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Extremal Results for Peg Solitaire on Graphs

A thesis

presented to

the faculty of

the Department of Mathematics and Statistics

East Tennessee State University

In partial fulfillment

of the requirements for the degree

Master of Science in Mathematical Sciences

by

Aaron D. Gray

December 2013

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Keywords: graph theory, peg solitaire, games on graphs, combinatorial games

ABSTRACT

Extremal Results for Peg Solitaire on Graphs

by

Aaron D. Gray

In a 2011 paper by Beeler and Hoilman, the game of peg solitaire is generalized to arbitrary boards. These boards are treated as graphs in the combinatorial sense. An open problem from that paper is to determine the minimum number of edges necessary for a graph with a fixed number of vertices to be solvable. This thesis provides new bounds on this number. It also provides necessary and sufficient conditions for two families of graphs to be solvable, along with criticality results, and the maximum number of pegs that can be left in each of the two graph families.

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DEDICATION

In memory of Paul Gray.

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I would like to thank my first graph theory teacher and advisor, Dr. Robert Beeler, for his guidance, encouragement, and inspiring brilliance. I would like to thank the other members of the thesis committee, Dr. Anant Godbole, Dr. Teresa Haynes, and Dr. Debra Knisley, for their help and advice. I would like to thank Perry and Donna Gray for all their love and support. I would like to thank Julie Simerly for her faithfulness and counsel. I would also like to thank Dr. Robert Gardner, Dr. Jeff Knisley, Prof. Rob Russell, Ms. Anne Elliott, and Prof. Marvin Glover. Lastly, I would like to thank the Great Mathematician for life, learning, and redemption.

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1 INTRODUCTION

1.1 Background

Peg solitaire is a one-player table game with its earliest recorded use in the late 17th century. The game is played on a board with a set number of holes. Pegs are placed in every hole but one. A peg is removed by jumping over it with an adjacent peg into an adjacent hole, as in Figure 1. This jump is similar to a jump in the game of checkers or draughts. The game ends when no further moves are possible. If only one peg remains on the board, then the board is considered solved and the game is won. Some game boards use stones or marbles and indentations instead of pegs and holes. In addition, a variation of the game requires the placement of the final peg in the central hole. Alternate names for peg solitaire include *Solitaire*, *Solo Noble*, *Hi-Q*, and *Brainvita*.

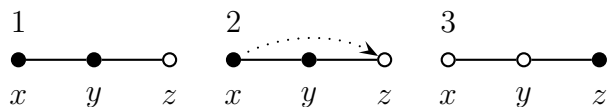


Figure 1: A typical jump in peg solitaire

In a 2011 paper, Beeler and Hoilman [8] generalize peg solitaire to arbitrary boards. These boards are treated as graphs in the combinatorial sense. In [8], Beeler and Hoilman also present an important open problem considering the set of connected graphs on n vertices and k edges, which they denote $G_{n,k}$. Given a fixed n , the problem is to determine the minimum k such that all graphs in $G_{n,k}$ are solvable. In this thesis, we consider the equivalent problem of determining the maximum k such

that there is at least one unsolvable graph in $G_{n,k}$. We provide a lower bound for this k and examine two graph families in which a single edge addition changes the solvability of the resulting graph. We establish necessary and sufficient conditions on the solvability of these graph families. We also determine the maximum number of pegs that can be left on these graph families with the restriction that a jump is made whenever possible.

1.2 Graph Theory Terminology

We now present several graph theory definitions. A *graph*, $G = (V, E)$, is a set of *vertices* V and a set of *edges* E . Figure 2 shows an example of a graph. A graph H is a *subgraph* of a graph G if $V(H) \subset V(G)$ and $E(H) \subset E(G)$. If $V(H) = V(G)$, then H is a *spanning subgraph* of G . A graph H is a *vertex-induced subgraph* of G if for any two vertices u and $v \in H$, $uv \in E(H)$ if and only if $uv \in E(G)$. If graph G does not contain a subgraph H , then G is *H -free*.

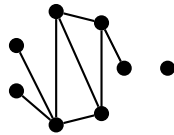


Figure 2: An example of a graph

The *order* of a graph is the cardinality of its vertex set. The *size* of a graph is the cardinality of its edge set. Two vertices, u and v , are *adjacent* if the edge uv is in the edge set of the graph. If uv is in the edge set, then v is *incident* to uv . A *loop* is an edge that is incident with only one vertex. The *degree* of a vertex v , denoted $deg(v)$,

is the number of edges incident to v . A *pendant* is a vertex with degree one.

A new graph H can be constructed from another graph G by adding vertices h_1, \dots, h_n to $V(G)$ and constructing edges incident with each h_i and a single vertex u in G . Thus each h_i is a pendant in H . We refer to this act as *appending* vertices. We refer to the pendants as a *cluster*. We refer to u as the cluster's *support vertex*. Figure 3 shows an example of appending vertices to a graph. When an edge is added between two nonadjacent vertices, u and v , in G , we denote the resulting graph by $G + uv$. An edge uv in graph G is *subdivided* if it is replaced with the new vertex w and the edges uw and wv . Figure 4 shows an example of subdividing an edge in a graph.

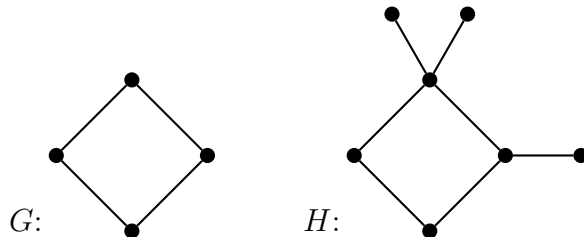


Figure 3: An example of appending vertices to graph G to form a new graph H

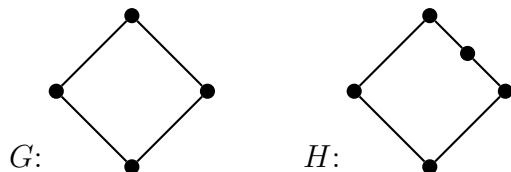


Figure 4: An example of subdividing an edge in graph G to form a new graph H

For an integer $n \geq 1$, the *path* is the graph with order n and size $n - 1$ whose vertices may be labeled v_1, \dots, v_n and whose edges are $v_i v_{i+1}$ for $i = 1, \dots, n - 1$. This

graph is denoted P_n . Figure 5 shows an example of the path. The *terminal vertices* of a path are the pendant vertices. If two vertices u and v are terminal vertices of a path subgraph, then the path subgraph is referred to as a $u - v$ path. Two vertices, u and v , in a graph G are *connected* if G has a $u - v$ path. A graph G is itself connected if G contains a $u - v$ path for every two vertices, u and v , in G . Figure 6 shows an example of a connected graph. The *distance* between two vertices, u and v , in a connected graph G is the minimum length of the all $u - v$ paths in G .



Figure 5: The path P_5

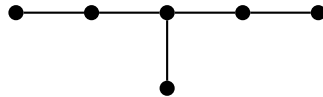


Figure 6: A connected graph

For an integer $n \geq 3$, the *cycle* is the graph with order and size n whose vertices may be labeled v_1, \dots, v_n and whose edges are v_1v_n and v_iv_{i+1} for $i = 1, \dots, n - 1$. This graph is denoted C_n . Figure 7 shows an example of the cycle. A *chord* of a cycle C_n is an edge incident with two vertices of C_n that are not adjacent in C_n . If G has a cycle as a spanning subgraph, then G is *hamiltonian*. A *tree* is a connected graph with no cycle subgraphs. The *empty graph* contains no edges. The *complete graph* contains all possible edges such that every two distinct vertices are adjacent. This graph is denoted K_n . Its vertex set is denoted $\{x_1, \dots, x_n\}$. Figure 8 shows an example of the

complete graph.

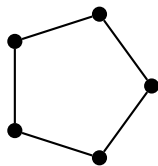


Figure 7: The cycle C_5

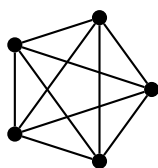


Figure 8: The complete graph K_5

A graph G is *bipartite* if $V(G)$ can be partitioned into two sets X and Y so that if $uw \in E(G)$, then $u \in X$ and $w \in Y$. We refer to $|X|$ and $|Y|$ as *vertex classes*. The *complete bipartite graph* can be partitioned so that $uw \in E(G)$ if and only if $u \in X$ and $w \in Y$. We denote the complete bipartite graph by $K_{s,t}$, where $s = |X|$ and $t = |Y|$. Figure 9 shows an example of the complete bipartite graph. These notions may be extended to k -partite graphs and k -partite complete graphs.

The *star* $K_{1,t}$ is a complete bipartite graph with $s = 1$. Figure 10 shows an example of the star. The *double star* $K_2(a_1, a_2)$ is formed by appending a_1 pendants to one vertex of K_2 and appending a_2 pendants to the second vertex of K_2 . Without loss of generality, we assume that $a_1 \geq a_2$. Figure 11 shows an example of the double star.

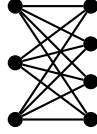


Figure 9: The complete bipartite graph $K_{3,4}$

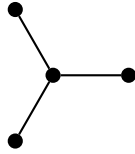


Figure 10: The star $K_{1,3}$

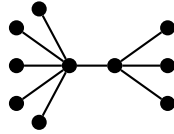


Figure 11: The double star $K_2(5,3)$

For a graph X and a graph Y with $x_1, \dots, x_{|V(X)|} \in V(X)$ and $y_1, \dots, y_{|V(Y)|} \in V(Y)$, the *Cartesian product* $G = X \square Y$ is the graph with $V(G) = V(X) \times V(Y)$. Two vertices (x_1, x_2) and (y_1, y_2) of G are adjacent if and only if either $x_1 = y_1$ and $x_2 y_2 \in E(X)$ or $x_2 = y_2$ and $x_1 y_1 \in E(Y)$. An example of a Cartesian product appears in Figure 12. The *mesh* is the Cartesian product $P_n \square P_m$. The *hypercube*, denoted Q_n , is the Cartesian product $\underbrace{P_2 \square \dots \square P_2}_{n \text{ times}}$.



Figure 12: The Cartesian product $P_2 \square P_3$

An *independent set* of vertices contains no adjacent vertices. The *independence number* of a graph G , denoted $\alpha(G)$, is the maximum number of vertices in an independent set of G . A *dominating set* of a graph G is a subset $D \subset V$ such that every vertex not in D is adjacent to some vertex in D . The *domination number* of G , denoted $\gamma(G)$ is the minimum cardinality of all dominating sets of G .

Two graphs G and H are *isomorphic*, denoted $G \cong H$, if there exists a bijection $\phi : V(G) \rightarrow V(H)$ such that two vertices u and v are adjacent in G if and only if $\phi(u)$ and $\phi(v)$ are adjacent in H . The bijection ϕ is an *isomorphism* from G to H . An *automorphism* of a graph G is an isomorphism from G to itself.

A bound is considered *sharp* if there exists a graph for which equality holds. Thus the bound may not be constrained further without excluding the aforementioned graph.

1.3 Peg Solitaire on Graphs Terminology

We now present several definitions specific to peg solitaire on graphs. Because of the restrictions of peg solitaire, we assume that all graphs are finite, undirected, connected graphs with no loops or multiple edges. If in a graph G , there are pegs in vertices x and y and a hole in vertex z , then the peg in x may jump over the peg in

y into the hole in z provided that xy and yz are edges in G . The peg in y is then removed (see Figure 1). We denote this jump with $x \cdot \overrightarrow{y} \cdot z$. Since the notions of rows and columns are not defined in graphs, ‘L’-shaped jumps are allowed.

Generally, the game begins with a *starting state* $S \subset V$, which is a set of vertices with holes. The game ends with a *terminal state* $T \subset V$, which is a set of vertices with pegs. A terminal state T is *associated* with a starting state S if T can be obtained from S by a series of jumps. Unless otherwise noted, we assume that S consists of a single vertex.

If a graph G has a starting state consisting of a single hole that is associated with a terminal state consisting of a single peg, then G is *solvable*. If G is solvable regardless of the placement of the initial hole, then G is *freely solvable*. It may not be possible to achieve a terminal state consisting of a single peg. If, beginning with a single hole, the minimum number of pegs in any associated terminal state consists of k vertices, then G is *k -solvable*. In particular, if the final two pegs are distance two apart, then G is *distance 2-solvable*. Figure 13 shows an example of a solvable (but not freely solvable) graph, Figure 14 shows an example of a freely solvable graph, and Figure 15 shows an example of a distance 2-solvable graph.

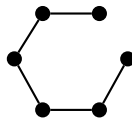


Figure 13: An example of a solvable (not freely solvable) graph

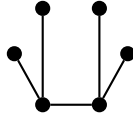


Figure 14: An example of a freely solvable graph

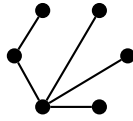


Figure 15: An example of a distance 2-solvable graph

2 LITERATURE REVIEW

In this section, we provide a survey of literature related to the traditional game of peg solitaire, extremal graph theory, combinatorial games, and other research on peg solitaire. We also provide an overview of the literature that explores peg solitaire on graphs under the paradigm of this thesis.

2.1 Traditional Peg Solitaire

A description of peg solitaire, including the board, rules, and problems appears in the August 1697 edition of the French literary magazine *Mercure Galant* [1]. Figure 16 shows a scan of the first pages of this article. Peg solitaire's rich history also includes its play in the court of King Louis XIV of France. Claude Auguste Berey's 1697 engraving, *Madame la Princesse de Soubise jouant au jeu de Solitaire* depicts the Princess of Soubise playing the game [14]. Figure 17 shows this engraving and the peg solitaire board depicted in it. To date, these are the earliest known records of the game [3, 4]. In 1710, mathematician Gottfried Wilhelm von Leibniz described the game in *Miscellanea Berolinensia*, as well as a variation in which the game is played in reverse with the following (quoted and translated in [4]):

Not so very long ago there became widespread an excellent kind of game, called *Solitaire*, where I play on my own, but as with a friend as witness and referee to see that I play correctly. A board is filled with stones set in

holes, which are removed in turn, but none (except the first, which may be chosen for removal at will) can be removed unless you are able to jump another stone across it into an adjacent empty place, when it is captured as in Draughts. He who removes all the stones right to the end according to this rule, wins; but he who is compelled to leave more than one stone still on the board, yields the palm. This game can more elegantly be played backwards, after one stone has been put at will on an empty board, by placing the rest with it, but the same rule being observed for the addition of stones as was stated just above for their removal. Thus we can either fill the board, or, what would be more clever, shape a predetermined figure from the stones...(Beasley xii)

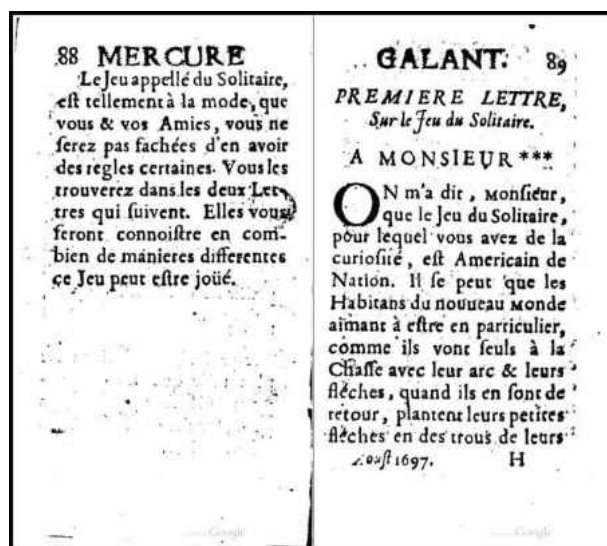


Figure 16: Scan of *Mercurie Galant* peg solitaire article, August 1697 [1]

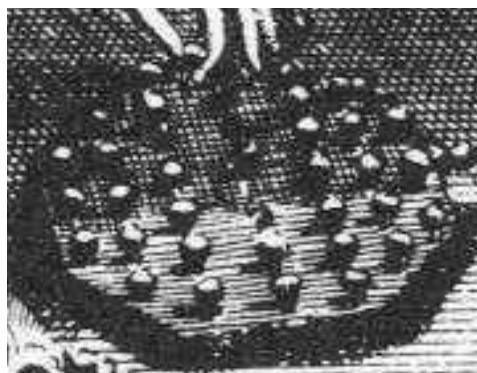


Figure 17: *Madame la Princesse de Soubise joüant au jeu de Solitaire* by Claude-Auguste Berrey, 1697 [14]

There are numerous variations of the peg solitaire board. The English board (also known as the standard board) is cross-shaped and has 33 holes. The European board (also known as the French or continental board) has an additional hole in each of the English board's four inside corners. Another example of the game is the 15 hole triangular board variation found in some wooden game sets and on the tables at Cracker Barrel® Old Country Store restaurants. Figure 18 shows the layout of the English board, Figure 19 shows the layout of the European board, and Figure 20 shows the layout of the 15 hole triangular board. In each figure, the white vertex denotes the usual location of the initial hole.

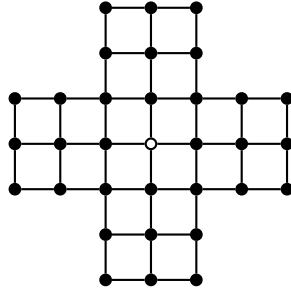


Figure 18: The English peg solitaire board

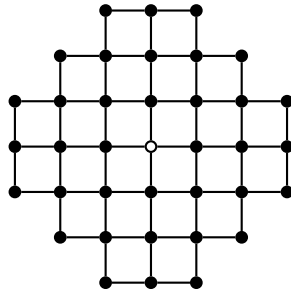


Figure 19: The European peg solitaire board

Berlekamp, Conway, and Guy [15] explore a helpful device for the elimination of pegs. They define a *package* as a known configuration of pegs that may be eliminated with a predetermined series of jumps. The elimination of these pegs is called a *purge*. A purge acts as a type of “shortcut” that can be used to efficiently progress the game.

Beasley [4] and Berlekamp, Conway, and Guy [15] share details of the traditional game’s rich history and solution techniques. The 15 hole variation is explored in depth by Bell [13].

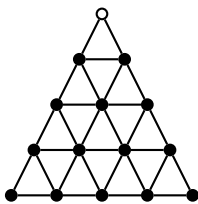


Figure 20: The 15 hole triangular peg solitaire board

Bruijn [23] explores the link between the English variation of peg solitaire and the finite field. In this work, pegs and holes are considered elements of the finite field with addition and multiplication operations. Hentzel [34] explores this concept on the triangular board using an abelian group under addition. In [37], peg solitaire is utilized in the context of artificial intelligence.

2.2 Extremal Graph Theory

Extremal graph theory is a branch of graph theory that investigates maximal or minimal graphs that maintain specific qualities. For example, all graphs with n vertices and n edges contain a cycle subgraph. Thus, trees are extremal or *edge critical* graphs that do not contain a cycle subgraph since any single edge addition would create a cycle subgraph. Extremal results are studied in the context of many graph theory topics. Such topics include order, size, connectivity, diameter, hamiltonicity, and domination. Dirac's Theorem and Turán's Theorem are two famous extremal graph theory results [21].

Theorem 2.1 (*Turán's Theorem [21]*) *Let G be K_{r+1} -free. Then the size of G is at most $\frac{(r-1)n^2}{2r}$.*

Theorem 2.2 (*Dirac's Theorem [21]*) *If G is a graph of order $n \geq 3$ such that $\deg(v) \geq \frac{n}{2}$ for each vertex v of G , then G is hamiltonian.*

According to [16], Turán initiated extremal graph theory as a subject in its own right, but the majority of the development of the field is credited to Paul Erdős because of his multitude of lectures, publications, and proposed problems on the subject. An extremal result from Erdős and Stone [25] appears below.

Theorem 2.3 (*Erdős-Stone Theorem [25]*) *For every $r \in \mathbb{N}$ and every $\epsilon > 0$, if n is sufficiently large and $m \geq t_r(n) + \epsilon n^2$, where $t_r(n)$ denotes the maximal size of a r -partite graph of order n , then every graph G with order n and size m contains a complete $(r + 1)$ -partite graph with arbitrarily large vertex classes.*

For more information on extremal graph theory, see Bollobás' book on the subject [16].

2.3 Games and Graphs

Numerous other extensions of combinatorial games to graph theory have proven to be valuable resources in areas beyond game or graph theory. We now survey a few examples.

In 1856, William Rowan Hamilton invented the Icosian Game [21]. The Icosian Game is a one-player board game in which a dodecahedron appears on the game board. A hole is in each vertex of the dodecahedron. The player uses pegs marked with letters to find a cycle within the dodecahedron that includes every hole on the board. The problem posed by the game led to the study of hamiltonicity in graphs [21]. The layout of the Icosian Game appears in Figure 21.

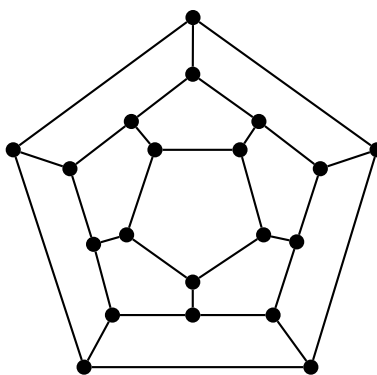


Figure 21: The Icosian Game

According to [21], the study of domination on graphs has its roots in the game of chess. In chess, a queen can move vertically, horizontally, or diagonally over any number of unoccupied spaces. In 1862, Carl Friedrich de Jaenisch investigated the minimum number of queens necessary for every space on an 8×8 chess board to be either occupied by a queen or reached by a queen in a single move. This number is 5. Likewise, the graph analog of possible moves by a queen on a chess board is called the *queen's graph*, and the domination number of the queen's graph on 64 vertices is 5. Figure 22 shows how 5 queens may dominate an 8×8 chess board.

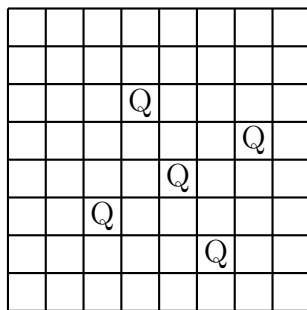


Figure 22: The domination of a chess board by 5 queens [21]

A number of other studies have continued the practice of extending games to graphs. In the two player game *Cops and Robbers*, one player is designated as the cop, C , and another is designated as the robber R . Each player occupies a distinct vertex on a graph G , then C and R alternate moves along the edges of G . If C occupies the same vertex as R , then C “captures” R , and C wins. Otherwise, R wins. The game’s play on graphs is introduced independently by Nowakowski and Winkler [40] and Quilliot’s Ph.D. dissertation [42]. Several families of graphs in which C or R has a winning strategy are characterized in [2] and [40].

In addition, many variations of Cops and Robbers have been studied, including those in which the robber may elect to not move during a turn [2] or in which the robber may move over a number of unoccupied vertices in a single turn [28]. Additional variations involve games in which the robber is only visible to the cop after a specific number of moves [20] or games with a single cop that can set up “road blocks” to cause the robber to lose a number of turns [35]. For more information on

the game, as well as its many variations, see Bonato and Nowakowski's book on the subject [17].

In *Nim*, two players take turns removing stones from at least three piles. During a turn, a player may take any number of stones from only a single pile. The player that removes the last stone or stones wins. In alternate versions, the loser is the player that removes the last stone or stones. The game is given its name and first studied in Bouton's turn of the twentieth century paper [18]. The game is discussed and the winning strategy is described in [26]. In [29] and [30], Nim is played on a graph with weights placed on each edge. Players subtract from these weights as they move along edges. Once the entire weight of an edge is removed, players may no longer move along it. The player that cannot make any additional moves is the loser. Nim on graphs is also studied in [19] and [26].

In *Sim*, two players take turns adding colored edges to an empty graph of order 6. Each player has a different color. The player that constructs a triangle in his or her color is the loser. Using Ramsey theory, a tie is impossible. The game is first created and named by Simmons in [43]. The game is analyzed by [24], and Mead, Rosa, and Huang [39] show that the second player has a winning strategy. In [44], the game is discussed and a variation on 18 vertices in which each player avoids creating a K_4 is studied. Note that the game of Sim may be played on graphs of other orders with other complete graphs as long as Ramsey theory is used to choose an order that prevents a tie.

In *Pebbling*, pebbles are placed on vertices of a graph. In a pebbling step, two pebbles are removed from a single vertex, and then one of the removed pegs is placed on an adjacent vertex. The other peg is discarded. The point of the game is to determine if it is possible to get a pebble to a specific vertex through a series of pebbling steps. Chung [22] first introduces the game in the literature and discusses its play on the hypercube. The pebbling numbers of odd cycles and squares of paths are found in [41]. In [36], notable pebbling results are surveyed and new variations are introduced.

Helleloid, et al. [33] establish graph pegging numbers and provide them for several classes of graphs. The pegging number of a graph is the minimum number of pegs k such that for every distribution of k pegs on the graph, any vertex of the graph can be reached by a sequence of jumps.

2.4 Peg Solitaire on Graphs

In [8], Beeler and Hoilman first generalize peg solitaire on graphs under the paradigm we use in this thesis. They establish the solvability of numerous families of graphs, including the path, the cycle, the complete graph, and the complete bipartite graph. The following results from [8] prove useful.

Theorem 2.4 [8]

- (i) *The path P_n is freely solvable iff $n = 2$; P_n is solvable iff n is even or $n = 3$; P_n is distance 2-solvable in all other cases.*

(ii) The cycle C_n is freely solvable iff n is even or $n = 3$; C_n is distance 2-solvable in all other cases.

(iii) For $n \geq 2$, the complete graph K_n is freely solvable.

(iv) The complete bipartite graph $K_{1,n}$ is $(n - 1)$ -solvable; The complete bipartite graph $K_{n,m}$ is freely solvable for $n, m \geq 2$.

Remark 2.5 [8] For $k \geq 2$, P_{2k} is solvable with the initial hole in 1 and the final peg in $(2k - 2)$. For $k \geq 2$, P_{2k+1} is distance 2-solvable with the initial hole in 1 and the final two pegs in $(2k - 2)$ and $(2k)$.

Observation 2.6 [8] If a graph G is k -solvable with the initial hole in s and a jump is possible, then there is a first jump; say $s'' \cdot \overrightarrow{s'} \cdot s$. Hence, if there are holes in s'' and s' and pegs elsewhere, then G can be k -solved from this configuration.

We also provide the following proposition from [8].

Proposition 2.7 [8] If G is a k -solvable spanning subgraph of H , then H is (at worst) k -solvable.

Because the contrapositive of this proposition is useful in our results, we list it below.

Proposition 2.8 [8] Suppose that H is a k -solvable graph and G is a spanning subgraph of H , then G is (at best) k -solvable.

The following theorem allows the completion of the game in reverse by exchanging the roles of pegs and holes. Beeler and Rodriguez [11] define the *dual* of a configuration of pegs on a graph as the arrangement of pegs when the roles of pegs and holes are reversed.

Theorem 2.9 [8] *Suppose that S is a starting state of G with associated terminal state T . Let S' and T' be the duals of S and T , respectively. It follows that T' is a starting state of G with associated terminal state S' .*

In [9], peg solitaire on the windmill and the double star is examined. The following results are established for the double star.

Theorem 2.10 [9] *The double star $K_2(a_1, a_2)$ is: (i) freely solvable if and only if $a_1 = a_2$ and $a_2 \neq 1$ (ii) solvable if and only if $a_1 \leq a_2 + 1$ (iii) distance 2-solvable if and only if $a_1 = a_2 + 2$ (iv) $(a_1 - a_2)$ -solvable if $a_1 \geq a_2 + 3$.*

While not explicitly stated in [9], a proof in that paper extends the notion of a purge to peg solitaire on graphs. In particular, it extends a purge to graphs that have a double star vertex-induced subgraph. We explicitly define this purge in the next paragraph.

Suppose that the graph G has a double star vertex-induced subgraph, where x_1 is a support vertex with a peg, X_1 is the cluster at x_1 , x_2 is the support vertex with a hole, and X_2 is the cluster at x_2 . The sequence of moves that removes c pegs from

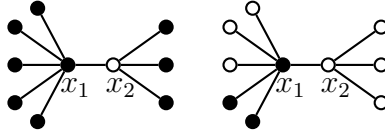


Figure 23: The double star $K_2(5,3)$ before and after $\mathcal{DS}(X_1, X_2, 3)$

each cluster is called a *double star purge*, denoted $\mathcal{DS}(X_1, X_2, c)$. Further, each step in this purge, in which one peg is used to eliminate another peg for a net loss of 2 pegs, is called an *exchange*. An example of a double star purge is given in Figure 23. We note that this purge extends to any graph with a double star subgraph, not just those with a vertex-induced subgraph.

In [7], Beeler, Gray, and Hoilman discuss ways of constructing solvable graphs from the ground up. In [5], Beeler and Gray provide the solvability of all graphs with seven or fewer vertices. In [12], Beeler and Walvoort establish the solvability of trees with diameter four with the following results.

Theorem 2.11 [12] *For the graph $G = K_{1,n}(c; a_1, \dots, a_n)$, where $a_1 \geq 2$, $a_1 \geq \dots \geq a_n \geq 1$, and $k = c - s + n$, with $s = \sum_{i=1}^n a_i$:*

- (i) *The graph G is solvable iff $0 \leq k \leq n + 1$;*
- (ii) *The graph G is freely solvable iff $1 \leq k \leq n$;*
- (iii) *The graph G is distance 2-solvable iff $k \in \{-1, n + 2\}$;*
- (iv) *The graph G is $(1 - k)$ -solvable if $k \leq -1$; The graph G is $(k - n)$ -solvable if $k \geq n + 2$.*

Theorem 2.12 [12]

The conditions for solvability of $K_{1,n}(c; 1, \dots, 1)$ are as follows:

(i) The graph $K_{1,2t}(c; 1, \dots, 1)$ is solvable iff $0 \leq c \leq 2t$ and $(t, c) \neq (1, 0)$. The

graph $K_{1,2t+1}(c; 1, \dots, 1)$ is solvable iff $0 \leq c \leq 2t + 2$.

(ii) The graph $K_{1,n}(c; 1, \dots, 1)$ is freely solvable iff $1 \leq c \leq n - 1$.

(iii) The graph $K_{1,2t}(c; 1, \dots, 1)$ is distance 2-solvable iff $c = 2t + 1$ or $(t, c) = (1, 0)$.

The graph $K_{1,2t+1}(c; 1, \dots, 1)$ is distance 2-solvable iff $c = 2t + 3$.

(iv) The graph $K_{1,2t}(c; 1, \dots, 1)$ is $(c - 2t + 1)$ -solvable if $c \geq 2t + 1$. The graph

$K_{1,2t+1}(c; 1, \dots, 1)$ is $(c - 2t - 1)$ -solvable if $c \geq 2t + 3$.

In [6], Beeler and Gray present families of graphs that are freely solvable, including the mesh.

In the peg solitaire variation *fool's solitaire*, the objective of the game is to leave as many pegs on the board as possible, with the caveat that a jump must be made whenever possible. In [11], Beeler and Rodriguez examine the maximum number of pegs in a terminal state with no adjacent pegs. This set of pegs in graph G is referred to as the *fool's solitaire solution* of G . The cardinality of the fool's solitaire solution of G is referred to as the *fool's solitaire number* of G and denoted $Fs(G)$. Loeb and Wise [38] explore fool's solitaire on graph products. The following results from [11] prove useful.

Corollary 2.13 [11] *On a graph G , there exists some vertex $s \in V(G)$ such that, when $S = \{s\}$, there exists some series of jumps that will yield T as a terminal state if and only if the dual T' of T is solvable to 1 peg.*

The following theorem establishes an upper bound for the fool's solitaire number.

Theorem 2.14 [11] *For any graph G , $Fs(G) \leq \alpha(G)$, where $\alpha(G)$ is the independence number of G .*

The following result provides the fool's solitaire number for the complete bipartite graph. Trivially, $Fs(K_n) = 1$.

Theorem 2.15 [11] *For $n, m \geq 2$, $Fs(K_{n,m}) = n - 1$.*

An extensive bibliography of combinatorial games, both on graphs and otherwise, is provided by [27].

3 TWO GRAPH FAMILIES

3.1 The Hairy Complete Graph

3.1.1 Construction

In this section, we consider a family of graphs that generalize the complete graph and the double star. The *hairy complete graph* is the graph on $n + a_1 + \dots + a_n$ vertices obtained from K_n by appending a_i pendant vertices to x_i for $i = 1, \dots, n$. We denote this graph $K_n(a_1, \dots, a_n)$. Without loss of generality, we assume that $a_1 \geq 1$, $a_1 \geq \dots \geq a_n$, and $n \geq 3$. We denote the a_i pendants adjacent to x_i as $x_{i,1}, \dots, x_{i,a_i}$. Let $X_i = \{x_{i,1}, \dots, x_{i,a_i}\}$, and let $X = \{x_1, \dots, x_n\}$. For $S \subset V(G)$, the function $\rho(S)$ gives the current number of pegs in S .

An example of a hairy complete graph is given in Figure 24.

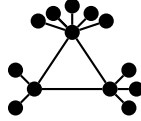


Figure 24: The hairy complete graph $K_3(5, 3, 2)$

3.1.2 Hairy Complete Purge

To aid in our results, we introduce a new purge for eliminating pegs in the hairy complete graph. For the hairy complete graph $K_n(a_1, \dots, a_n)$ with $a_n \geq d$ and a hole in x_n , we perform the jumps $x_{1,k} \cdot \overrightarrow{x_1} \cdot x_n$ and $x_{j,k} \cdot \overrightarrow{x_j} \cdot x_{j-1}$ for $j = 2, \dots, n$ and $k = 1, \dots, d$.

These jumps eliminate d pegs from each cluster of the graph, leave a peg in each x_i , where $i \in \{1, \dots, n-1\}$, and leave a hole in x_n . We note that these jumps work with the initial hole in any x_i , where $i \in \{1, \dots, n\}$. We refer to these jumps as the *hairy complete purge*, denoted $\mathcal{HC}(x_i, c, d)$, where there is a hole in x_i , c is the number of clusters involved in the purge, and d is the number of pegs eliminated from each cluster. Figure 25 shows an example of the hairy complete purge.



Figure 25: The hairy complete graph $K_3(5, 3, 2)$ before and after $\mathcal{HC}(x_1, 3, 2)$

3.1.3 Necessary and Sufficient Conditions

Theorem 3.1 *For the hairy complete graph $G = K_n(a_1, \dots, a_n)$:*

- (i) *The graph G is solvable iff $a_1 \leq \sum_{i=2}^n a_i + n - 1$;*
- (ii) *The graph G is freely solvable iff $a_1 \leq \sum_{i=2}^n a_i + n - 2$ and $(n, a_1, a_2, a_3) \neq (3, 1, 0, 0)$;*
- (iii) *The graph G is distance 2-solvable iff $a_1 = \sum_{i=2}^n a_i + n$;*
- (iv) *The graph G is $(a_1 - \sum_{i=2}^n a_i - n + 2)$ -solvable if $a_1 \geq \sum_{i=2}^n a_i + n$.*

Proof. We begin with the case of $G = K_3(1, 0, 0)$. Suppose that the initial hole is in vertex x_2 . Jump $x_{1,1} \cdot \overrightarrow{x_1} \cdot x_2$ and $x_2 \cdot \overrightarrow{x_3} \cdot x_1$ to solve the graph with the final peg in x_1 .

Thus, the graph is solvable. Suppose that the initial hole is in $x_{1,1}$. The first jump, $x_2 \cdot \overrightarrow{x_1} \cdot x_{1,1}$, is forced. This results in a peg in $x_{1,1}$ and a peg in x_3 , which are distance 2 apart. So the graph is not freely solvable.

Suppose that $G \neq K_3(1, 0, 0)$. To establish necessary conditions, we first examine the optimal method for solving the graph. The pegs in each cluster must be eliminated. Hence all pegs in X_1 must be removed. To do so, a peg must first be in x_1 . For this to occur, one of two jumps must be made, namely, $x_i \cdot \overrightarrow{x_j} \cdot x_1$, where $i \neq j$ and $i, j \neq 1$, or $x_{j,1} \cdot \overrightarrow{x_j} \cdot x_1$, where $i \neq j$. Therefore, one of two double star purges is necessary, namely $\mathcal{DS}(X_1, X_j, d)$, where $j \neq 1$ or $\mathcal{DS}(X_1, X - \{x_1, x_j\}, d)$. Each X_j can exchange a_j pegs with X_1 , and each x_i , where $i = 2, \dots, n$, can exchange 1 peg with X_1 . Hence $a_1 \leq \sum_{i=2}^n a_i + n - 1$ is necessary. Moreover, if $a_1 \geq \sum_{i=2}^n a_i + n$, then, at best, $a_1 - \sum_{i=2}^n a_i - n + 1$ pegs remain in the graph.

For sufficiency, let m be the greatest integer such that $a_1 - \sum_{i=m+1}^n a_i - 1 \leq a_2 + (m-2)a_m$. If no such integer exists, then let $m = 1$. Begin with the initial hole in x_j , for $j \neq 1$, and jump $x_{1,a_1} \cdot \overrightarrow{x_1} \cdot x_j$. For $i = 1, \dots, n - m$, perform the double star purge $\mathcal{DS}(X_1, X_{n-i+1}, a_{n-1+1})$.

Suppose that $m = 1$. Then $a_1 \geq \sum_{i=2}^n a_i + 1$. So, $\rho(X_1) = a_1 - \sum_{i=2}^n a_i - 1$. If $a_1 = \sum_{i=2}^n a_i + 1$, then the graph reduces to K_n with a hole in x_1 , which is solvable by Theorem 2.4. If $a_1 \geq \sum_{i=2}^n a_i + 2$, then perform the double star purge $\mathcal{DS}(X_1, X - \{x_1, x_j\}, \min(\rho(X_1), n - 2))$ to remove $\min(a_1 - \sum_{i=2}^n a_i, n - 2)$ pegs from X_1 . If $a_1 \leq \sum_{i=2}^n a_i + n - 2$, then this reduces the graph to K_r , where

$r = \sum_{i=2}^n a_i + n - a_1$, with a hole in x_1 , which is solvable. If $a_1 \geq \sum_{i=2}^n a_i + n - 1$, then there are $a_1 - \sum_{i=2}^n a_i - n + 1$ pegs in X_1 and one peg in x_j . Thus, the graph is $(a_1 - \sum_{i=2}^n a_i - n + 2)$ -solvable.

Suppose that $m \geq 2$. Then $a_1 < \sum_{i=2}^n a_i + 1$. So, $\rho(X_1) = a_1 - \sum_{i=m+1}^n a_i - 1$ and $\rho(X_i) = a_i$ for $i = 1, \dots, m$. Perform the double star purge $\mathcal{DS}(X_1, X_2, a_2 - a_3)$ to eliminate $a_2 - a_3$ pegs from X_1 and X_2 . For $j = 3, \dots, m - 1$, perform the hairy complete purge $\mathcal{HC}(x_1, j, a_j - a_{j+1})$ so that $\rho(X_1) = a_1 - \sum_{i=m+1}^n a_i - 1 - a_2 + a_m$.

Let $k = \lfloor \frac{\rho(X_1)}{m-1} \rfloor$. For $j = 1, \dots, m - 1$, perform the double star purge $\mathcal{DS}(X_1, X_{m-j+1}, k)$. Now perform the hairy complete purge $\mathcal{HC}(x_1, m, \min(\rho(X_1), \rho(X_2)))$. If $\rho(X_1) = \dots = \rho(X_n) = 0$, then this reduces the graph to K_n with hole in x_1 , which is solvable. If $\rho(X_1) = 0$ and $\rho(X_2) \geq 1$, then jump $x_{2,1} \cdot \overrightarrow{x_2} \cdot x_1$. Then perform the hairy complete purge $\mathcal{HC}(x_2, m - 1, \rho(X_2))$. For $j = 3, \dots, m$, jump $x_{j,1} \cdot \overrightarrow{x_j} \cdot x_{j-1}$. This reduces the graph to K_n with a hole in x_m , which is solvable. If $\rho(X_1) \geq 1$ and $\rho(X_i) = 0$, for $i \neq 1$, then perform the double star purge $\mathcal{DS}(X_1, X - \{x_1, x_j\}, \min(\rho(X_1), n - 2))$. This is solvable by the above arguments. If $\rho(X_1) \geq 1$ and $\rho(X_i) \geq 1$, for $i \neq 1$, then let $\rho(X_1) = \ell$. For $j = 1, \dots, \ell$, perform the double star purge $\mathcal{DS}(X_1, X_{m-j+1}, 1)$. This reduces the graph to a configuration that is solvable by the above arguments.

For the freely solvable result, we first show that $K_n(a_1, \dots, a_n)$ is not freely solvable if $a_1 = \sum_{i=2}^n a_i + n - 1$. Assume that the initial hole is in $x_{i,1}$, with $i \neq 1$. If $a_i \geq 2$, then we can jump $x_{i,2} \cdot \overrightarrow{x_i} \cdot x_{i,1}$. This reduces the graph to $K_n(a_1, \dots, a_i - 1, \dots, a_n)$ with a hole in x_i . This graph is not solvable by the above arguments. If $a_i = 1$, then

jump $x_{i-1} \cdot \overrightarrow{x_i} \cdot x_{i,1}$, $x_{1,1} \cdot \overrightarrow{x_1} \cdot x_i$, and $x_{i,1} \cdot \overrightarrow{x_i} \cdot x_1$. This reduces the graph to $K_{n-1}(a_1 - 1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$ with a hole in x_i , which is unsolvable by the above arguments.

We now show that $K_n(a_1, \dots, a_n)$ with $a_1 \leq \sum_{i=2}^n a_i + n - 2$ is solvable with the initial hole in any vertex, up to automorphism. Note that the graph is solvable with the initial hole in x_j for $j \neq 1$, as outlined above. Suppose the initial hole is in $x_{j,1}$ for $j \neq 1$. If $a_j = 1$, then jump $x_k \cdot \overrightarrow{x_j} \cdot x_{j,1}$, $x_{1,1} \cdot \overrightarrow{x_1} \cdot x_j$, and $x_{j,1} \cdot \overrightarrow{x_j} \cdot x_1$ for $k \neq 1$ and $k \neq j$. Ignoring x_j and $x_{j,1}$, this reduces the graph to $K_{n-1}(a_1 - 1, \dots, a_{j-1}, a_{j+1}, \dots, a_n)$ with a hole in x_k , which is solvable by the above arguments. If $a_j \geq 2$, then jump $x_{j,a_j} \cdot \overrightarrow{x_j} \cdot x_{j,1}$. This reduces the graph to $K_n(a_1, \dots, a_{j-1}, a_j - 1, a_{j+1}, \dots, a_n)$, with a hole in x_j , which is solvable by the above arguments.

Suppose the initial hole is in x_1 . If $a_2 = 0$, then jump $x_n \cdot \overrightarrow{x_2} \cdot x_1$. This reduces the graph to $K_{n-1}(a_1, \dots, a_{n-1})$ with a hole in x_2 , which is solvable by the above arguments. If $a_2 \geq 1$, then jump $x_{2,a_2} \cdot \overrightarrow{x_2} \cdot x_1$. This reduces the graph to $K_n(a_1, a_2 - 1, a_3, \dots, a_n)$ with a hole in x_2 , which is solvable by the above arguments.

Suppose the initial hole is in $x_{1,1}$. If $a_1 = 1$ and $a_n = 1$, then for $j = 2, \dots, n$ jump $x_{j,1} \cdot \overrightarrow{x_j} \cdot x_{j-1}$. This reduces the graph to K_n with a hole in x_n , which is solvable. If $a_1 = 1$ and $a_n = 0$, then let ℓ be the greatest integer such that $a_\ell = 1$. Jump $x_n \cdot \overrightarrow{x_1} \cdot x_{1,1}$ and $x_{\ell,1} \cdot \overrightarrow{x_\ell} \cdot x_1$. Ignoring x_n , this reduces the graph to $K_{n-1}(a_1, \dots, a_{\ell-1}, a_\ell - 1, a_{\ell+1}, \dots, a_{n-1})$ with a hole in x_ℓ , which is solvable by the above arguments. If $\ell = 1$ and $n \geq 4$, then jump $x_2 \cdot \overrightarrow{x_1} \cdot x_{1,1}$, $x_{n-1} \cdot \overrightarrow{x_n} \cdot x_1$, and $x_{1,1} \cdot \overrightarrow{x_1} \cdot x_2$. If $n = 4$, then the graph is solved. If $n \geq 5$, then this reduces the graph to K_{n-2} with a hole in x_1 , which is

solvable. If $a_1 \geq 2$, then jump $x_{1,a_1} \cdot \overrightarrow{x_1} \cdot x_{1,1}$. This reduces the graph to $K_n(a_1, \dots, a_n)$ with a hole in x_j after the first jump, which is solvable by Observation 2.6 and the above arguments. ■

The following corollary addresses an open problem in [5] of how to construct unsolvable graphs.

Corollary 3.2 *An unsolvable graph H can be constructed from graph G by the addition of at most $|V(G)|$ pendants to G .*

Proof. Pick any vertex $v \in G$. Append $|V(G)|$ pendants to vertex v . The resulting graph, H , is a spanning subgraph of $K_{|V(G)|}(|V(G)|, 0, \dots, 0)$, which is unsolvable by Theorem 3.1. By Proposition 2.8, H is unsolvable. ■

3.2 The Hairy Complete Bipartite Graph

3.2.1 Construction

In this section, we consider a family of graphs that generalize the complete bipartite graph and the double star. The *hairy complete bipartite graph*, denoted $K_{n,m}(a_1, \dots, a_n; b_1, \dots, b_m)$ is the graph on $n + m + a_1 + \dots + a_n + b_1 + \dots + b_m$ vertices obtained from $K_{n,m}$ by appending a_i pendant vertices to x_i for $i = 1, \dots, n$ and appending b_j pendant vertices to y_j for $j = 1, \dots, m$. Without loss of generality, we assume

that $n \geq 2$, $m \geq 2$, $a_1 \geq \dots \geq a_n$, $b_1 \geq \dots \geq b_m$, $a_1 \geq 1$, and $\sum_{i=1}^n a_i \geq \sum_{j=1}^m b_j$. Figure 26 shows an example of the hairy complete bipartite graph. We denote the a_i pendants adjacent to x_i with $x_{i,1}, \dots, x_{i,a_i}$. We denote the b_j pendants adjacent to y_j with $y_{j,1}, \dots, y_{j,b_j}$. Let $X_i = \{x_{i,1}, \dots, x_{i,a_i}\}$, let $Y_j = \{y_{j,1}, \dots, y_{j,b_j}\}$, and let $X = \{x_1, \dots, x_n\}$. For convenience of exposition, we refer to X_1, \dots, X_n as the *heavy clusters* and Y_1, \dots, Y_m as the *light clusters*. We define property \mathcal{P} as $n = 2$, m is even, and $a_2 = 0$. We define $(\sim \mathcal{P})$ as $n = 2$, m is odd or $n = 2$, m is even, and $a_2 \geq 1$ or $n \geq 3$.

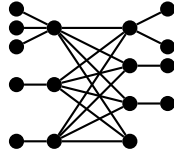


Figure 26: The hairy complete bipartite graph $K_{3,4}(3, 1, 1; 2, 1, 1, 0)$

3.2.2 Necessary and Sufficient Conditions

Theorem 3.3 For the hairy complete bipartite graph $G = K_{n,m}(a_1, \dots, a_n; b_1, \dots, b_m)$:

(i) If \mathcal{P} , then the graph G is solvable iff $\sum_{i=1}^n a_i \leq \sum_{j=1}^m b_j + n - 1$.

If $(\sim \mathcal{P})$, then the graph G is solvable iff $\sum_{i=1}^n a_i \leq \sum_{j=1}^m b_j + n$.

(ii) If \mathcal{P} , then the graph G is freely solvable iff $\sum_{i=1}^n a_i \leq \sum_{j=1}^m b_j + n - 2$.

If $(\sim \mathcal{P})$, then the graph G is freely solvable iff $\sum_{i=1}^n a_i \leq \sum_{j=1}^m b_j + n - 1$.

(iii) If \mathcal{P} , then the graph G is distance 2-solvable iff $\sum_{i=1}^n a_i = \sum_{j=1}^m b_j + n + 1$.

If $(\sim \mathcal{P})$, then the graph G is distance 2-solvable iff $\sum_{i=1}^n a_i = \sum_{j=1}^m b_j + n + 2$.

(iv) If \mathcal{P} , then graph G is $(\sum_{i=1}^n a_i - \sum_{j=1}^m b_j - n + 1)$ -solvable if $\sum_{i=1}^n a_i \geq \sum_{j=1}^m b_j + n$.

If $(\sim \mathcal{P})$, then the graph G is $(\sum_{i=1}^n a_i - \sum_{j=1}^m b_j - n)$ -solvable if $\sum_{i=1}^n a_i \geq \sum_{j=1}^m b_j + n + 1$.

Proof. To establish necessary conditions, we first examine the optimal method for solving the graph. The pegs in each cluster must be eliminated. Hence all pegs in each X_i must be removed. To do so, a peg must first be in x_i . For this to occur, one of two jumps is necessary, namely, $y_{j,k} \xrightarrow{\rightarrow} x_i$ or $x_\ell \xrightarrow{\rightarrow} x_i$, where $\ell \neq i$. Therefore one of two double star purges is necessary, namely $\mathcal{DS}(X_i, Y_j, d)$ or $\mathcal{DS}(X_i, X - \{x_i\}, d)$. Each Y_j can exchange b_j pegs with X_i , and each x_ℓ , where $\ell \in \{1, \dots, i-1, i+1, \dots, n\}$, can exchange with a peg in X_i . Hence $\sum_{i=1}^n a_i \leq \sum_{j=1}^m b_j + n$ is necessary. Moreover, if $\sum_{i=1}^n a_i \geq \sum_{j=1}^m b_j + n + 1$, then, at best, $\sum_{i=1}^n a_i - \sum_{j=1}^m b_j - n$ pegs remain in the graph.

We now show these conditions are sufficient using an algorithm for the elimination of pegs. We define a homomorphism $\phi : G \rightarrow G'$, where $G' = K_{n,1}(a_1, \dots, a_n; \sum_{j=1}^m b_j)$. Let $s_j = \sum_{k=1}^{j-1} b_k$. The homomorphism ϕ is defined by $\phi(y_j) = y'$, $\phi(y_{j,\ell}) = y'_{s_j+\ell}$, and $\phi(v) = v$ for all other vertices. Let Y' denote the set of all $y_{s_j+\ell}$.

This homomorphism has the effect of collapsing the support vertices of the light clusters. In addition, it allows the movement of a hole along each of the y_j . This

occurs because as each Y_j empties, the jumps $x_{i,1} \cdot \overrightarrow{x_i} \cdot y_j$ and $y_{j-1,1} \cdot \overrightarrow{y_j} \cdot x_i$, for $k \neq j$, result in a net loss of zero pegs for both X and Y .

Begin with the initial hole in y' . This corresponds to beginning with the initial hole in y_j for some j . Perform the double star purge $\mathcal{DS}(X_{n-i+1}, Y', \min(\rho(Y'), a_{n-i+1}))$, for $i = 1, \dots, n$.

We now consider G . Note that we now have no pegs in Y_j , for $j = 1, \dots, m$. We have $\sum_{i=1}^n a_i - \sum_{j=1}^m b_j$ pegs in $X_1 \cup \dots \cup X_n$. Further, we have n pegs in X and $m - 1$ pegs in Y . Without loss of generality, assume that there is a hole in y_m .

If $\sum_{i=1}^n a_i = \sum_{j=1}^m b_j$, then this reduces G to the complete bipartite graph with a hole in a single vertex, which is solvable with the final two pegs in x_i and y_j , for any i and j . Thus, the graph may be solved with the final peg in x_i , y_j , $x_{i,1}$, or $y_{j,1}$ for any i and j . Hence, by Theorem 2.9 G is freely solvable.

If $\sum_{i=1}^n a_i \geq \sum_{j=1}^m b_j + 1$, then let ℓ be the greatest integer such that $\rho(X_\ell) \geq 1$. For $i = 1, \dots, \ell$, perform the double star purge $\mathcal{DS}(X_{\ell-i+1}, X - \{x_{\ell-i+1}\}, \min(\rho(X_{\ell-i+1}), \rho(X - \{x_{\ell-i+1}\})))$. If $\sum_{i=1}^n a_i \geq \sum_{j=1}^m b_j + n$, then we now have $\sum_{i=1}^n a_i - \sum_{j=1}^m b_j - n + 1$ pegs in $X_1 \cup \dots \cup X_n$. We have no pegs in Y_j , for $j = 1, \dots, m$. In addition, we have 1 peg in X , and $m - 1$ pegs in Y .

If $\sum_{j=1}^m b_j + 1 \leq \sum_{i=1}^n a_i \leq \sum_{j=1}^m b_j + n - 1$, then this reduces the graph to the complete bipartite graph with a hole in y_m . By the same argument as above, we may solve the graph with the final peg in any vertex. Hence, by Theorem 2.9, the graph is freely solvable.

If $\sum_{i=1}^n a_i \geq \sum_{j=1}^m b_j + n$, then let ℓ be the greatest integer such that $\rho(X_\ell) \geq 1$.

Without loss of generality, assume there is a peg in x_ℓ . If $n \geq 3$ and m is even, then let $\ell' \neq \ell$, $\ell'' \neq \ell$, and $\ell'' \neq \ell'$. Jump $x_\ell \cdot y_{m-1} \overrightarrow{\cdot} x_{\ell'}$, $x_{\ell'} \cdot y_{m-2} \overrightarrow{\cdot} x_{\ell''}$, and $x_{\ell''} \cdot y_{m-3} \overrightarrow{\cdot} x_\ell$.

Let $\mu = \rho(Y)$. For $k = 1, \dots, \lfloor \frac{\mu}{2} \rfloor$, jump $x_\ell \cdot y_{2k-1} \overrightarrow{\cdot} x_{\ell'}$ and $x_{\ell'} \cdot y_{2k} \overrightarrow{\cdot} x_\ell$. If $n = 2$ and m is odd or $n \geq 3$, then jump $x_{\ell,1} \cdot \overrightarrow{x_\ell} \cdot y_j$. If $\sum_{i=1}^n a_i = \sum_{j=1}^m b_j + n$, then the graph is solved with the final peg in y_j . If $\sum_{i=1}^n a_i \geq \sum_{j=1}^m b_j + n + 1$, then $\sum_{i=1}^n a_i - \sum_{j=1}^m b_j - n$ pegs remain in $X_1 \cup \dots \cup X_n$ and one peg remains in y_j . In particular, if $\sum_{i=1}^n a_i = \sum_{j=1}^m b_j + n + 2$, then one peg remains in X_1 and one peg remains in y_j .

If \mathcal{P} , then we have one peg in x_1 , one peg in Y , and $\sum_{i=1}^n a_i - \sum_{j=1}^m b_j - n + 1$ pegs in X_1 . In any case, the peg in x_1 will be removed. If $n = 2$, m is even, and $a_2 \geq 1$, then we perform one less double star purge during the homomorphism so that $\rho(X_2) = 1$. Instead, we perform the additional double star purge $\mathcal{DS}(Y_1, X_1, 1)$ during the homomorphism. Then before removing the pegs in Y , we perform the double star purge jump $\mathcal{DS}(X_2, X - \{x_2\}, 1)$ and jump $x_2 \cdot y_{m-1} \overrightarrow{\cdot} x_1$. After removing the pegs in Y , we make the final jump $x_{1,1} \cdot \overrightarrow{x_1} \cdot y_1$. For the distance 2-solvable result, if \mathcal{P} , then we make the final jump $x_{1,1} \cdot \overrightarrow{x_1} \cdot y_2$.

For the freely solvable result, we first show that if \mathcal{P} , then $K_{n,m}(a_1, \dots, a_n; b_1, \dots, b_m)$ with $\sum_{i=1}^n a_i \leq \sum_{j=1}^m b_j + n - 1$ is not freely solvable. Note that if $\sum_{i=1}^n a_i \leq \sum_{j=1}^m b_j + n - 2$, then $K_{n,m}(a_1, \dots, a_n; b_1, \dots, b_m)$ is solvable. Thus, it suffices to consider the case where $\sum_{i=1}^n a_i = \sum_{j=1}^m b_j + n - 1$. Assume that the initial hole is in $y_{k,1}$ for some k .

If we jump $y_{k,2} \cdot \overrightarrow{y_k} \cdot y_{k,1}$, then $\sum_{i=1}^n a_i$ pegs remain in $X_1 \cup \dots \cup X_n$, but $\sum_{j=1}^m b_j - 1$ pegs remain in $Y_1 \cup \dots \cup Y_m$. Since one fewer peg can exchange with the pegs in $X_1 \cup \dots \cup X_n$, the graph is unsolvable. If we jump $x_i \cdot \overrightarrow{y_k} \cdot y_{k,1}$, then $\sum_{i=1}^n a_i$ pegs remain in $X_1 \cup \dots \cup X_n$, but $n - 1$ pegs remain in $X - \{x_i\}$. Since one fewer peg can exchange with the pegs in $X_1 \cup \dots \cup X_n$, the graph is unsolvable. The argument is similar for $n = 2$ where m is odd or $n = 2$, where m is even, and $a_2 \geq 1$ or $n \geq 3$ with $\sum_{i=1}^n a_i \leq \sum_{j=1}^m b_j + n$.

We now show that if $(\sim \mathcal{P})$, then $K_{n,m}(a_1, \dots, a_n; b_1, \dots, b_m)$ with $\sum_{i=1}^n a_i \leq \sum_{j=1}^m b_j + n - 1$ is solvable with the initial hole in any vertex, up to automorphism. Note that the graph is solvable with the initial hole in y_j , as outlined above.

Suppose the initial hole is in $y_{j,1}$ for some j . If $b_j = 1$ and $a_n = 0$, then jump $x_n \cdot \overrightarrow{y_j} \cdot y_{j,1}$. This reduces the graph to $K_{n-1,m}(a_1, \dots, a_{n-1}; b_1, \dots, b_m)$ with a hole in y_j , which is solvable by the above arguments. If $b_j = 1$ and $a_n \geq 1$, then jump $x_n \cdot \overrightarrow{y_j} \cdot y_{j,1}$, $y_{j-1,1} \cdot \overrightarrow{y_{j-1}} \cdot x_n$, $x_{n,1} \cdot \overrightarrow{x_n} \cdot y_j$, and $y_{j,1} \cdot \overrightarrow{y_j} \cdot x_n$. This reduces the graph to $K_{n,m-1}(a_1, \dots, a_n; b_1, \dots, b_{j-1}, b_{j+1}, \dots, b_{m-1})$ with a hole in y_{j-1} , which is solvable by the above arguments. If $b_j \geq 2$, then jump $y_{j,b_j} \cdot \overrightarrow{y_j} \cdot y_{j,1}$. This reduces the graph to $K_{n,m}(a_1, \dots, a_n; b_1, \dots, b_{j-1}, b_j - 1, b_{j+1}, \dots, b_m)$ with a hole in y_j , which is solvable by the above arguments.

Suppose the initial hole is in x_i for some i . Jump $y_{j,1} \cdot \overrightarrow{y_j} \cdot x_i$, where j is such that $b_j \geq 1$. This reduces the graph to $K_{n,m}(a_1, \dots, a_n; b_1, \dots, b_{j-1}, b_j - 1, b_{j+1}, \dots, b_m)$, with a hole in y_j , which is solvable by the above arguments.

Suppose the initial hole is in $x_{i,1}$. If $a_i = 1$, then jump $y_j \cdot \overrightarrow{x_i} \cdot x_{i,1}$, $y_{\ell, b_\ell} \cdot \overrightarrow{y_\ell} \cdot x_i$, and $x_{i,1} \cdot \overrightarrow{x_i} \cdot y_\ell$, where $\ell \neq j$ and $b_\ell \geq 1$. This reduces the graph to $K_{n-1, m}(a_1, \dots, a_{n-1}; b_1, \dots, b_{\ell-1}, b_\ell - 1, b_{\ell+1}, \dots, b_m)$, with a hole in y_j , which is solvable by the above arguments. If $a_i \geq 2$, then jump $x_{i, a_i} \cdot \overrightarrow{x_i} \cdot x_{i,1}$ and $y_{j,1} \cdot \overrightarrow{y_j} \cdot x_i$. This reduces the graph to $K_{n, m}(a_1, \dots, a_{i-1}, a_i - 1, a_{i+1}, \dots, a_n; b_1, \dots, b_{j-1}, b_j - 1, b_{j+1}, \dots, b_m)$, with a hole in y_j , which is solvable by the above arguments.

■

4 FOOL'S SOLITAIRE RESULTS

In this section, we present fool's solitaire results for the hairy complete graph and the hairy complete bipartite graph.

4.1 The Hairy Complete Graph

Theorem 4.1 *For the hairy complete graph $G = K_n(a_1, \dots, a_n)$:*

(i) *If $n \geq 3$ and $a_n = 0$, then $Fs(G) = \sum_{i=1}^n a_i + 1 = \alpha(G)$;*

(ii) *If $n = 3$ and $a_n = 1$ or $n \geq 4$ and $a_n \geq 1$, then $Fs(G) = \sum_{i=1}^n a_i = \alpha(G)$;*

(iii) *If $n = 3$ and $a_n \geq 2$, then $Fs(G) = \sum_{i=1}^n a_i - 1 = \alpha(G) - 1$.*

Proof. Suppose that $n \geq 3$ and $a_n = 0$. The maximum independent set is $A = X_1 \cup \dots \cup X_{n-1} \cup \{x_n\}$. The dual of A is $A' = \{x_1, \dots, x_{n-1}\}$. Solve A' by solving the K_n subgraph with a hole in x_n , which is freely solvable by Theorem 2.4.

Suppose that $n = 3$ and $a_n = 1$. We take $A = X_1 \cup X_2 \cup \{x_3\}$ for the maximum independent set. The dual of A is $A' = \{x_1, x_2, x_{3,1}\}$. Jump $x_1 \cdot \overrightarrow{x_2} \cdot x_3$ and $x_{3,1} \cdot \overrightarrow{x_3} \cdot x_1$ to solve A' .

Suppose that $n = 3$ and $a_n \geq 2$. The maximum independent set is $A = X_1 \cup X_2 \cup X_3$. However, the dual of this set, $A' = \{x_1, x_2, x_3\}$, is unsolvable since the forced jump $x_1 \cdot \overrightarrow{x_3} \cdot x_{3,1}$ leaves pegs in x_2 and $x_{3,1}$, which are distance 2 apart. Consider the set $T = (X_1 - \{x_{1,1}\}) \cup X_2 \cup X_3$. The dual of T is $T' = \{x_{1,1}, x_1, x_2, x_3\}$. Jump $x_3 \cdot \overrightarrow{x_2} \cdot x_{2,1}$, $x_{1,1} \cdot \overrightarrow{x_1} \cdot x_2$, and $x_{2,1} \cdot \overrightarrow{x_2} \cdot x_1$ to solve T' .

Suppose that $n \geq 4$ and $a_n \geq 1$. The maximum independent set is $A = X_1 \cup \dots \cup X_n$. The dual of A is $A' = X$. To solve A' , jump $x_{n-1} \cdot \overrightarrow{x_n} \cdot x_{n,1}$, $x_{n-3} \cdot \overrightarrow{x_{n-2}} \cdot x_n$, and $x_{n,1} \cdot \overrightarrow{x_n} \cdot x_{n-3}$. If $n = 4$, then A' is solved with the final peg in x_{n-3} . If $n \geq 5$, then this reduces A' to K_{n-2} with a hole in x_{n-2} , which is solvable by Theorem 2.4. \blacksquare

4.2 The Hairy Complete Bipartite Graph

Note that for the following result, we parameterize differently.

Theorem 4.2 *For $G = K_{n,m}(a_1, \dots, a_n; b_1, \dots, b_m)$ with $a_i = 0$ for $i \geq n - \ell + 1$, $b_j = 0$ for $j \geq m - \lambda + 1$, and $\ell \geq \lambda$:*

(i) *If $\ell = 0$, then $Fs(G) = \sum_{i=1}^n a_i + \sum_{j=1}^m b_j = \alpha(G)$;*

(ii) *If $1 \leq \ell \leq n - 1$, then $Fs(G) = \sum_{i=1}^n a_i + \sum_{j=1}^m b_j + \ell = \alpha(G)$;*

(iii) *If $\ell = n$, then $Fs(G) = \sum_{i=1}^n a_i + \sum_{j=1}^m b_j + \ell - 1 = \alpha(G) - 1$.*

Proof. Suppose that $\ell = 0$. Thus $a_i \geq 1$ for all i and $b_j \geq 1$ for all j . The maximum independent set is $A = X_1 \cup \dots \cup X_n \cup Y_1 \cup \dots \cup Y_m$. The dual of A is $A' = X \cup Y$. To solve A' , jump $y_1 \cdot \overrightarrow{x_1} \cdot x_{1,1}$, $x_2 \cdot \overrightarrow{y_2} \cdot x_1$, and $x_{1,1} \cdot \overrightarrow{x_1} \cdot y_2$. If $n = 2$ and $m = 2$, then A' is solved with the final peg in y_2 . If $n = 3$ and $m = 2$, then jump $x_3 \cdot \overrightarrow{y_2} \cdot x_1$ to solve A' . If $n \geq 4$ and $m \geq 2$, then this reduces A' to $K_{n-2,m}$ with a hole in y_1 , which is freely solvable by Theorem 2.4.

Suppose that $1 \leq \ell \leq n - 1$. The maximum independent set is $A = X_1 \cup \dots \cup X_n \cup Y_1 \cup \dots \cup Y_m \cup \{x_{n-\ell+1}, \dots, x_n\}$. The dual of A is $A' = \{x_1, \dots, x_{n-\ell}\} \cup Y$. To solve A' , solve the $K_{n-\ell+1, m}$ subgraph with a hole in $x_{n-\ell+1}$. This is freely solvable by Theorem 2.4.

Suppose that $\ell = n$. The maximum independent set is $A = X \cup Y_1 \cup \dots \cup Y_m$. The dual of this set is $A' = Y$. Since no pegs are adjacent in A' , it is not solvable. Thus, at least one peg must be added to the dual. Suppose that we add x_1 to obtain $T' = Y \cup \{x_1\}$. To solve T' , solve the $K_{2, m}$ subgraph with a hole in x_2 , which is freely solvable by Theorem 2.4. ■

5 EXTREMAL AND CRITICALITY RESULTS

In this section, we provide extremal and criticality results. We first define some useful terms. A graph G is *edge k -critical* if G is k -solvable, but the addition of any edge reduces the number of pegs at the end of the game. In particular, G is *edge critical* if G is not solvable, but the addition of any edge results in a solvable graph. We call an unsolvable (solvable but not freely solvable) graph G a *critical graph* if the addition of any edge to G results in a solvable (freely solvable) graph.

In [5], Beeler and Gray present the solvability of all 996 non-isomorphic connected graphs with seven vertices or less. The graphs are obtained from the appendix of Harary [32] and a small graph database [31]. The solvability of the graphs is determined using an exhaustive computer search algorithm [10]. In Figures 27, 29, 30, and 31, a black vertex indicates that the graph can be solved with the initial hole in that vertex. Graphs that are not solvable have the minimum number of pegs that can be obtained in a terminal state associated with a single vertex starting state listed. If a graph is distance 2-solvable, then this is indicated with a D, and a black vertex indicates that the graph can be distance 2-solved with the initial hole in that vertex. Since all hamiltonian graphs of even order are freely solvable by [8], these graphs are omitted. Figure 27 gives all non-isomorphic connected graphs with order four or less.

A chorded cycle, denoted $C(n, m)$, is obtained from a cycle on n vertices, which are labeled $0, 1, \dots, n-1$ in the usual way. An edge from 0 to m is inserted to form the

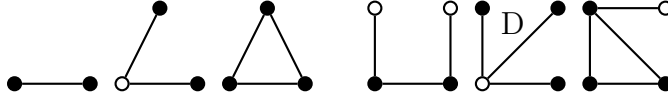


Figure 27: The solvability of all graphs with four vertices or less [5]

chord. By [5], the chorded five cycle $C(5, 2)$ is freely solvable. This result is extended in a later theorem. Figure 28 shows the chorded five cycle, and Figure 29 gives the solvability of all non-isomorphic connected graphs with five vertices that do not have the chorded five cycle as a spanning subgraph [5].

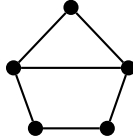


Figure 28: The chorded five cycle $C(5, 2)$

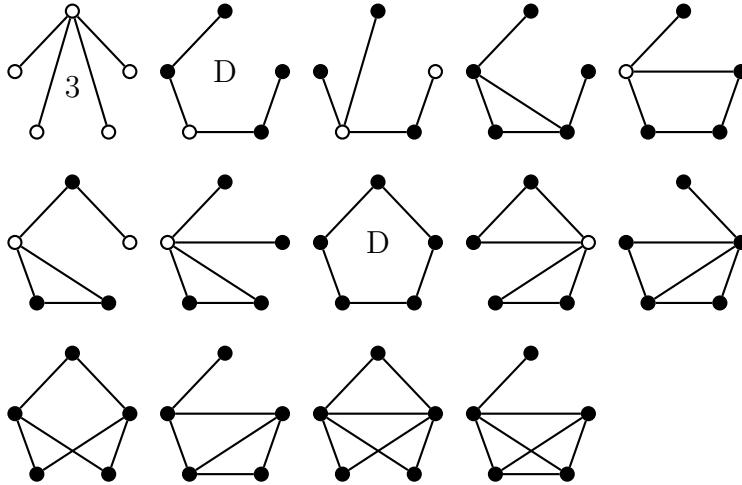


Figure 29: The solvability of graphs with five vertices [5]

Of the 112 non-isomorphic connected graphs with six vertices, nine are not freely

solvable. Figure 30 lists only those graphs that are *not* freely solvable [5].

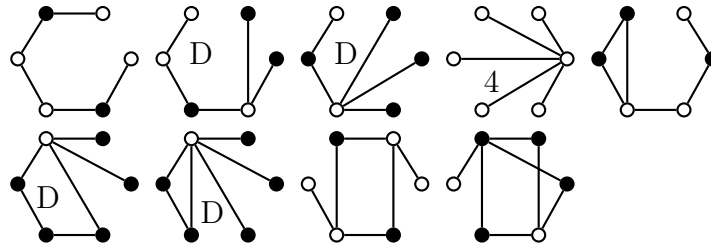


Figure 30: Graphs with six vertices that are not freely solvable [5]

There are 853 non-isomorphic connected graphs with seven vertices. Of these, thirty-three are not freely solvable. Figure 31 lists only those graphs with seven vertices that are *not* freely solvable [5].

Tables 1 and 2 show the percentages of connected graphs on seven vertices that are solvable and freely solvable. Note that as the number of edges increases, the percentages of solvable and freely solvable graphs increase as well.

Graphs on Seven Vertices					
Edges	6	7	8	9	10
Percent Solvable	54.5%	87.9%	98.5%	100%	100%

Table 1: Percentage of solvable graphs with order seven

Graphs on Seven Vertices					
Edges	6	7	8	9	10
Percent Freely Solvable	0%	53.1%	92.3%	98.1%	100%

Table 2: Percentage of freely solvable graphs with order seven

Beeler and Gray [6] present all freely solvable trees with order 10 or less. Figure 32 shows these trees.

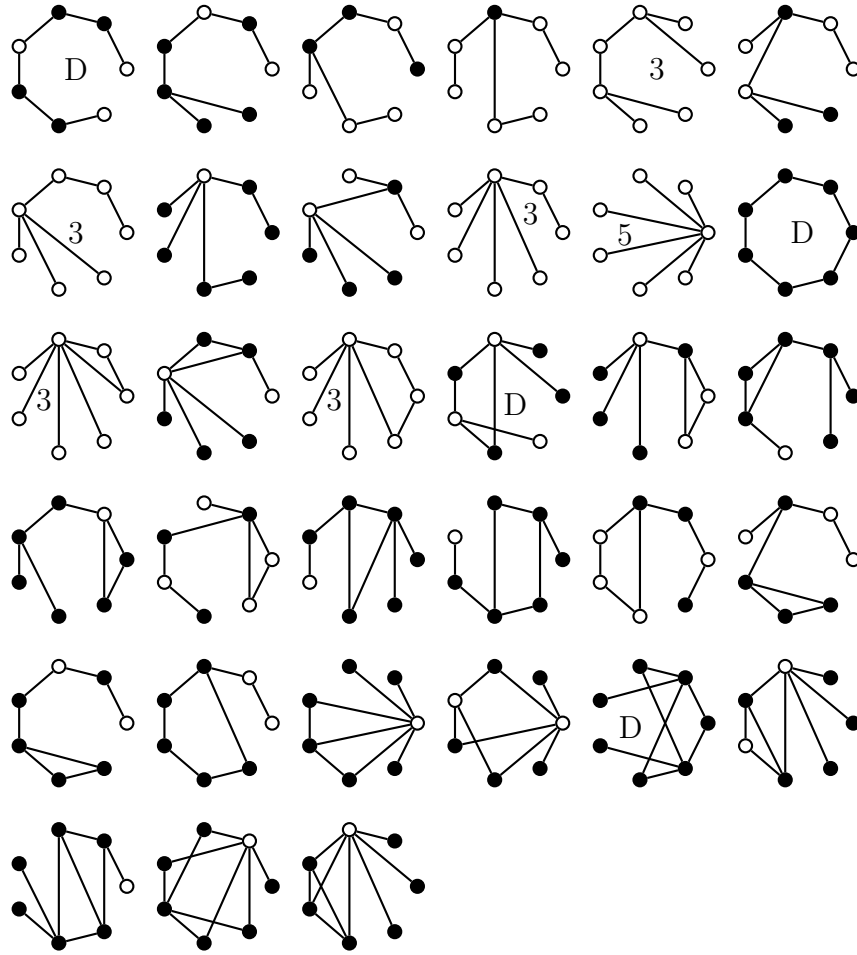


Figure 31: Graphs with seven vertices that are not freely solvable [5]

Using [10], Beeler and Gray [5] determined that $C(2n+1, m)$ is freely solvable for all $n \leq 9$ and $m \leq n$. To what extent *all* chorded odd cycles are freely solvable is not yet known. However, an important result is given in the following theorem.

Theorem 5.1 [5] *For all n and $m \leq n$, the chorded odd cycle $C(2n+1, m)$ is solvable.*

Proof. If $m = 3$, then begin with the initial hole in 1. Jump $2n \cdot \overrightarrow{0} \cdot 1$ and $2 \cdot \overrightarrow{3} \cdot 0$. For the next series of $n - 2$ jumps, the i th jump is $2i + 1 \cdot \overrightarrow{2i} \cdot 2i - 1$ for $i = 2, \dots, n - 1$. For

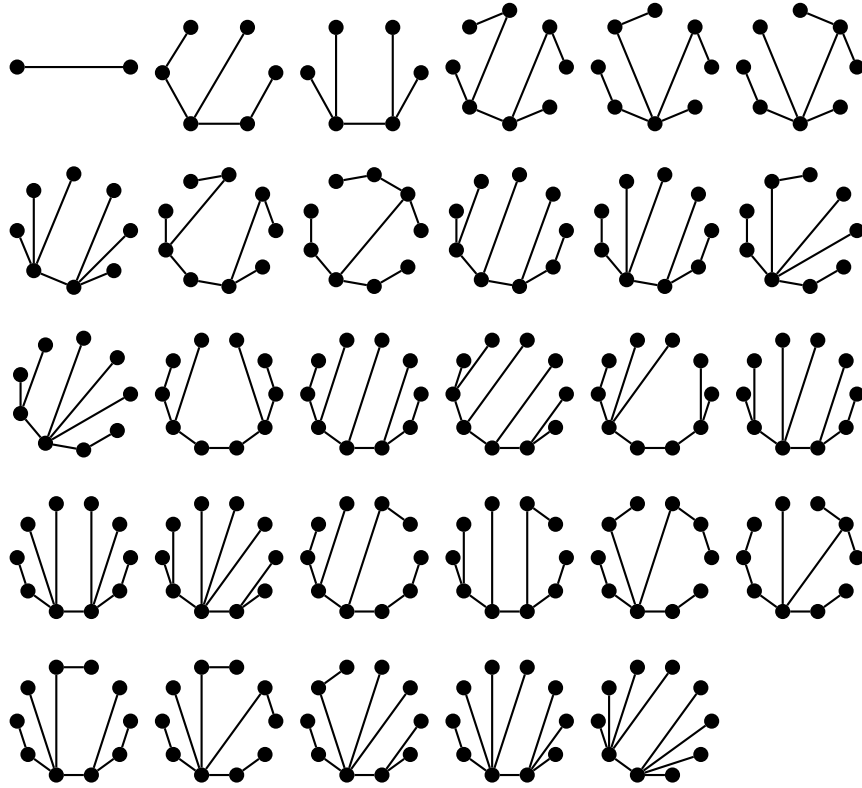


Figure 32: All freely solvable trees with order 10 or less [6]

the final series of $n - 1$ jumps, the j th jump is $2j \cdot 2j + 1 \cdot 2j + 2$ for $j = 0, 1, \dots, n - 2$, until a single peg remains in $2n - 2$.

If $m \neq 3$, then note that if we ignore 0 and m , then the graph consists of a path with an odd number of vertices and a path with an even number of vertices. If m is odd, then relabel 0 as α_0 and m as α_{2n+1-m} . Relabel the vertices of the odd path so that vertices $2n, 2n - 1, \dots, j$ are labeled $\alpha_1, \alpha_2, \dots, \alpha_{2n+1-j}$, respectively. Relabel the vertices of the even path so that vertices $m - 1, m - 2, \dots, 1$ are labeled $\alpha_{2n-m+2}, \dots, \alpha_{2n}$, respectively.

Begin with the initial hole in α_{2n-3} . Treating $\alpha_{2n-m}, \alpha_{2n+1-m}, \dots, \alpha_{2n-2}$ as an even

path, solve it as described in Remark 2.5 so that the remaining peg on the path is in α_{2n+1-m} . For the next series of $n - 3$ jumps, the i th jump is $\alpha_{2n-m-2i} \cdot \overrightarrow{\alpha_{2n-m-2i+1}} \cdot \alpha_{2n-m-2i+2}$ for $i = 1, 2, \dots, n - \frac{m+1}{2}$. Now jump $\alpha_{2n+1-m} \cdot \overrightarrow{\alpha_0} \cdot \alpha_1$ and $\alpha_{2n-1} \cdot \overrightarrow{\alpha_{2n}} \cdot \alpha_0$. For the final series of $n - 2$ jumps, the j th jump is $\alpha_{2j} \cdot \overrightarrow{\alpha_{2j+1}} \cdot \alpha_{2j+2}$ for $j = 0, 1, \dots, n - \frac{m+1}{2}$, until a single peg remains in α_{2n+1-m} .

If m is even, then begin with the initial hole in $2n - 3$. Treating $m - 1, m, \dots, 2n - 2$ as an even path, solve it as described in Remark 2.5 so that the remaining peg on the path is in m . For the next series of $\frac{m}{2} - 1$ jumps, the i th jump is $(m - 2i - 1) \cdot \overrightarrow{(m - 2i)} \cdot (m - 2i + 1)$ for $i = 1, 2, \dots, \frac{m}{2} - 1$. Now jump $m \cdot \overrightarrow{0} \cdot 1$ and $(2n - 1) \cdot \overrightarrow{2n} \cdot 0$. For the final series of $\frac{m}{2}$ jumps, the j th jump is $2j \cdot \overrightarrow{(2j + 1)} \cdot (2j + 2)$ for $j = 0, \dots, \frac{m}{2} - 1$, until a single peg remains in m . ■

Beeler and Gray [6] expand the results on chorded odd cycles while examining the cycle with a subdivided chord. The cycle with a subdivided chord, denoted $CSC(n, m)$, is formed from a cycle on n vertices, labeled $0, 1, \dots, n - 1$ in the usual way, by adding an edge from vertex 0 to vertex m to form a chord. This edge is then subdivided once. The resulting vertex on the chord is labeled c . Figure 33 shows a cycle with a subdivided chord.

The following result shows that $CSC(n, m)$ is solvable with the initial hole in several vertices. However, these graphs may be solvable from additional vertices as well.

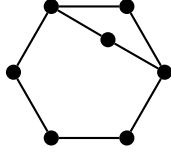


Figure 33: The cycle with a subdivided chord $CSC(6, 2)$

Lemma 5.2 [6] *The graph $CSC(2k, m)$ is solvable with the initial hole in $0, 1, 2, 3, m - 3, m - 2, m - 1, m, m + 1, m + 2, m + 3, 2k - 3, 2k - 2, 2k - 1$, or c . The graph $CSC(2k + 1, m)$ is solvable with the initial hole in $0, 2, m - 2, m, m + 2$, or $2k - 1$.*

Proof. Note that the vertices i and $i + m \pmod{n}$ are symmetric.

For $CSC(2k, m)$, suppose that the initial hole is in 0 . Jump $m \cdot \overrightarrow{c} \cdot 0$. An even path subgraph is formed by $m - 1, m, \dots, 2k - 1, 0, \dots, m - 2$ with a hole in m . This subgraph is solvable with the final peg in $m - 3$ by Remark 2.5. Alternately, an even path is formed by $m + 1, m, m - 1, \dots, 0, 2k - 1, \dots, m + 2$ with a hole in m . Solve this path with the final peg in $m + 3$. By Theorem 2.9, the graph may also be solved with the initial hole in $m - 3$ or $m + 3$. A similar argument holds if the initial hole is in $m, 3$, and $2k - 3$.

Suppose that the initial hole is in 1 . Jump $c \cdot \overrightarrow{0} \cdot 1$. Solve the even cycle formed by the remaining pegs with a hole in 0 . A similar argument holds when the initial hole is in $m - 1, m + 1$, or $2k - 1$.

Suppose that the initial hole is in 2 . Jump $0 \cdot \overrightarrow{1} \cdot 2$. Solve the even path formed by the remaining pegs with a hole in 0 . A similar argument holds for the case when the

initial hole is in $m - 2$, $m + 2$, or $2k - 2$.

Suppose that the initial hole is in c . Jump $1 \cdot \overrightarrow{0} \cdot c$. Solve the even path formed by the remaining pegs with a hole in 0.

For $CSC(2k + 1, m)$, suppose that the initial hole is in vertex 0. An even path is formed by $c, 0, 1, \dots, 2k$ with a hole in 0. Solve this path with the final peg in $2k - 1$. Alternately, an even path is formed by $c, 0, 2k, 2k - 1, \dots, 1$ with a hole in 0. Solve this path with the final peg in 2. By Theorem 2.9, the graph may also be solved with the initial hole in $2k - 1$ or 2. A similar argument holds when the initial hole is in m , $m - 2$, or $m + 2$. ■

We now provide a freely solvable example of this type of graph.

Theorem 5.3 [6] *The graph $CSC(2k, 1)$ is freely solvable.*

Proof. If the initial hole is in $\{2k - 3, 2k - 1, 0, 1, 2, 4\}$, then the graph is solvable by Lemma 5.2.

Suppose that the initial hole is in i , where i is even and $6 \leq i \leq 2k - 4$. An even path is formed by $i + 1, i, \dots, 1, c$ with a hole in i . Solve this path, ending in 1. Then jump $1 \cdot \overrightarrow{0} \cdot c$. Finally, solve the even path with a hole in 1, formed by $c, 1, \dots, i + 2$.

Suppose that the initial hole is in j , where j is odd and $3 \leq j \leq 2k - 5$. An even path is formed by $j - 1, j, \dots, 2k - 1, 0, c$ with a hole in j . Solve this path, ending in 0. Now jump $1 \cdot \overrightarrow{0} \cdot c$. Finally, solve the even path with a hole in 1, formed by

$c, 1, \dots, j - 2$. ■

Note that $CSC(2k, 1)$ is isomorphic to $C(2k + 1, 2)$. Thus, this provides partial progress on the open question from [5] as to whether all chorded odd cycles are freely solvable.

The results from Theorem 5.1 and Theorem 5.3 establish the next result.

Theorem 5.4 *The odd cycle C_{2k+1} is a critical graph.*

Proof. The odd cycle C_{2k+1} is distance 2-solvable by Theorem 2.4. The addition of an edge results in either a solvable graph by Theorem 5.1 or a freely solvable graph by Theorem 5.3. ■

A goal of this thesis is to improve the bounds on the maximum possible number of edges for an unsolvable connected graph on a fixed number of vertices. We denote this number $\tau(n)$, where n is the number of vertices in a graph. We use $\mathcal{J}(n)$ in the freely solvable case. Such results reveal more information about the necessary number of edges needed to guarantee that a graph is solvable (or freely solvable), regardless of its structure. They also provide progress on a peg solitaire analog to Turán's Theorem and an open problem posed in [8]. From the work in [5], the value of $\tau(n)$ and $\mathcal{J}(n)$ is known for several values of n . Table 3 shows $\tau(n)$ and $\mathcal{J}(n)$ for $4 \leq n \leq 7$.

n	4	5	6	7
$\tau(n)$	3	5	6	8
$\mathcal{J}(n)$	4	6	7	9

Table 3: $\tau(n)$ and $\mathcal{J}(n)$ for $4 \leq n \leq 7$

5.1 The Hairy Complete Graph

The graphs $K_n(n, 0, \dots, 0)$, $K_{n+1}(n, 0, \dots, 0)$, and $K_n(n+1, 0, \dots, 0)$ are three special cases of the hairy complete graph. Figure 34 shows an example of $K_n(n, 0, \dots, 0)$.

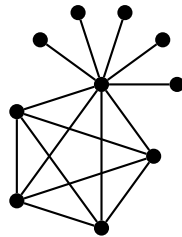


Figure 34: The hairy complete graph $K_5(5, 0, 0, 0, 0)$

Theorem 5.5 *For $n \geq 3$, the hairy complete graphs $K_n(n, 0, \dots, 0)$, $K_{n+1}(n, 0, \dots, 0)$, and $K_n(n+1, 0, \dots, 0)$ are critical graphs.*

Proof. If the initial hole is in x_j , where $j \neq 1$, then $K_n(n, 0, \dots, 0)$ is distance 2-solvable, $K_{n+1}(n, 0, \dots, 0)$ is solvable, and $K_n(n+1, 0, \dots, 0)$ is 3-solvable by Theorem 3.1. For all three graphs, up to automorphism, an additional edge may be inserted between two pendants, say $x_{1,1}$ and $x_{1,2}$, or between a pendant and a support vertex of the graph, say $x_{1,1}$ and x_j .

- (i) For $K_n(n, 0, \dots, 0)$, begin with the initial hole in x_1 , and perform the double star purge $\mathcal{DS}(X_1, X - \{x_1, x_j\}, n - 2)$, with $j \neq 1$, until pegs remain in $x_{1,1}$, $x_{1,2}$, and x_j .

For $K_n(n, 0, \dots, 0) + x_{1,1}x_{1,2}$, jump $x_{1,1} \cdot \overrightarrow{x_{1,2}} \cdot x_1$ and $x_j \cdot \overrightarrow{x_1} \cdot x_{1,1}$ to solve the graph with the final peg in $x_{1,1}$. Note that the final jump could also be $x_j \cdot \overrightarrow{x_1} \cdot x_{1,3}$ or $x_j \cdot \overrightarrow{x_1} \cdot x_k$, for $k \neq j$ and $k \neq 1$. Thus, by Theorem 2.9, $K_n(n, 0, \dots, 0) + x_{1,1}x_{1,2}$ may also be solved with the initial hole in $x_{1,1}$, $x_{1,3}$, or x_k .

For $K_n(n, 0, \dots, 0) + x_{1,1}x_j$, jump $x_{1,1} \cdot \overrightarrow{x_j} \cdot x_1$ and $x_{1,2} \cdot \overrightarrow{x_1} \cdot x_j$. Note that the final jump could also be $x_{1,2} \cdot \overrightarrow{x_1} \cdot x_{1,1}$ or $x_{1,2} \cdot \overrightarrow{x_1} \cdot x_k$ for $k \neq 1$ and $k \neq j$. Thus, by Theorem 2.9, $K_n(n, 0, \dots, 0) + x_{1,1}x_j$ may also be solved with the initial hole in x_j , $x_{1,1}$, or x_k .

- (ii) For $K_{n+1}(n, 0, \dots, 0) + x_{1,1}x_{1,2}$, suppose that $n = 3$. Begin with the initial hole in x_1 and jump $x_3 \cdot \overrightarrow{x_4} \cdot x_1$, $x_{1,3} \cdot \overrightarrow{x_1} \cdot x_3$, and $x_{1,1} \cdot \overrightarrow{x_{1,2}} \cdot x_1$. This reduces the graph to K_4 with a hole in x_4 , which is solvable with the final peg in $x_{1,1}$, $x_{1,3}$, x_1 , or x_k , with $k \neq 1$ and $k \neq 4$. By Theorem 2.9, $K_4(3, 0, \dots, 0) + x_{1,1}x_{1,2}$ may also be solved with the initial hole in $x_{1,1}$, $x_{1,3}$, x_1 , or x_k .

Suppose that $n \geq 4$. Begin with the initial hole in x_j , and jump $x_{1,3} \cdot \overrightarrow{x_1} \cdot x_j$. Perform the double star purge $\mathcal{DS}(X_1, X - \{x_1, x_j\}, n - 3)$, and jump $x_{1,1} \cdot \overrightarrow{x_{1,2}} \cdot x_1$. This reduces the graph to K_{n+1} with a hole in x_k , for $k \neq 1$ and $k \neq j$. This is solvable with the final peg in $x_{1,1}$, $x_{1,3}$, or x_1 . Thus, by Theorem 2.9,

$K_{n+1}(n, 0, \dots, 0) + x_{1,1}x_{1,2}$ may also be solved with the initial hole in $x_{1,1}$, $x_{1,3}$, or x_1 .

For $K_{n+1}(n, 0, \dots, 0) + x_{1,1}x_j$ begin with the initial hole in $x_{1,2}$ and jump $x_{1,3} \cdot \overrightarrow{x_1} \cdot x_{1,2}$. Perform the double star purge $\mathcal{DS}(X_1, X - \{x_1, x_j\}, n - 2)$, until pegs remain in $x_{1,1}$, x_j , and x_k , with $k \neq 1$ and $k \neq j$. Then jump $x_{1,1} \cdot \overrightarrow{x_j} \cdot x_1$ and $x_k \cdot \overrightarrow{x_1} \cdot x_{1,1}$ to solve the graph with the final peg in $x_{1,1}$. Note that the final two jumps could also be $x_{1,1} \cdot \overrightarrow{x_j} \cdot x_\ell$ and $x_\ell \cdot \overrightarrow{x_k} \cdot x_1$, for $\ell \notin \{1, j, k\}$. Thus, by Theorem 2.9, $K_{n+1}(n, 0, \dots, 0) + x_{1,i}x_j$ may also be solved with the initial hole in $x_{1,1}$, x_1 , or x_j .

(iii) For $K_n(n + 1, 0, \dots, 0)$, begin with the initial hole in x_j and solve the graph as described in the proof of Theorem 3.1 until pegs remain in $x_{1,1}$, $x_{1,2}$ and x_j . For $K_n(n + 1, 0, \dots, 0) + x_{1,1}x_{1,2}$, jump $x_{1,2} \cdot \overrightarrow{x_{1,1}} \cdot x_1$ and $x_j \cdot \overrightarrow{x_1} \cdot x_{1,1}$ to solve the graph with the final peg in $x_{1,1}$. For $K_n(n + 1, 0, \dots, 0) + x_{1,i}x_j$, jump $x_{1,1} \cdot \overrightarrow{x_j} \cdot x_1$ and $x_{1,2} \cdot \overrightarrow{x_1} \cdot x_{1,1}$ to solve the graph with the final peg in $x_{1,1}$.

■

Using Theorem 5.5, we can now give nontrivial lower bounds on $\tau(n)$ and $\mathcal{J}(n)$.

Corollary 5.6 *For $k \in \mathbb{Z}^+$, $\tau(2k) \geq \frac{k(k+1)}{2}$ and $\tau(2k + 1) \geq \frac{k(k+1)}{2} + 1$. Further, these bounds are sharp.*

Proof. By Theorem 3.1, the hairy complete graphs $K_k(k, 0, \dots, 0)$ and $K_k(k + 1, 0, \dots, 0)$ are not solvable. By Theorem 5.5, the addition of any single edge to either graph results in a solvable graph. The size of $K_k(k, 0, \dots, 0)$ is $\frac{k(k+1)}{2}$. The size of $K_k(k + 1, 0, \dots, 0)$ is $\frac{k(k+1)}{2} + 1$. ■

Corollary 5.7 *For $k \in \mathbb{Z}^+$, $\mathcal{J}(2k) \geq \frac{k(k+1)}{2}$ and $\mathcal{J}(2k + 1) \geq \frac{k(k+3)}{2}$. Further, these bounds are sharp.*

Proof. By Theorem 3.1 the hairy complete graphs $K_k(k, 0, \dots, 0)$ and $K_{k+1}(k, 0, \dots, 0)$ are not freely solvable. By Theorem 5.5, the addition of any single edge to either graph results in a freely solvable graph. The size of $K_k(k, 0, \dots, 0)$ is $\frac{k(k+1)}{2}$. The size of $K_{k+1}(k, 0, \dots, 0)$ is $\frac{k(k+3)}{2}$. ■

Corollary 5.6 states that a lower bound for $\tau(n)$ is approximately $\frac{n^2}{8}$. Because the trivial upper bound for $\tau(n)$ is $\frac{n(n-1)}{2}$, it may be difficult to find a better lower bound. This leads us to the following conjecture.

Conjecture 5.8 *For all $k \in \mathbb{Z}^+$, $\tau(2k) = \frac{k(k+1)}{2}$.*

Theorem 5.9 *Among all unsolvable graphs with a double star spanning subgraph and order $n + m$, the graph $K_n(m, 0, \dots, 0)$, with $m \geq n$, is the one with maximum size.*

Proof. Consider the double star $K_2(n, n - 2)$, where $n \geq 3$. Adding any number of edges to the set $\{x_1, x_2, x_{2,1}, \dots, x_{2,n-2}\}$ results in a spanning subgraph of

$K_n(n, 0, \dots, 0)$. Since this is not solvable by Theorem 3.1, this results in an unsolvable graph by Proposition 2.8. For $n \geq 4$, up to automorphism, there are three remaining places an edge can be added.

- (i) An edge is inserted between $x_{1,1}$ and $x_{1,2}$. Begin with the initial hole in x_1 . Perform the double star purge $\mathcal{DS}(X_1, X_2, n-2)$ so that the remaining pegs are in $x_{1,1}$, $x_{1,2}$, and x_2 . Jump $x_{1,1} \cdot \overrightarrow{x_{1,2}} \cdot x_1$ and $x_2 \cdot \overrightarrow{x_1} \cdot x_{1,1}$ to solve the graph with the final peg in $x_{1,1}$.
- (ii) An edge is inserted between $x_{1,1}$ and x_2 . Begin with the initial hole in x_1 . Perform the double star purge $\mathcal{DS}(X_1, X_2, n-2)$ so that the remaining pegs are in $x_{1,1}$, $x_{1,2}$, and x_2 . Jump $x_{1,1} \cdot \overrightarrow{x_2} \cdot x_1$ and $x_{1,2} \cdot \overrightarrow{x_1} \cdot x_{1,1}$ to solve the graph with the final peg in $x_{1,1}$.
- (iii) An edge is inserted between $x_{1,1}$ and $x_{2,1}$. Begin with the initial hole in x_2 . Perform the double star purge $\mathcal{DS}(X_2, X_1, n-3)$, so that the remaining pegs are in $x_{1,1}$, $x_{1,2}$, $x_{1,3}$, x_1 and $x_{2,2}$. Jump $x_{1,3} \cdot \overrightarrow{x_1} \cdot x_2$, $x_{2,2} \cdot \overrightarrow{x_2} \cdot x_{2,1}$, $x_{2,1} \cdot \overrightarrow{x_{1,1}} \cdot x_1$, and $x_{1,2} \cdot \overrightarrow{x_1} \cdot x_2$ to solve the graph with the final peg in x_2 .

For $n = 3$, edge addition between $x_{1,1}$ and $x_{2,1}$, results in $K_{2,2}(2, 0; 0, \dots, 0)$, which is not solvable. Figure 35 shows $K_{2,2}(2, 0; 0, \dots, 0)$. However, $K_{2,2}(2, 0; 0, \dots, 0)$ has the same size as $K_3(3, 0, 0)$. Further, in the next section, we show that $K_{2,2}(2, 0; 0, \dots, 0)$

is a critical graph. ■

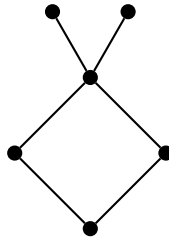


Figure 35: The hairy complete bipartite graph $K_{2,2}(2, 0; 0, \dots, 0)$

5.2 The Hairy Complete Bipartite Graph

The graph $K_{2,m}(a_1, a_2; 0, \dots, 0)$ is a special case of the hairy complete bipartite graph. Figure 36 shows an example of $K_{2,m}(a_1, a_2; 0, \dots, 0)$.

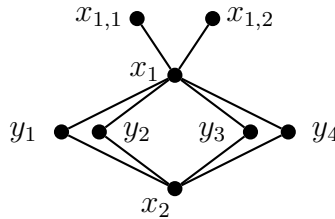


Figure 36: The hairy complete bipartite graph $K_{2,4}(2, 0; 0, 0, 0, 0)$

Theorem 5.10 *The hairy complete bipartite graph $G = K_{2,m}(a_1, a_2; 0, \dots, 0)$ is a critical graph.*

Proof. The graph G is unsolvable by Theorem 3.3, with $a_1 + a_2 - 1$ pegs remaining in $X_1 \cup X_2$. If \mathcal{P} , then a peg also remains in x_2 . If $(\sim \mathcal{P})$, then, instead, this peg is

in y_1 . Up to automorphism, an additional edge can be inserted in one of nine places:

- (i) An edge is inserted between $x_{1,1}$ and $x_{1,2}$. Solve the graph as described above. Then jump $x_{1,1} \cdot \overrightarrow{x_{1,2}} \cdot x_1$. This removes an additional peg. The argument is similar for an edge inserted between $x_{2,1}$ and $x_{2,2}$.
- (ii) An edge is inserted between $x_{1,1}$ and $x_{2,1}$. Solve the graph as described above. The jump $x_{2,1} \cdot \overrightarrow{x_{1,1}} \cdot x_1$ removes an additional peg.
- (iii) An edge is inserted between $x_{1,1}$ and x_2 . By relabeling $x_{1,1}$ as y_{m+1} , the graph is isomorphic to $K_{2,m+1}(a_1 - 1, a_2; 0, \dots, 0)$. As noted in the proof of Theorem 3.3, the peg in y_{m+1} may be removed while removing the pegs in Y and $X - \{x_1\}$. Thus, this is an additional peg that can be removed. The argument is similar for an edge inserted between $x_{2,1}$ and x_1 .
- (iv) An edge is inserted between x_1 and x_2 . The graph now has a double star spanning subgraph.
- (v) An edge is inserted between y_1 and y_2 . Before removing the pegs in Y and $X - \{x_1\}$, jump $y_1 \cdot \overrightarrow{y_2} \cdot x_1$ and $x_{1,1} \cdot \overrightarrow{x_1} \cdot y_1$. This removes an additional peg from X_1 .
- (vi) An edge is inserted between $x_{1,1}$ and y_1 . If $(\sim \mathcal{P})$, then solve the graph as described above. Then jump $x_{1,1} \cdot \overrightarrow{y_1} \cdot x_1$. This removes an additional peg. If \mathcal{P} , then with the initial hole in x_1 , first jump $x_{1,1} \cdot \overrightarrow{y_1} \cdot x_1$ before solving the graph

as described above. This removes an additional peg. The argument is similar for an edge inserted between $x_{2,1}$ and y_1 .

Note that each addition of an edge listed above may result in the elimination of multiple pegs. Also, the arguments are similar for freely solvable criticality.

■

Using Theorem 5.10, we can now give nontrivial lower bounds on $\tau(n)$ and $\mathcal{T}(n)$.

Corollary 5.11 *For $k \in \mathbb{Z}^+$, $\tau(2k + 5) \geq 4k + 4$ and $\tau(2k + 6) \geq 4k + 5$. Further, these bounds are sharp.*

Proof. By Theorem 3.3, the hairy complete bipartite graphs $K_{2,2k+1}(2, 1; 0, \dots, 0)$ and $K_{2,2k+1}(1, 1; 0, \dots, 0)$ are not solvable. By Theorem 5.10, any addition of a single edge to either graph results in a solvable graph. The size of $K_{2,2k+1}(2, 1; 0, \dots, 0)$ is $4k + 5$. The size of $K_{2,2k+1}(1, 1; 0, \dots, 0)$ is $4k + 4$.

■

Corollary 5.12 *For $k \in \mathbb{Z}^+$, $\mathcal{T}(2k + 3) \geq 4k + 1$ and $\mathcal{T}(2k + 4) \geq 4k + 3$. Further, these bounds are sharp.*

Proof. By Theorem 3.3, the hairy complete bipartite graphs $K_{2,2k}(1, 0; 0, \dots, 0)$ and $K_{2,2k+1}(1, 0; 0, \dots, 0)$ are not freely solvable. By Theorem 5.10, any addition of a single edge to either graph results in a freely solvable graph. The size of $K_{2,2k}(1, 0; 0, \dots, 0)$

is $2k + 3$. The size of $K_{2,2k+1}(1, 0; 0, \dots, 0)$ is $2k + 4$. ■

Table 4 and Table 5 summarize the established sharp lower bounds on $\tau(n)$ and $\mathcal{J}(n)$ for numerous values of n .

n	3	4	5	6	7	8	9	10	11	12	13
Actual Value	2	3	5	6	8	10	≥ 12	≥ 15	≥ 16	≥ 21	≥ 22
Corollary 5.6	2	3	4	6	7	10	11	15	16	21	22
Corollary 5.11	n/a	n/a	n/a	n/a	8	9	12	13	16	17	20

Table 4: Sharp lower bounds for $\tau(n)$

n	3	4	5	6	7	8	9	10	11	12	13
Actual Value	2	4	6	7	9	11	≥ 14	≥ 15	≥ 20	≥ 21	≥ 27
Corollary 5.7	2	3	5	6	9	10	14	15	20	21	27
Corollary 5.12	n/a	n/a	5	7	9	11	13	15	17	19	21

Table 5: Sharp lower bounds for $\mathcal{J}(n)$

6 CONCLUDING REMARKS

In this section, we conclude our discussion with open problems for future research. Theorem 3.1 and Theorem 3.3 establish the necessary and sufficient conditions for the hairy complete graph and the hairy complete bipartite graph. What are the necessary and sufficient conditions for the solvability of other graph families? Also, the proof of Theorem 3.3 utilizes a homomorphism. For what other graphs does a homomorphism prove helpful to eliminate pegs?

Theorem 5.4, Theorem 5.5, and Theorem 5.10 discuss critical graphs. What other graphs are critical graphs? In addition, how much can edge addition improve the solvability of a graph? What specific edge additions improve the solvability of a graph the most? What specific edge additions improve the solvability of a graph the least?

Corollary 5.6, Corollary 5.7, Corollary 5.11 and Corollary 5.12 provide lower bounds for $\tau(n)$ and $\mathcal{T}(n)$. Can we find nontrivial upper bounds for $\tau(n)$ and $\mathcal{T}(n)$? In addition, under what circumstances does $\mathcal{T}(n) = \tau(n)$? Under what circumstances does $\mathcal{T}(n) = \tau(n) + 1$?

The following open problems are also included in other studies of peg solitaire on graphs.

Corollary 3.2 provides a way to construct unsolvable graphs. As asked in [7], are there other methods of constructing unsolvable graphs?

Figure 27 , Figure 29, Figure 30, and Figure 31 include the solvability of all graphs with seven vertices or fewer. As asked in [5], what is the solvability of all graphs with eight vertices?

Theorem 5.3 discusses the solvability of a cycle with a subdivided chord. As asked in [6], what is the solvability of a cycle with a chord that is subdivided multiple times?

Theorem 4.1 and Theorem 4.2 concern fool's solitaire. As asked in [11], how much can edge deletion lower the fool's solitaire number of a graph? Also, what are the criticality results for fool's solitaire?

We discuss many graphs with pendants. As asked in [6], is the corona of a (freely) solvable graph likewise (freely) solvable?

In some variants of the traditional peg solitaire game, the location of the initial hole and the final peg is required to be the same. As asked in [6], for what graphs is it possible to start with a specific initial jump and end with another specific jump? Suppose that we want to start with the initial hole in any vertex s and end with the final peg in any vertex t . For what graphs is this possible?

In *peg duotaire* two players take turns making peg solitaire jumps. The player that is left without a jump loses. As asked in [8], for which graphs does Player One have a winning strategy? For which graphs does Player Two have a winning strategy? What if we consider peg solitaire moves rather than jumps, where a *move* is a series of jumps made with a single peg?

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