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Alliance Partitions in Graphs.

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Alliance Partitions in Graphs

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by

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ABSTRACT

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For a graph $G = (V, E)$, a nonempty subset S contained in V is called a *defensive alliance* if for each v in S, there are at least as many vertices from the closed neighborhood of v in S as in $V - S$. If there are strictly more vertices from the closed neighborhood of v in S as in $V - S$, then S is a *strong defensive alliance*. A (strong) defensive alliance is called *global* if it is also a dominating set of G. The *alliance partition number* (respectively, *strong alliance partition number*) is the maximum cardinality of a partition of V into defensive alliances (respectively, strong defensive alliances). The *global (strong) alliance partition number* is defined similarly. For each parameter we give both general bounds and exact values. Our major results include exact values for the alliance partition number of grid graphs and for the global alliance partition number of caterpillars.

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1 INTRODUCTION

In general, an alliance is simply some group which is united by a common interest or collective property. For example, countries may unite in an alliance by agreeing to mutually defend each other in the event of an attack on a member nation. We can model this type of situation using a graph where each country of interest is represented by a vertex and an edge joins countries which are related (by say, a common border, or some other common interest). We can then use properties of the graph to decide whether a proposed alliance is viable (according to some criteria). We might, for example, require that each country in a defensive alliance have at least as many allies as potential enemies. Motivated by examples such as this, alliances in graphs were first defined and studied in [11]. In this thesis, we study several types of alliances in graphs.

1.1 Preliminary Definitions

We begin by giving some basic graph theory definitions and terminology, generally following [1] and [9]. A graph G is a nonempty set $V(G)$ of *vertices*, together with a (possibly empty) set $E(G)$ of unordered pairs of distinct vertices (called *edges*). If it is clear which graph is under consideration, we will simply write V for the vertex set, E for the edge set, and we write $G = (V, E)$ to indicate that G is a graph with vertex set V and edge set E. An edge $\{u, v\} \in E(G)$ is usually denoted simply uv. We denote the *order* of a graph G as $n = |V|$ (assumed to be finite, unless otherwise stated) and the *size* as $m = |E|$. If $uv \in E$ we say that u and v are *adjacent* and that the edge uv is *incident* to u and to v. For a vertex $v \in V$, the *open neighborhood* of v is the set $N(v) = \{u \mid uv \in E\}$, while the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex v is $|N(v)|$, denoted $deg(v)$. The *minimum degree* of a graph G is $\delta(G) = \min\{deg(v) \mid v \in V\}$. Similarly, the *maximum degree* of G is $\Delta(G) = \max\{deg(v) \mid v \in V\}$.

A *path* in a graph is an alternating sequence of distinct vertices and edges, $v_1, v_1v_2, v_2, v_2v_3, ..., v_{k-1}v_k, v_k$. The *length* of a path is the number of edges present. If we have the additional property that $v_1v_k \in E$, then the sequence forms a *cycle*. Paths and cycles will usually be indicated by simply listing the vertices, as the edges present will then be apparent. The graph which consists of a path on n vertices is denoted P_n, while the graph which consists of a cycle on n vertices is denoted C_n. A graph is *complete* if every pair of vertices are adjacent. The complete graph of order n is written K_n.

Two vertices $u, v \in V$ are *connected* if there exists at least one path between them. The *distance* from u to v, denoted $d(u, v)$, is the length of a shortest path between u and v. A graph G is said to be *connected* if each pair of vertices in the graph are connected. A *tree* is a connected graph with no cycles. A vertex of degree one in a tree is called a *leaf* (or an *endvertex*), while a vertex adjacent to a leaf is called a *support vertex*. A vertex adjacent to at least two leaves is called a *strong support vertex*.

For a graph G, we write $\langle S \rangle$ to indicate the subgraph *induced* by a set of vertices S. That is, $\langle S \rangle$ is the graph consisting of the vertices in S along with all the edges of G which are incident to two vertices in S.

The *complement* of a graph G, denoted $\overline{G}$, has $V(\overline{G}) = V(G)$ and $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$. The *union* of two graphs, $G_1 \cup G_2$, has vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. The *cartesian product* of two graphs $G_1 \Box G_2$, has vertex set $V(G_1) \times V(G_2)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if either $u_1 = u_2$ and $v_1v_2 \in E(G_2)$ or $v_1 = v_2$ and $u_1u_2 \in E(G_1)$.

For a graph $G = (V, E)$, a subset $S \subseteq V$ is a *dominating set* if for every $v \in V$, $|N[v] \cap S| \geq 1$. That is, every vertex of the graph is either in S or is adjacent to a vertex in S. The minimum cardinality among all dominating sets of G is called the *domination number*, denoted $\gamma(G)$.

A *partition* of V is a collection of pairwise disjoint nonempty sets $\{V_1, V_2, ..., V_k\}$ of vertices, such that $V = \bigcup_{i=1}^{k} V_i$.

1.2 Defensive Alliances

Alliances in graphs were first proposed and studied by Kristiansen, Hedetniemi, and Hedetniemi in [11]. They introduced several types of alliances, including the defensive alliances we consider here. Let $G = (V, E)$ be a graph. A nonempty subset $S \subseteq V$ is a *defensive alliance* if for every $v \in S$, $|N[v] \cap S| \geq |N(v) \cap (V - S)|$. If u and v belong to an alliance S, we say u is *allied* with v and that u and v are *allies*. It will be convenient to consider a vertex v to be its own ally. Then, if S is a defensive alliance, we have that for each $v \in S$, v has at least as many allies in S as it has neighbors outside S. Thus, by strength of numbers, the vertices in S are *defended* from possible attack by neighboring vertices outside of S. If for every $v \in S$, $|N[v] \cap S| > |N(v) \cap (V - S)|$, then S is a *strong defensive alliance* and we say that the vertices of S are *strongly defended*. The *alliance number*, $a(G)$ (respectively, *strong alliance number*, $\hat{a}(G)$), is the minimum cardinality of a defensive alliance (respectively, strong defensive alliance) in G. A simple, but important, observation made in [11] is that any minimum alliance must induce a connected subgraph of G, for otherwise each component of the alliance is a strictly smaller alliance. In this thesis, we study several types of defensive alliances, and in what follows, 'alliance' should be understood to mean 'defensive alliance'.

An alliance which affects every vertex of the graph is said to be *global*. Thus, a global (strong) defensive alliance is a dominating set. Domination in graphs has been studied extensively [9, 10]. The minimum cardinality of a global (strong) defensive alliance in G is the *global alliance number*, $\gamma_a(G)$ (respectively, *global strong alliance number*, $\gamma_{\hat{a}}(G)$). Unlike in the case of ordinary defensive alliances, here we cannot assume that each (minimum) global alliance is connected (for example, the endvertices of P_4 form a minimum global alliance) [8].

In their introductory paper, Kristiansen, Hedetniemi, and Hedetniemi [11] conjectured the following upper bounds for the alliance number and strong alliance number, which were later proved by Fricke, Lawson, Haynes, Hedetniemi, and Hedetniemi [4].

Theorem 1.1 [4] *For any graph G,*
(i) $a(G) \leq \left\lceil \frac{n}{2} \right\rceil$, *and*
(ii) $\hat{a}(G) \leq \left\lfloor \frac{n}{2} \right\rfloor + 1$.

Haynes, Hedetniemi, and Henning [8] established the following upper and lower bounds for the global alliance number and global strong alliance number.

Theorem 1.2 [8] *For any graph G,*

(i) $\dfrac{\sqrt{4n+1}-1}{2} \le \gamma_a(G) \le n - \left\lceil \dfrac{\delta(G)}{2} \right\rceil$, *and*

(ii) $\sqrt{n} \le \gamma_{\hat{a}}(G) \le n - \left\lfloor \dfrac{\delta(G)}{2} \right\rfloor$.

Exact values of the (strong) alliance number and global (strong) alliance number are given for some classes of graphs in [8] and [11]. We summarize several of these in Table 1.

Table 1: Alliance Numbers.

Class	K_n	P_n $(n \ge 3)$	C_n $(n \ge 3)$
$a(G)$	$\lfloor \frac{n+1}{2} \rfloor$	1	2
$\hat{a}(G)$	$\lceil \frac{n+1}{2} \rceil$	2	2
$\gamma_a(G)$	$\lfloor \frac{n+1}{2} \rfloor$	$\lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{4} \rceil - \lfloor \frac{n}{4} \rfloor - 1$, if $n \equiv 2 \bmod 4$ $\lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{4} \rceil - \lfloor \frac{n}{4} \rfloor$, otherwise	$\lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{4} \rceil - \lfloor \frac{n}{4} \rfloor$
$\gamma_{\hat{a}}(G)$	$\lceil \frac{n+1}{2} \rceil$	$\lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{4} \rceil - \lfloor \frac{n}{4} \rfloor$	$\lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{4} \rceil - \lfloor \frac{n}{4} \rfloor$

1.3 The Alliance Partition Numbers

In this thesis, we will study partitions of the vertex set of a graph into different types of defensive alliances. We are now prepared to define these alliance partition parameters. The *alliance partition number*, $\psi_a(G)$, is the maximum cardinality of a partition of V into defensive alliances. Similarly, the *strong alliance partition number*, $\psi_{\hat{a}}(G)$, is the maximum cardinality of a partition of V into strong alliances. The maximum cardinality of a partition of V into global alliances (respectively, global strong alliances) is the *global alliance partition number*, $\psi_g(G)$ (respectively, *global strong alliance partition number*, $\psi_{\hat{g}}(G)$). A partition of V into defensive alliances is called an *alliance partition* and an alliance partition Π with $|\Pi| = \psi_a(G)$ will be called a *ψ_a-partition*. We use similar notation for strong, global, and global strong alliance partitions. Some of the questions we consider for alliance partitions in graphs have recently been studied independently by Eroh and Gera and we discuss selected results from [2, 3] in Chapters 2 and 3 of this thesis.

We note that for any graph G, V is an (strong, global) alliance. Thus, every graph has an (strong, global, global strong) alliance partition number. Furthermore, no partition of V contains more than n sets. Several inequalities involving these parameters follow immediately from the definitions, and we state these now.

Observation 1.3 *For any graph G,*

(i) $1 \leq \psi_{\hat{g}}(G) \leq \psi_{\hat{a}}(G) \leq \psi_a(G) \leq n$, *and*

(ii) $1 \leq \psi_{\hat{g}}(G) \leq \psi_g(G) \leq \psi_a(G) \leq n$.

Example 1.4 *Alliance Partition Numbers of P_6.*

We illustrate some of these ideas with an example. Consider the path P_6. Notice that the endvertices may each form an alliance of cardinality one, while each internal vertex must be allied with at least one neighbor. Thus, $\psi_a(P_6) \leq 4$. A partition into four alliances is shown in Figure 1(a). Thus, $\psi_a(P_6) = 4$. Any strong alliance in P_6 has at least two vertices, implying that $\psi_{\hat{a}}(P_6) \leq 3$. The path P_6 has a partition into three strong alliances, as shown in Figure 1(b), and hence, $\psi_{\hat{a}}(P_6) = 3$. Next,

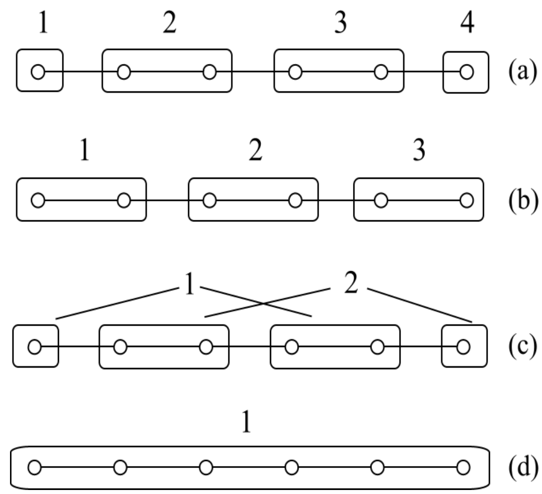

Figure 1: Alliance Partitions of P_6.

we claim that any global alliance in P_6 has at least three vertices. Suppose to the contrary that u and v form a global alliance. Since u and v form a dominating set, it is easy to see that at least one of u and v must be an internal vertex, say u, and that furthermore, $uv \notin E$. However, in this case, we see that u is not defended. Thus, $\gamma_a(P_6) \geq 3$, implying that $\psi_g(P_6) \leq 2$. Since P_6 can be partitioned into two global alliances (see Figure 1(c)), we have $\psi_g(P_6) = 2$. Finally, notice that since any strong alliance has at least two vertices, any global strong alliance containing an endvertex v also contains the adjacent support vertex. Thus, no other alliance can dominate v, implying that $\psi_{\hat{g}}(P_6) = 1$ (see Figure 1(d)).

Though in the preceding example we have $\psi_g(G) \leq \psi_{\hat{a}}(G)$, this inequality does not hold in general. In fact, the global alliance partition number and strong alliance partition number are incomparable, as can be seen by considering K_2, with $\psi_g(K_2) = 2 > \psi_{\hat{a}}(K_2) = 1$. More generally, we will show that for any positive integer $k > 2$, we have $\psi_g(P_{2k}) = 2 < \psi_{\hat{a}}(P_{2k}) = k$ and $\psi_g(K_{2k}) = 2 > \psi_{\hat{a}}(K_{2k}) = 1$.

1.4 Similar Concepts

Several authors have studied problems closely related to alliance partitions. A concept closely related to the strong alliance partition number has been studied by Gerber and Kobler [6]. Let G be a graph with vertex set V. For a subset $S \subseteq V$, a vertex $v \in S$ is said to be *satisfied with respect to* S, if it has at least as many neighbors in S as in $V - S$. It is easy to see that in this case, S is a strong defensive alliance. A graph is said to be *satisfiable* if there exists a partition of V into two nonempty disjoint sets such that every vertex is satisfied with respect to the set in which it occurs. So, using our notation, we see that a graph is satisfiable if and only if $\psi_{\hat{a}}(G) \geq 2$. They have shown that all graphs of girth at least five are satisfiable and they characterized all triangle-free graphs G for which the line graph of G has a satisfactory partition [6].

The global alliance partition number can be considered a variation of the well studied parameter, *domatic number* [9, 10]. For a graph $G = (V, E)$, the domatic number $d(G)$ is the maximum cardinality of a partition of V into dominating sets. Therefore, since every global alliance is a dominating set, for any graph G, the global alliance partition number is no greater than the domatic number. We now state a well known upper bound for the domatic number of a graph.

Theorem 1.5 [9] *For any graph G, $d(G) \leq \delta(G) + 1$.*

Graphs which achieve this bound are said to be *domatically full*. For example, all trees are known to be domatically full [9], though (as we will show) not all trees have $\psi_g(T) = \delta(T) + 1$. We will however, in Chapter 3, give a family of trees for which $\psi_g(T) = d(T)$.

2 ALLIANCE PARTITIONS

In this chapter, we give bounds and exact values for the alliance partition number and strong alliance partition number. Our major result in this chapter is a formula for the alliance partition number of any grid graph [7].

2.1 Bounds

Eroh and Gera [2] established sharp bounds on the alliance partition number of general graphs, and we begin by stating two of these.

Theorem 2.1 [2] *Let G be a connected graph of order $n \geq 3$. Then*

$$1 \leq \psi_a(G) \leq \left\lfloor n + \frac{3}{2} - \frac{\sqrt{1+4n}}{2} \right\rfloor .$$

Theorem 2.2 [2] *For any graph G,*

$$\psi_a(G) \leq \left\lfloor \frac{n}{\left\lceil \frac{\delta(G)+1}{2} \right\rceil} \right\rfloor .$$

Clearly, for any graph G, $\psi_{\hat{a}}(G)$ is bounded above by n. Furthermore, since a strong alliance partition of any graph with $\delta(G) \geq 1$ must have at least one alliance of two or more vertices, we have a characterization of graphs which attain this bound.

Theorem 2.3 *For any graph G, $\psi_{\hat{a}}(G) \leq n$, with equality if and only if G is $\overline{K_n}$.*

Motivated by Theorem 2.2, we give a similar bound involving minimum degree for the strong alliance partition number.

Theorem 2.4 *For any graph G,*

$$\psi_{\hat{a}}(G) \leq \left\lfloor \frac{n}{\left\lceil \frac{\delta(G)+2}{2} \right\rceil} \right\rfloor,$$

and this bound is sharp.

Proof. Let v be a vertex of minimum degree in a strong alliance A. Then, since v is strongly defended, v is allied with at least $\lceil deg(v)/2 \rceil = \lceil \delta(G)/2 \rceil$ neighbors. Thus $|A| \geq 1 + \lceil \delta(G)/2 \rceil = \lceil (\delta(G)+2)/2 \rceil$. Therefore, any strong alliance partition contains at most $\left\lfloor \frac{n}{\lceil (\delta(G)+2)/2 \rceil} \right\rfloor$ sets.

This bound is sharp for the graphs $K_t \Box K_t$, for any positive integer t. Since $\delta(K_t \Box K_t) = 2(t-1)$, we have

$$\psi_{\hat{a}}(K_t \Box K_t) \leq \left\lfloor \frac{n}{\left\lceil \frac{2(t-1)+2}{2} \right\rceil} \right\rfloor = \left\lfloor \frac{n}{t} \right\rfloor = \left\lfloor \frac{t^2}{t} \right\rfloor = t.$$

Since each copy of K_t forms a strong defensive alliance in $K_t \Box K_t$, we can partition the graph into t alliances, and so, $\psi_{\hat{a}}(K_t \Box K_t) = t$. $\Box$

As a corollary, we have a straightforward upper bound in terms of order for graphs with no isolated vertices.

Corollary 2.5 *If G is a graph with no isolates, then*

$$\psi_{\hat{a}}(G) \leq \left\lfloor \frac{n}{2} \right\rfloor.$$

Notice that the bound of Corollary 2.5 is sharp for tK_2, that is, the disjoint union of any t copies of K_2.

2.2 Examples

We now determine the alliance partition number and strong alliance partition number of several classes of graphs.

Proposition 2.6 *For any $n \geq 1$,*

$$\psi_a(K_n) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Let Π be a ψ_a-partition of V and let $V_i \in \Pi$. Let $v \in V_i$. Since V_i is a defensive alliance, at least $\lfloor deg(v)/2 \rfloor = \lfloor (n-1)/2 \rfloor$ neighbors of v belong to V_i. Thus, $|V_i| \geq 1 + \lfloor (n-1)/2 \rfloor$. If n is odd, we have

$$|V_i| \geq 1 + \frac{n-1}{2} = \frac{n+1}{2},$$

implying $\psi_a(K_n) = |\Pi| = 1$.

Let n be even. Then

$$|V_i| \geq 1 + \left\lfloor \frac{n-1}{2} \right\rfloor = \frac{n}{2},$$

and hence $|\Pi| \leq 2$. Moreover, any set of $n/2$ vertices is a defensive alliance in K_n and hence $\psi_a(K_n) = 2$. $\square$

Proposition 2.7 *For any $n \geq 1$, $\psi_{\hat{a}}(K_n) = 1$.*

Proof. Let V_i be an alliance of a $\psi_{\hat{a}}$-partition of V and let $v \in V_i$. Since V_i is strong, v is allied with at least $\lceil deg(v)/2 \rceil = \lceil (n-1)/2 \rceil$ neighbors. Thus,

$$|V_i| \geq \left\lceil \frac{n-1}{2} \right\rceil + 1 = \left\lceil \frac{n+1}{2} \right\rceil > \frac{n}{2},$$

implying $|\Pi| = \psi_{\hat{a}}(K_n) = 1$. $\square$

Proposition 2.8 *For any $n \geq 1$, $\psi_a(P_n) = \left\lceil \frac{n+1}{2} \right\rceil$.*

Proof. Let $P_n = v_1, v_2, ..., v_n$. Notice that each endvertex is an alliance of cardinality one, whereas the vertices $v_2, ..., v_{n-1}$ must each belong to an alliance of cardinality at least two. Hence, $\psi_a(P_n) \leq \left\lfloor \frac{n-2}{2} \right\rfloor + 2 = \left\lceil \frac{n+1}{2} \right\rceil$. We consider two cases depending on the parity of n.

If n is even, let $\Pi = \{V_1, V_2, ..., V_t\}$, where $V_1 = \{v_1\}$, $V_t = \{v_n\}$, and $V_i = \{v_{2i}, v_{2i+1}\}$ for $1 \leq i \leq \frac{n-2}{2}$. Then each V_i is a defensive alliance and so

$$\psi_a(P_n) \geq |\Pi| = t = 2 + \left(\frac{n-2}{2}\right) = \frac{n+2}{2} = \left\lceil \frac{n+1}{2} \right\rceil \text{(since } n \text{ is even).}$$

If n is odd, let $\Pi = \{V_1, V_2, ..., V_t\}$, where $V_1 = \{v_1\}$ and $V_i = \{v_{2i}, v_{2i+1}\}$, for $1 \leq i \leq \frac{n-1}{2}$. Then each V_i is a defensive alliance and so

$$\psi_a(P_n) \geq |\Pi| = t = 1 + \frac{n-1}{2} = \frac{n+1}{2}.$$

Hence, $\psi_a(P_n) = \left\lceil \frac{n+1}{2} \right\rceil$. $\square$

Proposition 2.9 *For a nontrivial path, $\psi_{\hat{a}}(P_n) = \left\lfloor \frac{n}{2} \right\rfloor$*

Proof. Let $P_n = v_1, v_2, ..., v_n$. Notice that any strong alliance in P_n has at least two vertices, implying that $\psi_{\hat{a}}(P_n) \leq \left\lfloor \frac{n}{2} \right\rfloor$. A partition of V into $\left\lfloor \frac{n}{2} \right\rfloor$ strong alliances can be obtained as follows. If n is even, let $V_i = \{v_{2i-1}, v_{2i} \mid 1 \leq i \leq \frac{n}{2}\}$. If $n \geq 3$ is odd, let $V_1 = \{v_1, v_2, v_3\}$ and let $V_i = \{v_{2i}, v_{2i+1} \mid 2 \leq i \leq \frac{n-1}{2}\}$. Then, in each case, each V_i is a strong alliance, and we have a partition of V into $\left\lfloor \frac{n}{2} \right\rfloor$ strong alliances. $\square$

Proposition 2.10 *For any $n \geq 3$, $\psi_a(C_n) = \psi_{\hat{a}}(C_n) = \left\lfloor \frac{n}{2} \right\rfloor$.*

Proof. Let $C_n = v_1, v_2, ..., v_n$. Since $\delta(C_n) = 2$, any alliance in C_n has at least two vertices. Thus, $\psi_{\hat{a}}(C_n) \leq \psi_a(C_n) \leq \lfloor \frac{n}{2} \rfloor$. Next we show $\psi_a(C_n) \geq \psi_{\hat{a}}(C_n) \geq \lfloor \frac{n}{2} \rfloor$.

If n is even, let $V_i = \{v_{2i-1}, v_{2i} \mid 1 \leq i \leq \frac{n}{2}\}$. If n is odd, let $V_1 = \{v_1, v_2, v_3\}$ and let $V_i = \{v_{2i}, v_{2i+1} \mid 2 \leq i \leq \frac{n-1}{2}\}$. In each case, each V_i is an (strong) alliance, and so we have a partition of V into $\lfloor \frac{n}{2} \rfloor$ (strong) alliances. $\square$

2.3 Grid Graphs

Here we determine the alliance partition number of grid graphs $G_{r,c} = P_r \square P_c$, a problem posed in [12]. We denote the order of $G_{r,c}$ as $n = rc$, and let $V = \{v_{1,1},\ v_{1,2},\ ...,\ v_{1,c},\ v_{2,1},\ v_{2,2},\ ...,\ v_{2,c},\ ...,\ v_{r,1},\ v_{r,2},\ ...,\ v_{r,c}\}$. Note that if we think of the vertices of the grid as entries in an $r \times c$ matrix, $v_{i,j}$ is the vertex in the i^{th} row and j^{th} column. The set of vertices in row i (respectively, column j) is written as R_i (respectively, C_j). We denote the *border* vertices of the grid as $B = \Big\{ v_{i,j} \mid i \in \{1, r\} \text{ or } j \in \{1, c\} \Big\}$ and call the vertices in $I = V - B$ *interior* vertices. In what follows we will assume $2 \leq r \leq c$.

Theorem 2.11 *For any $c > 1$, $\psi_a(G_{2,c}) = c$.*

Proof. Since the minimum degree of $G_{2,c}$ is two, any alliance in $G_{2,c}$ has at least two vertices. Thus $\psi_a(G_{2,c}) \leq \frac{n}{2} = c$. Since the vertices in a column form an alliance, $\Pi = \{C_i \mid 1 \leq i \leq c\}$ is an alliance partition of $G_{2,c}$ and $\psi_a(G_{2,c}) \geq c$. $\square$

Theorem 2.12 *For any $c \geq 3$,*

$$\psi_a(G_{3,c}) = \begin{cases} c & \text{if } c \text{ is odd} \\ c+1 & \text{if } c \text{ is even.} \end{cases}$$

Proof. Let Π be a ψ_a-partition of $G_{3,c}$, and let $V_i \in \Pi$. Clearly $|V_i| \geq 2$. We first show that $|V_i \cap B| \geq 2$. Suppose to the contrary that $|V_i \cap B| \leq 1$. Then there exists a vertex $v_{2,j} \in V_i$, where $2 \leq j < c$, with exactly one neighbor in V_i. Thus, $|N[v_{2,j}] \cap V_i| = 2 < 3 = |N[v_{2,j}] \cap (V - V_i)|$, contradicting the fact that V_i is an alliance. Hence $|V_i \cap B| \geq 2$ implying that

$$\psi_a(G_{3,c}) \leq \left\lfloor \frac{2c+2}{2} \right\rfloor = c+1.$$

If c is even, then the tiling in Figure 2 shows that $\psi_a(G_{3,c}) \geq c+1$ and hence $\psi_a(G_{3,c}) = c+1$.

Let c be odd. The partition $\Pi = \{C_i \mid 1 \leq i \leq c\}$ is an alliance partition of $G_{3,c}$ implying that $\psi_a(G_{3,c}) \geq c$. We next show that $\psi_a(G_{3,c}) \leq c$. Suppose to the contrary that $\psi_a(G_{3,c}) \geq c+1$, and let Π be a $\psi_a(G_{3,c})$-partition. Then $\psi_a(G_{3,c}) = c+1$. Since $|V_i \cap B| \geq 2$ for each $V_i \in \Pi$ and $|B| = 2c+2$, we have that $|V_i \cap B| = 2$ for each $V_i \in \Pi$. Since $\langle V_i \rangle$ is connected, the pair of border vertices in V_i must either be adjacent or connected by a path whose internal vertices are interior vertices of $G_{3,c}$.

Let V_i be an alliance in Π that contains an interior vertex of $G_{3,c}$. We claim V_i contains an even number of interior vertices. Suppose to the contrary V_i contains an odd number of interior vertices. Now V_i has exactly two border vertices, say u and v. Without loss of generality, we consider four cases, depending on u and v.

Case 1 $u, v \in R_2$. This implies that V_i is R_2. But then since c is odd, R_1 (respectively, R_3) cannot be partitioned into alliances of cardinality two.

Case 2 $u \in R_2$, $v \in R_1$. Assume $u = v_{2,1}$. Since the number of interior vertices is odd, we have that $v = v_{1,2k}$ for some $1 \leq k \leq \frac{c-1}{2}$. But then the set of border vertices $\{v_{1,1}, v_{1,2}, ..., v_{1,2k-1}\}$ has odd cardinality and cannot be partitioned into alliances of cardinality two.

Case 3 $u, v \in R_1$. Since the number of interior vertices is odd, u and v belong to columns whose indices have the same parity, say $u = v_{1,k}$ and $v = v_{1,j}$. Then there is again an odd number of border vertices in $\{v_{1,k+1}, v_{1,k+2}, ..., v_{1,j-1}\}$ that cannot be partitioned into alliances of cardinality two.

Case 4 $u \in R_1$, $v \in R_3$. As in Case 3, $u = v_{1,k}$ and $v = v_{3,j}$ and j and k have the same parity. Again, we have an odd number of border vertices in $\{v_{1,k-1}, v_{1,k-2}, ..., v_{1,1}, v_{2,1}, v_{3,1}, v_{3,2}, ..., v_{3,j-1}\}$ implying that the partition is impossible.

There is a contradiction in each of the four cases. Hence if V_i contains an interior vertex, then it contains an even number of interior vertices. But this is impossible since there are $c-2$ interior vertices and $c-2$ is odd. Thus $\psi_a(G_{3,c}) \leq c$ for odd c and our result follows. $\square$

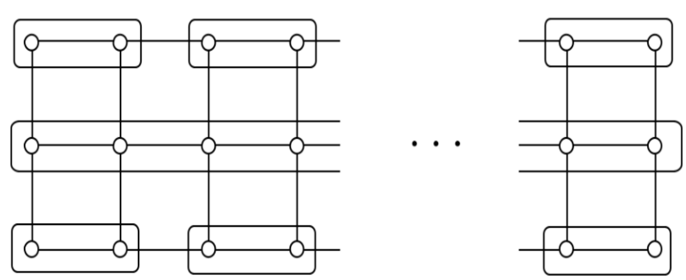

Figure 2: $G_{3,c}$, c is Even.

We now assume $4 \le r \le c$. The following lemma will be helpful in establishing our main result.

Lemma 2.13 *Let* Π *be an alliance partition of* $G_{r,c}$*, where* $4 \le r \le c$*, and let* V_i *be an alliance in* Π*. Let* $u \in B$ *and let* $v \in I$*. Then*

(i) $|V_i| \ge 2$,

(ii) *if* $v \in V_i$*, then* $|V_i| \ge 4$,

(iii) *if* $u, v \in V_i$*, then*

(a) $|V_i \cap B| \ge 2$*, or*

(b) $V_i - \{u\}$ *contains an alliance* V_i'*, where* $V_i' \cap B = \emptyset$.

Proof. *(i)* Since the minimum degree of $G_{r,c}$ is two, any alliance in $G_{r,c}$ has at least two vertices.

(ii) Let $v \in V_i \cap I$. Then, since $deg(v) = 4$, $|N[v] \cap V_i| \ge 3$ and so $|V_i| \ge 3$. If $|V_i| = 3$, then the subgraph induced by V_i is a path $P_3 = x, v, w$, where x and w are both adjacent to v and are border vertices (otherwise there is an interior vertex in V_i with fewer than two neighbors in V_i). Since $4 \le r \le c$, x and y must be adjacent to a corner vertex of the grid. Without loss of generality, assume that x and y are adjacent to $v_{1,1}$, and so $V_i = \{v_{1,2}, v_{2,2}, v_{2,1}\}$. But then $v_{1,1}$ is an isolate in the partition contradicting (i). Hence $|V_i| \ge 4$.

(iii) Assume that V_i contains border vertex u and interior vertex v. Suppose that the induced subgraph $\langle V_i \rangle$ is acyclic, and root it at u. Consider an endvertex x of a longest path from u in $\langle V_i \rangle$. Necessarily x is a leaf in $\langle V_i \rangle$ implying that x is a border vertex in $G_{r,c}$. Hence $|V_i \cap B| \ge 2$.

Now suppose $\langle V_i \rangle$ contains a cycle C. If u is on the cycle, then at least one of the neighbors of u on C must be a border vertex establishing (a). Thus, assume that C contains only interior vertices. Since each vertex on C has two neighbors on C, the vertices of C are a defensive alliance of $G_{r,c}$, and $V(C) \cap B = \emptyset$. $\square$

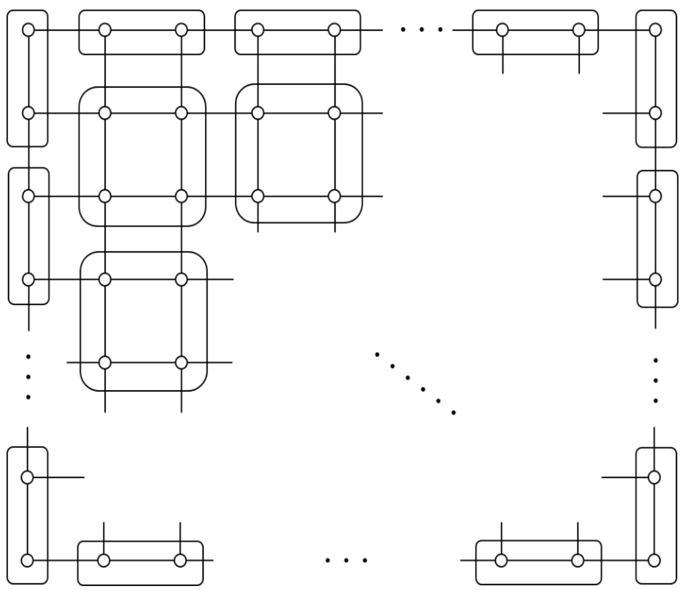

Figure 3: $G_{r,c}$, r Even, c Even.

Theorem 2.14 *For* $4 \leq r \leq c$,

$$\psi_a(G_{r,c}) = \left\lfloor \frac{r-2}{2} \right\rfloor \left\lfloor \frac{c-2}{2} \right\rfloor + r + c - 2.$$

Proof. We give four partitions of V, depending on the parities of r and c. These partitions are illustrated in Figures 3, 4, and 5. It is straightforward to verify that

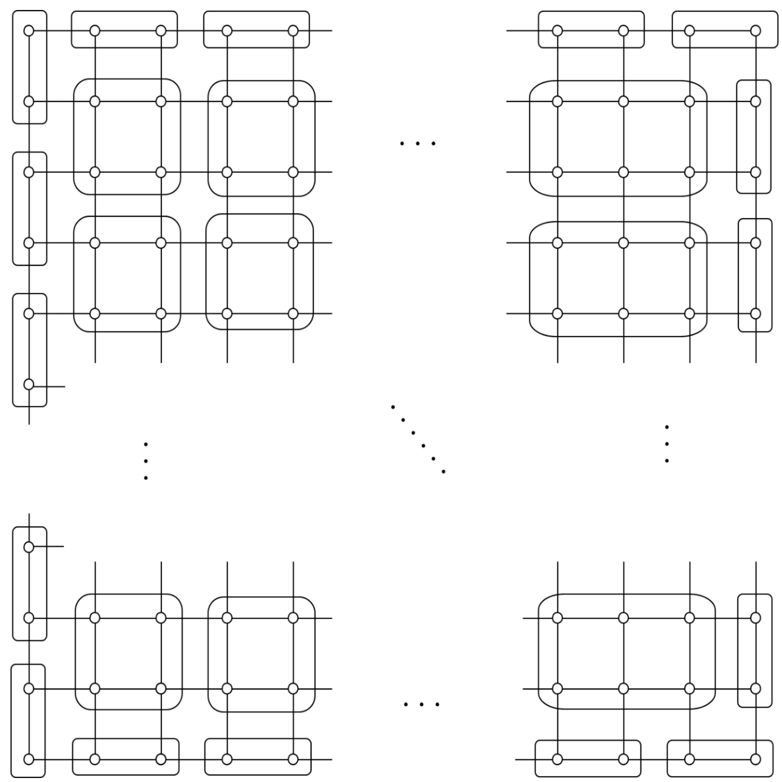

Figure 4: $G_{r,c}$, r Even, c Odd (Rotate 90° for r Odd, c Even).

each is an alliance partition of V with cardinality $\lfloor \frac{r-2}{2} \rfloor \lfloor \frac{c-2}{2} \rfloor + r + c - 2$. Hence $\psi_a(G_{r,c}) \geq \lfloor \frac{r-2}{2} \rfloor \lfloor \frac{c-2}{2} \rfloor + r + c - 2$.

Let Π be a ψ_a-partition of $V(G_{r,c})$. Suppose Π has x alliances consisting of only border vertices, y alliances consisting of only interior vertices, and z alliances consisting of both interior and border vertices. We proceed with the aid of the following two claims.

Claim 1. $x \leq r + c - 2$ *and* $y \leq \lfloor \frac{r-2}{2} \rfloor \lfloor \frac{c-2}{2} \rfloor$.

Proof. *(Claim 1)* By Lemma 2.13(i), $x \leq \frac{2(r+c)-4}{2} = r + c - 2$.

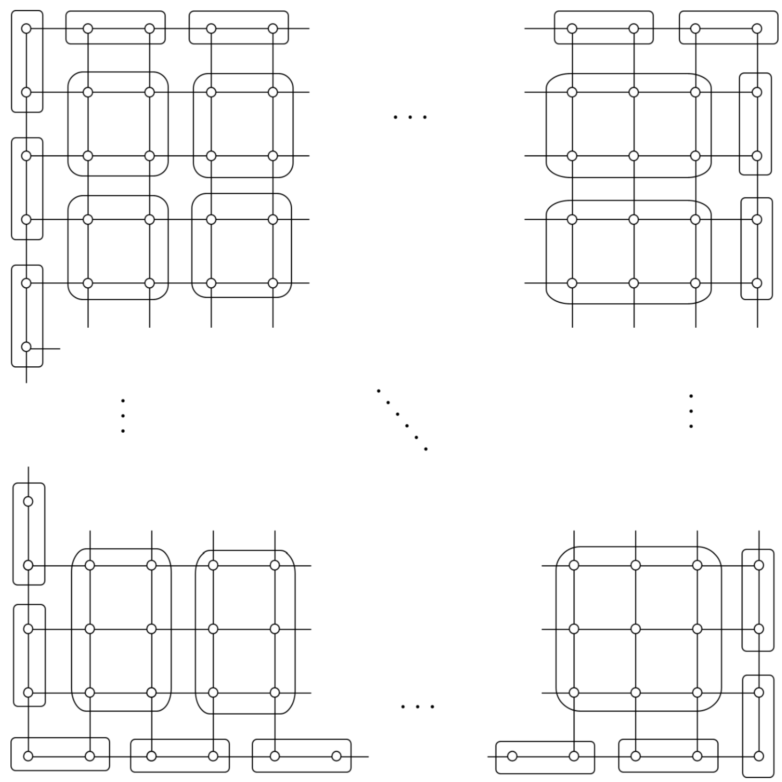

Figure 5: $G_{r,c}$, r Odd, c Odd.

Let $V_i \in \Pi$. The proof of Lemma 2.13(ii) implies that any vertex of degree one in the induced subgraph $\langle V_i \rangle$ is a border vertex. Thus if V_i contains no border vertices, every vertex of V_i must be on a cycle in $\langle V_i \rangle$. Furthermore, since $G_{r,c}$ has no triangles, each cycle has at least four vertices. Moreover a cycle of $\langle V_i \rangle$ includes at least two vertices of some row and at least two vertices of some column. Hence we have that $y \leq \left\lfloor \frac{r-2}{2} \right\rfloor \left\lfloor \frac{c-2}{2} \right\rfloor$, completing the proof of Claim 1.

Claim 2. *There exists a* ψ_a*-partition* Π *of* V *where no alliance in* Π *contains exactly one border vertex.*

Proof. *(Claim 2)* Among all ψ_a-partitions of V, let Π be one such that the number of alliances in the partition containing exactly one border vertex is minimized. If no alliance in Π contains exactly one border vertex, then our claim holds. Thus assume to the contrary that $V_i \in \Pi$ and $V_i \cap B = \{v\}$. It suffices to show that there exists a ψ_a-partition of V having fewer alliances containing exactly one border vertex to reach a contradiction.

By Lemma 2.13(i) and our hypothesis, V_i contains an interior vertex. Hence by Lemma 2.13(iii), $V_i - \{v\}$ contains an alliance consisting of only interior vertices. Among all such alliances in $V_i - \{v\}$, let V_i' be one of maximum cardinality.

For a vertex x and an alliance S, define

$$d(x, S) = \min\{d(x, w) \mid w \in S\}.$$

We proceed by induction on $d(v, V_i')$. Suppose $d(v, V_i') = 1$. Without loss of generality, assume $v \in C_1$. Note that $V_i = V_i' \cup \{v\}$. Also note that v has a neighbor in C_1 that is in an alliance, say A, where $A \neq V_i \in \Pi$. Let $\Pi' = (\Pi - \{A, V_i\}) \cup \{A \cup \{v\}, V_i'\}$. Then Π' is an ψ_a-partition of $G_{r,c}$ having fewer alliances with exactly one border vertex.

We now assume that for $d(v, V_i') \leq k - 1$ (where $k \geq 2$), we can form a new partition Π' such that $|\Pi'| = |\Pi|$ and Π' has fewer alliances than Π containing exactly one border vertex.

Suppose that $d(v, V_i') = k$. Note that $\langle V_i - V_i' \rangle$ is a path (this follows from our choice of V_i'), say $v = v_1, v_2, ..., v_k$. Since v_{k-1} and v_k are adjacent, we may assume without loss of generality that $v_{k-1}, v_k \in C_j$, say $v_k = v_{i,j}$ and $v_{k-1} = v_{i-1,j}$. We consider two cases.

Case 1 $v_{i+1,j} \in V_i'$. Then since $|N[v_{i+1,j}] \cap V_i'| \geq 3$, at least one of $v_{i+1,j-1}$ and $v_{i+1,j+1}$ is in V_i'. Without loss of generality we assume the latter case. By our choice of V_i', we know $v_{i,j+1} \notin V_i'$ (for otherwise v_k and $v_{i,j+1}$ could be included in V_i'). Now $v_{i,j+1}$ belongs to some alliance, say $A \neq V_i$. Since $|N[v_{i,j+1}] \cap A| \geq 3$, $v_{i+1,j+1} \notin A$, and $v_{i,j} \notin A$, we must have $\{v_{i-1,j+1}, v_{i,j+2}\} \subseteq A$. We form a new partition $\Pi' = (\Pi - \{A, V_i\}) \cup \{V_i', A'\}$, where $A' = A \cup (V_i - V_i')$. We note that Π' is an alliance partition of V and $|\Pi'| = |\Pi|$. If A contains a border vertex, then we have established the claim. If A does not contain a border vertex, we note that $d(v, A) \leq k - 1$ and apply our inductive hypothesis to obtain a new alliance partition Π'', where $|\Pi''| = |\Pi'|$ and Π'' has fewer alliances than Π' containing exactly one border vertex. Since Π' has the same number of alliances containing exactly one border vertex as Π, the claim is established.

Case 2 $v_{i+1,j} \notin V_i'$. We assume without loss of generality that $v_{i,j+1} \in V_i'$. We know by our choice of V_i', that $v_{i-1,j+1} \notin V_i'$. Therefore $v_{i+1,j+1} \in V_i'$. Now $v_{i+1,j}$ belongs to some alliance of Π, say $A \neq V_i$ (by our choice of V_i'). Note that $|N[v_{i+1,j}] \cap A| \geq 3$, implying that $\{v_{i+1,j-1}, v_{i+2,j}\} \subseteq A$. Let $A' = A \cup (V_i - V_i')$ and $\Pi' = (\Pi - \{A, V_i\}) \cup \{V_i', A'\}$. If A contains a border vertex, then Π' is a ψ_a-partition of $G_{r,c}$ having the desired property. Thus, assume that A has no border vertex. Note that Π' has the same number of alliances containing exactly one border vertex as Π. Let A'' be a maximum subgraph of A' containing only interior vertices. If $d(v, A'') \leq k - 1$, then we apply our inductive hypothesis obtaining a new partition Π'', where $|\Pi''| = |\Pi'|$ and Π'' has fewer alliances than Π' containing exactly one border vertex, establishing our claim. Assume that $d(v, A'') = k$. Then $A'' = A$ and the new partition Π' is equivalent to the partition Π in Case 1, with A' in the role of V_i and A'' in the role of V_i'. This establishes Claim 2.

Clearly $|\Pi| = x + y + z$. It follows from the two claims that

$$x \le \frac{2r + 2c - 4 - 2z}{2} = r + c - 2 - z$$

and

$$y \le \left\lfloor \frac{r-2}{2} \right\rfloor \left\lfloor \frac{c-2}{2} \right\rfloor .$$

So,

$$\begin{aligned}\psi_a(G_{r,c}) = |\Pi| &\le r + c - 2 - z + \left\lfloor \frac{r-2}{2} \right\rfloor \left\lfloor \frac{c-2}{2} \right\rfloor + z \\ &= \left\lfloor \frac{r-2}{2} \right\rfloor \left\lfloor \frac{c-2}{2} \right\rfloor + r + c - 2.\end{aligned}$$

Hence

$$\psi_a(G_{r,c}) = \left\lfloor \frac{r-2}{2} \right\rfloor \left\lfloor \frac{c-2}{2} \right\rfloor + r + c - 2. \quad \square$$

Let $G_{\infty,\infty}$ be the 4-regular infinite grid graph with vertex set $V = \mathbf{Z} \times \mathbf{Z}$ and $N((i,j)) = \{(i-1,j), (i+1,j), (i,j-1), (i,j+1)\}$. Let G_k be the $2k \times 2k$ induced subgraph of $G_{\infty,\infty}$, with vertex set $V(G_k) = \{(i,j) \mid -2k < i \le 2k, -2k < j \le 2k\}$. We define

$$\psi_a\%(G_{\infty,\infty}) = \max\Big\{ \lim_{k\to\infty} \frac{|\{V_i \mid V_i \in \Pi \text{ and } V_i \subseteq V(G_k)\}|}{(2k)^2} \mid \\ \Pi \text{ is a partition of } V(G_{\infty,\infty}) \text{ into defensive alliances} \Big\}.$$

Theorem 2.15 *For the infinite grid graph $G_{\infty,\infty}$,*

$$\psi_a\%(G_{\infty,\infty}) = \frac{1}{4}.$$

Proof. Each alliance V_i in a partition Π of $V(G_{\infty,\infty})$ has cardinality at least four. Therefore $\psi_a\%(G_{\infty,\infty}) \leq \frac{1}{4}$. Figure 6 shows that $\psi_a\%(G_{\infty,\infty}) \geq \frac{1}{4}$. $\square$

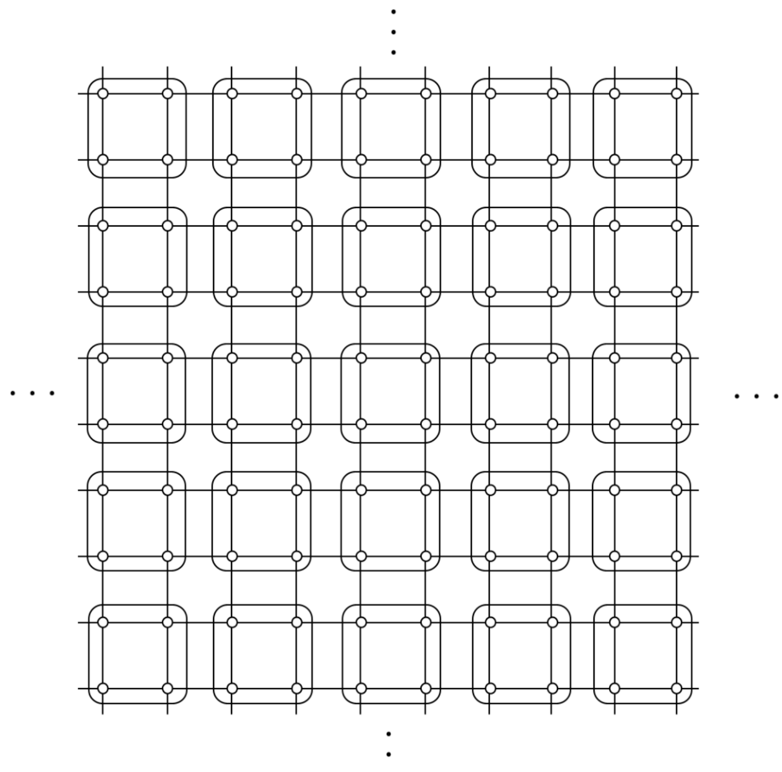

Figure 6: Alliance Partition of $G_{\infty,\infty}$.

3 GLOBAL ALLIANCE PARTITIONS

In this chapter we consider the global alliance partition number and global strong alliance partition number of graphs. We give bounds for general graphs and exact results for several classes of graphs. We conclude with a discussion of the global alliance partition number of certain classes of trees.

3.1 Bounds

In this section we establish bounds on the global alliance partition number and global strong alliance partition number of general graphs. For any graph G, $V(G)$ is trivially a global (strong) alliance, and hence every graph has a global (strong) alliance partition number (of at least one). Notice that any global strong alliance is a global alliance and that a global alliance is a dominating set. This leads to our first observation.

Observation 3.1 *For any graph G,*

$$1 \leq \psi_{\hat{g}}(G) \leq \psi_g(G) \leq d(G) \leq \delta(G) + 1.$$

Note that for any graph with isolated vertices we have equality throughout this inequality chain.

Theorem 3.2 *If G is a graph of order n, then*

$$\psi_g(G) \leq \frac{\sqrt{4n+1}+1}{2},$$

and this bound is sharp.

Proof. Let $\Pi = \{V_1, V_2, ..., V_k\}$ be a partition of V into global defensive alliances with $|\Pi| = k = \psi_g(G)$. It is given in [8] that $\gamma_a(G) \geq (\sqrt{4n+1}-1)/2$. Thus, for each i $(1 \leq i \leq k)$, $|V_i| \geq (\sqrt{4n+1}-1)/2$, which implies

$$\psi_g(G) = k \leq \frac{n}{(\sqrt{4n+1}-1)/2} = \frac{\sqrt{4n+1}+1}{2}.$$

That the bound is sharp may be seen by considering the cartesian product $K_t \Box K_{t+1}$, for $t \geq 1$. Notice that $n = t(t+1)$, so $\psi_g(K_t \Box K_{t+1}) \leq (\sqrt{4t(t+1)+1}+1)/2 = t+1$. For each i $(1 \leq i \leq t+1)$, the set $\{(u_j, v_i) \mid 1 \leq j \leq t\}$ is a global defensive alliance in $K_t \Box K_{t+1}$, and so, $\psi_g(K_t \Box K_{t+1}) \geq t+1$. $\Box$

Theorem 3.3 *If G is a graph of order n, $\psi_{\hat{g}}(G) \leq \sqrt{n}$, and this bound is sharp.*

Proof. Let $\Pi = \{V_1, ..., V_k\}$ be a $\psi_{\hat{g}}$-partition of V. It is given in [8] that $\gamma_{\hat{a}}(G) \geq \sqrt{n}$. Thus each alliance $V_i \in \Pi$ has cardinality at least $\sqrt{n}$. It follows that $\psi_{\hat{g}}(G) = |\Pi| \leq n/\sqrt{n} = \sqrt{n}$.

To see that this bound is sharp, consider the family of graphs $K_t \Box K_t$, for $t \geq 1$. Since $K_t \Box K_t$ has order $n = t^2$, $t = \sqrt{n}$. For each i $(1 \leq i \leq t)$, the set $\{(u_i, v_j) \mid 1 \leq j \leq t\}$ is a global strong defensive alliance in $K_t \Box K_t$. Thus $\psi_{\hat{g}}(K_t \Box K_t) \geq t$ and equality follows. $\Box$

Next, we give sharp upper bounds, in terms of minimum degree, which improve that of Observation 3.1.

Theorem 3.4 *For any graph G,*

$$\psi_g(G) \leq \left\lceil \frac{\delta(G)}{2} \right\rceil + 1,$$

and this bound is sharp.

Proof. Let v be a vertex of degree $\delta(G)$. Let $\Pi = \{V_1, V_2, ..., V_k\}$ be a $\psi_g(G)$-partition and assume without loss of generality that $v \in V_1$. Since V_1 is a defensive alliance, it follows that at least $\lfloor \delta(G)/2 \rfloor$ neighbors of v are in V_1. Furthermore, since v is dominated by each set in Π, it follows that

$$\psi_g(G) = |\Pi| \leq 1 + \delta(G) - \left\lfloor \frac{\delta(G)}{2} \right\rfloor = \left\lceil \frac{\delta(G)}{2} \right\rceil + 1.$$

Notice that the graphs $K_t \Box K_{t+1}$ described in the proof of Theorem 3.2 attain this bound with $\delta(K_t \Box K_{t+1}) = 2t - 1$ and $\psi_g(K_t \Box K_{t+1}) = t + 1$. $\Box$

Corollary 3.5 *If G is r-regular,*

$$\psi_g(G) \leq \left\lceil \frac{r}{2} \right\rceil + 1.$$

Theorem 3.6 *For any graph G,*

$$\psi_{\hat{g}}(G) \leq \left\lfloor \frac{\delta(G)}{2} \right\rfloor + 1,$$

and this bound is sharp.

Proof. Let v be a vertex of degree $\delta(G)$ and let $\Pi = \{V_1, ..., V_k\}$ be a $\psi_{\hat{g}}$-partition of V. Assume $v \in V_1$. Then, since V_1 is a strong alliance, at least $\lceil \delta(G)/2 \rceil$ neighbors of v are in V_1. Since v is dominated by each alliance of Π, we have

$$\psi_{\hat{g}}(G) = |\Pi| \leq 1 + \delta(G) - \left\lceil \frac{\delta(G)}{2} \right\rceil = \left\lfloor \frac{\delta(G)}{2} \right\rfloor + 1.$$

Note that the graphs $K_t \Box K_t$ described in the proof of Theorem 3.3 attain this bound with $\delta(K_t \Box K_t) = 2(t-1)$ and $\psi_{\hat{g}}(K_t \Box K_t) = t$. $\Box$

Corollary 3.7 *If G is r-regular,*

$$\psi_{\hat{g}}(G) \leq \left\lfloor \frac{r}{2} \right\rfloor + 1.$$

We note that the bounds given in Theorems 3.2 and 3.4 and Corollary 3.5 have also been obtained independently by Eroh and Gera [3, 5].

3.2 Examples

In this section we consider some specific families of graphs, namely, complete graphs, paths, cycles, and grid graphs. We begin by giving the global alliance partition number and global strong alliance partition number of complete graphs. Since $\psi_g(G) \leq \psi_a(G)$ and $\psi_{\hat{g}}(G) \leq \psi_{\hat{a}}(G)$, and any alliance in K_n is necessarily global, our next two results follow easily from Proposition 2.6 and Proposition 2.7.

Proposition 3.8 *For any $n \geq 1$,*

$$\psi_g(K_n) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

Proposition 3.9 *For any $n \geq 1$, $\psi_{\hat{g}}(K_n) = 1$.*

Next, we consider paths and cycles.

Proposition 3.10 *For any $n \geq 1$,*

$$\psi_g(P_n) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Let $P_n = v_1, v_2, ..., v_n$. Since $\delta(P_n) = 1$, $\psi_g(P_n) \leq 2$, by Theorem 3.4. We consider two cases depending on the parity of n.

Case 1 n is even.

If $n \equiv 2 \bmod 4$, let $\Pi = \{V_1, V_2\}$, where

$$\begin{aligned} V_1 &= \{v_1\} \cup \left(\bigcup_{i=1}^{(n-2)/4} \{v_{4i}, v_{4i+1}\} \right) \\ V_2 &= V - V_1 \end{aligned}$$

If $n \equiv 0 \bmod 4$, let $\Pi = \{V_1, V_2\}$, where

$$\begin{aligned} V_1 &= \{v_1, v_n\} \cup \left(\bigcup_{i=1}^{(n-4)/4} \{v_{4i}, v_{4i+1}\} \right) \\ V_2 &= V - V_1 \end{aligned}$$

Notice that in both cases Π is a partition of V into global defensive alliances with $|\Pi| = 2$. Hence, $\psi_g(P_n) \geq 2$ and equality follows.

Case 2 n is odd.

Since $V(P_n)$ is trivially a global defensive alliance, $\psi_g(P_n) \geq 1$. We claim $\psi_g(P_n) = 1$. Suppose to the contrary that $\Pi = \{V_1, V_2\}$ is a partition of V into global defensive alliances.

Suppose without loss of generality that $v_1 \in V_1$. Then, since v_1 must have a neighbor in V_2 (because V_2 is a dominating set), we have $v_2 \in V_2$. A similar argument for v_n shows that v_1 and v_n are isolated vertices in $\langle V_1 \rangle \cup \langle V_2 \rangle$.

For each $v_i \notin \{v_1, v_n\}$, $|N[v]| = 3$, and hence v must be allied with at least one neighbor. Also, $v_i \in V_1$ (respectively, $v_i \in V_2$) must have a neighbor in V_2

(respectively, V_1). It follows that each component of $\langle V_1\rangle \cup \langle V_2\rangle - \{v_1, v_n\}$ is a K_2. But this is impossible because n is odd. Thus $\psi_g(P_n) = 1$. $\square$

A consequence of Theorem 3.6 is that any path (indeed, any tree), has global strong alliance partition number one.

Proposition 3.11 *For any $n \geq 1$, $\psi_{\hat{g}}(P_n) = 1$.*

Next, we show that cycles have equal global alliance partition number and global strong alliance partition number.

Proposition 3.12 *For any $n \geq 3$,*

$$\psi_g(C_n) = \psi_{\hat{g}}(C_n) = \begin{cases} 2 & \text{if } n \equiv 0 \bmod 4, \\ 1 & \text{otherwise.} \end{cases}$$

Proof. Let $C_n = v_1, v_2, ..., v_n$. By Theorem 3.4, $\psi_{\hat{g}}(C_n) \leq \psi_g(C_n) \leq 2$. If $n \equiv 0$ mod 4, then the partition defined by

$$\begin{aligned} V_1 &= \bigcup_{i=0}^{(n-4)/4} \{v_{4i+1}, v_{4i+2}\} \\ V_2 &= V - V_1 \end{aligned}$$

is a $\psi_{\hat{g}}$-partition of V and so, $\psi_g(C_n) = \psi_{\hat{g}}(C_n) = 2$.

We now assume $4 \nmid n$ and suppose by way of contradiction that $\Pi = \{V_1, V_2\}$ is a partition of V into global defensive alliances ($V_1 \neq V_2$). For each $v_i \in V$, $|N[v_i]| = 3$, and hence, v_i must be allied with at least one neighbor and dominated by a neighbor in another alliance. Hence each component of $\langle V_1\rangle \cup \langle V_2\rangle$ is a K_2. Thus, if n is odd, we have obtained a contradiction and $\psi_g(C_n) = \psi_{\hat{g}}(C_n) = 1$.

Thus, we now assume $n \equiv 2 \bmod 4$. Without loss of generality, suppose $\{v_1, v_2\} \subseteq V_1$. Then, since V_2 is a global alliance, $\{v_3, v_4\} \subseteq V_2$. Similarly, we must have $\{v_5, v_6\} \subseteq V_1$. If $n = 6$, we have reached a contradiction, since no vertex of V_2 dominates v_1. If $n > 6$, continuing in a similar manner, since $4 \nmid n$, we conclude that $v_n \in V_1$, again contradicting the fact that V_2 is a global alliance. Thus $\psi_g(C_n) = \psi_{\hat{g}}(C_n) = 1$. $\square$

Finally, we consider grid graphs $G_{r,c}$. We follow the notation of Chapter 2 and assume in what follows that $2 \leq r \leq c$.

Proposition 3.13 *For the grid graph $G_{r,c}$, $2 \leq r \leq c$, $\psi_g(G_{r,c}) = 2$.*

Proof. By Theorem 3.4, $\psi_g(G_{r,c}) \leq 2$. Consider the partition $\Pi = \{V_1, V_2\}$ with

$$\begin{aligned} V_1 &= \bigcup_{i=1}^{\lfloor r/2 \rfloor} R_{2i} \\ V_2 &= \bigcup_{j=0}^{\lfloor (r-1)/2 \rfloor} R_{2j+1} \end{aligned}$$

Then Π is a partition of V into global defensive alliances, and so $\psi_g(G_{r,c}) \geq |\Pi| = 2$. It follows that $\psi_g(G_{r,c}) = 2$. $\square$

Proposition 3.14 *If $n = rc$ is even, then $\psi_{\hat{g}}(G_{r,c}) = 2$.*

Proof. By Theorem 3.3 with $\delta(G) = 2$, $\psi_{\hat{g}}(G_{r,c}) \leq 2$. Since n is even, at least one of r and c is even, say r.

Case 1 $r \equiv 0 \bmod 4$.

Let $\Pi = \{V_1, V_2\}$, where

$$\begin{aligned} V_1 &= R_1 \cup R_r \cup \left(\bigcup_{i=1}^{(r-4)/4} (R_{4i} \cup R_{4i+1}) \right) \\ V_2 &= V - V_1 \end{aligned}$$

Then Π is a partition of V into two global strong alliances and hence $\psi_{\hat{g}}(G_{r,c}) = 2$.

Case 2 $r \equiv 2 \bmod 4$.

Let $\Pi = \{V_1, V_2\}$, where

$$\begin{aligned} V_1 &= R_1 \cup \left(\bigcup_{i=1}^{(r-2)/4} (R_{4i} \cup R_{4i+1}) \right) \\ V_2 &= V - V_1 \end{aligned}$$

Again, Π is a partition of V into two global strong alliances and hence $\psi_{\hat{g}}(G_{r,c}) = 2$.

□

An obvious question is: What is the global strong alliance partition number of grids of odd order? We make the following conjecture.

Conjecture 3.15 *If $n = rc$ is odd, then $\psi_{\hat{g}}(G_{r,c}) = 1$.*

As in Chapter 2, we denote by $G_{\infty,\infty}$ the 4-regular infinite grid graph with vertex set $V = \mathbf{Z} \times \mathbf{Z}$ and $N((i,j)) = \{(i-1,j), (i+1,j), (i,j-1), (i,j+1)\}$.

Proposition 3.16 *For the infinite grid graph $G_{\infty,\infty}$,*

$$\psi_g(G_{\infty,\infty}) = \psi_{\hat{g}}(G_{\infty,\infty}) = 3.$$

Proof. Since $G_{\infty,\infty}$ is 4-regular, $\psi_{\hat{g}}(G_{\infty,\infty}) \leq \psi_g(G_{\infty,\infty}) \leq 3$. A partition of $V(G_{\infty,\infty})$ into three global strong alliances is shown in Figure 7 (where the three alliances are indicated by open, solid, and ringed vertices, respectively). Hence $\psi_g(G_{\infty,\infty}) = \psi_{\hat{g}}(G_{\infty,\infty}) = 3$. $\square$

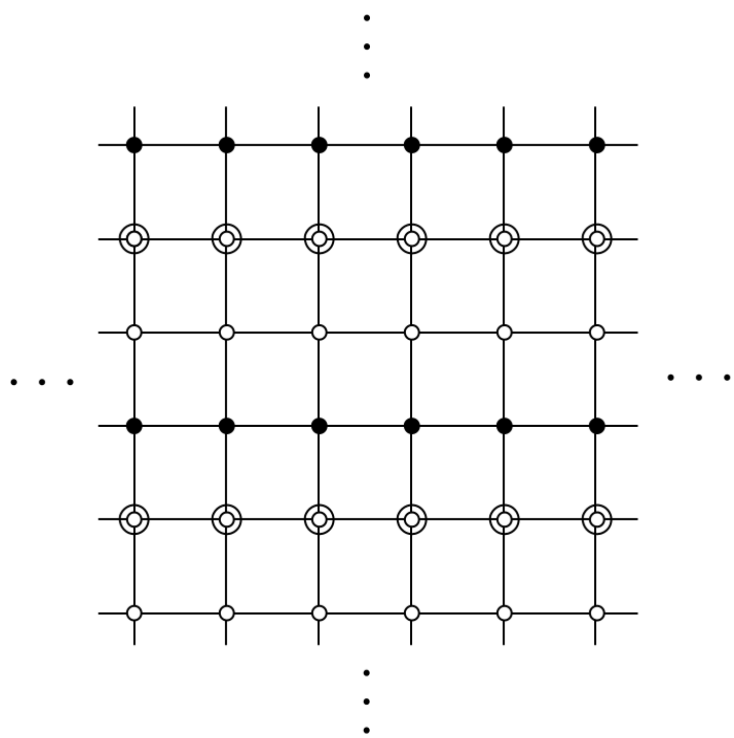

Figure 7: Global (Strong) Alliance Partition of $G_{\infty,\infty}$.

3.3 Trees

We have already observed that the global strong alliance partition number of any tree is one. The situation is more complicated, however, for the global alliance partition number. It is well known that the vertex set of any (nontrivial) tree can be partitioned into two disjoint dominating sets [9]. It is also possible to partition the vertex set into two disjoint defensive alliances; for example (if $n \geq 2$), the set V_1 consisting of a single leaf and the set V_2 consisting of all other vertices. However, not all trees have two disjoint alliances which are also dominating sets (odd paths, for example). As a consequence of Theorems 3.4 and 3.6, we have the following.

Corollary 3.17 *For any tree T,*

(i) $1 \leq \psi_g(T) \leq 2$, *and*

(ii) $\psi_{\hat{g}}(T) = 1$.

Proposition 3.10 illustrates the sharpness of the bounds in Corollary 3.17(i). Ideally, we would like to characterize those trees which achieve these bounds, though this seems to be a difficult problem. Eroh and Gera have studied the global alliance partition number of trees [3], and following their notation, we say a tree T is of *Class 1* if $\psi_g(T) = 1$, or of *Class 2* if $\psi_g(T) = 2$. They give some sufficient conditions for a tree to be of Class 1 or Class 2 and show that every tree is the induced subgraph of some Class 2 tree. A *binary tree* is a tree of maximum degree at most 3. Eroh and Gera have characterized binary trees [3].

Theorem 3.18 [3] *Let T be a binary tree of order $n \geq 3$. Then T is Class 2 if and only if there exist a pair of endvertices in T that are an odd distance from one another.*

We will take a similar approach and limit our attention to certain classes of trees. The *star* S_r is the complete bipartite graph $K_{1,s}$. A *double star* is a tree which has exactly two vertices which are not leaves. Let $S_{r,s}$ denote the double star with support vertices v_1 and v_2 adjacent to r and s leaves, respectively. A *caterpillar* is a tree which has the property that the removal of the endvertices results in a path. The resulting path $v_1, ..., v_t$ is called the *spine* of the caterpillar. Note that if the spine is trivial, the caterpillar is a star, and if the spine is K_2, the caterpillar is a double star. For vertices v_i, v_j on the spine, if $i < j$ we say v_i is to the left of v_j and v_j is to the right of v_i. We will characterize the Class 2 caterpillars, but first we prove a lemma which provides a necessary condition for a tree to be Class 2.

For a tree T, denote the set of leaves as $L = \{v \in V(T) \mid deg(v) = 1\}$, and for a vertex $v \in V(T)$, let L_v denote the set of leaves adjacent to v.

Lemma 3.19 *Let T be a Class 2 tree. Then for each $v \in V$,*

$$|L_v| \leq \frac{|N[v]|}{2}.$$

Proof. Let T be a Class 2 tree with a ψ_g-partition $\Pi = \{V_1, V_2\}$ and let $v \in V_1$.

First, notice if u is a leaf adjacent to v, then $u \in V_2$ (for if $u \in V_1$, no vertex of V_2 dominates u). Thus $|N(v) \cap V_2| \geq |L_v|$.

Since V_1 is a defensive alliance, $|N[v] \cap V_1| \geq |N(v) \cap (V - V_1)| = |N(v) \cap V_2|$. Since Π partitions V, we have

$$\begin{aligned} |N[v]| &= |N[v] \cap V_1| + |N(v) \cap V_2| \\ &\geq 2\,|N(v) \cap V_2| \\ &\geq 2\,|L_v|\,. \end{aligned}$$

Hence, $\frac{|N[v]|}{2} \geq |L_v|$. $\square$

We note that a condition similar to that of Lemma 3.19 is given in [3]. Lemma 3.19 allows us to easily characterize the Class 2 stars and double stars.

Corollary 3.20 *The star S_r is Class 2 if and only if $r = 1$.*

Proof. Lemma 3.19 implies if S_r is Class 2, then $r = 1$. Conversely, $S_1 = K_2$ is Class 2 by Proposition 3.8. $\square$

Corollary 3.21 *The double star $S_{r,s}$ is Class 2 if and only if $1 \leq r \leq s \leq 2$.*

Proof. If $S_{r,s}$ is Class 2, then Lemma 3.19 implies that $r \leq s \leq 2$. For the converse, suppose $r \leq s \leq 2$. Then $\{L, V - L\}$ is a partition of V into two global alliances. Hence $S_{r,s}$ is Class 2. $\square$

The Class 2 stars and double stars are illustrated in Figure 8. Global alliance partitions of each are shown, where the two global alliances are indicated by solid and open vertices, respectively.

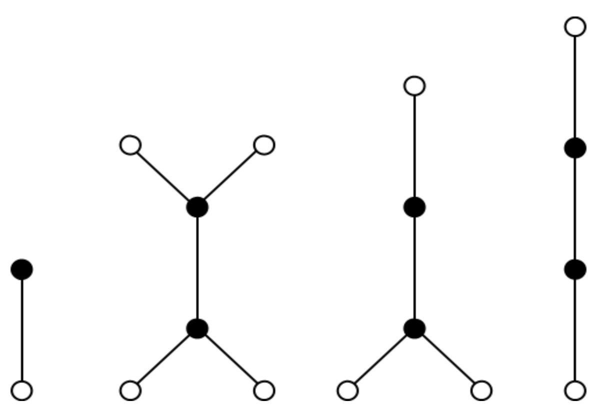

Figure 8: The Class 2 Stars and Double Stars.

In order to characterize the (general) Class 2 caterpillars we introduce a family $\mathcal{T}$.

Definition 3.22 *Let T be a caterpillar with spine $v_1, v_2, ..., v_t$, $t \geq 2$, and let S be the set of strong support vertices, together with the endvertices of the spine, that is, $S = \{v_i \mid deg(v_i) \geq 4\} \cup \{v_1, v_t\}$.*

For vertices $v_i \in S$, we define a left-distance *and* right-distance *as follows:*

$$d_l(v_i) = \begin{cases} \min\{d(v_i, v_j) \mid v_j \in S \cap \{v_1, ..., v_{i-1}\}\} & \text{if } i \neq 1 \\ 0 & \text{if } i = 1 \end{cases}$$

$$d_r(v_i) = \begin{cases} \min\{d(v_i, v_j) \mid v_j \in S \cap \{v_{i+1}, ..., v_t\}\} & \text{if } i \neq t \\ 0 & \text{if } i = t \end{cases}$$

Then $T \in \mathcal{T}$ if and only if T satisfies the following three properties:

(i) $\Delta(T) \leq 5$,

(ii) $deg(v_i) \leq 3$ *for* $i \in \{1, t\}$, *and*

(iii) *If $v_i \in S$ and $0 \neq d_l(v_i) = d(v_i, v_j)$ (respectively, $0 \neq d_r(v_i) = d(v_i, v_j)$) is even, then there exists a vertex v_k, where $j < k < i$ (respectively, $i < k < j$), such that $deg(v_k) = 3$ and $d(v_i, v_k)$ is odd.*

We claim the family of Class 2 caterpillars is precisely $\mathcal{T}$. For ease of presentation, as in [3], we investigate this problem from a vertex coloring perspective. Specifically, we seek a mapping $c : V(T) \to \{1, 2\}$ (that is, $c(v) = i$ if v has color i) such that for each $v \in V(T)$, at least one, but no more than half, of the vertices of $N[v]$ have a different color than v. That is, v is defended in its color and dominated by the other. Clearly, a tree T has such a coloring if and only if T is Class 2, and we refer to such a coloring as a *Class 2 coloring* of T. For $c(v) \in \{1, 2\}$, we define the *complement color* of color $c(v)$, denoted $\overline{c(v)}$, to be the element of the set $\{1, 2\} - c(v)$.

For a caterpillar $T \in \mathcal{T}$, we present an algorithm which produces a Class 2 coloring c of T. We first give an algorithm which two colors an even path.

Algorithm 3.23 : EVENPATH2COLOR

Input: An even path $P_{2p} = u_1, u_2, ..., u_{2p}$ and a color $X \in \{1, 2\}$.

Output: A two coloring c of P_{2p} with $c(u_1) = X$.

Begin

1. *For* $i = 1$ *to* p *do*

   1.1. *Let* $c(u_{2i-1}) = X$.

   1.2. *Let* $c(u_{2i}) = \overline{X}$.

   1.3. *Let* $X = \overline{X}$.

2. *Output* c.

End

Algorithm 3.24 : COLORCATERPILLAR

Input: A caterpillar $T \in \mathcal{T}$ with spine labeled $v_1, v_2, ...v_t$, $t \geq 2$, and set of leaves L.

Output: A Class 2 coloring c of T.

Begin

1. *Let* $c(v_1) = 1$.

2. *Let* $S = \{v_1, v_t\}$.

3. *For* $j = 2$ *to* $t - 1$ *do*

3.1. *If* $deg(v_j) \geq 4$, *then let* $S = S \cup \{v_j\}$.

4. *While* $|S| \geq 2$ *do*

4.1. *Let* $i = \min\{i \mid v_i \in S\}$.

4.2. *If* $d_r(v_i) = k$ *is odd, then*

4.2.1. *if* $k = 1$, *then goto Step 4.4,*

4.2.2. *otherwise EVENPATH2COLOR*$(v_{i+1}, v_{i+2}, ..., v_{i+k-1}; c(v_i))$.

4.3. *If* $d_r(v_i) = k$ *is even, then*

4.3.1. *let* $j = \min\{j \mid j > i$ *and* $deg(v_j) = 3$ *and* $d(v_i, v_j)$ *is odd*$\}$,

4.3.2. *if* $d(v_i, v_j) = 1$, *then let* $c(v_j) = c(v_i)$,

4.3.3. *otherwise*

4.3.3.1. *EVENPATH2COLOR*$(v_{i+1}, ..., v_{j-1}; c(v_i))$,

4.3.3.2. *let* $c(v_j) = c(v_{j-1})$,

4.3.4. *if* $j = i + k - 1$, *then goto Step 4.4,*

4.3.5. *otherwise EVENPATH2COLOR*$(v_{j+1}, ..., v_{i+k-1}; c(v_j))$.

4.4. *Let* $c(v_{i+k}) = c(v_{i+k-1})$.

4.5. *Let* $S = S - \{v_i\}$.

5. *For each support vertex* v *on the spine of* T *do*

5.1. *Color each leaf in* L_v, *the complement color of* $c(v)$.

6. *Output* c.

End

We now verify the validity of Algorithm 3.24.

Theorem 3.25 *Algorithm 3.24 produces a Class 2 coloring of a caterpillar $T \in \mathcal{T}$.*

Proof. Let $T \in \mathcal{T}$ be a caterpillar with spine $v_1, v_2, ..., v_t$, $t \geq 2$, and let c be the coloring given by Algorithm 3.24. We show that c is a Class 2 coloring of T.

We first consider the vertices in the set S, as defined in Definition 3.22. By property (ii), $2 \leq deg(v_1) \leq 3$. Since $c(v_1) = c(v_2) = 1$, and v_1 is adjacent to either one or two leaves of color 2, v_1 is defended in color 1 and dominated by color 2. Similarly, v_t is both defended and dominated in the coloring c. Notice, for each $v_i \in S - \{v_1, v_t\}$, we have $c(v_{i-1}) = c(v_i) = c(v_{i+1})$. By property (i) (and the definition of set S), we have $4 \leq deg(v_i) \leq 5$. Thus v_i is adjacent to at least two, but no more than three, leaves of color $\overline{c(v_i)}$. Hence, v_i is defended in its color and dominated by its complement color.

Let $v_j \in \{v_1, v_2, ..., v_t\} - S$. We consider two possibilities.

Case 1 v_j is a vertex selected at Step 4.3.1 of Algorithm 3.24.

Notice that the existence of such a vertex v_j, where $i < j < i + k$, is guaranteed by property (iii). Since $deg(v_j) = 3$ and $c(v_{j-1}) = c(v_j) = c(v_{j+1})$, v_j is defended in its color. The leaf adjacent to v_j has color $\overline{c(v_j)}$, and hence v_j is dominated by its complement color.

Case 2 v_j is not a vertex selected at Step 4.3.1 of Algorithm 3.24.

In this case, v_j is colored by Algorithm 3.23, and hence, v_j has one neighbor on the spine with color $c(v_j)$ and one neighbor on the spine with color $\overline{c(v_j)}$. Since $v_j \notin S$, we have $deg(v_j) \leq 3$, and so, v_j is both defended in its color and dominated by its complement color in c.

Finally, notice that Step 5 implies that each leaf $u \in L$ is both defended (by itself) and dominated by its complement color (by its support vertex).

Hence c is a Class 2 coloring. $\square$

Theorem 3.26 *A caterpillar T is Class 2 if and only if $T \in \mathcal{T}$.*

Proof. Let T be a caterpillar with spine $v_1, v_2, ..., v_t$, $t \geq 2$. The result holds by Corollary 3.21 if $t = 2$, thus we assume $t \geq 3$.

If $T \in \mathcal{T}$, then Theorem 3.25 implies that T is Class 2.

Conversely, suppose T is Class 2, and let c be a Class 2 coloring of T using colors 1 and 2. We show $T \in \mathcal{T}$ by demonstrating that T satisfies properties (i), (ii), and (iii) of Definition 3.22. The necessity of (i) and (ii) follows from Lemma 3.19. Suppose (iii) is false — that is, there exists $v_i \in S$ where (say) $d_r(v_i) = d(v_i, v_j) = k$ is even, for some $i < j$, and (iii) fails. That is, if $deg(v_r) = 3$ (where $i < r < j$), then $d(v_i, v_r)$ is even.

Without loss of generality, suppose $c(v_i) = 1$. Then we must have $c(v_i) = c(v_{i+1}) = 1$. Similarly, $c(v_j) = c(v_{j-1})$. If $d(v_i, v_j) = 2$, we have a contradiction since v_{i+1} is not dominated by any vertex of color 2 (because $deg(v_{i+1}) = 2$). If $d(v_i, v_j) \geq 4$, then $c(v_{i+2}) = 2$, since v_{i+1} must must be dominated by color 2. Thus, since v_{i+2} is defended in color 2, we must have $c(v_{i+3}) = 2$. Notice that if $d(v_i, v_j) = 4$, we again have a contradiction, since no color 1 vertex dominates v_{i+3}.

Similarly, for $d(v_i, v_j) = 2k$ (for any integer $k \geq 2$), we have $c(v_j) = c(v_{j-1}) = c(v_{j-2})$. By hypothesis, $deg(v_{j-1}) = 2$, and hence, v_{j-1} has no neighbor with a different color, a contradiction. Therefore, we may conclude that (iii) holds, and so, $T \in \mathcal{T}$. $\square$

Notice that Theorem 3.26 provides a sufficient condition for a tree to have equal global alliance partition number and domatic number.

Corollary 3.27 *If $T \in \mathcal{T}$, then $\psi_g(T) = d(T)$.*

Our final result provides an operation which allows us to construct a new Class 2 tree from two given Class 2 trees.

Theorem 3.28 *If T_1 and T_2 are Class 2 trees, then the tree T, obtained by joining (with an edge) any vertex of T_1 to any vertex of T_2, is Class 2.*

Proof. Suppose $\Pi_1 = \{V_1, V_2\}$ and $\Pi_2 = \{W_1, W_2\}$ are ψ_g-partitions of $V(T_1)$ and $V(T_2)$, respectively. Suppose $u \in V(T_1)$, $v \in V(T_2)$, and assume without loss of generality that $u \in V_1$ and $v \in W_1$. Let T be the tree with vertex set $V(T) = V(T_1) \cup V(T_2)$ and edge set $E(T) = E(T_1) \cup E(T_2) \cup \{uv\}$. We define a ψ_g-partition of $V(T)$ as follows: let $\Pi = \{X_1, X_2\}$, where $X_1 = V_1 \cup W_1$ and $X_2 = V_2 \cup W_2$. Clearly, each vertex $x \notin \{u, v\}$ inherits the property of being both defended in its respective set and dominated by the other set. Thus, we need only show that u and v are defended in X_1 and dominated by X_2. It suffices to show this for u. Since u is defended in T_1, it must be defended in T, as v is its ally in T. Furthermore, since (in T_1) u is dominated by a vertex of V_2 it is dominated by the same vertex of X_2 (in T), thus establishing the theorem. $\square$

Obviously, Theorem 3.28 allows us to construct an infinite family of Class 2 trees from a given set of Class 2 trees. However, not all Class 2 trees can be obtained in this manner. Consider, for example, the tree obtained from the star S_7 by subdividing three edges (see Figure 9). Hence, characterizing in general which trees are Class 2 remains an open problem.

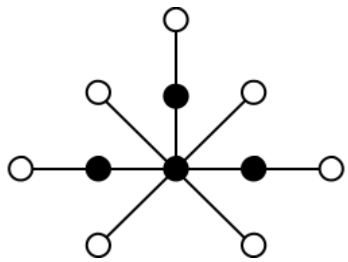

Figure 9: A Class 2 Subdivided Star.

4 CONCLUSIONS

We conclude by summarizing some exact values for the alliance partition numbers and listing some open problems.

Table 2: Alliance Partition Numbers.

Class	$\psi_a(G)$	$\psi_{\hat{a}}(G)$	$\psi_g(G)$	$\psi_{\hat{g}}(G)$
K_n $n \geq 1$	2 if n is even 1 if n is odd	1	2 if n is even 1 if n is odd	1
P_n $n \geq 1$	$\lceil \frac{n+1}{2} \rceil$	$\max\{1, \lfloor \frac{n}{2} \rfloor\}$	2 if n is even 1 if n is odd	1
C_n $n \geq 3$	$\lfloor \frac{n}{2} \rfloor$	$\lfloor \frac{n}{2} \rfloor$	2 if $n \equiv 0 \bmod 4$ 1 otherwise	2 if $n \equiv 0 \bmod 4$ 1 otherwise
$G_{2,c}$ $c \geq 2$	c		2	2
$G_{3,c}$ $c \geq 3$	$c+1$ if c is even c if c is odd		2	2 if c is even
$G_{r,c}$ $4 \leq r \leq c$	$\lfloor \frac{r-2}{2} \rfloor \lfloor \frac{c-2}{2} \rfloor +$ $r+c-2$		2	2 if rc is even

4.1 Open Problems

1. Determine the strong alliance partition number of grids $P_r \Box P_c$.

2. Determine the global strong alliance partition number of grids $P_r \Box P_c$ of odd order.

3. Determine the algorithmic complexity of finding a maximum (strong, global) alliance partition.

4. Characterize the trees T for which $\psi_g(T) = 2$.

5. Determine the alliance partition number of *cylinders* $P_j \Box C_k$.

6. Determine the alliance partition number of *toruses* $C_j \Box C_k$.

BIBLIOGRAPHY

[1] G. Chartrand and L. Lesniak, *Graphs & Digraphs: Fourth Edition.* Chapman & Hall/CRC, Boca Raton, FL (2004).

[2] L. Eroh and R. Gera, Alliance partition number in graphs, submitted for publication, January 2007.

[3] L. Eroh and R. Gera, Global alliance partition in trees, submitted for publication, January 2007.

[4] G. H. Fricke, L. M. Lawson, T. W. Haynes, S. M. Hedetniemi, and S. T. Hedetniemi, A note on defensive alliances in graphs, *Bull. Inst. Combin. Appl.* 38 (2003) 37-41.

[5] R. Gera, personal communication.

[6] M. U. Gerber and D. Kobler, Classes of graphs that can be partitioned to satisfy all their vertices, *Australas. J. Combin.* 29 (2004) 201-214.

[7] T. W. Haynes and J. A. Lachniet, The alliance partition number of grid graphs, to appear in *AKCE Int. J. Graphs Comb.*

[8] T. W. Haynes, S. T. Hedetniemi, and M. A. Henning, Global defensive alliances in graphs, *Electron. J. Combin.* 10 (2003), Research Paper 47, 13pp.

[9] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater, *Fundamentals of Domination in Graphs.* Marcel Dekker, Inc. New York (1998).

[10] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater, *Domination in Graphs: Advanced Topics.* Marcel Dekker, Inc. New York (1998).

[11] S. M. Hedetniemi, S. T. Hedetniemi, and P. Kristiansen, Alliances in graphs, *J. Combin. Math. Combin. Comput.* 48 (2004) 157-177.

[12] P. Kristiansen, S. M. Hedetniemi, and S. T. Hedetniemi, Introduction to alliances in graphs, *Proc. 17th Internat. Symp. Comput. Inform. Sci. (I. Cicekli, N. K. Cicekli, and Gelenbe, Eds.).* ISCIS xvii, October 28-30, 2002, Orlando, FL, CRC Press, pp. 308-312.

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