a $\gamma(G)$-set, and a 2-packing set with cardinality $\rho(G)$ is called a $\rho(G)$-set. A dominating set S of G is called an *efficient dominating set* if it is also a 2-packing of G. It was shown by Bange et al. in [1] that if a graph G has an efficient dominating set S, then $|S| = \gamma(G)$.

A *coloring* of a graph G is a partitioning of the vertex set V into color classes. A *proper coloring* of the vertices of a graph G assigns a color to each vertex of G in such a way that no two adjacent vertices have the same color. The *chromatic number* $\chi(G)$ is the minimum number of colors required in any proper coloring of G. Similarly, a *proper achromatic coloring* of a graph G assigns colors to each vertex of G such that for each color class C_i, $N[C_i]$ contains representatives of every color class. The maximum number of color classes in a proper achromatic partition of G is the achromatic number of G, and is denoted $\psi(G)$.

Let $\pi = \{V_1, V_2, \ldots, V_k\}$ be a partition of the vertices V of a graph G into distinct color classes V_i. For ease of discussion, if the vertices of a set S are assigned colors, then we say that S *contains* these assigned colors. Let $deg_\pi[v] = |\{i : N[v] \cap V_i \neq \emptyset\}|$; that is, $deg_\pi[v]$ equals the number of different colors assigned to vertices in the closed neighborhood of v by the partition π. A (neighborhood-restricted) $[\leq k]$*-coloring* of G is a π partition of the vertices of G wherein $deg_\pi[v] \leq k$ for all $v \in V$ [5]; that is, every closed neighborhood contains at most k different colors. Figure 1 is an example of a $[\leq k]$-coloring. The $[\leq k]$*-achromatic number* $\psi_{[\leq k]}(G)$ is the maximum order of a $[\leq k]$-coloring of G; that is, $\psi_{[\leq k]}(G)$ is the maximum number of colors in any $[\leq k]$-coloring of G. If π is a $[\leq k]$-coloring of G with $\psi_{[\leq k]}(G)$ colors, then we say that

π is a $\psi_{[\leq k]}(G)$-coloring. Note that the trivial partition $\pi = \{V\}$ is a $[\leq k]$-coloring for every integer $k \geq 1$, so $\psi_{[\leq k]}(G) \geq 1$ is defined for all graphs G and all positive integers k.

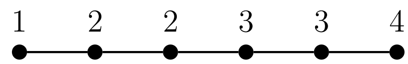

Figure 1: Achromatic coloring of the graph P_6

The main focus in this thesis is to consider the special case of $[\leq k]$-colorings where $k = 2$. We develop a Nordhaus-Gaddum type result for $\psi_{[\leq 2]}(G)$ and improve upon a known upper bound for $\psi_{[\leq 2]}(G)$. We further characterize all extremal trees in terms of a previously established upper bound on $\psi_{[\leq 2]}(G)$ in terms of n.

Bujtás, Sampathkumar, Tuza, Subramanya, and Dominic [3] considered *3-consecutive C-colorings*, which they defined to be a mapping $\phi : V(G) \to \mathbb{N}$ such that there exists no 3-colored path in G. This restriction is equivalent to our restriction of the number of distinct colors present in the closed neighborhood of a vertex v for the special case where $k = 2$. They gave the following upper bound on $\psi_{[\leq 2]}(G)$.

Theorem 2.1 [3] *For any graph $G = (V, E)$ of order n and minimum degree δ, we have $\psi_{[\leq 2]}(G) \leq \left\lfloor \frac{2n}{\delta+1} \right\rfloor$.*

In a graph $G = (V, E)$, a set $S \subset V$ is a *neighborhood set* if $\cup_{v \in S} \langle (N[v]) \rangle = G$, where $\langle N(v) \rangle$ is the subgraph induced by $N[v]$, the closed neighborhood of v. The *neighborhood number* of a graph G, denoted by $n_0(G)$, is the minimum cardinality of a neighborhood set in G.

Theorem 2.2 [3] *Let G be a connected graph. Then, $\psi_{[\leq 2]}(G) \leq n_0(G) + 1$. Further, for a tree T, $\psi_{[\leq 2]}(T) = n_0(T) + 1$.*

Theorem 2.3 [3] *For any connected graph G, $\psi_{[\leq 2]}(G) \leq 2\gamma(G)$.*

Theorem 2.4 [3] *A connected graph $G = (V, E)$ has a 3-consecutive C-coloring with exactly three colors; that is, $\psi_{[\leq 2]}(G) \geq 3$ if and only if its diameter is at least 3.*

And finally, Bujtás et al. in [3] showed that determining whether a graph G has $\psi_{[\leq 2]}(G) = 3$ or $\psi_{[\leq 2]}(G) = 4$ is solvable in polynomial time.

Bujtás, Sampathkumar, Tuza, Dominic, and Pushpalatha [2] considered the case where the star subgraph for each vertex v contains at most k colors. This restriction is equivalent to our restriction on the number of colors present in $N[v]$ for all $v \in G$, $k \in \mathbb{N}$.

Goddard and Xu [6] expanded on the work in [3], calling the colorings forbidden rainbow colorings. A subgraph is said to be *rainbow* if under a given coloring, its vertices receive distinct colors. A coloring having no rainbow subgraph F is called a *no-rainbow-F coloring* [6]. In the particular case where F is a P_3, a no-rainbow-P_3 coloring is equivalent to a neighborhood-restricted $[\leq 2]$-achromatic coloring. More generally, for $F = K_{1,k}$, a no-rainbow-$K_{1,k}$ coloring is equivalent to a neighborhood-restricted $[\leq k]$ achromatic coloring. Goddard and Xu [6] defined the maximum cardinality of a no-rainbow-F coloring of a graph G as the *F-upper chromatic number* of G, denoted $NR_F(G)$. Thus, $NR_{K_{1,k}}(G) = \psi_{[\leq k]}(G)$, and $NR_{P_3}(G) = \psi_{[\leq 2]}(G)$. Goddard and Xu [6] gave the following bound on $\psi_{[\leq 2]}(G)$ in terms of the diameter of G and the order of G.

Theorem 2.5 [6] *For any graph G, $\psi_{[\leq 2]}(G) \geq \frac{diam(G)}{2} + 1$, and for any non-empty graph G, $\psi_{[\leq 2]}(G) \geq \rho(G) + 1$.*

Theorem 2.6 [6] *For a connected graph G of order n, $\psi_{[\leq 2]}(G) \leq \lfloor n/2 \rfloor + 1$.*

Theorem 2.7 [6] *For a connected graph G of order n, then $\psi_{[\leq 2]}(cor(G)) = |n| + 1$.*

To build on the previous complexity result in [3], Goddard and Xu [6] showed that computing the P_3-upper chromatic number of G is equivalent to computing the packing number of G. Thus, computing $NR_{P_3}(G)$ is NP-hard.

3 MAIN RESULTS

3.1 Background and Aims

The following bounds in terms of diameter are known.

Observation 3.1 [5, 6] *For any connected graph G with diameter $diam(G)$,*

(i) $\psi_{[\leq 2]}(G) \geq \lceil diam(G)/2 \rceil + 1$, *and*

(ii) $\psi_{[\leq 3]}(G) \geq diam(G) + 1$.

Theorem 3.2 [3] *A nontrivial connected graph G has $\psi_{[\leq 2]}(G) = 2$ if and only if $diam(G) \leq 2$.*

In Section 2, we consider the diameter of graphs and determine some Nordhaus-Gaddum type results for $\psi_{[\leq 2]}(G)$. Another lower bound in terms of the 2-packing number is found in [6].

Theorem 3.3 [6] *For a graph G, $\psi_{[\leq 2]}(G) \geq \rho(G) + 1$.*

The graphs attaining the bound of Theorem 3.3 were characterized in [5] as follows.

Theorem 3.4 [5] *For any isolate-free graph G, $\psi_{[\leq 2]}(G) \geq \rho(G) + 1$ with equality if and only if G has a $\psi_{[\leq 2]}(G)$-coloring in which at least one color class dominates G.*

The following upper bound on $\psi_{[\leq 2]}(G)$ in terms of the domination number is given in [3].

Theorem 3.5 [3] *For any graph G, $\psi_{[\leq 2]}(G) \leq 2\gamma(G)$.*

It is known [7] that the 2-packing number is a lower bound on the domination number of any graph G, that is, $\rho(G) \leq \gamma(G)$. In this section, we will characterize the graphs attaining the bound of Theorem 3.5 and improve the bound by showing that, in fact, $\psi_{[\leq 2]}(G) \leq 2\rho(G)$. Hence, we have that $\rho(G) + 1 \leq \psi_{[\leq 2]}(G) \leq 2\rho(G)$. We show every value in this range can be achieved by trees.

An upper bound on $\psi_{[\leq 2]}(G)$ in terms of the order n of a graph G was determined by Goddard, et al. [6].

Theorem 3.6 [6] *For a connected graph G of order n, $\psi_{[\leq 2]}(G) \leq \lfloor (n+2)/2 \rfloor$.*

Figure 2 gives another example of a $[\leq k]$-coloring of the graph $K_4 \circ K_1$. Since $\rho(K_4 \circ K_1) = 4$ and $n = 8$, Theorem 3.6 and Theorem 3.3 give that $\psi_{[\leq 2]}(K_4 \circ K_1) = 5$. Thus, the coloring in Figure 2 is also a $\psi_{[\leq k]}(G)$-coloring.

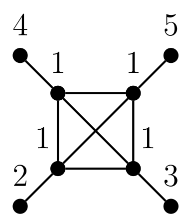

Figure 2: Achromatic coloring of the corona graph $K_4 \circ K_1$

In Section 3, we give a constructive characterization of the extremal trees for the bound of Theorem 3.6. Finally, in Section 4, we close with some open problems.

3.2 Diameter

First we obtain a bound on the $[\leq 2]$-achromatic number of G by considering the diameter of its complement $\overline{G}$. Note that the diameter of a disconnected graph G is defined to be $\operatorname{diam}(G) = \infty$.

Proposition 3.7 *If G is a graph and $diam(\overline{G}) \geq 3$, then $\psi_{[\leq 2]}(G) \leq 3$.*

Proof. Since $\operatorname{diam}(\overline{G}) \geq 3$, there exists two vertices, say u and v, in $\overline{G}$ that are at least distance 3 apart. In G, u and v are adjacent and $\{u, v\}$ dominates G. Let π be any $\psi_{[\leq 2]}(G)$-coloring. If u and v are colored the same color, say c_1, then any vertex of $N(u)$ can be colored at most one color different from c_1 and likewise for any vertex in $N(v)$. Hence, $\psi_{[\leq 2]}(G) \leq 3$. If u and v are colored different colors, say c_1 and c_2, then every vertex of $N(u) \cup N(v)$ must be colored c_1 or c_2 as well. Thus, $\psi_{[\leq 2]}(G) < 3$.
□

Theorem 3.2 and Proposition 3.7 imply the following.

Corollary 3.8 *If G is a nontrivial graph, then $\psi_{[\leq 2]}(G) = 2$ or $\psi_{[\leq 2]}(\overline{G}) \leq 3$.*

Our next result establishes a limit on the number of color classes in any $\psi_{[\leq 2]}(G)$-coloring that can be dominating sets.

Proposition 3.9 *For any $\psi_{[\leq 2]}(G)$-coloring of a graph G, at most two color classes are dominating sets of G. Furthermore, if two color classes dominate a connected graph G, then $\psi_{[\leq 2]}(G) = 2$.*

Proof. Clearly, if three color classes in any $\psi_{[\leq 2]}(G)$-coloring are dominating sets of G, every vertex in G has a least three different colors in its closed neighborhood. Thus, no $\psi_{[\leq 2]}(G)$-coloring has more than two color classes that dominate.

Assume that a $\psi_{[\leq 2]}(G)$-coloring has two dominating color classes, say V_1 and V_2. Then each vertex in V_i has a neighbor in V_{3-i}, implying that no vertex in V_i for $i \in \{1, 2\}$ has a neighbor in $V \setminus (V_1 \cup V_2)$. Since G is connected, it follows that $V \setminus (V_1 \cup V_2) = \emptyset$, and so $\{V_1, V_2\}$ is a partition of V. Hence, $\psi_{[\leq 2]}(G) = 2$. □

Proposition 3.9 and Theorem 3.2 imply that for a connected graph G with $\text{diam}(G) \geq 3$, a $\psi_{[\leq 2]}(G)$-coloring has at most one color class that dominates G.

Notice the operation of adding a new vertex and joining it to every vertex in an existing graph H yields a new graph G with $\psi_{[\leq 2]}(G) = 2$. Thus, for any graph H with $\psi_{[\leq 2]}(H) \geq 3$, there exists a graph G having H as an induced subgraph and $\psi_{[\leq 2]}(G) = 2 < \psi_{[\leq 2]}(H)$. On the other hand, let H be a graph having $\text{diam}(H) = 2$. By Theorem 3.2, $\psi_{[\leq 2]}(H) = 2$. Let u and v be vertices at distance 2 apart in H and add a new vertex, say v', and edge vv', to form graph G. Then $\text{diam}(G) \geq 3$, and by Theorem 3.2, $\psi_{[\leq 2]}(G) \geq 3 > \psi_{[\leq 2]}(H)$. Hence, there is no inequality between the $[\leq 2]$-achromatic number of a graph G and the $[\leq 2]$-achromatic number of an induced subgraph of G.

The following Nordhaus-Gaddum type results are proved for general k in [2]. We state the theorem for the special case of $k = 2$.

Theorem 3.10 [2] *For a graph G of order n and its complement $\overline{G}$, $\psi_{[\leq 2]}(G) + \psi_{[\leq 2]}(\overline{G}) \leq n + 3$.*

We note that if G is non-trivial, and both G and $\overline{G}$ are connected, then an improved Nordhaus-Gaddum type result follows directly from Theorem 3.6 and Corollary 3.8:

Corollary 3.11 *If G is non-trivial, and G and $\overline{G}$ are connected graphs of order $n \geq 2$, then $\psi_{[\leq 2]}(G) + \psi_{[\leq 2]}(\overline{G}) \leq \lfloor (n+2)/2 \rfloor + 3$.*

3.3 2-Packing Number

First we characterize the graphs attaining the bound of Theorem 3.5.

Theorem 3.12 *A graph G has $\psi_{[\leq 2]}(G) = 2\gamma(G)$ if and only if every $\gamma(G)$-set S is an efficient dominating set such that for every vertex $v \in S$, the following hold:*

1. *if $u \in N(v)$, then u is distance 2 from at most one vertex in $S \setminus \{v\}$, and*
2. *there exists a vertex $u \in N(v)$ such that $d(u, x) \geq 3$ for every $x \in V \setminus N[v]$.*

Proof. To characterize graphs attaining the bound of $2\gamma(G)$, assume that G is a graph with $\psi_{[\leq 2]}(G) = 2\gamma(G)$. Let $S = \{v_1, v_2, ..., v_\gamma\}$ be any $\gamma(G)$-set, and let π be a $\psi_{[\leq 2]}(G)$-coloring. Since S dominates G and every vertex of S can have at most two colors from π in its closed neighborhood, it follows that $N[v_i]$ contains exactly two colors and these colors are not contained in $V \setminus N[v_i]$ for $1 \leq i \leq \gamma(G)$. Hence, $N[v_i] \cap N[v_j] = \emptyset$ for all $v_i, v_j \in S$ for $i \neq j$. In other words, S is a 2-packing, and so S is an efficient dominating set. Among the vertices in $N(v_i)$ colored different from v_i, select one, say u_i. Since u_i and v_i are colored differently under π, every neighbor of u_i must be colored one of the two colors assigned to u_i and v_i, that is, $N[u_i] \subseteq N[v_i]$. In particular, u_i has no neighbor in $V \setminus N[v_i]$. To see that $d(u_i, x) \geq 3$ for all $x \in V \setminus N[v_i]$, note that if $d(u_i, x) = 2$ for some vertex $x \in V \setminus N[v_i]$, then the common neighbor of u_i and x, say w, is in $N(v_i)$. But then $N(w)$ contains three different colors under π, a contradiction. Now suppose that some vertex, say y, in

$N(v_i)$ is adjacent to a vertex in $N(v_j)$ and a vertex in $N(v_k)$, where i, j, and k are distinct. Then y has at least three colors in its closed neighborhood, a contradiction. Hence, no vertex in $N(v_i)$ is at distance 2 from two or more vertices in $S \setminus \{v_i\}$ for $1 \leq i \leq \gamma(G)$.

For the sufficiency, assume that $S = \{v_1, v_2, ..., v_k\}$ is an efficient dominating set of G. As proved in [1], $|S| = k = \gamma(G)$ and S is a packing. Assume that S satisfies the property of the theorem, that is, no vertex in $N(v_i)$ is distance 2 from two or more vertices in $S \setminus \{v_i\}$ for $1 \leq i \leq \gamma(G)$, and for every $v_i \in S$, there exists some $u_i \in N(v_i)$ such that $d(u_i, x) \geq 3$ for every $x \in V \setminus N[v_i]$. For $1 \leq i \leq k$, select such a u_i for v_i and assign the color i to the vertices in $N[v_i] \setminus \{u_i\}$ and the color $i+k$ to the vertex u_i. Note that for $1 \leq i \leq k$, $N[v_i]$ and $N[u_i]$ contain only the colors i and $i+k$. We claim that every vertex in $N(v_i) \setminus \{u_i\}$ also has at most two colors in its closed neighborhood. To see this, assume that $x_i \in N(v_i) \setminus \{u_i\}$. Clearly, if $N[x_i] \subseteq N[v_i]$, then $N[x_i]$ contains only the colors i and $i + k$ and the claim holds. First assume that x_i is adjacent to u_i. Since u_i is at distance three or more from every vertex in $V \setminus N[v_i]$, it follows that x_i has no neighbor in $V \setminus N[v_i]$, that is, $N[x_i] \subseteq N[v_i]$. Next assume that x_i is not adjacent to u_i. Thus, every vertex in $N[x_i] \cap N[v_i]$ is colored i. If x_i has no neighbor in $V \setminus N[v_i]$, then the claim holds. Thus, assume x_i has a neighbor $w_j \in N[v_j]$ for some $j \neq i$. Since S is a packing and x_i is at distance 2 from at most one vertex in $S \setminus \{v_i\}$, it follows that $N[x_i] \subseteq (N[v_i] \setminus \{u_i\}) \cup N(v_j)$. Further, by our choice of u_j, we deduce that $w_j \neq u_j$. Therefore, every vertex in $N[x_i]$ is colored either i or j, so $N[x_i]$ contains at most two colors. Hence, this coloring is a $[\leq 2]$-coloring with order $2|S| = 2\gamma(G)$. $\square$

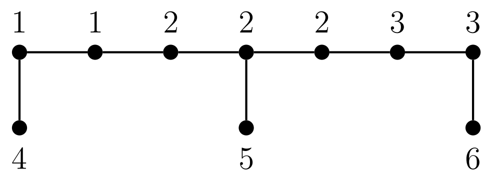

Figure 3: The graph G_3

For an example of a graph attaining the bound, consider the following graph G_k for $k \geq 2$ constructed as follows. Begin with the corona $P_k \circ K_1$ and subdivide each edge of the P_k exactly twice. See Figure 3 for an example of G_3. Then $\gamma(G_k) = k$ and the set of support vertices forms a $\gamma(G_k)$-set. Let $v_1, v_2, ..., v_k$ denote the support vertices. Coloring each v_i and its non-leaf neighbors color i for $1 \leq i \leq k$, and assigning color $k+i$ to the leaf neighbor of v_i yields an $\psi_{[\leq 2]}(G)$-coloring with $2k = 2\gamma(G)$ colors.

Recall that as mentioned in the introduction, the 2-packing number $\rho(G)$ is a lower bound on the domination number $\gamma(G)$ for any graph G. Next we improve the upper bound of Theorem 3.5.

Theorem 3.13 *For any graph G, $\psi_{[\leq 2]}(G) \leq 2\rho(G)$.*

Proof. Let S be a $\rho(G)$-set and π be a $\psi_{[\leq 2]}(G)$-coloring. Suppose, to the contrary, that $\psi_{[\leq 2]}(G) \geq 2\rho(G)+1$. We note that the vertices of S contain at most $\rho(G)$ colors of π. Accordingly, there are at least $\rho(G)+1$ color classes of π that do not contain a vertex of S. Let $V_1, V_2, \ldots, V_k$ where $k \geq \rho(G)+1$ denote the color classes of π that do not contain a vertex of S. We form a set A by selecting one vertex, say v_i, from each V_i, for $1 \leq i \leq k$, as follows: if $V_i \cap N(S) \neq \emptyset$, then let $v_i \in V_i \cap N(S)$, else let v_i be an arbitrary vertex of V_i. Thus, $|A| = k \geq \rho(G)+1$.

Note that since S is a maximum 2-packing, every vertex $v_i \in A$ is either in $N(S)$ or has a neighbor, say x_i, in $N(S)$. Let $v_i \in V_i$ and $v_j \in V_j$ be two arbitrary vertices of A. To show that A is a packing, we show that $d(v_i, v_j) \geq 3$. Let c_i denote the color of vertex v_i for all $v_i \in A$, and let $c(u)$ denote the color of vertex u, for all $u \notin A$.

Since $c_i \neq c_j$ and π is a $\psi_{[\leq 2]}(G)$-coloring, it follows that any common neighbor x of v_i and v_j must be colored either c_i or c_j; else $N[x]$ would contain at least three colors. We consider three cases:

Case 1. $\{v_i, v_j\} \subseteq N(S)$. Let $u \in N(v_i) \cap S$ and $w \in N(v_j) \cap S$. Since no vertex of V_i is in S, we have that $c(u) \neq c_i$. Thus, every vertex in $N(v_i)$ must be colored either $c(u)$ or c_i. Similarly, every vertex in $N(v_j)$ is colored either c_j or $c(w)$. Since $c_j \notin \{c_i, c(u)\}$ and $c_i \notin \{c_j, c(w)\}$, it follows that v_i and v_j are not adjacent. Further, if x is a common neighbor of v_i and v_j, then $c(x) \in \{c_i, c_j\}$. But $c_i \notin \{c_j, c(w)\}$ and $c_j \notin \{c_i, c(u)\}$, contradicting that x is a common neighbor of v_i and v_j. See Figure 4.

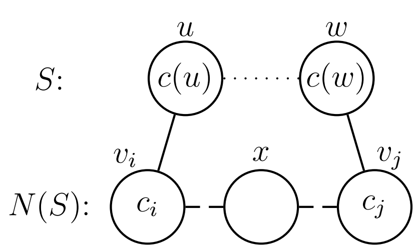

Figure 4: Theorem 3.13, Case 1

Case 2. Without loss of generality, $v_i \in N(S)$ and $v_j \in V \setminus N[S]$. Note that since $v_j \in V \setminus N[S]$, by the manner in which we constructed set A, $V_j \cap N[S] = \emptyset$, so no vertex of $N[S]$ is colored c_j. Since $v_i \in N(S)$, there exists some vertex $u \in S$ that is adjacent to v_i and $c(u) \notin \{c_i, c_j\}$. Further, every vertex in $N[v_i]$ is assigned either color c_i or $c(u)$ under π. Since $c_j \notin \{c_i, c(u)\}$, v_i and v_j are not adjacent. Moreover, v_j has neighbor x_j in $N(S)$ and $c_j \neq c(x_j)$, implying that every vertex in $N[v_j]$ is colored either c_j or $c(x_j)$. Also note that $c(x_j) \neq c_i$, else the neighbor of x_j in S must be colored either c_i or c_j, a contradiction. Now $c_i \notin \{c_j, c(x_j)\}$ and $c_j \notin \{c_i, c(u)\}$, implying that v_i and v_j have no common neighbor, z. See Figure 5.

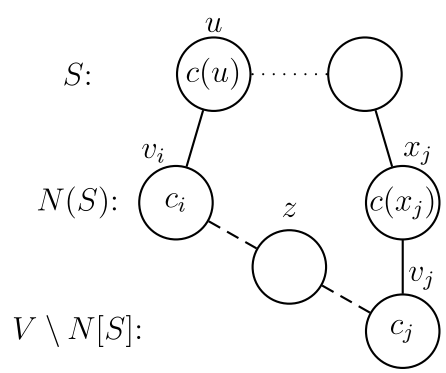

Figure 5: Theorem 3.13, Case 2

Case 3. Consider where $\{v_i, v_j\} \subseteq V \setminus N[S]$. By our construction of A, no vertex of $N[S]$ can be colored c_i or c_j. Again, v_i has a neighbor x_i in $N(S)$ and v_j has a neighbor x_j in $N(S)$. Since $c(x_i) \neq c_i$, every vertex of $N[v_i]$ is colored either c_i or $c(x_i)$. Similary, every vertex of $N[v_j]$ is colored either c_j or $c(x_j)$. Again, v_i and v_j are not adjacent, and since $c_i \notin \{c_j, c(x_j)\}$ and $c_j \notin \{c_i, c(x_i)\}$, they have no common neighbor, z. See Figure 6.

Therefore, in all three cases, $d(v_i, v_j) \geq 3$. Thus, A is a 2-packing of G with cardinality $k \geq \rho(G) + 1$, a contradiction. Hence, we conclude that $\psi_{[\leq 2]}(G) \leq 2\rho(G)$.

$\square$

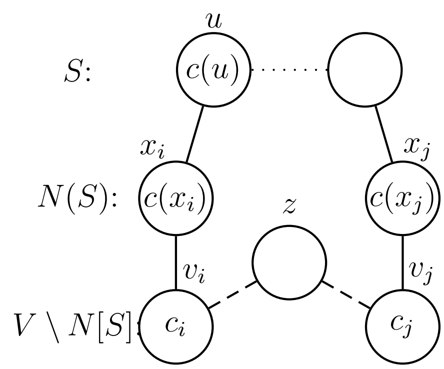

Figure 6: Theorem 3.13, Case 3

Together, Theorems 3.4 and 3.13 yield the following corollary.

Corollary 3.14 *For any graph G, $\rho(G) + 1 \leq \psi_{[\leq 2]}(G) \leq 2\rho(G)$.*

We next show that trees exist with $[\leq 2]$-achromatic number for every value in the range established by the bounds of Corollary 3.14.

Theorem 3.15 *Let a and b be positive integers such that $1 \leq a \leq b$. There exists a tree T such that $\rho(T) = b$ and $\psi_{[\leq 2]}(T) = a + b$.*

Proof. Let a and b be positive integers such that $1 \leq a \leq b$. Let T be the tree obtained from a $P_{3a} = v_1, v_2, ..., v_{3a}$ by adding a leaf vertex b_i to each v_i where $i \equiv 2 \,(mod\, 3)$ and attaching $b - a$ copies of P_2 attached to v_{3a}. See Figure 7 for an example where $a = 2$ and $b = 5$. It is straightforward to see that $\rho(T) = b$. Let π be an $\psi_{[\leq 2]}(T)$-coloring. Let B be the set of leaves labeled b_i. Note that $N[v_i]$ can contain at most two colors of π for each i where $i \equiv 2 \,(mod\, 3)$. Thus, at most $2a$ colors can be used on the vertices in $\{v_1, v_2, ..., v_{3a}\} \cup B$. For the added P_2's adjacent to v_{3a}, at most $b - a$ new colors are possible. Hence, $\psi_{[\leq 2]}(T) \leq 2a + b - a = a + b$.

Consider the $[\leq 2]$-coloring of T where the vertices of the P_{3a} are colored sequentially as follows 111222...aaa, the vertices of B are colored $a+1$ to $2a$, and the remaining vertices in the $N(v_{3a})$ are colored a while their adjacent leaves are colored $b-a$ new distinct colors. See Figure 7. This coloring has $a+a+b-a=a+b$ colors, implying that $\psi_{[\leq 2]}(T) \geq a+b$, and so, $\psi_{[\leq 2]}(T) = a+b$. □

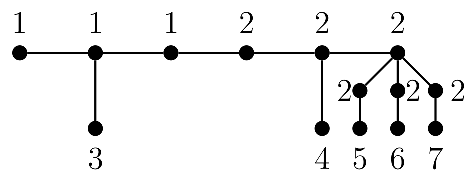

Figure 7: The tree T where $a=2$ and $b=5$

3.4 Extremal Trees for Theorem 3.6

In this section, we characterize the trees attaining the upper bound of Theorem 3.6. We say that two vertex sets $S, T \in V(G)$ are adjacent if there exists vertices $s \in S$ and $t \in T$ such that $st \in E(G)$. We first give two lemmas. We say that a vertex v is a *monochromatic vertex* under a coloring π if every vertex in $N[v]$ is in the same color class of π.

Lemma 3.16 *A graph G of order n for which $\psi_{[\leq 2]}(G) = \lfloor (n+2)/2 \rfloor$ has at most one monochromatic vertex in any $\psi_{[\leq 2]}(G)$-coloring.*

Proof. Suppose to the contrary that there exists some graph G of order n where $\psi_{[\leq 2]}(G) = \lfloor (n+2)/2 \rfloor$ and G has a $\psi_{[\leq 2]}(G)$-coloring π with monochromatic vertices

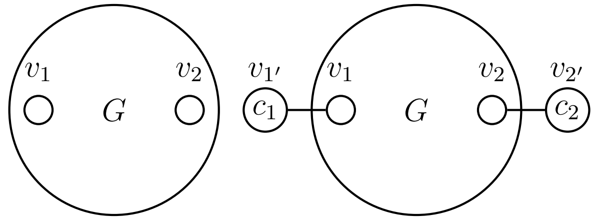

Figure 8: Consequences of having two monochromatic vertices

v_1 and v_2. We build the graph G' from G by adding vertices v_1' and v_2' and edges v_1v_1' and v_2v_2'. Then the coloring π for the vertices of G along with a new color each for v_1' and v_2' yields a $[\leq 2]$-coloring of G' with $\psi_{[\leq 2]}(G)+2 = \lfloor (n+2)/2 \rfloor + 2$ colors. See Figure 8. Thus, G' has order $n+2$ and $\psi_{[\leq 2]}(G') \geq \lfloor (n+2)/2 \rfloor + 2 > \lfloor ((n+2)+2))/2 \rfloor$, contradicting Theorem 3.6. $\square$

Lemma 3.17 *A tree T of order n with $\psi_{[\leq 2]}(T) = \lfloor (n+2)/2 \rfloor$ has at most one strong support vertex and that vertex supports exactly two leaves.*

Proof. Assume to the contrary that there exists some tree T of order n for which $\psi_{[\leq 2]}(T) = \lfloor (n+2)/2 \rfloor$, and T has either two strong support vertices or some support vertex adjacent to at least 3 leaves. Let π be a $\psi_{[\leq 2]}(T)$-coloring.

Case 1. T has two or more strong support vertices, say v_1 and v_2. Let $v_{i,1}$ and $v_{i,2}$ be two leaf vertices adjacent to v_i for $i \in \{1, 2\}$. By Lemma 3.16, we have that T has at most one monochromatic vertex under π. If a support vertex is monochromatic, then the adjacent leaves are also monochromatic, so neither v_1 nor v_2 is monochromatic. Moreover, at most one of their adjacent leaves is monochromatic. Hence, we may assume, without loss of generality, that each of $v_{1,2}$, $v_{2,1}$, and $v_{2,2}$ has at least two colors in their neighborhoods. This implies that v_2 is a different color from each of

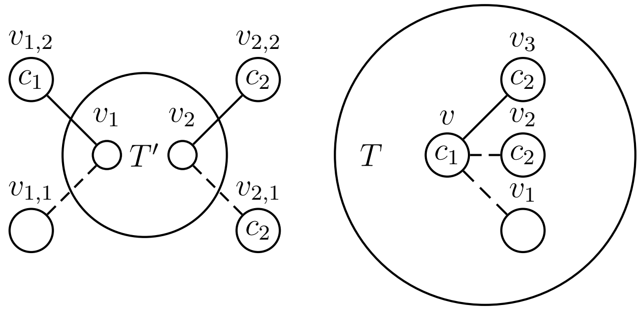

Figure 9: Consequences of strong support

$v_{2,1}$ and $v_{2,2}$. Thus, $v_{2,1}$ and $v_{2,2}$ are in the same color class. Also, v_1 and $v_{1,2}$ are in different color classes in π, and $v_{1,1}$ is in the same color class as either v_1 or $v_{1,2}$.

We now build T' from T by removing the two leaves, $v_{1,1}$ and $v_{2,1}$. See Figure 9. Let π' be the restriction of π on T'. Note that π' is an $[\leq 2]$-coloring of T'. Since $v_{1,1}$ is in the same color class as either v_1 or $v_{1,2}$, that color class is still represented in π'. Similarly, $v_{2,1}$ and $v_{2,2}$ are in the same color class in π, so that color class is also present in π'. Thus, $|\pi'| = |\pi| = \psi_{[\leq 2]}(T)$. Hence, $\psi_{[\leq 2]}(T') \geq |\pi'| = \psi_{[\leq 2]}(T) = \lfloor (n+2)/2 \rfloor$. However, by Theorem 3.6, we have $\psi_{[\leq 2]}(T') \leq \lfloor [(n+2)-2]/2 \rfloor = \lfloor n/2 \rfloor < \lfloor (n+2)/2 \rfloor = \psi_{[\leq 2]}(T)$, which is a contradiction. Thus, T does not have two or more strong support vertices.

Case 2. Let T have a unique strong support vertex v with at least three leaf neighbors, say v_1, v_2, and v_3. By Lemma 3.16, at most one of v_1, v_2, and v_3 is monochromatic. Without loss of generality, assume that at least v_2 and v_3 are not monochromatic. Hence, under π, v is in a different color class than v_2 and v_3, implying that v_2 and v_3 are in the same color class. Moreover, v_1 is either in the same color class as v or the same color class as v_2 and v_3.

Now we will construct T' from T by removing v_1 and v_2. See Figure 9. Let π' be π restricted to T'. Since v_1 is in the same color class under π as either v or v_3, that color class is still represented in π'. Similarly, v_2 and v_3 are in the same color class, so that color class is also present in π'. Thus, $\psi_{[\leq 2]}(T') \geq |\pi'| = |\pi| = \psi_{[\leq 2]}(T) = \lfloor (n+2)/2 \rfloor$. As before, $\psi_{[\leq 2]}(T') \leq \lfloor [(n-2)+2]/2 < \lfloor (n+2)/2 \rfloor = \psi_{[\leq 2]}(T)$, yielding the contradiction. Therefore, if T has a strong support vertex, then it is adjacent to exactly two leaves. $\square$

Definition. Let $f(T, v)$ be the function where v is a vertex of T and we add a P_2 with vertices v_a and v_b to T via edge vv_a. Let $\mathcal{F}$ be the smallest family of graphs such that: $\mathcal{F}$ contains K_1 and K_2, and is closed under f.

Theorem 3.18 *The family $\mathcal{F}$ is precisely the family of trees for which $\psi_{[\leq 2]}(T) = \lfloor (n+2)/2 \rfloor$.*

Proof. Note that K_1 and K_2 can trivially be colored with one and two colors, respectively, and $\psi_{[\leq 2]}(K_1) = 1 = \lfloor (1+2)/2 \rfloor$ and $\psi_{[\leq 2]}(K_2) = 2 = \lfloor (2+2)/2 \rfloor$. To show that every tree in $\mathcal{F}$ satisfies the equality, we proceed by induction. Assume T is a tree of order n in $\mathcal{F}$ with $\psi_{[\leq 2]}(T) = \lfloor (n+2)/2 \rfloor$. Let π be a $\psi_{[\leq 2]}(T)$-coloring, and let v be an arbitrary vertex of T. Form T' from T by applying $f(T, v)$, that is, adding a P_2 with vertices v_a and v_b to T via edge vv_a. Then T' is in $\mathcal{F}$ and T' has order $n' = n + 2$. Let v_a be in the same color class as v under π, and let v_b be in some new color class, say C_{v_b}. This produces a $[\leq 2]$-coloring for T' having $\psi_{[\leq 2]}(T) + 1$ colors, so $\psi_{[\leq 2]}(T') \geq \psi_{[\leq 2]}(T) + 1$. See Figure 10. By Theorem 3.6, $\psi_{[\leq 2]}(T') \leq \lfloor (n+4)/2 \rfloor = \lfloor (n+2)/2 \rfloor + 1 = \psi_{[\leq 2]}(T) + 1$, implying that $\psi_{[\leq 2]}(T') =$

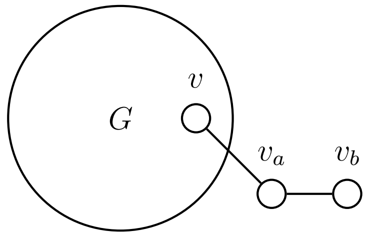

Figure 10: Tree characterization, Part 1

$\lfloor((n+2)+2)/2\rfloor$. Thus, f clearly preserves trees having $\psi_{[\leq 2]}(T) = \lfloor(n+2)/2\rfloor$, and every tree in $\mathcal{F}$ has $\psi_{[\leq 2]}(T) = \lfloor(n+2)/2\rfloor$.

To show that every tree that has $\psi_{[\leq 2]}(T) = \lfloor(n+2)/2\rfloor$ is in $\mathcal{F}$, we proceed by induction on the order of T. Since K_1 and K_2 are in $\mathcal{F}$, and $f(K_1, v) = P_3$ (with $\psi_{[\leq 2]}(P_3) = \lfloor(3+2)/2\rfloor = 2$), let T be a tree of order at least 4 with $\psi_{[\leq 2]}(T) = \lfloor(n+2)/2\rfloor$.

By Theorem 3.2, $\psi_{[\leq 2]}(G) = 2 < \lfloor(n+2)/2\rfloor$ for any star of order $n \geq 4$. Hence, we may assume that T is not a star, that is, $diam(T) \geq 3$. Assume that any smaller tree for which $\psi_{[\leq 2]}(T) = \lfloor(n+2)/2\rfloor$ is in $\mathcal{F}$. We next identify a set P of vertices in T that can be pruned to leave a tree T_p with $\psi_{[\leq 2]}(T_p) = \lfloor(n(T_p)+2)/2\rfloor$, and show that $f(T_p, v) = T$.

Choose a diametral path in T, labeling the vertices of this path as $v_1, v_2, ..., v_k$. If v_2 is a strong support vertex, then from Lemma 3.17, it is the only such vertex. In this case, relabel the diametral path with $v_1 = v_k, v_2 = v_{k-1}, ..., v_{k-1} = v_2, v_k = v_1$. We now observe that the degree of v_2 is 2, because v_2 has only v_1 as a leaf neighbor since it is not a strong support vertex and any neighbor other than v_3 would contradict our choice of a diametral path. Since T has at most one monochromatic neighborhood,

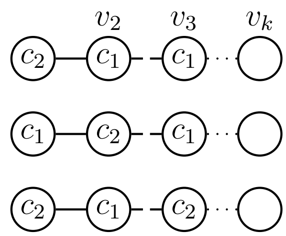

Figure 11: Tree characterization, Part 2

v_2 is not monochromatic. Thus, either v_1 and v_2 are in the same color class, or one of $\{v_1, v_2\}$ is in the same color class as v_3.

Let $P = \{v_1, v_2\}$. Then $T - P$ is a tree, say T_p, with order $n - 2$. In removing set P, we have removed exactly two vertices and at most one color class from a coloring of T, since either v_1 and v_2 are in the same color class or v_3 is a representative of the color class of either v_1 or v_2. If removing set P did not remove at least one color class, then $\psi_{[\leq 2]}(T_p) \geq \psi_{[\leq 2]}(T) = \lfloor (n+2)/2 \rfloor$. But $\psi_{[\leq 2]}(T_p) \leq \lfloor ((n-2)+2)/2 \rfloor = \lfloor n/2 \rfloor < \lfloor (n+2)/2 \rfloor$. Thus, removing P removed exactly one color class from T, so T_p can be colored with $\psi_{[\leq 2]}(T) - 1$ colors, implying that $\psi_{[\leq 2]}(T_p) \geq \psi_{[\leq 2]}(T) - 1 = \lfloor (n+2)/2 \rfloor - 1 = \lfloor n/2 \rfloor$. Since $\psi_{[\leq 2]}(T_p) \leq \lfloor ((n-2)+2)/2 \rfloor = \lfloor n/2 \rfloor$, by Theorem 3.6, $\psi_{[\leq 2]}(T_p) = \lfloor n/2 \rfloor = \lfloor (n(T_p)+2)/2 \rfloor$. See Figure 11.

Now clearly $T \in \mathcal{F}$, since $f(T_p, v_3) = T$, with $v_a = v_2$ and $v_b = v_1$. $\square$

4 CONCLUDING REMARKS

For future study, we are interested in characterizing the connected graphs G attaining $\psi_{[\leq 2]}(G) = \lceil \mathrm{diam}(G)/2 \rceil + 1$, and characterizing the graphs G attaining $\psi_{[\leq 2]}(G) = 2\rho(G)$. We are also interested in determining bounds on $\psi_{[\leq k]}(G)$ in terms of $\rho(G)$ for other values of k. And finally, we are interested in studying $[\geq k]$ *chromatic colorings* wherein we require at least k colors to be present in each closed neighborhood.

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