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Packings and Coverings of Complete Graphs with a Hole with the 4-Cycle with a Pendant Edge

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Packings and Coverings of Complete Graphs with a Hole with the 4-Cycle with a Pendant Edge

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ABSTRACT

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In this thesis, we consider packings and coverings of various complete graphs with the 4-cycle with a pendant edge. We consider both restricted and unrestricted coverings. Necessary and sufficient conditions are given for such structures for (1) complete graphs K_v, (2) complete bipartite graphs $K_{m,n}$, and (3) complete graphs with a hole $K(v, w)$.

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1 INTRODUCTION, MOTIVATION, AND HISTORY

In this chapter, we will introduce definitions, give examples and verify different types of graph decompositions, packings, and coverings with background and history.

A *g-decomposition of graph* G is a set of subgraphs of G, $\gamma = \{g_1, g_2, \ldots, g_n\}$, where $g_i \cong g$ for $i \in \{1, 2, \ldots, n\}$, $E(g_i) \cap E(g_j) = \varnothing$ for $i \neq j$, and $\cup_{i=1}^{n} E(g_i) = E(G)$. The g_i are called *blocks* of the decomposition. When G is a complete graph, the g-decomposition is often called a *graph design*. See Figure 1 for a 5-cycle decomposition of K_5.

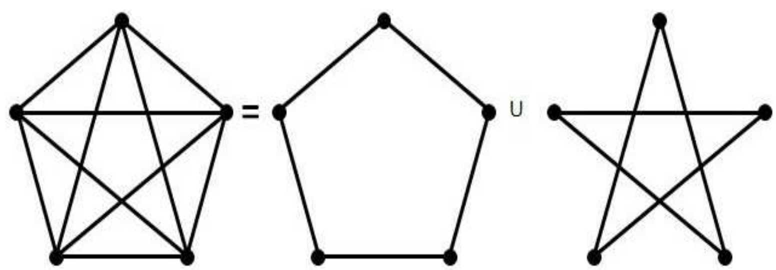

Figure 1: A C_5 decomposition of K_5

The concept of a graph decomposition lies in the general area of *design theory*. Consider the following experiment: Suppose we have a collection of v samples and we want to compare a property of the samples. However, these samples are compared by running them in the special machine three-at-a-time. Due to the cost of running the machine, the machine cannot be calibrated from run to run. So the only way two samples can be compared is to run them together in the machine. When can all of the v samples be optimally compared to each other by running the machine $\binom{v}{2}/3$ times? The solution to this question is equivalent to finding a K_3-decomposition of K_v, where each vertex of K_v represents a sample, an edge joining two vertices represents a comparison of the two corresponding samples, and a copy of K_3 represents a run of the machine. A K_3-decomposition of K_v exists if and only if $v \equiv 1$ or $3 \pmod 6$. Such a structure is called a *Steiner triple system* [15]. See Figure 2 for a K_3.

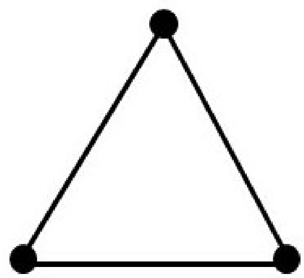

Figure 2: The complete graph on three vertices, K_3

In the event that a g-decomposition of G does not exist, we can still consider a set of isomorphic copies of graphs g which "approximate" a decomposition. There are two ways to approach this. We describe the two approaches in terms of the sample comparison analogy. In the first approach, we can try comparing as many of the samples as possible, without repetition of comparisons. From the above, we want to find a collection of runs of the machine (represented by copies of K_3) which do not repeat pairs of samples run together. That is, the copies of K_3 are edge disjoint. This minimizes the number of pairs of samples which are omitted. That is, the cardinality of the set of edges in K_v which are in none of the copies of K_3 is minimal. Such an experimental design is related to a maximal graph packing. A *maximal packing* of a simple graph G with isomorphic copies of a graph g is a set $\{g_1, g_2, \ldots, g_n\}$ where

$g_i \cong g$ and $V(g_i) \subset V(G)$ for all i, $E(g_i) \cap E(g_j) = \varnothing$ for $i \neq j$, $\cup_{i=1}^{n} g_i \subset G$, and $|E(G) \setminus \cup_{i=1}^{n} E(g_i)|$ is minimal. This experimental design corresponds to a K_3-packing of K_v. Such designs are explored in [17]. Other packings of complete graphs have also been studied. For example, 4-cycle-packings [18], K_4-packings [3], and 6-cycle-packings [10, 11]. See Figure 3 for a packing of K_5.

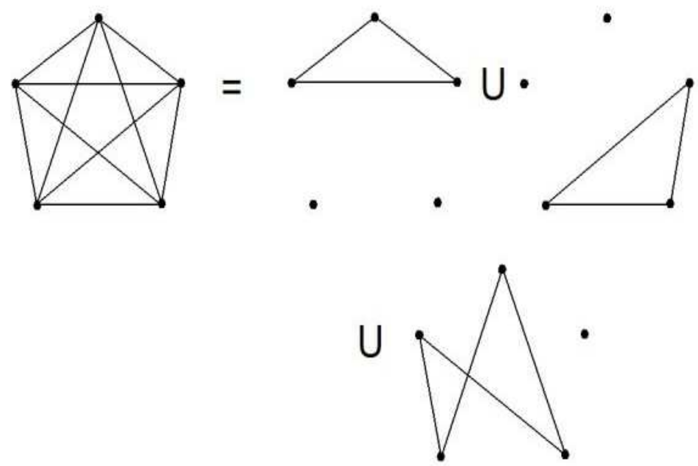

Figure 3: A packing of K_5 with 3-cycles has a leave L with 4 edges

A second approach involves comparing *all* of the samples to each other, but with minimal repetitions of compared samples. So we might assume that the machine must have three samples in it during each run. This experimental design is related to a minimal graph covering. A *minimal covering* of a simple graph G with isomorphic copies of a graph g is a set $\{g_1, g_2, \ldots, g_n\}$ where $g_i \cong g$, $V(g_i) \subset V(G)$, $E(g_i) \subset E(G)$ for all i, $G \subset \cup_{i=1}^{n} g_i$, and $|\cup_{i=1}^{n} E(g_i) \setminus E(G)|$ is minimal (when considering coverings, the graph $\cup_{i=1}^{n} g_i$ may not be simple and $\cup_{i=1}^{n} E(g_i)$ may be a multiset). The machine analogy in this case corresponds to a K_3-covering of K_v. Such designs are explored in [8]. Coverings of K_v have also been explored, for example, for 4-cycles [17] and

6-cycles [12]. See Figure 4 for a covering of K_5.

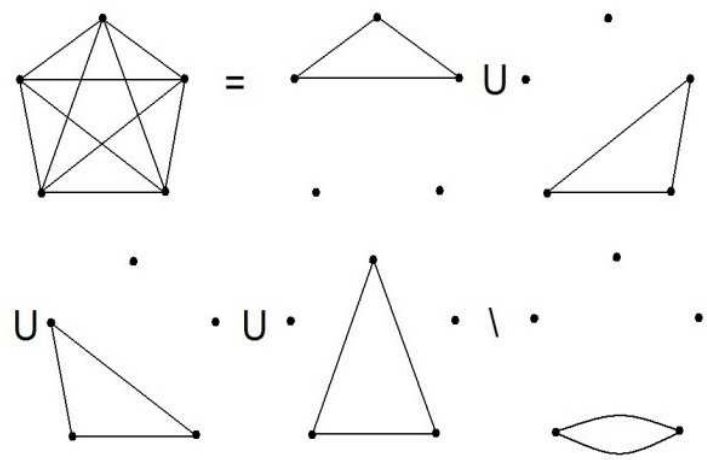

Figure 4: A covering of K_5 with 3-cycles has a padding of $2 \times K_2$

In terms of graph decompositions, several studies have centered on g-decompositions of complete graphs into copies of a given graph g with a small number of vertices [1, 2, 13, 14].

The 3-cycle with a pendant edge is denoted L. See Figure 5 for an L. The graph L is sometimes called the *lollipop*. An L-decomposition of K_v exists if and only if $v \equiv 0$ or 1 (mod 8) [2].

The 4-cycle with a pendant edge is denoted H. See Figure 6 for an H. The graph H is sometimes called a *kite*. We consider a single graph g, the 4-cycle with a pendant edge, and explore packings and coverings of several graphs related to the complete graph. We denote the 4-cycle with a pendant edge as $H = [a, b, c, d; e]$ where $V(H) = \{a, b, c, d, e\}$ and $E(H) = \{(a, b), (b, c), (c, d), (a, d), (a, e)\}$. An H-decomposition of K_v exists if and only if $v \equiv 0$ or 1 (mod 5), $v \geq 10$ [1].

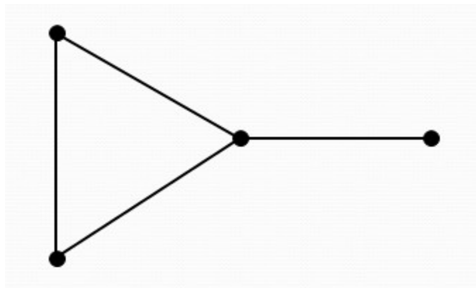

Figure 5: The 3-cycle with a pendant edge, denoted L

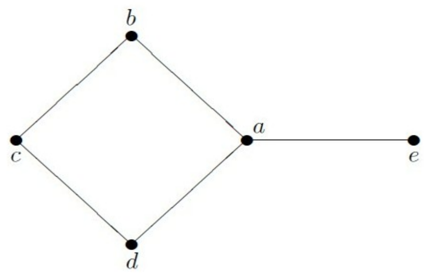

Figure 6: The 4-cycle with a pendant edge, denoted H

Suppose we have a collection of m samples from one population and a collection of n samples from a second population. We are interested in comparing the two populations and hence in comparing each sample from m to each sample from n. This motivates us to consider decompositions, packings, and coverings of complete bipartite graphs. An H-decomposition of the complete bipartite graph, $K_{m,n}$, exists if and only if $mn \equiv 0 \pmod 5$, $m \geq 5$, and $n \geq 2$ [6].

Another graph related to the complete graph is the complete graph with a hole,

$K(v,w)$. The complete graph on v vertices with a hole of size w is the graph with vertex set $V(K(v,w)) = V_{v-w} \cup V_w$, where $|V_{v-w}| = v - w$ and $|V_w| = w$, and edge set $E(K(v,w)) = \{(a,b) \mid a, b \in V(K(v,w)), \{a,b\} \not\subset V_w\}$. Necessary and sufficient conditions for the decomposition of $K(v,w)$ into m-cycles are known for $m \in \{3,4,5,6,7,8,10,12,14\}$ [4, 5, 16]. There is an H-decomposition of $K(v,w)$ if and only if $|E(K(v,w))| \equiv 0 \pmod 5$, $v - w \geq 4$, and $(v,w) \notin \{(5,1),(6,1)\}$ [6].

The graph $K(v,w)$ relates to the experimental design story as follows. Suppose we have performed comparisons on a collection of w samples and then receive an additional collection of samples (say, $v - w$ new samples). We now wish to compare the $v-w$ new samples to each other and to the original w samples. In the case of the machine described above, this would correspond to a K_3 decomposition of $K(v,w)$. In the event that a decomposition does not exist, we need to look into packings and coverings of $K(v,w)$. With a graph g-packing of G, we require that each copy of g is a subgraph of G. The definition given above for a g-covering also involves the condition that each copy of g is a subgraph of G. Most studies of coverings have involved $G = K_v$, so the condition that the copies of g are subgraphs of G is trivially satisfied. But when G is not a complete graph, there is no obvious reason to impose the subgraph condition.

Returning to the testing-of-samples story, we see no reason to disallow, the testing (or re-testing) of two samples in the hole of $K(v,w)$. Therefore, we are motivated to refine the definition of a graph covering into two cases—one case in which edges that are not in G are forbidden from use in the copies of g, and a case in which these edges are not forbidden. A *minimal unrestricted covering* of graph G with isomorphic copies of a graph g is a set $\{g_1, g_2, \ldots, g_n\}$ where $g_i \cong g$, $V(g_i) \subset V(G)$, $G \subset \cup_{i=1}^n g_i$, and $| \cup_{i=1}^n E(g_i) \setminus G|$ is minimal. The graph $\cup_{i=1}^n g_i$ may not be simple and $\cup_{i=1}^n E(g_i)$ may be a multiset. The definition given above for a g-covering also involves the condition that each copy of g is a subgraph of G. Most studies of coverings have involved $G = K_v$. A *minimal restricted covering* of graph G with isomorphic copies of a graph g is a set $\{g_1, g_2, \ldots, g_n\}$ where $g_i \cong g$, $V(g_i) \subset V(G)$, $G \subset \cup_{i=1}^n g_i$, and $| \cup_{i=1}^n E(g_i) \setminus G|$ is minimal, and $E(g_i) \subset E(G)$ for all i. Notice that in the event that G is a complete graph, there is no distinction between a minimal restricted and minimal unrestricted covering. See Figure 7 for an unrestricted covering of $K(5,3)$. In $K(5,3)$, $H = [0_0, a, b, c; 1_0]$ is bipartite with partite sets $\{0_0, 1_0\}$ and $\{a, c, 1_0\}$. So, $(0_0, 1_0)$ cannot be the pendant edge. Also, $(0_0, 1_0)$ cannot be an edge in the 4-cycle since then the other 3 edges have to be in the bipartite $K_{2,3}$, a contradiction occurs. Therefore, the restricted covering of $K(5,3)$ does not exist.

Figure 7: An unrestricted H-covering of $K(5,3)$

The purpose of this thesis is to give H-packings of $K(v, w)$, and restricted and unrestricted H-coverings of $K(v, w)$, and unrestricted coverings of $K_{m,n}$. We also present some related results in Chapter 2 on H-packings and H-coverings of K_v and of $K_{m,n}$.

2 PREVIOUS KNOWN FINDINGS

In this chapter, the results are due to Brandon Coker, Gary Coker, and Robert Gardner [7]. They will be combined with the results of the following chapters and submitted for publication. We include these results for completeness.

2.1 Packing and Covering K_v

In this section, when necessary, we assume the vertex set of K_v is $V(K_v) = \{0, 1, 2, \ldots, v-1\}$. Since H has 5 vertices, we only consider $v \geq 5$.

Theorem 2.1 *A maximal H-packing of K_v, $v \geq 5$, has leave L where $|E(L)| = |E(K_v)|$ (mod 5), except when $v \in \{5, 6\}$ in which case $|E(L)| = 5$.*

Proof. Since $|E(H)| = 5$, then it is necessary that in any H-packing of K_v with leave L, $|E(L)| \equiv |E(K_v)| \pmod 5$. Therefore, such a packing with $|E(L)| = |E(K_v)| \pmod 5|$ would be maximal. If $v \in \{5, 6\}$, then $|E(K_v)| \equiv 0 \pmod 5$, but there is not an H-decomposition of K_v [1]. So for $v \in \{5, 6\}$, an H-packing of K_v with leave L where $|E(L)| = 5$ would be maximal.

Case 1. Suppose $v = 5$. Then $|V(K_5)| = 10$ and, since there is no H-decomposition of K_5, then a packing with $|E(L)| = 5$ would be maximal. The set $\{[0, 1, 2, 3; 4]\}$ is a maximal packing of K_5 with leave L where $E(L) = \{(0, 2), (1, 3), (1, 4), (2, 4), (3, 4)\}$.

Case 2. Suppose $v = 6$. Then $|V(K_6)| = 15$ and, since there is no H-decomposition of K_6, then a packing with $|E(L)| = 5$ would be maximal. The set $\{[0, 1, 2, 3; 4], [1, 3, 4, 5; 4]\}$ is a maximal packing of K_6 with leave L where $E(L) = \{(0, 2), (0, 5), (2, 5), (2, 4),$

$(3,5)\}$.

Case 3. Suppose $v \equiv 2$ or $4 \pmod 5$, $v \geq 9$. Since $|E(K_v)| \equiv 1 \pmod 5$, $|E(L)| = 1$ would be optimal. Now $K(v,2)$ can be decomposed [6], so $|E(L)| = 1$.

Case 4. Suppose $v \equiv 3 \pmod 5$, $v \geq 8$. Since $|E(K_v)| \equiv 3 \pmod 5$, $|E(L)| = 3$ would be optimal. Now $K(v,3)$ can be decomposed [6], so $|E(L)| = 3$. $\square$

Theorem 2.2 *A minimal H-covering of K_v, $v \geq 5$, has padding P where $|E(P)| = -|E(K_v)|$ (mod 5), except when $v \in \{5,6\}$ in which case $|E(P)| = 5$.*

Proof. Since $|E(H)| = 5$, then it is necessary that in any H-covering of K_v with padding P, that $|E(K_n)| + |E(P)| \equiv 0 \pmod 5$, or that $|E(P)| \equiv -|E(K_v)| \pmod 5$. So if $|E(P)| = -|E(K_v)| \pmod 5$, then the covering is minimal. In the case $v \in \{5,6\}$, then $-|E(K_v)| \equiv 0 \pmod 5$, but there is no H-decomposition of K_v [1]. So for $v \in \{5,6\}$, an H-covering of K_v with padding P where $|E(P)| = 5$ would be minimal.

Case 1. Suppose $v = 5$. Then $|V(K_5)| = 10$. Since there is no H-decomposition of K_5, then a covering with $|E(P)| = 5$ would be minimal. The set $\{[0,1,2,3;4], [1,2,0,4;3], [4,1,0,2;3]\}$ is a minimal covering of K_5 with padding P where $E(P) = \{(0,1), (0,2), (0,4), (1,2), (1,4)\}$. So $|E(P)| = 5$.

Case 2. Suppose $v = 6$. Then $|V(K_6)| = 15$. Since there is no H-decomposition of K_6, then a covering with $|E(P)| = 6$ would be minimal. The set $\{[0,1,2,3;4], [5,0,2,4;1], [5,3,\ 4,\ 1;2], [3,4,5,2;1]\}$ is a minimal covering of K_6 with padding P where $E(P) = \{(1,5), (2,3), (2,5), (3,4), (4,5)\}$. So $|E(P)| = 5$.

Case 3. Suppose $v \equiv 2$ or $4 \pmod 5$, $v \geq 7$. Since $|E(K_v)| \equiv 1 \pmod 5$, $|E(P)| = 4$

would be optimal. There is a H-decomposition of $K(v,2)$ [6]. Take such a decomposition, along with another copy of H which includes the edge of the hole of $K(v,2)$. This gives a covering of K_v with padding P where $|E(P)| = 4$ and so the covering is optimal.

Case 4. Suppose $v \equiv 3 \pmod 5$, $v \geq 8$. Since $|E(K_v)| \equiv 3 \pmod 5$, $|E(P)| = 2$ would be optimal. A H-covering of K_8 is given by $\{[0,1,2,7;3], [1,3,5,7;6], [4,5,6,3;7], [2,4,6,0;5], [1, 4,0,5;2], [7,3,2,6;0]\}$ with padding P where $E(P) = \{(0,7),(1,2)\}$ and the covering is optimal. For $v \geq 13$, $K_v = K(v,8) \bigcup K_8$, $K(v,8)$ can be decomposed [6], and K_8 can be covered with padding P where $|E(P)| = 2$. Therefore, there is an optimal H-covering of K_v with padding P where $|E(P)| = 2$. $\square$

2.2 Packing and Covering $K_{m,n}$

In this section, we consider H-packings and H-coverings of complete bipartite graphs $K_{m,n}$. We assume the partite sets of $K_{m,n}$ are $\{0_0, 1_0, \ldots, (m-1)_0\}$ and $\{0_1, 1_1, \ldots, (n-1)_1\}$.

Theorem 2.3 *A maximal H-packing of $K_{m,n}$ has leave L where*

(1) $|E(L)| = mn$ if $m = 1$ or $n = 1$, or if $m = n = 2$, or

(2) $|E(L)| = |E(K_{m,n})| \pmod 5$ otherwise.

Proof. First, if m or n equals 1, then H is not a subgraph of $K_{m,n}$ and the leave must have mn edges. Similarly, the leave of a packing of $K_{2,2}$ has $mn = 4$ edges. For $m \geq 2$ and $n \geq 3$, as in the proof of Theorem 2.1, an H-packing of $K_{m,n}$ with leave L where $|E(L)| = |E(K_{m,n})| \pmod 5$ would be maximal. Next, for $m \geq 2$ and $n \geq 3$

we observe that if there is a packing of $K_{m,n}$ with leave L, then there is a packing of $K_{m+5i,n+5j}$ with leave L for all $i,j \in \mathbb{N}$. This is because $K_{m+5i,n+5j} = K_{m,n} \cup K_{m,5j} \cup K_{5i,n} \cup K_{5i,5j}$ where the partite sets of $K_{m+5i,n+5j}$ are $\{0_0, 1_0, \ldots, (m-1+5i)_0\}$ and $\{0_1, 1_1, \ldots, (n-1+5j)_1\}$, the partite sets of $K_{m,n}$ are $\{0_0, 1_0, \ldots, (m-1)_0\}$ and $\{0_1, 1_1, \ldots, (n-1)_1\}$, the partite sets of $K_{m,5j}$ are $\{0_0, 1_0, \ldots, (m-1)_0\}$ and $\{n_1, (n+1)_1, \ldots, (n-1+5j)_1\}$, the partite sets of $K_{5i,n}$ are $\{m_0, (m+1)_0, \ldots, (m-1+5i)_0\}$ and $\{0_1, 1_1, \ldots, (n-1)_1\}$, and the partite sets of $K_{5i,5j}$ are $\{m_0, (m+1)_0, \ldots, (m-1+5i)_0\}$ and $\{n_1, (n+1)_1, \ldots, (n-1+5j)_1\}$. There is an H-decomposition of $K_{m,5j}$, $K_{5i,n}$, and $K_{5i,5j}$ [6]. The packings given in Table 1, combined with the decompositions of

Table 1: Packings of $K_{m,n}$ for small values of m and n

$(m,n) \pmod 5$	$K_{m,n}$	Packing	Leave
$(1,1)$	$K_{6,6}$	$\{[0_0,0_1,1_0,1_1;2_1],[0_0,5_1,1_0,4_1;3_1],$ $[2_0,0_1,3_0,1_1;2_1],[2_0,4_1,3_0,5_1;3_1],$ $[4_0,0_1,5_0,1_1;2_1],[4_0,4_1,5_0,5_1;3_1],$ $[2_1,1_0,3_1,3_0;5_0]\}$	$\{(5_0,3_1)\}$
$(1,2)$	$K_{6,2}$	$\{[0_1,0_0,1_1,1_0;2_0],[1_1,5_0,0_1,4_0;3_0]\}$	$\{(2_0,1_1),(3_0,0_1)\}$
$(1,3)$	$K_{6,3}$	$\{[3_0,0_1,2_0,1_1;2_1],[1_0,0_1,0_0,1_1;2_1]$ $[5_0,0_1,4_0,1_1;2_1]\}$	$\{(0_0,2_1),(2_0,2_1),$ $(4_0,2_1)\}$
$(1,4)$	$K_{6,4}$	$\{[0_0,0_1,1_0,1_1;2_1],[3_1,2_0,2_1,3_0;1_0],$ $[2_1,4_0,1_1,5_0;1_0],[3_1,4_0,0_1,5_0;0_0]\}$	$\{(2_0,0_1),(2_0,1_1),$ $(3_0,0_1),(3_0,1_1)\}$
$(2,2)$	$K_{2,2}$	$\varnothing$	$\{(0_0,0_1),(0_0,1_1),$ $(1_0,0_1),(1_0,1_1)\}$
$(2,3)$	$K_{3,2}$	$\{[1_1,1_0,0_1,0_0;2_0]\}$	$\{(2_0,0_1)\}$
$(2,4)$	$K_{4,2}$	$\{[1_1,1_0,0_1,0_0;2_0]\}$	$\{(2_0,0_1),(3_0,0_1),$ $(3_0,1_1)\}$
$(3,3)$	$K_{3,3}$	$\{[1_1,1_0,0_1,0_0;2_0]\}$	$\{(0_0,2_1),(1_0,2_1),$ $(2_0,0_1),(2_0,2_1)\}$
$(3,4)$	$K_{4,3}$	$\{[0_0,0_1,1_0,1_1;2_1],[2_0,3_1,1_0,2_1;1_1]\}$	$\{(0_0,3_1),(2_0,0_1)\}$
$(4,4)$	$K_{4,4}$	$\{[1_0,1_1,0_0,0_1;2_1],[3_0,1_1,2_0,0_1;2_1],$ $[3_1,2_0,2_1,0_0;3_0]\}$	$\{(1_0,3_1)\}$

complete bipartite graphs mentioned above, yield the result. □

Theorem 2.4 *A minimal restricted H-covering of $K_{m,n}$, where neither m nor n equals 1 and $m+n \geq 5$, has padding P where $|E(P)| = -|E(K_{m,n})|$ (mod 5).*

Proof. For $K_{1,n}$, H is not a subgraph and so a restricted H-covering does not exist. Similar to the argument in Theorem 2.2, an H-covering of $K_{m,n}$ with padding P where $|E(P)| = -|E(K_{m,n})|$ (mod 5) would be minimal. As in Theorem 2.3, for $m \geq 2$ and $m \geq 3$, if there is a restricted covering of $K_{m,n}$ with padding P, then there is a restricted covering of $K_{m+5i,n+5j}$ with padding P for all $i, j \in \mathbb{N}$. The coverings in Table 2, combined with the decompositions of complete graphs mentioned in Theorem 2.3, yield the result. □

Table 2: Coverings of $K_{m,n}$ for small values of m and n

(m,n) (mod 5)	$K_{m,n}$	Covering	Padding
$(1,1)$	$K_{6,6}$	$\{[3_1, 0_0, 0_1, 1_0; 5_0]\}$	$\{(0_0,3_1),(0_0,0_1),(1_0,0_1),(1_0,3_1)\}$
$(1,2)$	$K_{6,2}$	$\{[0_1, 2_0, 1_1, 1_0; 3_0]\}$	$\{(1_0,0_1),(1_0,1_1),(2_0,0_1)\}$
$(1,3)$	$K_{6,3}$	$\{[2_1, 0_0, 1_1, 2_0; 4_0]\}$	$\{(0_0,1_1),(2_0,1_1)\}$
$(1,4)$	$K_{6,4}$	$\{[2_0, 0_1, 3_0, 1_1; 2_1]\}$	$\{(2_0,2_1)\}$
$(2,2)$	$K_{7,2}$	$\{[0_1, 0_0, 1_1, 1_0; 6_0],$ $[1_1, 2_0, 0_1, 3_0; 6_0],$ $[1_1, 5_0, 0_1, 4_0; 0_0]\}$	$\{(0_0,1_1)\}$
$(2,3)$	$K_{3,2}$	$\{[0_1, 1_0, 1_1, 0_0; 2_0]\}$	$\{(0_0,0_1),(0_0,1_1),(1_0,0_1),(1_0,1_1)\}$
$(2,4)$	$K_{4,2}$	$\{[0_1, 3_0, 1_1, 0_0; 2_0]\}$	$\{(0_0,0_1),(0_0,1_1)\}$
$(3,3)$	$K_{3,3}$	$\{[2_1, 2_0, 0_1, 1_0; 0_0]\}$	$\{(1_0,0_1)\}$
$(3,4)$	$K_{4,3}$	$\{[0_1, 0_0, 3_1, 2_0; 1_0]\}$	$\{(0_0,0_1),(1_0,0_1),(2_0,3_1)\}$
$(4,4)$	$K_{4,4}$	$\{[3_1, 0_0, 0_1, 3_0; 1_0]\}$	$\{(0_0,3_1),(0_0,0_1),(3_0,0_1),(3_0,3_1)\}$

Theorem 2.5 *A minimal unrestricted H-covering of $K_{m,n}$ has padding P where*

(1)*when $m > 1$, and $n > 1$, $|E(P)| = -|E(K_{m,n})|(mod\,5)$, and*

(2)*when $m = 1$,*

$|E(P)| = (2/3)n$ *for* $n \equiv 0 \ (mod\,3)$,

$|E(P)| = 2(n+5)/3$ *for* $n \equiv 1 \ (mod\,3)$,

$|E(P)| = (2n+5)/3$ *for* $n \equiv 2 \ (mod\,3)$.

Proof. For $m > 1$ and $n > 1$, the necessary condition follows as in the proof of Theorem 2.4. In this case, sufficiency also follows from Theorem 2.4.

When $m = 1$, a copy of H where $V(H) \subset V(K_{1,n})$ has at most 3 edges in $E(K_{1,n})$ and at least 2 edges in the padding. So in an H-covering of $K_{1.n}$ there are at least $\lceil n/3 \rceil$ copies of H. Now $\lfloor n/3 \rfloor$ copies of H can have at most $3\lfloor n/3 \rfloor$ edges in $E(K_{1,n})$ and at least $2\lfloor n/3 \rfloor$ edges in the padding. If $n \equiv 1$ (mod 3), then to completely cover $K_{1,n}$ we must add one more copy of H which has at most 1 edge in $E(K_{1,n})$ and at least 4 edges in the padding. If $n \equiv 2$ (mod 3), then to completely cover $K_{1,n}$ we must add one more copy of H which has at most 2 edges in $E(K_{1,n})$ and at least 3 edges in the padding. This yields the necessary conditions for $m = 1$. We now establish sufficiency for $m = 1$.

Case 1. Suppose $m = 1$ and $n \equiv 0$ (mod 3), where $n \geq 6$. Consider the blocks $\{[0_0, 0_1, 3_1, 1_1; 2_1]\} \cup \{[0_0, (3k)_1, 2_1, (3k+1)_1; (3k+2)_1] | k = 1, 2, ..., (n/3)-1\}$. This is a covering of $K_{m,n}$ with padding $P = \{(0_1, 3_1), (1_1, 3_1)\} \cup \{(2_1, (3k)_1), (2_1, (3k+1)_1) | k = 1, 2, ..., (n/3) - 1\}$, where $|E(P)| = 2 + 2((n/3) - 1) = (2/3)n$.

Case 2. Suppose $m = 1$ and $n \equiv 1$ (mod 3), where $n \geq 4$. From Case 1, there is a covering of $K_{1,n-1}$, where the partite sets of $K_{1,n-1}$ are $\{0_0\}$, and $M_n \setminus \{(n-1)_1\}$ with padding P_1 where $|E(P_1)| = 2(n-1)/3$. This covering along with $\{[0_0, 0_1, 2_1, 1_1; 3_1]\}$, is an unrestricted of $K_{m,n}$ with padding $P_2 = P_1 \cup \{(0_0, 0_1), (0_1, 2_1), (2_1, 1_1), (0_0, 1_1)\}$ and so $|E(P_2)| = 2(n-1)/3 + 4 = 2(n+5)/3$.

Case 3. Suppose $m = 1$ and $n \equiv 2 \pmod 3$, where $n \geq 5$. From Case 1, there is a covering of $K_{1,n-2}$, where the partite sets of $K_{1,n-2}$ are $\{0_0\}$, and $M_n \setminus \{(n-2)_1\}$ with padding P_1 where $|E(P_1)| = 2(n-2)/3$. This covering along with $\{[0_0, (n-1)_1, 0_1, (n-2)_1; 1_1]\}$, is an unrestricted of $K_{m,n}$ with padding $P_2 = P_1 \cup \{(0_1, (n-1)_1), (0_1, (n-2)_1), (0_0, 1_1)\}$ and so $|E(P_2)| = 2(n-2)/3 + 3 = (2n+5)/3$. $\square$

In this chapter we have given necessary and sufficient conditions for an H-packing of K_v (Theorem 2.1), an H-covering of K_v (Theorem 2.2), an H-packing of $K_{m,n}$ (Theorem 2.3), a restricted covering of $K_{m,n}$ (Theorem 2.4), and an unrestricted covering of $K_{m,n}$ (Theorem 2.5). Theorem 2.1 through 2.4 are due to Brandon Coker, Gary Coker, and Robert Gardner [7] and are included for completeness.

3 PACKINGS AND COVERINGS OF $K(v, w)$

In this chapter, we give necessary and sufficient conditions for H-packing and H-covering (both restricted and unrestricted) of the complete graph on v vertices with a hole of size w, $K(v, w)$.

3.1 Packing $K(v, w)$

In this section, we assume the vertex set of $K(v, w)$ is $V(K(v, w)) = V_{v-w} \cup v_w$ as described in Section 1, where $V_{v-w} = \{0_0, 1_0, \ldots, (v-w-1)_0\}$ and $V_w = \{0_1, 1_1, \ldots, (w-1)_1\}$.

Theorem 3.1 *A maximal H-packing of $K(v, w)$ has leave L where $|E(L)| = |E(K(v, w))|$ (mod 5), and $v - w \geq 2$ is necessary.*

Proof. When $v = w + 1$, H is not a subgraph of $K(v, w)$ and so there is no packing. Therefore, $v - w \geq 2$ is necessary for the existence of a packing.

Case 1. If $v - w = 6$, then $K(v, w) = K_6 \cup K_{6,w}$ where the vertex set of K_6 is V_{v-w} and the partite sets of $K_{6,w}$ are V_{v-w} and V_w. There exists a packing $K_{6,w}$ with leave L_2 such that $|E(L_2)| \in \{1, 2, 3, 4\}$. Without loss of generality, $(0_0, 0_1) \in E(L_2)$. Consider $\{[4_0, 5_0, 2_0, 3_0; 1_0], [0_0, 3_0, 1_0, 5_0; 2_0], [0_0, 1_0, 2_0, 4_0; 0_1]\}$, $L = \{(3_0, 5_0)\} \cup E(L_2) \setminus \{(0_0, 0_1)\}$. So, $|E(L)| = |E(L_2)|$.

Case 2. Suppose $v \equiv 0 \pmod 5$ and $w \equiv 2 \pmod 5$. Consider $K(v, w) = K_{v-w} \cup K_{v-w,w}$ where $V(K_{v-w})$=V_{v-w} and the partite sets of $K_{v-w,w}$ are V_{v-w} and V_w. We have $v - w \equiv 3 \pmod 5$, and $w \equiv 2 \pmod 5$. There is a maximal packing of K_{v-w} where $v - w \equiv 3 \pmod 5$ with $|E(L_1)| = 3$ by Theorem 2.1 and a maximal packing of $K_{v-w,w}$ with $|E(L_2)| = 1$ by Theorem 2.3. Therefore, there is a maximal packing of $K(v, w)$ has leave L where $|E(L)| = |E(K(v, w))| \pmod 5 = 4$.

Case 3. Suppose $v \equiv 0 \pmod 5$ and $w \equiv 3 \pmod 5$, or $v \equiv 3 \pmod 5$ and $w \equiv 4 \pmod 5$. Consider $K(v, w) = K_{v-w} \cup K_{v-w,w}$ as in Case 2, where $v - w \equiv 2 \pmod 5$ and $w \equiv 3 \pmod 5$, or $v - w \equiv 4 \pmod 5$ and $w \equiv 4 \pmod 5$. There is a maximal packing of K_{v-w} where $v - w \equiv 2 \pmod 5$ or $v - w \equiv 4 \pmod 5$ with $|E(L_1)| = 1$ by Theorem 2.1 and there is a maximal packing of $K_{v-w,w}$ with $|E(L_2)| = 1$ by Theorem 2.3. Therefore, there is a maximal packing of $K(v, w)$ with leave L where $|E(L)| = |E(K(v, w))| \pmod 5 = 2$.

Case 4. Suppose $v \equiv 0 \pmod 5$ and $w \equiv 4 \pmod 5$. When $v - w \geq 11$, consider $K(v, w) = K_{v-w} \cup K_{v-w,w}$, as in Case 2, where $v - w \equiv 1 \pmod 5$, and $w \equiv 4 \pmod 5$. There is a maximal packing of $K_{v-w,w}$ with $|E(L_2)| = 4$ by Theorem 2.3 and K_{v-w} where $v - w \equiv 1 \pmod 5$ is decomposable [1]. Therefore, there is a maximal packing of $K(v, w)$ has leave L where $|E(L)| = |E(K(v, w))| \pmod 5 = 4$.

Case 5. Suppose $v \equiv 1 \pmod 5$ and $w \equiv 2 \pmod 5$, or $v \equiv 1 \pmod 5$ and $w \equiv 4 \pmod 5$. Consider $K(v, w) = K_{v-w} \cup K_{v-w,w}$, as in Case 2, where $v - w \equiv 4 \pmod 5$ and $w \equiv 2 \pmod 5$, or $v - w \equiv 2 \pmod 5$ and $w \equiv 4 \pmod 5$. There is a maximal packing of K_{v-w} where $v - w \equiv 4 \pmod 5$ or $v - w \equiv 2 \pmod 5$ with $|E(L_1)| = 1$ by Theorem 2.1 and there is a maximal packing of $K_{v-w,w}$ with $|E(L_2)| = 3$ by Theorem 2.3. Therefore, there is a maximal packing of $K(v, w)$ has leave L where $|E(L)| = |E(K(v, w))| \pmod 5 = 4$.

Case 6. Suppose $v \equiv 1 \pmod 5$ and $w \equiv 3 \pmod 5$. Consider $K(v, w) = K_{v-w+1} \cup K_{v-w,w-1}$ where $V(K_{v-w+1}) = V_{v-w} \cup \{w_1\}$ and the partite sets of $K_{v-w,w-1}$

are $V_{v-w} \cup \{w_1\}$ and $V_w \setminus \{w_1\}$. Then there is a maximal packing of K_{v-w+1} with leave L_1 where $|E(L_1)| = 1$ by Theorem 2.1 and there is a maximal packing of $K_{v-w,w-1}$ with leave L_2 where $|E(L_2)| = 1$ by Theorem 2.3. Therefore, there is a maximal packing of $K(v, w)$ has leave L where $|E(L)| = |E(K(v, w))| \pmod 5 = 2$.

Case 7. Suppose $v \equiv 2 \pmod 5$ and $w \equiv 0 \pmod 5$, or $v \equiv 4 \pmod 5$ and $w \equiv 0 \pmod 5$. Consider $K(v, w) = K_{v-w} \cup K_{v-w,w}$, as in Case 2, where $v - w \equiv 2 \pmod 5$ and $w \equiv 0 \pmod 5$, or $v - w \equiv 4 \pmod 5$ and $w \equiv 0 \pmod 5$. There is a maximal packing of K_{v-w} where $v - w \equiv 2 \pmod 5$ or $v - w \equiv 4 \pmod 5$ with $|E(L_1)| = 1$ by Theorem 2.1 and $K_{v-w,w}$ is decomposable [6]. Therefore, there is a maximal packing of $K(v, w)$ has leave L where $|E(L)| = |E(K(v, w))| \pmod 5 = 1$.

Case 8. Suppose $v \equiv 2 \pmod 5$ and $w \equiv 1 \pmod 5$. When $v - w \geq 11$, consider $K(v, w) = K_{v-w} \cup K_{v-w,w}$, as in Case 2, where $v - w \equiv 1 \pmod 5$, and $w \equiv 1 \pmod 5$. There is a maximal packing of $K_{v-w,w}$ with $|E(L_2)| = 1$ by Theorem 2.3 and K_{v-w} where $v - w \equiv 1 \pmod 5$ is decomposable [1]. Therefore, there is a maximal packing of $K(v, w)$ has leave L where $|E(L)| = |E(K(v, w))| \pmod 5 = 1$.

Case 9. Suppose $v \equiv 2 \pmod 5$ and $w \equiv 3 \pmod 5$ or $v \equiv 3 \pmod 5$ and $w \equiv 1 \pmod 5$. Consider $K(v, w) = K_{v-w} \cup K_{v-w,w}$, as in Case 2, where $v - w \equiv 4 \pmod 5$ and $w \equiv 3 \pmod 5$, or $v - w \equiv 2 \pmod 5$ and $w \equiv 1 \pmod 5$. There is a maximal packing of K_{v-w} where $v - w \equiv 4 \pmod 5$ or $v - w \equiv 2 \pmod 5$ with $|E(L_1)| = 1$ by Theorem 2.1 and there is a maximal packing of $K_{v-w,w}$ with $|E(L_2)| = 2$ by Theorem 2.3. Therefore, there is a maximal packing of $K(v, w)$ has leave L where $|E(L)| = |E(K(v, w))| \pmod 5 = 3$.

Case 10. Suppose $v \equiv 3 \pmod 5$ and $w \equiv 0 \pmod 5$. Consider $K(v, w) =$

$K_{v-w} \cup K_{v-w,w}$, as in Case 2, where $v - w \equiv 3 \pmod 5$, and $w \equiv 0 \pmod 5$. There is a maximal packing of K_{v-w} where $v - w \equiv 3 \pmod 5$ with $|E(L_1)| = 3$ by Theorem 2.1 and $K_{v-w,w}$ is decomposable [6]. Therefore, there is a maximal packing of $K(v, w)$ has leave L where $|E(L)| = |E(K(v, w))| \pmod 5 = 3$.

Case 11. Suppose $v \equiv 3 \pmod 5$ and $w \equiv 2 \pmod 5$. Similar to Case 4, when $v - w \geq 11$, consider $K(v, w) = K_{v-w} \cup K_{v-w,w}$ where $v - w \equiv 1 \pmod 5$, and $w \equiv 2 \pmod 5$. There is a maximal packing of $K_{v-w,w}$ with $|E(L_2)| = 2$ by Theorem 2.3 and K_{v-w} where $v - w \equiv 1 \pmod 5$ is decomposable [1]. Therefore, there is a maximal packing of $K(v, w)$ has leave L where $|E(L)| = |E(K(v, w))| \pmod 5 = 2$.

Case 12. Suppose $v \equiv 4 \pmod 5$ and $w \equiv 1 \pmod 5$. As in Case 6, we have $K(v, w) = K_{v-w+1} \cup K_{v-w,w-1}$. Then there is a maximal packing of K_{v-w+1} with leave L_1 where $|E(L_1)| = 1$ by Theorem 2.1 and $K_{v-w,w-1}$ is decomposable [6]. Therefore, there is a maximal packing of $K(v, w)$ has leave L where $|E(L)| = |E(K(v, w))| \pmod 5$ $= 1$.

Case 13. Suppose $v \equiv 4 \pmod 5$ and $w \equiv 3 \pmod 5$. Similar to Case 4, when $v - w \geq 11$, consider $K(v, w) = K_{v-w} \cup K_{v-w,w}$ where $v - w \equiv 1 \pmod 5$, and $w \equiv 3 \pmod 5$. There is a maximal packing of $K_{v-w,w}$ with $|E(L_2)| = 3$ by Theorem 2.3 and K_{v-w} where $v - w \equiv 1 \pmod 5$ is decomposable [1]. Therefore, there is a maximal packing of $K(v, w)$ has leave L where $|E(L)| = |E(K(v, w))| \pmod 5 = 3$. $\square$

For the sake of illustration, as in Case 6, we have $K(v, w) = K_{v-w+1} \cup K_{v-w,w-1}$. Then there is a maximal packing of K_{v-w+1} with leave L_1 where $|E(L_1)| = 1$ by Theorem 2.1 and there is a maximal packing of $K_{v-w,w-1}$ with leave L_2 where $|E(L_2)| = 1$ by Theorem 2.3. Therefore, there is a maximal packing of $K(v, w)$ has leave L where

$|E(L)| = |E(K(v,w))| \pmod 5) = 2$. See Figure 8 for an H packing of $K(v,w)$.

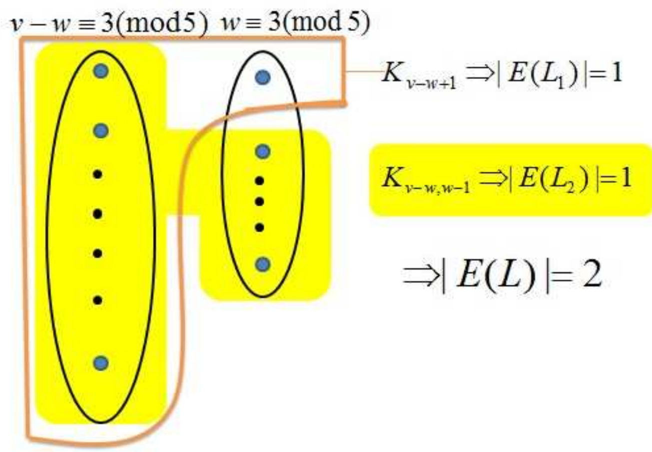

Figure 8: An H-packing of $K(v,w)$ when $v \equiv 1 \pmod 5$ and $w \equiv 3 \pmod 5$

3.2 Restricted Covering $K(v,w)$

As in the previous section, we assume the vertex set of $K(v,w)$ is $V(K(v,w)) = V_{v-w} \cup V_w$, where $V_{v-w} = \{0_0, 1_0, \ldots, (v-w-1)_0\}$ and $V_w = \{0_1, 1_1, \ldots, (w-1)_1\}$.

Theorem 3.2 *A minimal restricted H-covering of $K(v,w)$ has padding P where $|E(P)| = -|E(K(v,w))| (mod\ 5)$ when $v - w > 2$.*

Proof. First, suppose $v - w = 2$. Consider the edge $(0_0, 1_0)$. If $(0_0, 1_0)$ is the pendant edge of an H, say $H = [0_0, a, b, c; 1_0]$, then $0_0, 1_0$, and b must be distinct vertices in V_{v-w}. But $|V_{v-w}| = 2$, so this cannot happen. If $(0_0, 1_0)$ is an edge in the 4-cycle of some of H, then there must be an edge in the 4-cycle of the form (a_1, b_1), a contradiction to the restricted covering. So, $v - w > 2$ is necessary.

Similar to the argument in Theorem 2.2, an H-covering of $K(v,w)$ with padding P where $|E(P)| = -|E(K(v,w)|$ (mod 5) would be minimal.

Case 1. Suppose $v \equiv 0 \pmod 5$ and $w \equiv 2 \pmod 5$. First, $K(5,2)$ can be covered with $\{[0_0, 0_1, 2_0, 1_0; 1_1], [2_0, 0_1, 1_0, 1_1; 0_0]\}$ and this has a padding P with $E(P) = \{(2_0, 0_1)\}$ and so $|E(P)| = 1$. In general, $K(v,w) = K(5,2) \cup K_{v-w-3,3} \cup K_{v-w,w-2}$ where the partite sets of $K(5,2)$ are $\{0_0, 1_0, 2_0\}$ and $\{0_1, 1_1\}$, the partite sets of $K_{v-w-3,3}$ are $\{3_0, 4_0, ..., (v-w-1)_0\}$ and $\{0_0, 1_0, 2_0\}$, and the partite sets of $K_{v-w,w-2}$ are V_{v-w} and $\{2_1, 3_1, ..., (w-1)_1\}$. Now, $K_{v-w-3,3}$ and $K_{v-w,w-2}$ can be decomposed [6]. Taking these decompositions along with the above covering of $K(5,2)$ yields a covering of $K(v,w)$ with padding P where $E(P) = \{(2_0, 0_1)\}$ and so $|E(P)| = 1$.

Case 2. Suppose $v \equiv 0 \pmod 5$ and $w \equiv 3 \pmod 5$, or $v \equiv 3 \pmod 5$ and $w \equiv 4 \pmod 5$. Consider $K(v,w) = K_{v-w} \cup K_{v-w,w}$, where $V(K_{v-w})$=V_{v-w} and the partite sets of $K_{v-w,w}$ are V_{v-w} and V_w, and $v - w \equiv 2 \pmod 5$ and $w \equiv 3 \pmod 5$, or $v - w \equiv 4 \pmod 5$ and $w \equiv 4 \pmod 5$. There is a maximal packing of K_{v-w} where $v - w \equiv 2 \pmod 5$ or $v - w \equiv 4 \pmod 5$ with $|E(L_1)| = 1$ by Theorem 2.1. There is a maximal packing of $K_{v-w,w}$ with $|E(L_2)| = 1$ by Theorem 2.3. Therefore, there is a minimal covering of $K(v,w)$ with padding P where $|E(P)| = 3$.

Case 3. Suppose $v \equiv 0 \pmod 5$ and $w \equiv 4 \pmod 5$. Consider $K(v,w) = K_{v-w} \cup K_{v-w,w}$, as in Case 2, where $v - w \equiv 1 \pmod 5$, and $w \equiv 4 \pmod 5$. There is a minimal covering of $K_{v-w,w}$ with padding P where $|E(P)| = 1$ by Theorem 2.4 and K_{v-w} where $v - w \equiv 1 \pmod 5$ is decomposable [1]. Therefore, there is a minimal covering of $K(v,w)$ with padding P where $|E(P)| = 1$.

Case 4. Suppose $v \equiv 1 \pmod 5$ and $w \equiv 2 \pmod 5$. Consider $K(v,w) =$

$K_{v-w} \cup K_{v-w,w}$, as in Case 2, where $v-w \equiv 4 \pmod 5$ and $w \equiv 2 \pmod 5$. There is a maximal packing of K_{v-w} with leave L_1 where $|E(L_1)| = 1$ by Theorem 2.1. Without loss of generality, assume $E(L_1)= \{(0_0, 2_0)\}$. There is a maximal packing of $K_{v-w,w}$ with leave L_2 where $|E(L_2)| = 3$ and $E(L_2) = \{(2_0, 0_1), (0_1, 3_0), (3_0, 1_1)\}$ by Theorem 2.3. These two packings combined with $\{[2_0, 0_1, 3_0, 1_1; 0_0]\}$ form a covering of $K(v, w)$ with padding P where $|E(P)| = 1$ and $E(P) = \{(2_0, 1_1)\}$.

Case 5. Suppose $v \equiv 1 \pmod 5$ and $w \equiv 4 \pmod 5$. Consider $K(v, w) = K_{v-w} \cup K_{v-w,w}$, as in Case 2, where $v-w \equiv 2 \pmod 5$ and $w \equiv 4 \pmod 5$. There is a maximal packing of K_{v-w} with leave L_1 where $|E(L_1)| = 1$ by Theorem 2.1. Without loss of generality, assume $E(L_1)= \{(0_1, 2_1)\}$. There is a maximal packing of $K_{v-w,w}$ with leave L_2 where $|E(L_2)| = 3$ and $E(L_2) = \{(2_0, 0_1), (0_1, 3_0), (3_0, 1_1)\}$ by Theorem 2.3. These two packings combined with $\{[0_1, 3_0, 1_1, 2_0; 2_1]\}$ form a covering of $K(v, w)$ with padding P where $|E(P)| = 1$ and $E(P) = \{(2_0, 1_1)\}$.

Case 6. Suppose $v \equiv 1 \pmod 5$ and $w \equiv 3 \pmod 5$. Consider $K(v, w) = K_{v-w+1} \cup K_{v-w,w-1}$ where $V(K_{v-w+1})$=$V_{v-w} \cup \{w_1\}$ and the partite sets of $K_{v-w,w-1}$ are $V_{v-w} \cup \{w_1\}$ and $V_w \setminus \{w_1\}$. Then, there is a maximal packing of K_{v-w+1} with leave L_1 where $|E(L_1)| = 1$ by Theorem 2.1, and there is a maximal packing of $K_{v-w,w-1}$ with leave L_2 where $|E(L_2)| = 1$ by Theorem 2.3. Therefore, we can add an additional copy of H which includes the edges in L_1 and L_2. So, there is a minimal covering of $K(v, w)$ with padding P where $|E(P)| = 3$.

Case 7. Suppose $v \equiv 2 \pmod 5$ and $w \equiv 0 \pmod 5$, or $v \equiv 4 \pmod 5$ and $w \equiv 0 \pmod 5$. Consider $K(v, w) = K_{v-w} \cup K_{v-w,w}$, as in Case 2, where $v - w \equiv 2 \pmod 5$ and $w \equiv 0 \pmod 5$, or $v-w \equiv 4 \pmod 5$ and $w \equiv 0 \pmod 5$. There is a minimal covering of K_{v-w} with padding P where $|E(P)| = 4$ by Theorem 2.2 and $K_{v-w,w}$ is decomposable [6]. Therefore, there is a minimal covering of $K(v, w)$ with padding P where $|E(P)| = 4$.

Case 8. Suppose $v \equiv 2 \pmod 5$ and $w \equiv 1 \pmod 5$. Consider $K(v, w) = K_{v-w} \cup K_{v-w,w}$, as in Case 2, where $v - w \equiv 1 \pmod 5$, and $w \equiv 1 \pmod 5$. There is a minimal covering of $K_{v-w,w}$ with padding P where $|E(P)| = 4$ by Theorem 2.4 and K_{v-w} where $v - w \equiv 1 \pmod 5$ is decomposable [1]. Therefore, there is a minimal covering of $K(v, w)$ with padding P where $|E(P)| = 4$.

Case 9. Suppose $v \equiv 2 \pmod 5$ and $w \equiv 3 \pmod 5$. Consider $K(v, w) = K_{v-w} \cup K_{v-w,w}$, as in Case 2, where $v - w \equiv 4 \pmod 5$ and $w \equiv 3 \pmod 5$. There is a maximal packing of K_{v-w} with leave L_1 where $|E(L_1)| = 1$ by Theorem 2.1 and, without loss of generality, $E(L_1)= \{(0_0, 1_0)\}$. There is a maximal packing of $K_{v-w,w}$ with leave L_2 where $|E(L_2)| = 2$ and $E(L_2) = \{(0_0, 3_1), (2_0, 0_1)\}$ by Theorem 2.3. These two packings combined with $\{[0_0, 0_1, 2_0, 3_1; 1_0]\}$ form a covering of $K(v, w)$ with padding P where $|E(P)| = 2$ and $E(P) = \{(0_0, 0_1), (2_0, 3_1)\}$.

Case 10. Suppose $v \equiv 3 \pmod 5$ and $w \equiv 1 \pmod 5$. Consider $K(v, w) = K_{v-w} \cup K_{v-w,w}$, as in Case 2, where $v - w \equiv 2 \pmod 5$ and $w \equiv 1 \pmod 5$. There is a maximal packing of K_{v-w} with leave L_1 where $|E(L_1)| = 1$ by Theorem 2.1. Without loss of generality, assume $E(L_1)= \{(1_0, 2_0)\}$. There is a maximal packing of $K_{v-w,w}$ with leave L_2 where $|E(L_2)| = 2$ and $E(L_2) = \{(2_0, 1_1), (3_0, 0_1)\}$ by Theorem 2.3. These two packings combined with $\{[2_0, 0_1, 3_0, 1_1; 1_0]\}$ form a covering of $K(v, w)$ with padding P where $|E(P)| = 2$ and $E(P) = \{(2_0, 0_1), (3_0, 1_1)\}$.

Case 11. Suppose $v \equiv 3 \pmod 5$ and $w \equiv 0 \pmod 5$. Consider $K(v, w) =$

$K_{v-w} \cup K_{v-w,w}$, as in Case 2, where $v-w \equiv 3 \pmod 5$, and $w \equiv 0 \pmod 5$. There is a minimal covering of K_{v-w} with padding P where $|E(P)| = 2$ by Theorem 2.2 and $K_{v-w,w}$ is decomposable [6]. Therefore, there is a minimal covering of $K(v,w)$ with padding P where $|E(P)| = 2$.

Case 12. Suppose $v \equiv 3 \pmod 5$ and $w \equiv 2 \pmod 5$. Consider $K(v,w) = K_{v-w} \cup K_{v-w,w}$, as in Case 2, where $v-w \equiv 1 \pmod 5$, and $w \equiv 2 \pmod 5$. There is a minimal covering of $K_{v-w,w}$ with padding P where $|E(P)| = 3$ by Theorem 2.4 and K_{v-w} where $v-w \equiv 1 \pmod 5$ is decomposable [1]. Therefore, there is a minimal covering of $K(v,w)$ with padding P where $|E(P)| = 3$.

Case 13. Suppose $v \equiv 4 \pmod 5$ and $w \equiv 1 \pmod 5$. As in Case 6, we have $K(v,w) = K_{v-w+1} \cup K_{v-w,w-1}$. Then there is a minimal covering of K_{v-w+1} with padding P where $|E(P)| = 4$ by Theorem 2.2 and $K_{v-w,w-1}$ is decomposable [6]. Therefore, there is a minimal covering of $K(v,w)$ with padding P where $|E(P)| = 4$.

Case 14. Suppose $v \equiv 4 \pmod 5$ and $w \equiv 3 \pmod 5$. Consider $K(v,w) = K_{v-w} \cup K_{v-w,w}$, as in Case 2, where $v-w \equiv 1 \pmod 5$, and $w \equiv 3 \pmod 5$. There is a minimal covering of $K_{v-w,w}$ with padding P where $|E(P)| = 2$ by Theorem 2.4 and K_{v-w} where $v-w \equiv 1 \pmod 5$ is decomposable [1]. Therefore, there is a minimal covering of $K(v,w)$ with padding P where $|E(P)| = 2$. $\square$

For the sake of illustration, as in Case 1, we have $K(v,w) = K(5,2) \cup K_{v-w-3,3} \cup K_{v-w,w-2}$. First, $K(5,2)$ can be covered with $\{[0_0, 0_1, 2_0, 1_0; 1_1], [2_0, 0_1, 1_0, 1_1; 0_0]\}$ and this has a padding P with $E(P) = \{(2_0, 0_1)\}$ and so $|E(P)| = 1$. $K_{v-w-3,3}$ and $K_{v-w,w-2}$ can be decomposed [6]. Taking these decompositions along with the above covering of $K(5,2)$ yields a covering of $K(v,w)$ with padding P where $E(P) =$

$\{(2_0, 0_1)\}$ and so $|E(P)| = 1$. See Figure 9 for a restricted H-covering of $K(v, w)$.

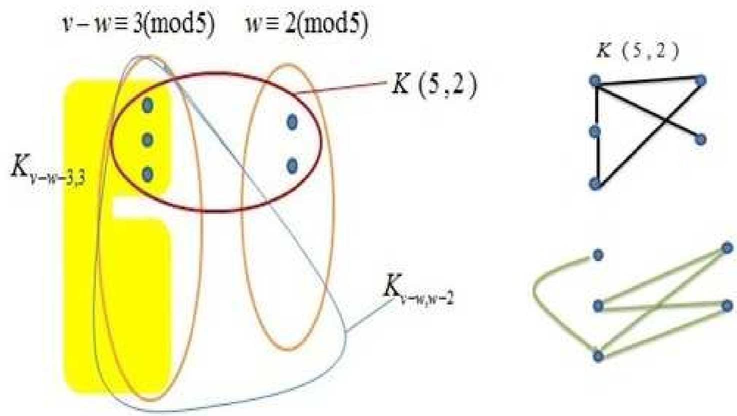

Figure 9: A restricted H-covering of $K(v, w)$ when $v \equiv 0 \pmod 5$ and $w \equiv 2$ (mod 5)

3.3 Unrestricted Covering $K(v, w)$

We assume the same vertex set for $K(v, w)$ as given previously.

Theorem 3.3 *A minimal unrestricted H-covering of $K(v, w)$ has padding P where*
(1) when $v - w > 2$, $|E(P)| = -|E(K(v, w))|$ $(mod\ 5)$,
(2) when $v - w = 1$, $|E(P)| = (2/3)w$ for $w \equiv 0$ $(mod\ 3)$, $|E(P)| = 2(w + 5)/3$ for $w \equiv 1$ $(mod\ 3)$, $|E(P)| = (2w + 5)/3$ for $w \equiv 2$ $(mod\ 3)$, and
(3) when $v - w = 2$, $|E(P)| = 5 - \ell$ where $\ell = |E(K(v, w)|$ $(mod\ 5)$ for $v \neq 6$, and $|E(P)| = 6$ for $v = 6$.

Proof. When $v - w > 2$, the necessary and sufficient conditions follow from Theorem 3.2. When $v - w = 1$, $K(v, w) \cong K_{1,w}$ and the necessary and sufficient conditions follow from Theorem 2.5.

When $v - w = 2$, similar to the argument in Theorem 2.2, an H-covering of $K(v, w)$ with padding P must satisfy $|E(P)| \equiv -|E(K(v, w)|$ (mod 5). Since an H-decomposition of $K(v, w)$ does not exist for $w \equiv 2$ (mod 5) [6], the necessary conditions follow for $v - w = 2$ and $v \neq 6$. For $v = 6$, since $|E(K(6, 4))| = 9$, then an unrestricted H-covering of $K(6, 4)$ with padding P where $|E(P)| = 1$ would be minimal. However, in such a covering, there are only two copies of H. Edge $(0_0, 1_0)$ cannot be the pendant edge of a copy of H in such a covering since this copy would have 2 edges in the padding. If edge $(0_0, 1_0)$ is in a copy of H and is not the pendant edge, then this copy of H must be of the form $[0_0, 1_0, a_1, b_1; c_1]$ for some distinct $a_1, b_1, c_1 \in \{0_1, 1_1, 2_1, 3_1\}$. However, the complement of this graph in $K(6, 4)$ is not a copy of H. Therefore, no such H-covering of $K(6, 4)$ exists, and a minimal unrestricted H-covering of $K(6, 4)$ with padding P where $|E(P)| = 6$ would be minimal. The set $\{[1_0, 1_1, 0_1, 0_0; 2_1], [0_0, 2_1, 1_0, 3_1; 1_1], [1_0, 3_1, 0_0, 2_1; 0_1]\}$ is an unrestricted H-covering of $K(6, 4)$ with padding P where $E(P) = \{(0_1, 1_1), (1_0, 2_1), (0_0, 2_1), (1_0, 2_1), (1_0, 3_1), (0_0, 3_1)\}$. So $|E(P)| = 6$ and the covering is minimal.

Case 1. Suppose $v - w = 2$, and $w \equiv 0$ (mod 5), $w \geq 5$. Then $K(v, w) = K(7, 5) \cup K_{2,w-5}$ where the vertex set of $K(7, 5)$ is $\{0_0, 1_0, 0_1, 1_1, 2_1, 3_1, 4_1\}$ and the hole is on vertex set $\{0_1, 1_1, 2_1, 3_1, 4_1\}$, and the partite sets of $K_{2,w-5}$ are $\{0_0, 1_0\}$ and $\{5_1, 6_1, \ldots, (w - 1)_1\}$. There is an H-decomposition of $K_{2,w-5}$ [6], and the set $\{[1_0, 1_1, 0_1, 0_0; 2_1], [0_0, 3_1, 1_0, 4_1; 2_1], [1_1, 0_1, 1_0, 2_1; 0_0]\}$ is an unrestricted H-covering of $K(7, 5)$ with padding P where $E(P) = \{(0_1, 1_1), (0_1, 1_1), (1_1, 2_1), (1_0, 2_1)\}$ and $|E(P)| = 4$. So, there is an unrestricted covering of $K(v, w)$ with padding P where

$|E(P)| = 4$.

Case 2. Suppose $v - w = 2$, and $w \equiv 1 \pmod 5$, $w \geq 6$. Then, as in Case 1, $K(v,w) = K(8,2) \cup K_{2,w-6}$. There is an H-decomposition of $K_{2,w-6}$ [6], and the set $\{[1_0, 1_1, 0_1, 0_0; 2_1], [1_0, 3_1, 0_0, 4_1; 0_1], [0_1, 2_1, 1_0, 5_1; 1_1]\}$ is an unrestricted H-covering of $K(8,6)$ with padding P where $E(P) = \{(0_1, 1_1), (1_0, 2_1)\}$ and $|E(P)| = 2$. So, there is an unrestricted covering of $K(v,w)$ with padding P where $|E(P)| = 2$.

Case 3. Suppose $v - w = 2$, and $w \equiv 2 \pmod 5$, $w \geq 7$. Then, as in Case 1, $K(v,w) = K(9,7) \cup K_{2,w-7}$. There is an H-decomposition of $K_{2,w-7}$ [6], and the set $\{[1_0, 1_1, 0_1, 0_0; 2_1], [1_0, 3_1, 0_0, 4_1; 0_1], [0_1, 2_1, 1_0, 5_1; 1_1], [0_0, 5_1, 1_0, 6_1; 0_1]\}$ is an unrestricted H-covering of $K(9,7)$ with padding P where $E(P) = \{(0_1, 1_1), (1_0, 2_1), (0_0, 0_1), (0_0, 5_1), (1_0, 5_1)\}$ and $|E(P)| = 5$. So, there is an unrestricted covering of $K(v,w)$ with padding P where $|E(P)| = 5$.

Case 4. Suppose $v - w = 2$, and $w \equiv 3 \pmod 5$. Then, as in Case 1, $K(v,w) = K(5,3) \cup K_{2,w-3}$. There is an H-decomposition of $K_{2,w-3}$ [6], and the set $\{[1_0, 0_0, 0_1, 1_1; 2_1], [0_1, 2_1, 0_0, 1_1; 1_0]\}$ is an unrestricted H-covering of $K(5,3)$ with padding P where $E(P) = \{(0_1, 1_1), (0_1, 1_1), (0_1, 2_1)\}$ and $|E(P)| = 3$. So, there is an unrestricted covering of $K(v,w)$ with padding P where $|E(P)| = 3$.

Case 5. Suppose $v - w = 2$, and $w \equiv 4 \pmod 5$, $w \geq 9$. Then, as in Case 1, $K(v,w) = K(11,9) \cup K_{2,w-9}$. There is an H-decomposition of $K_{2,w-9}$ [6], and the set $\{[1_0, 1_1, 0_1, 0_0; 2_1], [0_0, 7_1, 1_0, 8_1; 1_1], [0_0, 5_1, 1_0, 6_1; 2_1], [0_0, 3_1, 1_0, 4_1; 0_1]\}$ is an unrestricted covering of $K(11,9)$ with padding P where $E(P) = \{(0_1, 1_1)\}$ and $|E(P)| = 1$. So, there is an unrestricted covering of $K(v,w)$ with padding P where $|E(P)| = 1$. □

See Figure 10 for an unrestricted H-covering of $K(v, w)$ as in Case 5. As an illustration of Case 5, we notice that $K(v, w) = K(11, 9) \cup K_{2,w-9}$, as given in Figure 10. The colored copies of H in Figure 10 illustrate the H-covering of $K(11, 9)$. $K_{2,w-9}$ is decomposable as mentioned above.

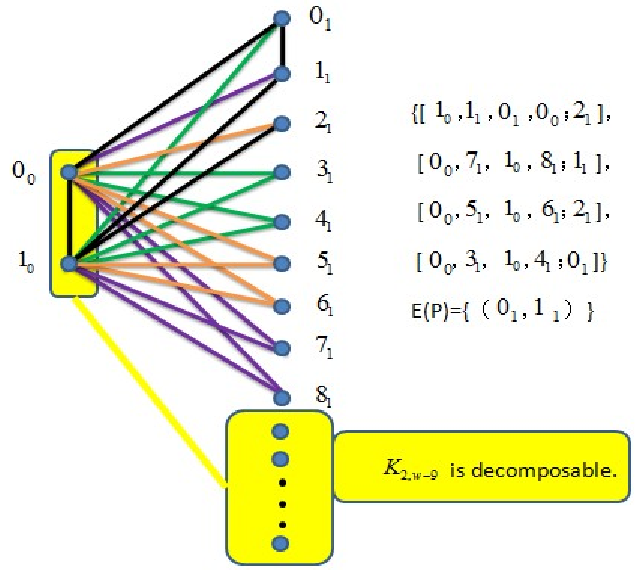

Figure 10: An unrestricted H-covering of $K(v, w)$ when $v - w = 2$ and $w \equiv 4 \pmod 5$, $w \geq 9$

4 CONCLUSION

Motivated by experimental designs and comparisons of samples, we have given necessary and sufficient conditions for H-packings and H-coverings of complete graphs, complete bipartite graphs, and complete graphs with a hole, where H is a 4-cycle with a pendant edge. For complete bipartite graphs and complete graphs with a hole, we have given both restricted and unrestricted coverings. In summary we have:

Theorem 2.1 *A maximal H-packing of K_v, $v \geq 5$, has leave L where $|E(L)| = |E(K_v)|$ (mod 5), except when $v \in \{5, 6\}$ in which cases $|E(L)| = 5$.*

Theorem 2.2 *A minimal H-covering of K_v, $v \geq 5$, has padding P where $|E(P)| = -|E(K_v)|$ (mod 5), except when $v \in \{5, 6\}$ in which cases $|E(P)| = 5$.*

Theorem 2.3 *A maximal H-packing of $K_{m,n}$ has leave L where*

(1) $|E(L)| = mn$ if $m = 1$ or $n = 1$, or if $m = n = 2$, or

(2) $|E(L)| = |E(K_{m,n})|$ (mod 5) otherwise.

Theorem 2.4 *A minimal restricted H-covering of $K_{m,n}$, where neither m nor n equals 1 and $m + n \geq 5$, has padding P where $|E(P)| = -|E(K_{m,n})|$ (mod 5).*

Theorem 2.5 *A minimal unrestricted H-covering of $K_{m,n}$ has padding P where*

(1) when $m > 1$, and $n > 1$, $|E(P)| = -|E(K_{m,n})|$(mod 5), and

(2) when $m = 1$, $|E(P)| = (2/3)n$ for $n \equiv 0$ (mod 3), $|E(P)| = 2(n + 5)/3$ for $n \equiv 1$ (mod 3), $|E(P)| = (2n + 5)/3$ for $n \equiv 2$ (mod 3).

Theorem 3.1 *A maximal H-packing of $K(v, w)$ has leave L where $|E(L)| = |E(K(v, w))|$ (mod 5), and $v - w \geq 2$ is necessary.*

Theorem 3.2 *A minimal restricted H-covering of $K(v, w)$ has padding P where $|E(P)| = -|E(K(v, w))|$(mod 5) when $v - w > 2$.*

Theorem 3.3 *A minimal unrestricted H-covering of $K(v, w)$ has padding P where*

(1) *when $v - w > 2$, $|E(P)| = -|E(K(v, w))|$ (mod 5),*

(2) *when $v - w = 1$, $|E(P)| = (2/3)w$ for $w \equiv 0$ (mod 3), $|E(P)| = 2(w + 5)/3$ for $w \equiv 1$ (mod 3), $|E(P)| = (2w + 5)/3$ for $w \equiv 2$ (mod 3), and*

(3) *when $v - w = 2$, $|E(P)| = 5 - \ell$ where $\ell = |E(K(v, w)|$ (mod 5) for $v \neq 6$, and $|E(P)| = 6$ for $v = 6$.*

We see future research concentrating on packings and coverings of various complete graphs with other "small" graphs. In particular, existing research on restricted versus unrestricted coverings is still limited.

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