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Packings and Coverings of Various Complete Digraphs with the Orientations of a 4-Cycle.

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Packings and Coverings of Various Complete Digraphs with the Orientations of a 4−Cycle

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by

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ABSTRACT

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by

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There are four orientations of cycles on four vertices. Necessary and sufficient conditions are given for covering complete directed digraphs D_v, packing and covering complete bipartite digraphs, $D_{m,n}$, and packing and covering the complete digraph on v vertices with hole of size w, $D(v, w)$, with three of the orientations of a 4-cycle, including C_4, X, and Y.

DEDICATION

This paper is dedicated to my parents, Ronnie and Audrey Cooper, for being the best cheerleaders in the world! From kindergarten to graduate school, they have always pushed me to do my best. Without their constant encouragement and support, I truly believe I would not have made it this far. To my mom, your Melzy Welzy finally made it. To my dad, thanks for helping find the light at the end of the tunnel. I love you!

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1 INTRODUCTION

Within the field of combinatorial mathematics exists a branch of mathematics known as design theory. Design theory can be used in professions related to computer science, telecommunications, traffic management, and environmental conservation [2]. More specifically, within design theory we can study decompositions, packings, and coverings of graphs.

A *graph* is a mathematical representation of a relationship. There are finite and infinite graphs. In this paper we only consider finite graphs. That is, a graph G consists of two sets—a nonempty finite set V of *vertices* and a finite set E of *edges* consisting of unordered pairs of distinct vertices from V. There also exist directed graphs D called *digraphs*. In these digraphs, we no longer have edges but instead we have *arcs* that are assigned a direction. In a non-directed graph, two vertices are *adjacent* if they have an edge in common. A graph on v vertices in which every vertex is adjacent to every other vertex is a *complete graph on v vertices* and is denoted K_v. For directed complete graphs, D_v, we replace each edge with two arcs of opposite orientations. The degree of a vertex, v, is defined as the number of edges adjacent to v. In directed graphs, each vertex has an *out degree*, $od(v)$, which is the number of vertices that v is adjacent to and *in degree*, $id(v)$, which is the number of vertices that v is adjacent from. Equivalently, $od(v)$ is the number of arcs that point away from v and $id(v)$ is the number of arcs that point toward v. The *total degree* of a vertex in a directed graph is $od(v)+id(v)$. In this paper, if we have two graphs, G and H, $G \cup H$ is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$.

A *decomposition* of a simple graph H with isomorphic copies of a graph G is a set

$\{G_1, G_2, \ldots, G_n\}$ where $G_i \cong G$ and $V(G_i) \subset V(H)$ for all i, $E(G_i) \bigcap E(G_j) = \emptyset$ if $i \neq j$, $\bigcup_{i=1}^{n} G_i \subset H$. The vertex set of a graph G is denoted $V(G)$ and the edge set of graph G is denoted $E(G)$. Figure 1 is an example of a basic graph decomposition. In Figure 1, we have decomposed K_5 into two 5-cycles.

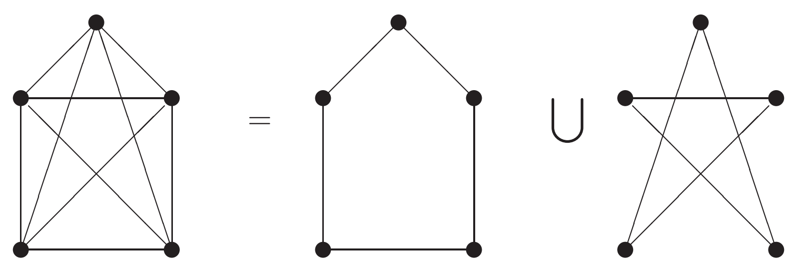

Figure 1: Decomposition of a K_5 into $5-$cycles

The first decompositions studied were the *Steiner triple systems* of order v [9], which are decompositions of K_v into isomorphic copies of $C_3's$. The $C_3's$ are non-oriented three cycles. We denote these as $STS(v)$. These systems exist if and only if $v \equiv 1$ or 3 (mod 6). Decompositions also exist for directed digraphs. However, instead of having an edge set, $E(G)$, we have an arc set $A(G)$. So orientations were given to the edges of the $C_3's$. There are two orientations of C_3, the $3-$circuit and the transitive triple as depicted in Figure 2.

This led to the study of *Mendelsohn triple systems* of order v, *MTS(v)* [10], which are decompositions of D_v into $3-$circuits, and *directed triple system* or order v, *DTS(v)*, which are decompositions of D_v into *transitive triples* [7]. A *MTS(v)* exists if and only if $v \equiv 0$ or 1 (mod 3), $v \neq 6$ [10]. A *DTS(v)* exists if and only if

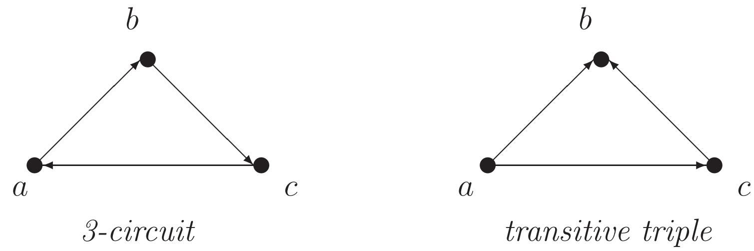

Figure 2: Orientations of a 3−cycle

$v \equiv 0$ or 1 (mod 3) [7].

Then several other directed and non-directed graphs were decomposed, among those the 4-cycle. In Figure 3, notice that there are 4 orientations of a 4-cycle.

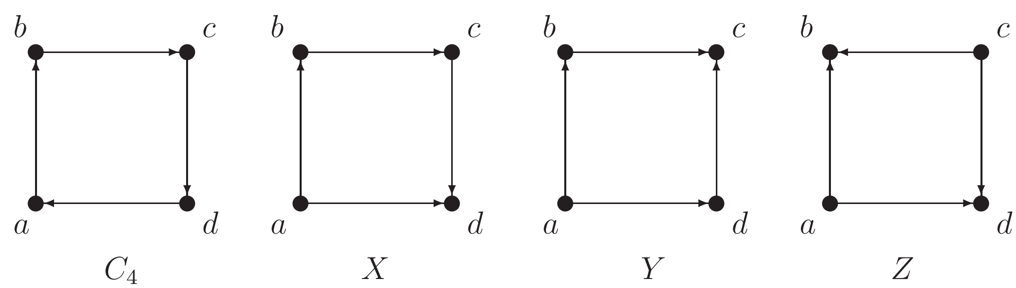

Figure 3: The Orientations of a 4-cycle.

These digraphs are denoted in the following way: $[a,b,c,d]_C$, $[a,b,c,d]_X$, $[a,b,c,d]_Y$, and $[a,b,c,d]_Z$ respectively. In this paper, we will only be considering C_4, X, and Y.

The following are the results of the decompositions of each of the orientations of the four cycle:

An X- decomposition of D_v exists if and only if $v \equiv 0$ or 1 (mod 4), $v \neq 5$ [6];

A Y-decomposition of D_v exists if and only if $v \equiv 0$ or 1 (mod 4), $v \notin \{4, 5\}$ [6];

A Z-decomposition of D_v exists if and only if $v \equiv 1$ (mod 4) [6]; and

A C_4-decomposition of D_v exists if and only if $v \equiv 0$ or $1 (\mathrm{mod}\ 4)$, $v \neq 4$ [12].

However, not all graphs can be decomposed. Now consider how close we can get to a graph decomposition. One way is to pack a graph. That is, we remove isomorphic copies of G without removing any repeated arcs until there are no more copies of G left. The arcs that remain are known as the *leave* of the graph. Formally, a *maximal packing* of a simple graph H with isomorphic copies of a graph G is a set $\{G_1, G_2, \ldots, G_n\}$ where $G_i \cong G$ and $V(G_i) \subset V(H)$ for all i, $E(G_i) \bigcap E(G_j) = \emptyset$ if $i \neq j$, $\bigcup_{i=1}^{n} G_i \subset H$, and

$$|E(L)| = \left| E(H) \setminus \bigcup_{i=1}^{n} E(G_i) \right|$$

is minimal. The set L is the *leave* of the packing. Figure 4 is an example of a graph packing. In this figure, we have packed K_5 with $C_3's$.

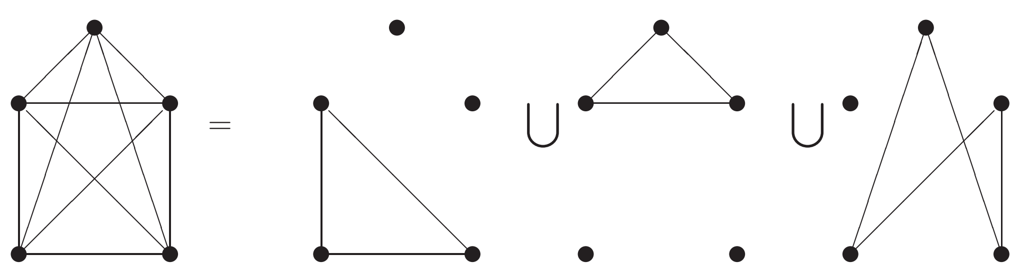

Figure 4: Packing of a K_5 with 3-cycles ($C_3's$)

Packings of the complete graph on v vertices of K_v, with a graph G, have been studied when G is a 3-cycle [11], a 4-cycle [12], a K_4[1] , and a 6-cycle [8]. Packings of the orientations of a 3-cycle, that is *DTS(v)* and *MTS(v)* have also been studied

[4].

We also know the packings of the orientations of a 4-cycle. They are as follows:

Theorem 1.1 *An optimal packing of D_v with copies of C_4 and leave L satisfies:*

1. $L=\emptyset$ *if* $v \equiv 0$ *or 1* $(mod\ 4)$, $v \neq 4$,

2. $|A(L)|=4$ *if* $v=4$,

3. $L=D_2$ *if* $v \equiv 2$ *or* $3\ (mod\ 4)$[5].

Theorem 1.2 *An optimal packing of D_v with copies of X and leave L satisfies:*

1. $L=\emptyset$ *if* $v \equiv 0$ *or* $1\ (mod\ 4)$, $v \neq 5$,

2. $|A(L)|=4$ *if* $v=5$,

3. $L=D_2$ *if* $v \equiv 2$ *or* $3\ (mod\ 4)$[5].

Theorem 1.3 *An optimal packing of D_v with copies of Y and leave L satisfies:*

1. $L=\emptyset$ *if* $v \equiv 0$ *or* $1\ (mod\ 4)$, $v \notin \{4,5\}$,

2. $|A(L)|=4$ *if* $v \in \{4,5\}$,

3. $L=D_2$ *if* $v \equiv 2$ *or* $3\ (mod\ 4)$[5].

Theorem 1.4 *An optimal packing of D_v with copies of Z and leave L satisfies:*

1. $L=\emptyset$ *if* $v \equiv 1\ (mod\ 4)$,

2. $|A(L)|=v$ *and the arcs of L are arranged in a collection of disjoint circuits if* $v \equiv 0$ *or* $2\ (mod\ 4)$,

3. $|A(L)|=6$ *and the arcs of L are arranged in such a way that each vertex of the leave has in-degree = out-degree* $\equiv 0\ (mod\ 2)$ *and* $v \equiv 3\ (mod\ 4)$[5].

Another option, if a graph cannot be decomposed, is to *cover* the graph. That is, we cover the graph with isomorphic copies of G until all arcs are covered. The arcs that are repeated are known as the *padding* of the graph. Formally, a *minimal covering* of a simple graph H with isomorphic copies of a graph G is a set $\{G_1, G_2, \ldots, G_n\}$ where $G_i \cong G$ and $V(G_i) \subset V(H)$ and $E(G_i) \subset E(H)$ for all i, $H \subset \bigcup_{i=1}^{n} G_i$, and

$$|E(P)| = \left| \bigcup_{i=1}^{n} E(G_i) \setminus E(H) \right|$$

is minimal. However, the graph $\bigcup_{i=1}^{n} G_i$ may not be simple and $\bigcup_{i=1}^{n} E(G_i)$ may be a multiset. The graph P is the *padding* of the covering. We can see what a graph covering is in Figure 5. In this figure, we cover K_5 with copies of C_3.

Coverings of the complete graph on v vertices of, K_v with a graph G have been studied when G is a 3-cycle [3], a 4-cycle [13], and a 6-cycle. Coverings of the orientations of a 3-cycle [8], that is *DTS(v)* and *MTS(v)*, have been studied as well [4].

In this paper, we will cover complete directed digraphs D_v, pack and cover complete bipartite digraphs, $D_{m,n}$, and pack and cover the complete digraph on v vertices with hole of size w, $D(v, w)$, with three of the orientations of a 4-cycle, including C_4, X, and Y.

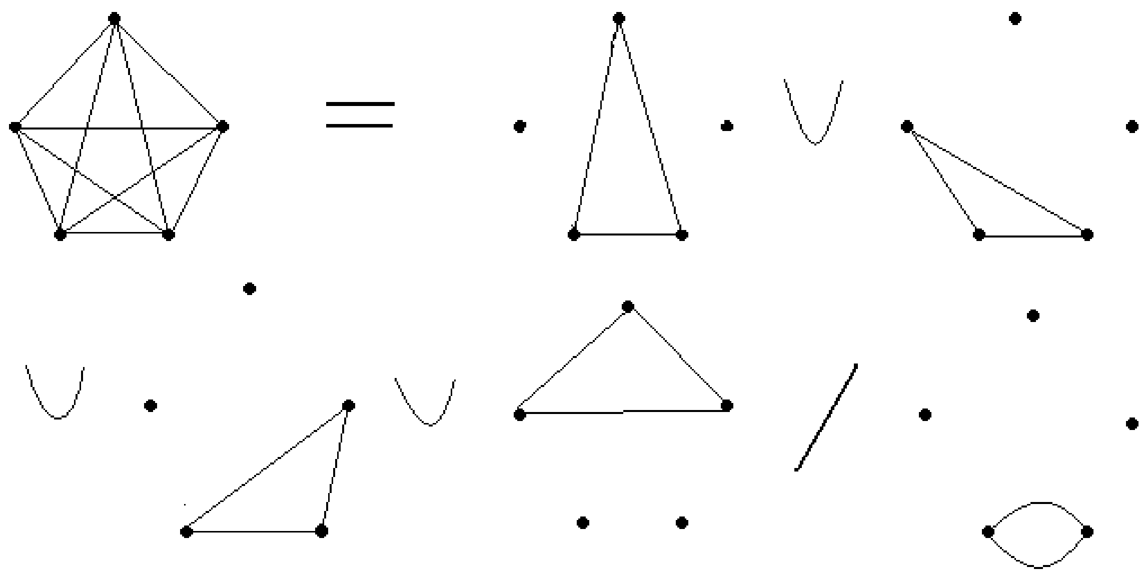

Figure 5: Covering a K_5 with 3−cycles ($C_3's$)

2 PACKING AND COVERING $D_{m,n}$ WITH THE C_4, X, AND Y ORIENTATIONS OF THE 4-CYCLE

A *complete bipartite digraph* is a graph that can be partitioned into two nonempty subsets V_m and V_n, called the partite sets, such that (a, b) is an arc of $D_{m,n}$, if and only if a and b belong to different partitions [2]. We use the notation V_m={0_1, 1_1, ..., $(m-1)_1$} and V_n={0_2, 1_2, ..., $(n-1)_2$}. Decompositions of these graphs have been studied in the past, with the following results, beginning with C_4.

Theorem 2.1 [5] *A* C_4 *decomposition of* $D_{m,n}$ *exists if and only if* $m, n \geq 2$ *and* $mn \equiv 0$ *(mod 2).*

We now want to know the results of packing these graphs.

Theorem 2.2 *A maximal* C_4 *packing of* $D_{m,n}$ *satisfies:*

1. $|A(L)| = 0$ *when* $mn \equiv 0$ *(mod 2),* $m, n \geq 2$*,*
2. $|A(L)| = 2$*, otherwise.*

For clarity, throughout this thesis we consider values of parameter m and n modulo 4. We do these modulo 4 because we are considering orientations of a 4-cycle.

Proof. Therefore, we present sixteen cases. Since $|A(C_4)| = 4$, it is necessary that the leave, L, satisfy $|A(L)| \equiv |A(D_{m,n})|$(mod 4). Therefore, for $m \equiv n \equiv 0$ (mod 4), $|A(L)| \geq 0$ and for the other cases $|A(L)| \geq 2$.

Case 1: If $m \equiv n \equiv 0$ (mod 4), then $mn \equiv 0$ (mod 2). Therefore, by Theorem 2.1, a decomposition exists.

Case 2: If $m \equiv 0$ (mod 4) and $n \equiv 1$ (mod 4), then $mn \equiv 0$ (mod 2). Thus, by Theorem 2.1, a decomposition exists.

Case 3: If $m \equiv 0$ (mod 4) and $n \equiv 2$ (mod 4), then $mn \equiv 0$ (mod 2). Therefore, by Theorem 2.1, a decomposition exists.

Case 4: If $m \equiv 0$ (mod 4) and $n \equiv 3$ (mod 4), then $mn \equiv 0$ (mod 2). Thus, by Theorem 2.1, a decomposition exists.

Case 5: If $m \equiv 1$ (mod 4) and $n \equiv 0$ (mod 4), then $mn \equiv 0$ (mod 2). Therefore, by Theorem 2.1, a decomposition exists.

Case 6: If $m \equiv n \equiv 1$ (mod 4), where $m, n \geq 2$, $D_{m,n}$=$D_{5,5} \cup D_{m-5,5} \cup D_{m,n-5} \cup D_{m-5,n-5}$. Now pack each smaller graph with C_4. For $D_{5,5}$, the partite sets are

$\{0_1, 1_1, 2_1, 3_1, 4_1\}$ and $\{0_2, 1_2, 2_2, 3_2, 4_2\}$. One possible way of packing $D_{5,5}$ so that the leave is minimal is: $\{[0_1, 0_2, 1_1, 1_2]_C, [0_1, 1_2, 2_1, 2_2]_C, [0_1, 2_2, 4_1, 4_2]_C, [0_1, 3_2, 4_1, 0_2]_C, [0_1, 4_2, 1_1, 3_2]_C, [1_1, 0_2, 4_1, 1_2]_C, [1_1, 4_2, 2_1, 3_2]_C, [2_1, 0_2, 3_1, 2_2]_C, [2_1, 1_2, 3_1, 0_2]_C, [2_1, 4_2, 3_1, 3_2]_C, [3_1, 1_2, 4_1, 3_2]_C, [3_1, 4_2, 4_1, 2_2]_C\}$ and $|A(L)|=\{(1_1, 2_2),(2_2, 1_1)\}$. The partite sets for $D_{m-5,5}$ are $\{5_1, 6_1, \ldots, (m-1)_1\}$ and $\{0_2, 1_2, 2_2, 3_2, 4_2\}$. For $D_{m,n-5}$, the partite sets are $\{0_1, 1_1, 2_1, 3_1, 4_1\}$ and $\{5_2, 6_2, \ldots, (n-1)_2\}$. Finally, for $D_{m-5,n-5}$, the partite sets are $\{5_1, 6_1, \ldots, (m-1)_1\}$ and $\{5_2, 6_2, \ldots, (n-1)_2\}$. Since $m \equiv 1 \pmod 4$ and $n \equiv 1 \pmod 4$ and $m, n \geq 2$, then $m-5 \equiv 0 \pmod 2$ and $n-5 \equiv 0 \pmod 2$. Therefore, $D_{m-5,5}$, $D_{m,n-5}$, and $D_{m-5,n-5}$ can be decomposed by Theorem 2.1. Thus, the leave L satisfies $|A(L)|=2$.

Case 7: If $m \equiv 1 \pmod 4$ and $n \equiv 2 \pmod 4$, then $mn \equiv 0 \pmod 2$. By Theorem 2.1, it follows that a decomposition exists.

Case 8: If $m \equiv 1 \pmod 4$ and $n \equiv 3 \pmod 4$, where $m, n \geq 2$, then $D_{m,n}=D_{5,3} \cup D_{m-5,3} \cup D_{m,n-3} \cup D_{m-5,n-3}$. Now pack each smaller graph with C_4. For $D_{5,3}$, the partite sets are $\{0_1, 1_1, 2_1, 3_1, 4_1\}$ and $\{0_2, 1_2, 2_2\}$. One possible way of packing $D_{5,3}$ so that the leave is minimal is: $\{[0_2, 0_1, 1_2, 4_1]_C, [0_1, 0_2, 1_1, 2_2]_C, [1_1, 0_2, 4_1, 1_2]_C, [0_2, 2_1, 2_2, 3_1]_C, [0_2, 3_1, 1_2, 2_1]_C, [2_2, 2_1, 1_2, 0_1]_C, [2_2, 1_1, 1_2, 3_1]_C\}$ and $|A(L)|=\{(4_1, 2_2),(2_2, 4_1)\}$. The partite sets for $D_{m-5,3}$ are $\{5_1, 6_1, \ldots, (m-1)_1\}$ and $\{0_2, 1_2, 2_2\}$. For $D_{m,n-3}$, the partite sets are $\{0_1, 1_1, 2_1, 3_1, 4_1\}$ and $\{3_2, 4_2, \ldots, (n-1)_2\}$. Finally, for $D_{m-5,n-3}$, the partite sets are $\{5_1, 6_1, \ldots, (m-1)_1\}$ and $\{3_2, 4_2, \ldots, (n-1)_2\}$. Since $m \equiv 1 \pmod 4$ and $n \equiv 3 \pmod 4$ and $m, n \geq 2$, then $m-5 \equiv 0 \pmod 2$ and $n-3 \equiv 0 \pmod 2$. Therefore, $D_{m-5,3}$, $D_{m,n-3}$, and $D_{m-5,n-3}$ can be decomposed by Theorem 2.1. Thus, the leave L satisfies $|A(L)|=2$.

Case 9: If $m \equiv 2 \pmod 4$ and $n \equiv 0 \pmod 4$, then $mn \equiv 0 \pmod 2$. Therefore, by Theorem 2.1, a decomposition exists.

Case 10: If $m \equiv 2 \pmod 4$ and $n \equiv 1 \pmod 4$, then $mn \equiv 0 \pmod 2$. By Theorem 2.1, it follows that a decomposition exists.

Case 11: If $m \equiv n \equiv 2 \pmod 4$, then $mn \equiv 0 \pmod 2$. Therefore, by Theorem 2.1, a decomposition exists.

Case 12: If $m \equiv 2 \pmod 4$ and $n \equiv 3 \pmod 4$, then $mn \equiv 0 \pmod 2$. Thus, by Theorem 2.1, a decomposition exists.

Case 13: If $m \equiv 3 \pmod 4$ and $n \equiv 0 \pmod 4$, then $mn \equiv 0 \pmod 2$. By Theorem 2.1, it follows that a decomposition exists.

Case 14: If $m \equiv 3 \pmod 4$ and $n \equiv 1 \pmod 4$, where $m, n \geq 2$, then $D_{m,n}$=$D_{3,5}$ $\cup$ $D_{m-3,5} \cup D_{m,n-5} \cup D_{m-3,n-5}$. Now pack each smaller graph with C_4. For $D_{3,5}$, the partite sets are $\{0_1, 1_1, 2_1\}$ and $\{0_2, 1_2, 2_2, 3_2, 4_2\}$. One possible way of packing $D_{3,5}$ so that the leave is minimal is: $\{[0_1, 0_2, 1_1, 4_2]_C$, $[0_2, 0_1, 1_2, 2_1]_C$, $[1_2, 0_1, 4_2, 1_1]_C$, $[0_1, 2_2, 2_1, 3_2]_C$, $[0_1, 3_2, 1_1, 2_2]_C$, $[2_1, 2_2, 1_1, 0_2]_C$, $[2_1, 1_2, 1_1, 3_2]_C\}$ and $|A(L)|$=$\{(2_1, 4_2),(4_2, 2_1)\}$. The partite sets for $D_{m-3,5}$ are $\{3_1, 4_1, \ldots, (m-1)_1\}$ and $\{0_2, 1_2, 2_2, 3_2, 4_2\}$. For $D_{m,n-5}$, the partite sets are $\{0_1, 1_1, 2_1\}$ and $\{5_2, 6_2, \ldots, (n-1)_2\}$. Finally, for $D_{m-3,n-5}$, the partite sets are $\{3_1, 4_1, \ldots, (m-1)_1\}$ and $\{5_2, 6_2, \ldots, (n-1)_2\}$. Since $m \equiv 3 \pmod 4$ and $n \equiv 1 \pmod 4$ and $m, n \geq 2$, then $m - 3 \equiv 0 \pmod 2$ and $n - 3 \equiv 0 \pmod 2$. Therefore, $D_{m-3,5}$, $D_{m,n-5}$, and $D_{m-3,n-5}$ can be decomposed by Theorem 2.1. Thus, the leave L satisfies $|A(L)|$=2.

Case 15: If $m \equiv 3 \pmod 4$ and $n \equiv 2 \pmod 4$, then $mn \equiv 0 \pmod 2$. Therefore, by Theorem 2.1 a decomposition exists.

Case 16: If $m \equiv n \equiv 3 \pmod 4$, where $m, n \geq 2$, then decompose $D_{m,n}$. That is $D_{m,n}$=$D_{3,3} \cup D_{m-3,3} \cup D_{m,n-3} \cup D_{m-3,n-3}$. Now pack each smaller graph with C_4. For $D_{3,3}$, the partite sets are $\{0_1, 1_1, 2_1\}$ and $\{0_2, 1_2, 2_2\}$. One possible way of packing $D_{3,3}$ so that the leave is minimal is: $\{[0_1, 0_2, 1_1, 1_2]_C, [0_1, 1_2, 2_1, 2_2]_C, [0_1, 2_2, 2_1, 0_2]_C, [1_1, 0_2, 2_1, 1_2]_C\}$ and $|A(L)|$=$\{(1_1, 2_2),(2_2, 1_1)\}$. The partite sets for $D_{m-3,3}$ are $\{3_1, 4_1, \ldots, (m-1)_1\}$ and $\{0_2, 1_2, 2_2\}$. For $D_{m,n-3}$, the partite sets are $\{0_1, 1_1, 2_1\}$ and $\{3_2, 4_2, \ldots, (n-1)_2\}$. Finally, for $D_{m-3,n-3}$, the partite sets are $\{3_1, 4_1, \ldots, (m-1)_1\}$ and $\{3_2, 4_2, \ldots, (n-1)_2\}$. Since $m \equiv 1 \pmod 4$ and $n \equiv 1 \pmod 4$ and $m, n \geq 2$, then $m-3 \equiv 0 \pmod 2$ and $n-3 \equiv 0 \pmod 2$. Therefore, $D_{m-3;3}$, $D_{m;n-3}$, and $D_{m-3,n-3}$ can be decomposed by Theorem 2.1 Thus, the leave L satisfies $|A(L)|$=2. *Q.E.D*

Thus, each case for C_4 satisfies the conditions of the theorem. Decomposing $D_{m,n}$ with copies of X has the following results.

Theorem 2.3 [5] *An X decomposition of $D_{m,n}$ exists if and only if either $m \equiv n \equiv 0 \pmod 2$ or $m \equiv 1 \pmod 2$, $m \geq 3$, and $n \equiv 0 \pmod 4$.*

Now we need to pack $D_{m,n}$ with copies of X.

Theorem 2.4 *A maximal X packing of $D_{m,n}$ satisfies:*

1. *$|A(L)| = 0$ when $m \equiv n \equiv 0 \pmod 2$ or $m \equiv 1 \pmod 2$, $m \geq 3$,*
2. *$|A(L)| = 2$, when $m \equiv 1 \pmod 2$ and $n \equiv 1 \pmod 2$, $m \geq 3$.*
3. *$|A(L)| = 4$, when $m \equiv 1 \pmod 2$ and $n \equiv 2 \pmod 4$, $m \geq 3$.*

Proof. As in Theorem 2.2, it is necessary that $|A(L)| \equiv |A(D_{m,n})| \pmod 4$. So for $m \equiv n \equiv 0 \pmod 2$, we have that $|A(L)| \geq 0$. For $m \equiv n \equiv 1 \pmod 2$, we have

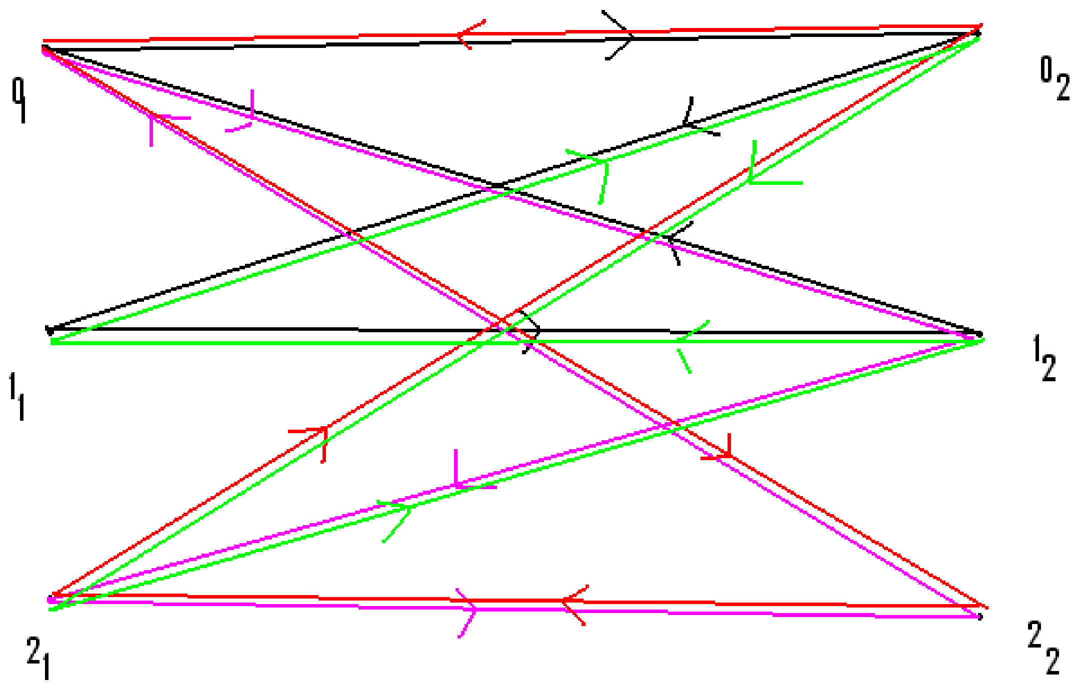

Figure 6: Packing $D_{3,3}$ with Copies of C_4

$|A(L)| \geq 2$. For the remaining cases, $|A(D_{m,n})| \equiv 0 \pmod 4$. However, by Theorem 2.3, a decomposition does not exist. Thus, $|A(L)| \geq 4$ in these cases.

Case 1: If $m \equiv n \equiv 0 \pmod 4$, then $m \equiv n \equiv 0 \pmod 2$. Therefore, by Theorem 2.3, a decomposition exists.

Case 2: If $m \equiv 0 \pmod 4$ and $n \equiv 1 \pmod 4$, then $m \equiv 0 \pmod 4$ and $n \equiv 1 \pmod 2$. By Theorem 2.3, it follows that a decomposition exists.

Case 3: If $m \equiv 0 \pmod 4$ and $n \equiv 2 \pmod 4$, then $m \equiv n \equiv 0 \pmod 2$. Therefore, by Theorem 2.3, a decomposition exists.

Case 4: If $m \equiv 0 \pmod 4$ and $n \equiv 3 \pmod 4$, then $m \equiv 0 \pmod 4$ and $n \equiv 1 \pmod 2$. Thus, by Theorem 2.3, a decomposition exists.

Case 5: If $m \equiv 1 \pmod 4$ and $n \equiv 0 \pmod 4$, then $m \equiv 1 \pmod 2$ and $n \equiv 0 \pmod 4$. Therefore, by Theorem 2.3, a decomposition exists.

Case 6: If $m \equiv n \equiv 1 \pmod 4$, where $m \geq 3$, then $D_{m,n}$=$D_{5,5} \cup D_{m-5,5} \cup D_{m,n-5} \cup D_{m-5,n-5}$. Now pack each smaller graph with X. For $D_{5,5}$, the partite sets are $\{0_1, 1_1, 2_1, 3_1, 4_1\}$ and $\{0_2, 1_2, 2_2, 3_2, 4_2\}$. One possible way of packing $D_{5,5}$ so that the leave is minimal is: $\{[0_1, 0_2, 1_1, 1_2]_X, [2_2, 3_1, 4_2, 4_1]_X, [1_2, 0_1, 4_2, 2_1]_X, [2_1, 3_2, 3_1, 2_2]_X, [0_2, 0_1, 3_2, 4_1]_X, [3_2, 0_1, 2_2, 2_1]_X, [4_1, 0_2, 2_1, 4_2]_X$ $[4_2, 1_1, 0_2, 3_1]_X, [3_1, 1_2, 0_1, 3_2]_X, [2_1, 1_2, 3_1, 0_2]_X, [2_2, 1_1, 4_2, 0_1]_X, [4_1, 3_2, 1_1, 2_2]_X\}$ and $|A(L)|$=$\{(4_1, 1_2),(1_2, 4_1)\}$. The partite sets for $D_{m-5,5}$ are $\{5_1, 6_1, \ldots, (m-1)_1\}$ and $\{0_2, 1_2, 2_2, 3_2, 4_2\}$. For $D_{m,n-5}$, the partite sets are $\{0_1, 1_1, 2_1, 3_1, 4_1\}$ and $\{5_2, 6_2, \ldots, (n-1)_2\}$. Finally, for $D_{m-5,n-5}$, the partite sets are $\{5_1, 6_1, \ldots, (m-1)_1\}$ and $\{5_2, 6_2, \ldots, (n-1)_2\}$. Since $m \equiv 1 \pmod 4$, $n \equiv 1 \pmod 4$, then $m-5 \equiv 0 \pmod 2$ and $n-5 \equiv 0 \pmod 2$. Therefore, $D_{m-5,5}$, $D_{m,n-5}$, and $D_{m-5,n-5}$ can be decomposed by Theorem 2.3. Thus, the leave, L, satisfies $|A(L)|$=2.

Case 7: If $m \equiv 1 \pmod 4$ and $n \equiv 2 \pmod 4$, where $m \geq 3$, then $D_{m,n}$=$D_{5,2} \cup D_{m-5,2} \cup D_{m,n-2} \cup D_{m-5,n-2}$. Now pack each smaller graph with X. For $D_{5,2}$, the partite sets are $\{0_1, 1_1, 2_1, 3_1, 4_1\}$ and $\{0_2, 1_2\}$. One possible way of packing $D_{5,2}$ so that the leave is minimal is: $\{[0_1, 0_2, 1_1, 1_2]_X, [2_1, 1_2, 4_1, 0_2]_X, [3_1, 1_2, 1_1, 0_2]_X, [0_2, 4_1, 1_2, 3_1]_X\}$ and $|A(L)|$=$\{(0_2, 0_1), (0_2, 2_1), (1_2, 0_1), (1_2, 2_1)\}$. The partite sets for $D_{m-5,2}$ are $\{5_1, 6_1, \ldots, (m-1)_1\}$ and $\{0_2, 1_2\}$. For $D_{m,n-2}$, the partite sets are $\{0_1, 1_1, 2_1, 3_1, 4_1\}$ and $\{2_2, 3_2, \ldots, (n-1)_2\}$. Finally, for $D_{m-5,n-2}$, the partite sets are $\{5_1, 6_1, \ldots, (m-1)_1\}$ and $\{2_2, 3_2, \ldots, (n-1)_2\}$. Since $m \equiv 1 \pmod 4$ and $n \equiv 3 \pmod 4$, then $m-5 \equiv 0 \pmod 2$ and $n-2 \equiv 0 \pmod 2$. Therefore, $D_{m-5,2}$, $D_{m,n-2}$, and $D_{m-5,n-2}$ can be decomposed by Theorem 2.3. Thus, the leave, L, satisfies $|A(L)|$=4.

Case 8: If $m \equiv 1 \pmod 4$ and $n \equiv 3 \pmod 4$, where $m \geq 3$, then $D_{m,n}$=$D_{5,3} \cup D_{m-5,3} \cup D_{m,n-3} \cup D_{m-5,n-3}$. Now pack each smaller graph with X. For $D_{5,3}$, the partite sets are $\{0_1, 1_1, 2_1, 3_1, 4_1\}$ and $\{0_2, 1_2, 2_2\}$. One possible way of packing $D_{5,3}$ so that the leave is minimal is: $\{[0_2, 0_1, 1_2, 4_1]_X, [2_2, 3_1, 4_2, 4_1]_X, [1_2, 0_1, 4_2, 2_1]_X, [2_1, 3_2, 3_1, 2_2]_X, [0_2, 0_1, 3_2, 4_1]_X, [3_2, 0_1, 2_2, 2_1]_X, [4_1, 0_2, 2_1, 4_2]_X\}$ and $|A(L)|$=$\{(4_1, 2_2),(2_2, 4_1)\}$ The partite sets for $D_{m-5,3}$ are $\{5_1, 6_1, \ldots, (m-1)_1\}$ and $\{0_2, 1_2, 2_2\}$. For $D_{m,n-3}$, the partite sets are $\{0_1, 1_1, 2_1, 3_1, 4_1\}$ and $\{3_2, 4_2, \ldots, (n-1)_2\}$. Finally, for $D_{m-5,n-3}$, the partite sets are $\{5_1, 6_1, \ldots, (m-1)_1\}$ and $\{3_2, 4_2, \ldots, (n-1)_2\}$. Now, note that since $m \equiv 1 \pmod 4$ and $n \equiv 3 \pmod 4$, then $m-5 \equiv 0 \pmod 2$ and $n-3 \equiv 0 \pmod 2$. Therefore $D_{m-5,3}$, $D_{m,n-3}$, and $D_{m-5,n-3}$ can be decomposed by Theorem 2.3. Therefore, the leave, L, satisfies $|A(L)|$=2.

Case 9: If $m \equiv 2 \pmod 4$ and $n \equiv 0 \pmod 4$, then $m \equiv n \equiv 0 \pmod 2$. Therefore, by Theorem 2.3, a decomposition exists.

Case 10: If $m \equiv 2 \pmod 4$ and $n \equiv 1 \pmod 4$, where $m \geq 3$, then $D_{m,n}$=$D_{2,5} \cup D_{m-2,5} \cup D_{m,n-5} \cup D_{m-2,n-5}$. Now pack each smaller graph with X. For $D_{2,5}$, the partite sets are $\{0_1, 1_1\}$ and $\{0_2, 1_2, 2_2, 3_2, 4_2\}$. One possible way of packing $D_{2,5}$ so that the leave is minimal is: $\{[0_2, 0_1, 1_2, 1_1]_X, [2_2, 1_1, 4_2, 0_1]_X, [3_2, 1_1, 1_2, 0_1]_X, [0_1, 4_2, 1_1, 3_2]_X\}$ and $|A(L)|$=$\{(0_1, 0_2),(0_1, 2_2), (1_1, 0_2), (1_1, 2_2)\}$. The partite sets for $D_{m-2,2}$ are $\{2_1, 3_1, \ldots, (m-1)_1\}$ and $\{0_2, 1_2, 2_2, 3_2, 4_2\}$. For $D_{m,n-5}$, the partite sets are $\{0_1, 1_1\}$ and $\{5_2, 6_2, \ldots, (n-1)_2\}$. Finally, for $D_{m-2,n-5}$, the partite sets are $\{2_1, 3_1, \ldots, (m-1)_1\}$ and $\{5_2, 6_2, \ldots, (n-1)_2\}$. Since $m \equiv 2 \pmod 4$ and $n \equiv 1 \pmod 4$, then $m-2 \equiv 0 \pmod 2$ and $n-5 \equiv 0 \pmod 2$. Therefore, $D_{m-2,5}$, $D_{m,n-5}$, and $D_{m-2,n-5}$ can be decomposed by Theorem 2.3. Thus, the leave, L, satisfies $|A(L)|$=4.

Case 11: If $m \equiv n \equiv 2 \pmod 4$, then $m \equiv n \equiv 0 \pmod 2$. Therefore, by Theorem 2.3, a decomposition exists.

Case 12: If $m \equiv 2 \pmod 4$ and $n \equiv 3 \pmod 4$, where $m \geq 3$, then $D_{m,n}$=$D_{2,3} \cup D_{m-2,3} \cup D_{m,n-3} \cup D_{m-2,n-3}$. Now pack each smaller graph with X. For $D_{2,3}$, the partite sets are $\{0_1, 1_1\}$ and $\{0_2, 1_2, 2_2\}$. One possible way of packing $D_{2,3}$ so that the leave is minimal is: $\{[0_2, 0_1, 1_2, 1_1]_X, [1_1, 1_2, 0_1, 2_2]_X\}$ and $|A(L)|$=$\{(2_2, 0_1), (2_2, 1_1), (0_1, 0_2), (1_1, 0_2)\}$. The partite sets for $D_{m-2,3}$ are $\{2_1, 3_1, \ldots, (m-1)_1\}$ and $\{0_2, 1_2, 2_2\}$. For $D_{m,n-3}$, the partite sets are $\{0_1, 1_1\}$ and $\{3_2, 4_2, \ldots, (n-1)_2\}$. Finally, for $D_{m-2,n-3}$, the partite sets are $\{2_1, 3_1, \ldots, (m-1)_1\}$ and $\{3_2, 4_2, \ldots, (n-1)_2\}$. Since $m \equiv 2 \pmod 4$ and $n \equiv 3 \pmod 4$, then $m-2 \equiv 0 \pmod 2$ and $n-3 \equiv 0 \pmod 2$. Therefore, $D_{m-2,3}$, $D_{m,n-3}$, and $D_{m-2,n-3}$ can be decomposed by Theorem 2.3. Thus, the leave, L, satisfies $|A(L)|$=4.

Case 13: If $m \equiv 3 \pmod 4$ and $n \equiv 0 \pmod 4$, then $m \equiv 1 \pmod 2$ and $n \equiv 0 \pmod 4$. Therefore, by Theorem 2.3, a decomposition exists.

Case 14: If $m \equiv 3 \pmod 4$ and $n \equiv 1 \pmod 4$, where $m \geq 3$, then $D_{m,n}$=$D_{3,5} \cup D_{m-3,5} \cup D_{m,n-5} \cup D_{m-3,n-5}$. Now pack each smaller graph with X. For $D_{3,5}$, the partite sets are $\{0_1, 1_1, 2_1\}$ and $\{0_2, 1_2, 2_2, 3_2, 4_2\}$. One possible way of packing $D_{3,5}$ so that the leave is minimal is: $\{[0_1, 0_2, 1_1, 4_2]_X, [0_2, 0_1, 1_2, 2_1]_X, [1_2, 0_1, 4_2, 1_1]_X, [0_1, 2_2, 2_1, 3_2]_X, [0_1, 3_2, 1_1, 2_2]_X, [2_1, 2_2, 1_1, 0_2]_X, [2_1, 1_2, 1_1, 3_2]_X\}$ and $|A(L)|$=$\{(2_1,$

$4_2),(4_2, 2_1)\}$. The partite sets for $D_{m-3,5}$ are $\{3_1, 4_1, \ldots, (m-1)_1\}$ and $\{0_2, 1_2, 2_2, 3_2, 4_2\}$. For $D_{m,n-5}$, the partite sets are $\{0_1, 1_1, 2_1\}$ and $\{5_2, 6_2, \ldots, (n-1)_2\}$. Finally, for $D_{m-3,n-5}$, the partite sets are $\{3_1, 4_1, \ldots, (m-1)_1\}$ and $\{5_2, 6_2, \ldots, (n-1)_2\}$. Since $m \equiv 3 \pmod 4$ and $n \equiv 1 \pmod 4$, then $m-3 \equiv 0 \pmod 2$ and $n-2 \equiv 0 \pmod 2$. Therefore, $D_{m-3,5}$, $D_{m,n-5}$, and $D_{m-3,n-5}$ can be decomposed by Theorem 2.3. Thus, the leave, L, satisfies $|A(L)|=2$.

Case 15: If $m \equiv 3 \pmod 4$ and $n \equiv 2 \pmod 4$, where $m \geq 3$, then $D_{m,n}=D_{3,2} \cup D_{m-3,2} \cup D_{m,n-2} \cup D_{m-3,n-2}$. Now pack each smaller graph with X. For $D_{3,2}$, the partite sets are $\{0_1, 1_1, 2_1\}$ and $\{0_2, 1_2\}$. One possible way of packing $D_{3,2}$ so that the leave is minimal is: $\{[0_1, 0_2, 1_1, 1_2]_X, [1_2, 1_1, 0_2, 2_1]_X\}$ and $|A(L)|=\{(2_1, 0_2),(2_1, 1_2), (0_2, 0_1), (1_2, 0_1)\}$. The partite sets for $D_{m-3,2}$ are $\{3_1, 4_1, \ldots, (m-1)_1\}$ and $\{0_2, 1_2\}$. For $D_{m,n-2}$, the partite sets are $\{0_1, 1_1, 2_1\}$ and $\{2_2, 3_2, \ldots, (n-1)_2\}$. Finally, for $D_{m-3,n-2}$, the partite sets are $\{3_1, 4_1, \ldots, (m-1)_1\}$ and $\{2_2, 3_2, \ldots, (n-1)_2\}$. Since $m \equiv 3 \pmod 4$ and $n \equiv 2 \pmod 4$, then $m-3 \equiv 0 \pmod 2$ and $n-2 \equiv 0 \pmod 2$. Therefore, $D_{m-3,2}$, $D_{m,n-2}$, and $D_{m-3,n-2}$ can be decomposed by Theorem 2.3. Thus, the leave, L, satisfies $|A(L)|=4$.

Case 16: If $m \equiv n \equiv 3 \pmod 4$, where $m \geq 3$, then $D_{m,n}=D_{3,3} \cup D_{m-3,3} \cup D_{m,n-3} \bigcup D_{m-3,n-3}$. Now pack each smaller graph with X. For $D_{3,3}$, the partite sets are $\{0_1, 1_1, 2_1\}$ and $\{0_2, 1_2, 2_2\}$. One possible way of packing $D_{3,3}$ so that the leave is minimal is: $\{[0_1, 0_2, 2_1, 1_2]_X, [1_1, 0_2, 0_1, 2_2]_X, [2_2, 2_1, 1_2, 1_1]_X, [1_2, 2_1, 2_2, 0_1]_X\}$ and $|A(L)|=\{(2_1, 0_2),(0_2, 2_1)\}$ The partite sets for $D_{m-3,3}$ are $\{3_1, 4_1, \ldots, (m-1)_1\}$ and $\{0_2, 1_2, 2_2\}$. For $D_{m,n-3}$, the partite sets are $\{0_1, 1_1, 2_1\}$ and $\{3_2, 4_2, \ldots, (n-1)_2\}$. Finally, for $D_{m-3,n-3}$, the partite sets are $\{3_1, 4_1, \ldots, (m-1)_1\}$ and $\{3_2, 4_2, \ldots,$

$(n-1)_2\}$. Since $m \equiv 3 \pmod 4$, $n \equiv 3 \pmod 4$, then $m-3 \equiv 0 \pmod 2$ and $n-3 \equiv 0 \pmod 2$. Therefore, $D_{m-3,3}$, $D_{m,n-3}$, and $D_{m-3,n-3}$ can be decomposed by Theorem 2.3. Thus, the leave, L, satisfies $|A(L)|$=2.$Q.E.D$

Thus, each case for X satisfies the conditions of the theorem. Decomposing $D_{m,n}$ with copies of Y has the following results.

Theorem 2.5 [5] *A Y decomposition of $D_{m,n}$ exists if and only if $m, n \geq 2$ and $mn \equiv 0$ (mod 2).*

Theorem 2.6 *A maximal Y packing of $D_{m,n}$ satisfies:*

1. $|A(L)|= 0$ *when* $mn \equiv 0$ *(mod 2)*, $m, n \geq 2$,
2. $|A(L)|= 2$, *otherwise.*

Proof. The necessary conditions follow as in Theorem 2.2.

Case 1: If $m \equiv n \equiv 0 \pmod 4$, then $mn \equiv 0 \pmod 2$. Therefore, by Theorem 2.5, a decomposition exists.

Case 2: If $m \equiv 0 \pmod 4$ and $n \equiv 1 \pmod 4$, then $mn \equiv 0 \pmod 2$. By Theorem 2.5, it follow that a decomposition exists.

Case 3: If $m \equiv 0 \pmod 4$ and $n \equiv 2 \pmod 4$, then $mn \equiv 0 \pmod 2$. Thus, by Theorem 2.5, a decomposition exists.

Case 4: If $m \equiv 0 \pmod 4$ and $n \equiv 3 \pmod 4$, then $mn \equiv 0 \pmod 2$. Therefore, by Theorem 2.5, a decomposition exists.

Case 5: If $m \equiv 1 \pmod 4$ and $n \equiv 0 \pmod 4$, then $mn \equiv 0 \pmod 2$. By Theorem 2.5, it follows that a decomposition exists.

Case 6: If $m \equiv n \equiv 1 \pmod 4$, where $m, n \geq 2$, then $D_{m,n}$=$D_{5,5} \cup D_{m-5,5} \cup D_{m,n-5} \cup D_{m-5,n-5}$. Now pack each smaller graph with Y. For $D_{5,5}$, the partite sets are $\{0_1, 1_1, 2_1, 3_1, 4_1\}$ and $\{0_2, 1_2, 2_2, 3_2, 4_2\}$. One possible way of packing $D_{5,5}$ so that the leave is minimal is: $\{[3_1, 0_2, 2_1, 3_2]_Y$, $[2_1, 3_2, 3_1, 4_2]_Y$, $[3_2, 0_1, 2_2, 4_1]_Y$, $[1_1, 1_2, 0_1, 2_2]_Y$, $[4_2, 1_1, 0_2, 4_1]_Y$, $[0_2, 0_1, 4_2, 1_1]_Y$, $[4_2, 0_1, 0_2, 2_1]_Y$, $[0_1, 1_2, 1_1, 3_2]_Y$, $[2_2, 1_1, 3_2, 4_1]_Y$, $[2_2, 2_1, 1_2, 3_1]_Y$, $[0_2, 4_1, 4_2, 3_1]_Y$, $[1_2, 2_1, 2_2, 3_1]_Y\}$ and $|A(L)|$=$\{(4_1, 1_2),(1_2, 4_1)\}$. The partite sets for $D_{m-5,5}$ are $\{5_1, 6_1, \ldots, (m-1)_1\}$ and $\{0_2, 1_2, 2_2, 3_2, 4_2\}$. For $D_{m,n-5}$, the partite sets are $\{0_1, 1_1, 2_1, 3_1, 4_1\}$ and $\{5_2, 6_2, \ldots, (n-1)_2\}$. Finally, for $D_{m-5,n-5}$, the partite sets are $\{5_1, 6_1, \ldots, (m-1)_1\}$ and $\{5_2, 6_2, \ldots, (n-1)_2\}$. Since $m \equiv 1 \pmod 4$, $n \equiv 1 \pmod 4$ and $m, n \geq 2$, then $m-5 \equiv 0 \pmod 2$ and $n-5 \equiv 0 \pmod 2$. Therefore, $D_{m-5,5}$, $D_{m,n-5}$, and $D_{m-5,n-5}$ can be decomposed by Theorem 2.5. Thus, the leave, L, satisfies $|A(L)|$=2.

Case 7: If $m \equiv 1 \pmod 4$ and $n \equiv 2 \pmod 4$, then $mn \equiv n \equiv 0 \pmod 2$. Therefore, by Theorem 2.5, a decomposition exists.

Case 8: If $m \equiv 1 \pmod 4$ and $n \equiv 3 \pmod 4$, where $m, n \geq 2$, then $D_{m,n}$=$D_{5,3} \cup D_{m-5,3} \cup D_{m,n-3} \cup D_{m-5,n-3}$. Now pack each smaller graph with Y. For $D_{5,3}$, the partite sets are $\{0_1, 1_1, 2_1, 3_1, 4_1\}$ and $\{0_2, 1_2, 2_2\}$. One possible way of packing $D_{5,3}$ so that the leave is minimal is: $\{[4_1, 0_2, 0_1, 1_2]_Y$, $[3_1, 0_2, 4_1, 2_2]_Y$, $[0_2, 3_1, 1_2, 2_1]_Y$, $[0_2, 3_1, 1_2, 2_1]_Y$, $[0_1, 0_2, 1_1, 1_2]_Y$, $[1_1, 1_2, 3_1, 2_2]_Y$, $[1_2, 2_1, 2_2, 4_1]_Y\}$ and $|A(L)|$=$\{(0_1, 2_2),(2_2, 0_1)\}$. The partite sets for $D_{m-5,3}$ are $\{5_1, 6_1, \ldots, (m-1)_1\}$ and $\{0_2, 1_2, 2_2\}$. For $D_{m,n-3}$, the partite sets are $\{0_1, 1_1, 2_1, 3_1, 4_1\}$ and $\{3_2, 4_2, \ldots, (n-1)_2\}$. Finally, for $D_{m-5,n-3}$, the partite sets are $\{5_1, 6_1, \ldots, (m-1)_1\}$ and $\{3_2, 4_2, \ldots, (n-1)_2\}$. Since $m \equiv 1 \pmod 4$ and $n \equiv 3 \pmod 4$ and $m, n \geq 2$, then $m-5 \equiv 0 \pmod 2$ and

$n-3 \equiv 0 \pmod 2$. Therefore, $D_{m-5,3}$, $D_{m,n-3}$, and $D_{m-5,n-3}$ can be decomposed by Theorem 2.5. Thus, the leave, L, satisfies $|A(L)|$=2.

Case 9: If $m \equiv 2 \pmod 4$ and $n \equiv 0 \pmod 4$, then $mn \equiv 0 \pmod 2$. By Theorem 2.5, it follows that a decomposition exists.

Case 10: If $m \equiv 2 \pmod 4$ and $n \equiv 1 \pmod 4$, then $mn \equiv 0 \pmod 2$. Therefore, by Theorem 2.5, a decomposition exists.

Case 11: If $m \equiv n \equiv 2 \pmod 4$, then $mn \equiv 0 \pmod 2$. Thus, by Theorem 2.5, a decomposition exists.

Case 12: If $m \equiv 2 \pmod 4$ and $n \equiv 3 \pmod 4$, then $mn \equiv 0 \pmod 2$. By Theorem 2.5, it follows that a decomposition exists.

Case 13: If $m \equiv 3 \pmod 4$ and $n \equiv 0 \pmod 4$, then $mn \equiv 0 \pmod 2$. Therefore, by Theorem 2.5 a decomposition exists.

Case 14: If $m \equiv 3 \pmod 4$ and $n \equiv 1 \pmod 4$, where $m, n \geq 2$, then $D_{m,n}$=$D_{3,5}$ $\cup$ $D_{m-3,5} \cup D_{m,n-5} \cup D_{m-3,n-5}$. Now pack each smaller graph with Y. For $D_{3,5}$, the partite sets are $\{0_1, 1_1, 2_1\}$ and $\{0_2, 1_2, 2_2, 3_2, 4_2\}$. One possible way of packing $D_{3,5}$ so that the leave is minimal is: $\{[4_2, 0_1, 0_2, 1_1]_Y, [3_2, 0_1, 4_2, 2_1]_Y, [0_1, 3_2, 1_1, 2_2]_Y, [0_2, 0_1, 1_2, 1_1]_Y, [1_2, 1_1, 3_2, 2_1]_Y, [1_1, 2_2, 2_1, 4_2]_Y, [2_1, 1_2, 0_1, 2_2]_Y\}$ and $|A(L)|$=$\{(2_1, 0_2),(0_2, 2_1)\}$. The partite sets for $D_{m-3,5}$ are $\{3_1, 4_1, \ldots, (m-1)_1\}$ and $\{0_2, 1_2, 2_2, 3_2, 4_2\}$. For $D_{m,n-5}$, the partite sets are $\{0_1, 1_1, 2_1\}$ and $\{5_2, 6_2, \ldots, (n-1)_2\}$. Finally, for $D_{m-3,n-5}$, the partite sets are $\{3_1, 4_1, \ldots, (m-1)_1\}$ and $\{5_2, 6_2, \ldots, (n-1)_2\}$. Since $m \equiv 3 \pmod 4$ and $n \equiv 1 \pmod 4$ and $m, n \geq 2$, then $m-3 \equiv 0 \pmod 2$ and $n-5 \equiv 0 \pmod 2$. Therefore, $D_{m-3,5}$, $D_{m,n-5}$, and $D_{m-3,n-5}$ can be decomposed by Theorem 2.5. Thus, the leave, L, satisfies $|A(L)|$=2.

Case 15: If $m \equiv 3 \pmod 4$ and $n \equiv 2 \pmod 4$, then $mn \equiv 0 \pmod 2$. By Theorem 2.5, it follows that a decomposition exists.

Case 16: If $m \equiv n \equiv 3 \pmod 4$, where $m, n \geq 2$, then $D_{m,n}$=$D_{3,3} \cup D_{m-3,3} \cup D_{m,n-3} \cup D_{m-3,n-3}$. Now pack each smaller graph with Y. For $D_{3,3}$, the partite sets are $\{0_1, 1_1, 2_1\}$ and $\{0_2, 1_2, 2_2\}$. One possible way of packing $D_{3,3}$ so that the leave is minimal is: $\{[2_1, 0_2, 0_1, 1_2]_Y, [1_2, 2_1, 2_2, 1_1]_Y, [2_2, 1_1, 1_2, 0_1]_Y, [0_1, 2_2, 2_1, 0_2]_Y\}$ and $|A(L)|$=$\{(1_1, 0_2),(0_2, 1_1)\}$. The partite sets for $D_{m-3,3}$ are $\{3_1, 4_1, \ldots, (m-1)_1\}$ and $\{0_2, 1_2, 2_2\}$. For $D_{m,n-3}$, the partite sets are $\{0_1, 1_1, 2_1\}$ and $\{3_2, 4_2, \ldots, (n-1)_2\}$. Finally, for $D_{m-3,n-3}$, the partite sets are $\{3_1, 4_1, \ldots, (m-1)_1\}$ and $\{3_2, 4_2, \ldots, (n-1)_2\}$. Now, note that since $m \equiv 3 \pmod 4$, $n \equiv 3 \pmod 4$ and $m, n \geq 2$, then $m - 3 \equiv 0 \pmod 2$ and $n - 3 \equiv 0 \pmod 2$. Therefore, $D_{m-3,3}$, $D_{m,n-3}$, and $D_{m-3,n-3}$ can be decomposed by Theorem 2.5. Thus, the leave, L, satisfies $|A(L)|$=2.$Q.E.D$

Thus, each case for Y satisfies the conditions of the theorem.

Now we will consider coverings of $D_{m,n}$.

Theorem 2.7 *A minimal C_4 covering of $D_{m,n}$ satisfies:*

1. $|A(P)|= 0$ *when* $mn \equiv 0 \pmod 2$, $m, n \geq 2$,
2. $|A(P)|= 2$, *otherwise.*

Proof. Since $|A(C_4)|$=4, it is necessary that the padding P satisfy $|A(D_{m,n})| + |A(P)| \equiv 0 \pmod 4$. Therefore, for $m \equiv n \equiv 0 \pmod 4$, $|A(P)| \geq 0$ and for the other cases $|A(P)| \geq 2$.

Case 1: If $m \equiv n \equiv 0 \pmod 4$, then $mn \equiv 0 \pmod 2$. Therefore, by Theorem 2.1, a decomposition exists.

Case 2: If $m \equiv 0 \pmod 4$ and $n \equiv 1 \pmod 4$, then $mn \equiv 0 \pmod 2$. Thus, by Theorem 2.1, a decomposition exists.

Case 3: If $m \equiv 0 \pmod 4$ and $n \equiv 2 \pmod 4$, then $mn \equiv 0 \pmod 2$. Therefore, by Theorem 2.1, a decomposition exists.

Case 4: If $m \equiv 0 \pmod 4$ and $n \equiv 3 \pmod 4$, then $mn \equiv 0 \pmod 2$. By Theorem 2.1, it follows that a decomposition exists.

Case 5: If $m \equiv 1 \pmod 4$ and $n \equiv 0 \pmod 4$, then $mn \equiv 0 \pmod 2$. Therefore, by Theorem 2.1 a decomposition exists.

Case 6: If $m \equiv n \equiv 1 \pmod 4$, where $m, n \geq 2$, then $D_{m,n}$=$D_{5,5} \cup D_{m-5,5} \cup D_{m,n-5} \cup D_{m-5,n-5}$. Now cover each smaller graph with C_4. For $D_{5,5}$, the partite sets are $\{0_1, 1_1, 2_1, 3_1, 4_1\}$ and $\{0_2, 1_2, 2_2, 3_2, 4_2\}$. One possible way of covering $D_{5,5}$ so that the padding is minimal is: $\{[0_1, 0_2, 1_1, 1_2]_C, [0_1, 1_2, 3_1, 2_2]_C, [0_1, 2_2, 2_1, 4_2]_C, [0_1, 3_2, 4_1, 0_2]_C, [1_1, 4_2, 2_1, 3_2]_C, [1_1, 2_2, 3_1, 1_2]_C, [0_1, 4_2, 1_1, 3_2]_C, [1_1, 0_2, 4_1, 2_2]_C, [2_1, 0_2, 3_1, 3_2]_C, [2_1, 2_2, 4_1, 1_2]_C, [3_1, 4_2, 4_1, 3_2]_C, [2_1, 1_2, 4_1, 0_2]_C, [4_1, 4_2, 3_1, 0_2]_C\}$ and $|A(P)|$=$\{(4_1, 0_2),(0_2, 4_1)\}$. The partite sets for $D_{m-5,5}$ are $\{5_1, 6_1, \ldots, (m-1)_1\}$ and $\{0_2, 1_2, 2_2, 3_2, 4_2\}$. For $D_{m,n-5}$, the partite sets are $\{0_1, 1_1, 2_1, 3_1, 4_1\}$ and $\{5_2, 6_2, \ldots, (n-1)_2\}$. Finally, for $D_{m-5,n-5}$, the partite sets are $\{5_1, 6_1, \ldots, (m-1)_1\}$ and $\{5_2, 6_2, \ldots, (n-1)_2\}$. Since $m \equiv 1 \pmod 4$, $n \equiv 1 \pmod 4$ and $m, n \geq 2$, then $m - 5 \equiv 0 \pmod 2$ and $n - 5 \equiv 0 \pmod 2$. Therefore, $D_{m-5,5}$, $D_{m,n-5}$, and $D_{m-5,n-5}$ can be decomposed by Theorem 2.1. Thus, the padding, P, satisfies $|A(P)|$=2.

Case 7: If $m \equiv 1 \pmod 4$ and $n \equiv 2 \pmod 4$, then $m \equiv n \equiv 0 \pmod 2$. Therefore, by Theorem 2.1 a decomposition exists.

Case 8: If $m \equiv 1 \pmod 4$ and $n \equiv 3 \pmod 4$, where $m, n \geq 2$, then $D_{m,n}$=$D_{5,3} \cup D_{m-5,3} \cup D_{m,n-3} \cup D_{m-5,n-3}$. Now cover each smaller graph with C_4. For $D_{5,3}$, the partite sets are $\{0_1, 1_1, 2_1, 3_1, 4_1\}$ and $\{0_2, 1_2, 2_2\}$. One possible way of covering $D_{5,3}$ so that the padding is minimal is: $\{[0_2, 0_1, 1_2, 4_1]_C, [0_2, 1_1, 2_2, 3_1]_C, [1_2, 1_1, 0_2, 2_1]_C, [1_2, 3_1, 2_2, 4_1]_C, [2_2, 0_1, 0_2, 4_1]_C, [1_2, 2_1, 2_2, 3_1]_C, [2_2, 2_1, 0_2, 3_1]_C, [1_2, 0_1, 2_2, 3_1]_C\}$ and $|A(P)|$=$\{(3_1, 2_2),(2_2, 3_1)\}$. The partite sets for $D_{m-5,3}$ are $\{5_1, 6_1, \ldots, (m-1)_1\}$ and $\{0_2, 1_2, 2_2\}$. For $D_{m,n-3}$, the partite sets are $\{0_1, 1_1, 2_1, 3_1, 4_1\}$ and $\{3_2, 4_2, \ldots, (n-1)_2\}$. Finally, for $D_{m-5,n-3}$, the partite sets are $\{5_1, 6_1, \ldots, (m-1)_1\}$ and $\{3_2, 4_2, \ldots, (n-1)_2\}$. Since $m \equiv 1 \pmod 4$ and $n \equiv 3 \pmod 4$ and $m, n \geq 2$, then $m - 5 \equiv 0 \pmod 2$ and $n - 3 \equiv 0 \pmod 2$. Therefore, $D_{m-5,3}$, $D_{m,n-3}$, and $D_{m-5,n-3}$ can be decomposed by Theorem 2.1. Thus, the padding, P, satisfies $|A(P)|$=2.

Case 9: If $m \equiv 2 \pmod 4$ and $n \equiv 0 \pmod 4$, then $mn \equiv 0 \pmod 2$. Therefore, by Theorem 2.1, a decomposition exists.

Case 10: If $m \equiv 2 \pmod 4$ and $n \equiv 1 \pmod 4$, then $mn \equiv 0 \pmod 2$. By Theorem 2.1, it follows that a decomposition exists.

Case 11: If $m \equiv n \equiv 2 \pmod 4$, then $mn \equiv 0 \pmod 2$. Thus, by Theorem 2.1, a decomposition exists.

Case 12: If $m \equiv 2 \pmod 4$ and $n \equiv 3 \pmod 4$, then $mn \equiv 0 \pmod 2$. Therefore, by Theorem 2.1, a decomposition exists.

Case 13: If $m \equiv 3 \pmod 4$ and $n \equiv 0 \pmod 4$, then $mn \equiv 0 \pmod 2$. By Theorem 2.1, it follows that a decomposition exists.

Case 14: If $m \equiv 3 \pmod 4$ and $n \equiv 1 \pmod 4$, where $m, n \geq 2$, then decompose $D_{m,n}$. That is $D_{m,n}$=$D_{3,5} \cup D_{m-3,5} \cup D_{m,n-5} \cup D_{m-3,n-5}$. Now cover each smaller graph with C_4. For $D_{3,5}$, the partite sets are $\{0_1, 1_1, 2_1\}$ and $\{0_2, 1_2, 2_2, 3_2, 4_2\}$. One possible way of covering $D_{3,5}$ so that the padding is minimal is: $\{[0_1, 0_2, 1_1, 4_2]_C, [0_1, 1_2, 2_1, 3_2]_C, [1_1, 1_2, 0_1, 2_2]_C, [1_1, 3_2, 2_1, 4_2]_C, [2_1, 0_2, 0_1, 4_2]_C, [1_1, 0_2, 2_1, 3_2]_C, [1_1, 2_2, 2_1, 3_2]_C, [2_1, 2_2, 0_1, 3_2]_C\}$ and $|A(P)|=\{(2_1, 3_2),(3_2, 2_1)\}$. The partite sets for $D_{m-3,5}$ are $\{3_1, 4_1, \ldots, (m-1)_1\}$ and $\{0_2, 1_2, 2_2, 3_2, 4_2\}$. For $D_{m,n-5}$, the partite sets are $\{0_1, 1_1, 2_1\}$ and $\{5_2, 6_2, \ldots, (n-1)_2\}$. Finally, for $D_{m-3,n-5}$, the partite sets are $\{3_1, 4_1, \ldots, (m-1)_1\}$ and $\{5_2, 6_2, \ldots, (n-1)_2\}$. Since $m \equiv 3 \pmod 4$ and $n \equiv 1 \pmod 4$ and $m, n \geq 2$, then $m-3 \equiv 0 \pmod 2$ and $n-5 \equiv 0 \pmod 2$. Therefore, $D_{m-3,5}$, $D_{m,n-5}$, and $D_{m-3,n-5}$ can be decomposed by Theorem 2.1. Thus, the padding, P, satisfies $|A(P)|=2$.

Case 15: If $m \equiv 3 \pmod 4$ and $n \equiv 2 \pmod 4$, then $mn \equiv 0 \pmod 2$. Therefore, by Theorem 2.1, a decomposition exists.

Case 16: If $m \equiv n \equiv 3 \pmod 4$, where $m, n \geq 2$, then $D_{m,n}=D_{3,3} \cup D_{m-3,3} \cup D_{m,n-3} \cup D_{m-3,n-3}$. Now cover each smaller graph with C_4. For $D_{3,3}$, the partite sets are $\{0_1, 1_1, 2_1\}$ and $\{0_2, 1_2, 2_2\}$. One possible way of covering $D_{3,3}$ so that the padding is minimal is: $\{[0_1, 0_2, 1_1, 1_2]_C, [2_2, 2_1, 1_2, 0_1]_C, [0_2, 0_1, 2_2, 1_1]_C, [1_2, 1_1, 0_2, 2_1]_C, [2_1, 0_2, 1_1, 2_2]_C\}$ and $|A(P)|=\{(0_1, 2_2),(2_2, 0_1)\}$. The partite sets for $D_{m-3,3}$ are $\{3_1, 4_1, \ldots, (m-1)_1\}$ and $\{0_2, 1_2, 2_2\}$. For $D_{m,n-3}$, the partite sets are $\{0_1, 1_1, 2_1\}$ and $\{3_2, 4_2, \ldots, (n-1)_2\}$. Finally, for $D_{m-3,n-3}$, the partite sets are $\{3_1, 4_1, \ldots, (m-1)_1\}$ and $\{3_2, 4_2, \ldots, (n-1)_2\}$. Since $m \equiv 3 \pmod 4$, $n \equiv 3 \pmod 4$ and $m, n \geq 2$, then $m-3 \equiv 0 \pmod 2$ and$n-3 \equiv 0 \pmod 2$. Therefore $D_{m-3,3}$, $D_{m,n-3}$, and $D_{m-3,n-3}$ can be decomposed by Theorem 2.1. Thus, the padding, P, satisfies $|A(P)|=2$. *Q.E.D.*

Thus, each case for C_4 satisfies the conditions of the theorem.

Theorem 2.8 *A minimal X covering of $D_{m,n}$ satisfies:*

1. $|A(P)|= 0$, *when $m \equiv n \equiv 0$ (mod 2) or $m \equiv 1$ (mod 2), and $m \geq 3$;*
2. $|A(P)|$=2, *when $m \equiv 1$ (mod 2) and $n \equiv 1$ (mod 2), and $m \geq 3$;*
3. $|A(P)|$=4, *when $m \equiv 1$ (mod 2) and $n \equiv 2$ (mod 4), and $m \geq 3$;*

Proof. As in Theorem 2.9, for $m \equiv n \equiv 0$ (mod 4) we have $|A(P)| \geq 0$. For $m \equiv n \equiv 1$ (mod 2), we have $|A(P)| \geq 2$. For $m \equiv 1$ (mod 2) and $n \equiv 2$ (mod 4), a decomposition does not exist for $|A(D_{m,n})| \equiv 0$ (mod 4). So $|A(P)| \geq 4$ is necessary.

Case 1: If $m \equiv n \equiv 0$ (mod 4), then $m \equiv n \equiv 0$ (mod 2). Therefore, by Theorem 2.3, a decomposition exists.

Case 2: If $m \equiv 0$ (mod 4) and $n \equiv 1$ (mod 4), then $m \equiv 0$ (mod 4) and $n \equiv 1$ (mod 2). Thus, by Theorem 2.3, a decomposition exists.

Case 3: If $m \equiv 0$ (mod 4) and $n \equiv 2$ (mod 4), then $m \equiv n \equiv 0$ (mod 2). Therefore, by Theorem 2.3, a decomposition exists.

Case 4: If $m \equiv 0$ (mod 4) and $n \equiv 3$ (mod 4), then $m \equiv 0$ (mod 4) and $n \equiv 1$ (mod 2). By Theorem 2.3, it follows that a decomposition exists.

Case 5: If $m \equiv 1$ (mod 4) and $n \equiv 0$ (mod 4), then $m \equiv 1$ (mod 2) and $n \equiv 0$ (mod 4). Therefore, by Theorem 2.3, a decomposition exists.

Case 6: If $m \equiv n \equiv 1$ (mod 4), where $m \geq 3$, then $D_{m,n}$=$D_{5,5} \cup D_{m-5,5} \cup D_{m,n-5}$ $\cup$ $D_{m-5,n-5}$. Now cover each smaller graph with X. For $D_{5,5}$, the partite sets are $\{0_1, 1_1, 2_1, 3_1, 4_1\}$ and $\{0_2, 1_2, 2_2, 3_2, 4_2\}$. One possible way of covering $D_{5,5}$ so that the padding is minimal is: $\{[0_1, 0_2, 1_1, 1_2]_X$, $[0_1, 2_2, 2_1, 3_2]_X$, $[2_2, 0_1, 4_2, 1_1]_X$, $[0_2, 2_1, 1_2, 0_1]_X$, $[3_2, 4_1, 4_2, 0_1]_X$, $[3_1, 1_2, 4_1, 0_2]_X$, $[3_1, 2_2, 4_1, 3_2]_X$, $[4_2, 2_1, 0_2, 3_1]_X$, $[4_2,$

4_1, 1_2, $1_1]_X$, $[3_2$, 3_1, 4_2, $1_1]_X$, $[1_1$, 2_1, 2_2, $3_1]_X$, $[1_1$, 3_2, 2_1, $4_2]_X$, $[1_1$, 0_2, 4_1, $2_2]_X\}$ and $|A(P)|=\{(1_1, 4_2),(4_2, 1_1)\}$. The partite sets for $D_{m-5,5}$ are $\{5_1, 6_1, \ldots, (m-1)_1\}$ and $\{0_2, 1_2, 2_2, 3_2, 4_2\}$. For $D_{m,n-5}$, the partite sets are $\{0_1, 1_1, 2_1, 3_1, 4_1\}$ and $\{5_2, 6_2, \ldots, (n-1)_2\}$. Finally, for $D_{m-5,n-5}$, the partite sets are $\{5_1, 6_1, \ldots, (m-1)_1\}$ and $\{5_2, 6_2, \ldots, (n-1)_2\}$. Since $m \equiv 1 \pmod 4$, $n \equiv 1 \pmod 4$ and $m, n \geq 5$, then $m-5 \equiv 0 \pmod 2$ and $n-5 \equiv 0 \pmod 2$. Therefore, $D_{m-5,5}$, $D_{m,n-5}$, and $D_{m-5,n-5}$ can be decomposed by Theorem 2.3. Thus, the padding, P, satisfies $|A(P)|$=2.

Case 7: If $m \equiv 1 \pmod 4$ and $n \equiv 2 \pmod 4$, where $m \geq 3$, then $D_{m,n}=D_{5,2} \cup D_{m-5,2} \cup D_{m,n-2} \cup D_{m-5,n-2}$. Now cover each smaller graph with X. For $D_{5,2}$, the partite sets are $\{0_1, 1_1, 2_1, 3_1, 4_1\}$ and $\{0_2, 1_2\}$. One possible way of covering $D_{5,2}$ so that the padding is minimal is: $\{[0_1, 0_2, 1_1, 1_2]_X$, $[2_1, 1_2, 4_1, 0_2]_X$, $[3_1, 1_2, 1_1, 0_2]_X$, $[0_2, 4_1, 1_2, 3_1]_X$, $[1_2, 0_1, 0_2, 2_1]_X$, $[0_2, 0_1, 1_2, 2_1]_X\}$ and $|A(P)|=\{(0_1, 0_2),(0_1, 1_2), (0_2, 2_1), (1_2, 2_1)\}$. The partite sets for $D_{m-5,2}$ are $\{5_1, 6_1, \ldots, (m-1)_1\}$ and $\{0_2, 1_2\}$. For $D_{m,n-2}$, the partite sets are $\{0_1, 1_1, 2_1, 3_1, 4_1\}$ and $\{2_2, 3_2, \ldots, (n-1)_2\}$. Finally, for $D_{m-5,n-2}$, the partite sets are $\{5_1, 6_1, \ldots, (m-1)_1\}$ and $\{2_2, 3_2, \ldots, (n-1)_2\}$. Since $m \equiv 1 \pmod 4$ and $n \equiv 2 \pmod 4$ and $m, n \geq 5$, then $m-5 \equiv 0 \pmod 2$ and $n-2 \equiv 0 \pmod 2$. Therefore, $D_{m-5,2}$, $D_{m,n-2}$, and $D_{m-5,n-2}$ can be decomposed by Theorem 2.3. Thus, the padding, P, satisfies $|A(P)|$=4.

Case 8: If $m \equiv 1 \pmod 4$ and $n \equiv 3 \pmod 4$, where $m \geq 3$, then $D_{m,n}=D_{5,3} \cup D_{m-5,3} \cup D_{m,n-3} \cup D_{m-5,n-3}$. Now cover each smaller graph with X. For $D_{5,3}$, the partite sets are $\{0_1, 1_1, 2_1, 3_1, 4_1\}$ and $\{0_2, 1_2, 2_2\}$. One possible way of covering $D_{5,3}$ so that the padding is minimal is: $\{[0_1, 0_2, 1_1, 1_2]_X$, $[2_2, 2_1, 0_2, 4_1]_X$, $[4_1, 0_2, 2_1, 1_2]_X$, $[3_1, 1_2, 1_1, 2_2]_X$, $[1_2, 4_1, 2_2, 0_1]_X$, $[1_2, 2_1, 2_2, 3_1]_X$, $[2_2, 1_1, 0_2, 3_1]_X$, $[3_1, 0_2, 0_1, 2_2]_X\}$

and $|A(P)|$={(3_1, 0_2),(0_2, 3_1)}. The partite sets for $D_{m-5,3}$ are {5_1, 6_1, ..., $(m-1)_1$} and {0_2, 1_2, 2_2}. For $D_{m,n-3}$, the partite sets are {0_1, 1_1, 2_1, 3_1, 4_1} and {3_2, 4_2, ..., $(n-1)_2$}. Finally, for $D_{m-5,n-3}$, the partite sets are {5_1, 6_1, ..., $(m-1)_1$} and {3_2, 4_2, ..., $(n-1)_2$}. Since $m \equiv 1 \pmod 4$ and $n \equiv 3 \pmod 4$ and $m, n \geq 5$, then $m-3 \equiv 0 \pmod 2$ and $n-3 \equiv 0 \pmod 2$. Therefore, $D_{m-5,3}$, $D_{m,n-3}$, and $D_{m-5,n-3}$ can be decomposed by Theorem 2.2. Thus, the padding, P, satisfies $|A(P)|$=2.

Case 9: If $m \equiv 2 \pmod 4$ and $n \equiv 0 \pmod 4$, then $m \equiv n \equiv 0 \pmod 2$. Therefore, by Theorem 2.3, a decomposition exists.

Case 10: If $m \equiv 2 \pmod 4$ and $n \equiv 1 \pmod 4$, where $m \geq 3$, then $D_{m,n}$=$D_{2,5} \cup D_{m-2,5} \cup D_{m,n-5} \cup D_{m-2,n-5}$. Now cover each smaller graph with X. For $D_{2,5}$, the partite sets are {0_1, 1_1} and {0_2, 1_2, 2_2, 3_2, 4_2}. One possible way of covering $D_{2,5}$ so that the padding is minimal is: {$[0_2, 0_1, 1_2, 1_1]_X$, $[2_2, 1_1, 4_2, 0_1]_X$, $[3_2, 1_1, 1_2, 0_1]_X$, $[0_1, 4_2, 1_1, 3_2]_X$, $[1_1, 0_2, 0_1, 2_2]_X$, $[0_1, 0_2, 1_1, 2_2]_X$} and $|A(P)|$={(0_2, 0_1),(0_2, 1_1), (0_1, 2_2), (1_1, 2_2)}. The partite sets for $D_{m-2,5}$ are {2_1, 3_1, ..., $(m-1)_1$} and {0_2, 1_2, 2_2, 3_2, 4_2}. For $D_{m,n-5}$, the partite sets are {0_1, 1_1} and {5_2, 6_2, ..., $(n-1)_2$}. Finally, for $D_{m-2,n-5}$, the partite sets are {2_1, 3_1, ..., $(m-1)_1$} and {5_2, 6_2, ..., $(n-1)_2$}. Since $m \equiv 2 \pmod 4$ and $n \equiv 1 \pmod 4$ and $m, n \geq 5$, then $m-2 \equiv 0 \pmod 2$ and $n-5 \equiv 0 \pmod 2$. Therefore, $D_{m-2,5}$, $D_{m,n-5}$, and $D_{m-2,n-5}$ can be decomposed by Theorem 2.3. Therefore, the padding, P, satisfies $|A(P)|$=4.

Case 11: If $m \equiv n \equiv 2 \pmod 4$, then $m \equiv n \equiv 0 \pmod 2$. Therefore, by Theorem 2.3, a decomposition exists.

Case 12: If $m \equiv 2 \pmod 4$ and $n \equiv 3 \pmod 4$, where $m \geq 3$, then $D_{m,n}$=$D_{2,3} \cup D_{m-2,3} \cup D_{m,n-3} \cup D_{m-2,n-3}$. Now cover each smaller graph with X. For $D_{2,3}$, the partite sets are $\{0_1, 1_1\}$ and $\{0_2, 1_2, 2_2\}$. One possible way of covering $D_{2,3}$ so that the padding is minimal is: $\{[0_2, 0_1, 1_2, 1_1]_X, [1_1, 1_2, 0_1, 2_2]_X, [2_2, 0_1, 0_2, 1_1]_X, [2_2, 1_1, 0_2, 0_1]_X\}$ and $|A(P)|=\{(2_2, 0_1),(2_2, 1_1), (0_1, 0_2), (1_1, 0_2)\}$ The partite sets for $D_{m-2,3}$ are $\{2_1, 3_1, \ldots, (m-1)_1\}$ and $\{0_2, 1_2, 2_2\}$. For $D_{m,n-3}$, the partite sets are $\{0_1, 1_1\}$ and $\{3_2, 4_2, \ldots, (n-1)_2\}$. Finally, for $D_{m-2,n-3}$, the partite sets are $\{2_1, 3_1, \ldots, (m-1)_1\}$ and $\{3_2, 4_2, \ldots, (n-1)_2\}$. Since $m \equiv 2 \pmod 4$ and $n \equiv 3 \pmod 4$ and $m, n \geq 5$, then $m-2 \equiv 0 \pmod 2$ and $n-3 \equiv 0 \pmod 2$. Therefore, $D_{m-2,3}$, $D_{m,n-3}$, and $D_{m-2,n-3}$ can be decomposed by Theorem 2.3. Thus, the padding, P, satisfies $|A(P)|$=4.

Case 13: If $m \equiv 3 \pmod 0$ and $n \equiv 0 \pmod 4$, then $m \equiv n \equiv 0 \pmod 2$. Therefore, by Theorem 2.3, a decomposition exists.

Case 14: If $m \equiv 3 \pmod 4$ and $n \equiv 1 \pmod 4$, where $m \geq 3$, then $D_{m,n}$=$D_{3,5} \cup D_{m-3,5} \cup D_{m,n-5} \cup D_{m-3,n-5}$. Now cover each smaller graph with X. For $D_{3,5}$, the partite sets are $\{0_1, 1_1, 2_1\}$ and $\{0_2, 1_2, 2_2, 3_2, 4_2\}$. One possible way of covering $D_{3,5}$ so that the padding is minimal is: $\{[0_1, 0_2, 1_1, 4_2]_X, [0_1, 1_2, 2_1, 3_2]_X, [1_1, 1_2, 0_1, 2_2]_X, [1_1, 3_2, 2_1, 4_2]_X, [2_1, 0_2, 0_1, 4_2]_X, [1_1, 0_2, 2_1, 3_2]_X, [1_1, 2_2, 2_1, 3_2]_X, [2_1, 2_2, 0_1, 3_2]_X\}$ and $|A(P)|=\{(2_1, 3_2),(3_2, 2_1)\}$. The partite sets for $D_{m-3,5}$ are $\{3_1, 4_1, \ldots, (m-1)_1\}$ and $\{0_2, 1_2, 2_2, 3_2, 4_2\}$. For $D_{m,n-5}$, the partite sets are $\{0_1, 1_1, 2_1\}$ and $\{5_2, 6_2, \ldots, (n-1)_2\}$. Finally, for $D_{m-3,n-5}$, the partite sets are $\{3_1, 4_1, \ldots, (m-1)_1\}$ and $\{5_2, 6_2, \ldots, (n-1)_2\}$. Since $m \equiv 3 \pmod 4$ and $n \equiv 1 \pmod 4$ and $m, n \geq 5$, then $m-3 \equiv 0 \pmod 2$ and $n-5 \equiv 0 \pmod 2$. Therefore, $D_{m-3,5}$, $D_{m,n-5}$, and $D_{m-3,n-5}$ can be decomposed by Theorem 2.3. Thus, the padding, P, satisfies $|A(P)|$=2.

Case 15: If $m \equiv 3 \pmod 4$ and $n \equiv 2 \pmod 4$, where $m \geq 3$, then $D_{m,n}$=$D_{3,2} \cup D_{m-3,2} \cup D_{m,n-2} \cup D_{m-3,n-2}$. Now cover each smaller graph with X. For $D_{3,2}$, the partite sets are $\{0_1, 1_1, 2_1\}$ and $\{0_2, 1_2\}$. One possible way of covering $D_{3,2}$ so that the padding is minimal is: $\{[0_1, 0_2, 1_1, 1_2]_X, [1_2, 1_1, 0_2, 2_1]_X, [2_1, 0_2, 0_1, 1_2]_X, [2_1, 1_2, 0_1, 0_2]_X\}$ and $|A(P)|$=$\{(2_1, 0_2),(2_1, 1_2), (0_2, 0_1), (1_2, 0_1)\}$. The partite sets for $D_{m-3,2}$ are $\{3_1, 4_1, \ldots, (m-1)_1\}$ and $\{0_2, 1_2\}$. For $D_{m,n-2}$, the partite sets are $\{0_1, 1_1, 2_1\}$ and $\{2_2, 3_2, \ldots, (n-1)_2\}$. Finally, for $D_{m-3,n-2}$, the partite sets are $\{3_1, 4_1, \ldots, (m-1)_1\}$ and $\{2_2, 3_2, \ldots, (n-1)_2\}$. Since $m \equiv 3 \pmod 4$ and $n \equiv 2 \pmod 4$ and $m, n \geq 5$, then $m-3 \equiv 0 \pmod 2$ and $n-2 \equiv 0 \pmod 2$. Therefore, $D_{m-3,2}$, $D_{m,n-2}$, and $D_{m-3,n-2}$ can be decomposed by Theorem 2.3. Thus, the padding, P, satisfies $|A(P)|$=4.

Case 16: If $m \equiv n \equiv 3 \pmod 4$, where $m \geq 3$, then $D_{m,n}$=$D_{3,3} \cup D_{m-3,3} \cup D_{m,n-3} \cup D_{m-3,n-3}$. Now cover each smaller graph with X. For $D_{3,3}$, the partite sets are $\{0_1, 1_1, 2_1\}$ and $\{0_2, 1_2, 2_2\}$. One possible way of covering $D_{3,3}$ so that the padding is minimal is: $\{[0_1, 0_2, 2_1, 1_2]_X, [2_2, 1_1, 1_2, 2_1]_X, [1_1, 0_2, 0_1, 2_2]_X, [1_2, 0_1, 0_2, 1_1]_X, [2_1, 2_2, 0_1, 0_2]_X\}$ and $|A(P)|$=$\{(0_1, 0_2),(0_2, 0_1)\}$. The partite sets for $D_{m-3,3}$ are $\{3_1, 4_1, \ldots, (m-1)_1\}$ and $\{0_2, 1_2, 2_2\}$. For $D_{m,n-3}$, the partite sets are $\{0_1, 1_1, 2_1\}$ and $\{3_2, 4_2, \ldots, (n-1)_2\}$. Finally, for $D_{m-3,n-3}$, the partite sets are $\{3_1, 4_1, \ldots, (m-1)_1\}$ and $\{3_2, 4_2, \ldots, (n-1)_2\}$. Since $m \equiv 3 \pmod 4$, $n \equiv 3 \pmod 4$ and $m, n \geq 5$, then $m-3 \equiv 0 \pmod 2$ and $n-3 \equiv 0 \pmod 2$. Therefore, D_{m-3_3}, $D_{m,n-3}$, and $D_{m-3,n-3}$ can be decomposed by Theorem 2.3. Thus, the padding, P, satisfies $|A(P)|$=2. *Q.E.D.*

Thus, each case for X satisfies the conditions of the theorem.

Theorem 2.9 *A minimal Y covering of $D_{m,n}$ satisfies:*

1. $|A(P)|=0$, *when* $mn \equiv 0$ *(mod 2),* $m,n \geq 2$,
2. $|A(P)|$=2, *otherwise.*

Proof. The necessary conditions follow as in Theorem 2.9.

Case 1: If $m \equiv n \equiv 0$ (mod 4), then $mn \equiv 0$ (mod 2). Therefore, by Theorem 2.5 a decomposition exists.

Case 2: If $m \equiv 0$ (mod 4) and $n \equiv 1$ (mod 4), then $mn \equiv 0$ (mod 2). By Theorem 2.5, it follows that a decomposition exists.

Case 3: If $m \equiv 0$ (mod 4) and $n \equiv 2$ (mod 4), then $mn \equiv 0$ (mod 2). Therefore, by Theorem 2.5, a decomposition exists.

Case 4: If $m \equiv 0$ (mod 4) and $n \equiv 3$ (mod 4), then $mn \equiv 0$ (mod 2). Thus, by Theorem 2.5, a decomposition exists.

Case 5: If $m \equiv 1$ (mod 4) and $n \equiv 0$ (mod 4), then $mn \equiv 0$ (mod 2). Therefore, by Theorem 2.5, a decomposition exists.

Case 6: If $m \equiv n \equiv 1$ (mod 4), where $m, n \geq 2$, then $D_{m,n}$=$D_{5,5} \cup D_{m-5,5} \cup D_{m,n-5} \cup D_{m-5,n-5}$. Now cover each smaller graph with Y. For $D_{5,5}$, the partite sets are $\{0_1, 1_1, 2_1, 3_1, 4_1\}$ and $\{0_2, 1_2, 2_2, 3_2, 4_2\}$. One possible way of covering $D_{5,5}$ so that the padding is minimal is: $\{[2_1, 0_2, 3_1, 1_2]_{,Y}$ $[3_2, 0_1, 2_2, 4_1]_Y$, $[1_1, 1_2, 2_1, 3_2]_Y$, $[2_1, 2_2, 1_1, 3_2]_Y$, $[0_1, 0_2, 1_1, 1_2]_Y$, $[2_2, 3_1, 1_2, 4_1]_Y$, $[3_1, 0_2, 2_1, 3_2]_Y$, $[2_1, 3_2, 3_1, 4_2]_Y$, $[0_2, 0_1, 3_2, 4_1]_Y$, $[4_2, 4_1, 0_2, 1_1]_Y$, $[1_1, 4_2, 0_1, 2_2]_Y$, $[1_2, 0_1, 4_2, 4_1]_Y$, $[3_1, 2_2, 2_1, 4_2]_Y\}$ and $|A(P)|$=$\{(2_1, 3_2)$,$(3_2, 2_1)\}$. The partite sets for $D_{m-5,5}$ are $\{5_1, 6_1, \ldots, (m-1)_1\}$ and $\{0_2, 1_2, 2_2, 3_2, 4_2\}$. For $D_{m,n-5}$, the partite sets are $\{0_1, 1_1, 2_1, 3_1, 4_1\}$ and $\{5_2, 6_2, \ldots, (n-1)_2\}$. Finally, for $D_{m-5,n-5}$, the partite sets are $\{5_1, 6_1, \ldots, (m-1)_1\}$

and $\{5_2, 6_2, \ldots, (n-1)_2\}$. Since $m \equiv 1 \pmod 4$, $n \equiv 1 \pmod 4$ and $m, n \geq 2$, then $m-5 \equiv 0 \pmod 2$ and $n-5 \equiv 0 \pmod 2$. Therefore, $D_{m-5,5}$, $D_{m,n-5}$, and $D_{m-5,n-5}$ can be decomposed by Theorem 2.5. Thus, the padding, P, satisfies $|A(P)|$=2.

Case 7: If $m \equiv 1 \pmod 4$ and $n \equiv 2 \pmod 4$, then $mn \equiv 0 \pmod 2$. By Theorem 2.5, it follows that a decomposition exists.

Case 8: If $m \equiv 1 \pmod 4$ and $n \equiv 3 \pmod 4$, where $m, n \geq 2$, then $D_{m,n}$=$D_{5,3}$ $\cup$ $D_{m-5,3} \cup D_{m,n-3} \cup D_{m-5,n-3}$. Now cover each smaller graph with Y. For $D_{5,3}$, the partite sets are $\{0_1, 1_1, 2_1, 3_1, 4_1\}$ and $\{0_2, 1_2, 2_2\}$. One possible way of covering $D_{5,3}$ so that the padding is minimal is: $\{[0_2, 0_1, 1_2, 1_1]_Y, [1_2, 3_1, 2_2, 4_1]_Y, [2_2, 1_1, 0_2, 2_1]_Y, [2_1, 2_2, 0_1, 1_2]_Y, [4_1, 0_2, 2_1, 1_2]_Y, [1_1, 0_2, 3_1, 2_2]_Y, [3_1, 0_2, 1_1, 1_2]_Y, [0_1, 2_2, 4_1, 0_2]_Y\}$ and $|A(P)|$=$\{(1_1, 0_2),(0_2, 1_1)\}$. The partite sets for $D_{m-5,3}$ are $\{5_1, 6_1, \ldots, (m-1)_1\}$ and $\{0_2, 1_2, 2_2\}$. For $D_{m,n-3}$, the partite sets are $\{0_1, 1_1, 2_1, 3_1, 4_1\}$ and $\{3_2, 4_2, \ldots, (n-1)_2\}$. Finally, for $D_{m-5,n-3}$, the partite sets are $\{5_1, 6_1, \ldots, (m-1)_1\}$ and $\{3_2, 4_2, \ldots, (n-1)_2\}$. Since $m \equiv 1 \pmod 4$ and $n \equiv 3 \pmod 4$ and $m, n \geq 2$, then $m-5 \equiv 0 \pmod 2$ and $n-3 \equiv 0 \pmod 2$. Therefore, $D_{m-5,3}$, $D_{m,n-3}$, and $D_{m-5,n-3}$ can be decomposed by Theorem 2.5. Thus, the padding, P, satisfies $|A(P)|$=2.

Case 9: If $m \equiv 2 \pmod 4$ and $n \equiv 0 \pmod 4$, then $mn \equiv 0 \pmod 2$. Therefore, by Theorem 2.5, a decomposition exists.

Case 10: If $m \equiv 2 \pmod 4$ and $n \equiv 1 \pmod 4$, then $mn \equiv 0 \pmod 2$. Thus, by Theorem 2.5, a decomposition exists.

Case 11: If $m \equiv n \equiv 2 \pmod 4$, then $mn \equiv 0 \pmod 2$. Therefore, by Theorem 2.5, a decomposition exists.

Case 12: If $m \equiv 2 \pmod 4$ and $n \equiv 3 \pmod 4$, then $mn \equiv 0 \pmod 2$. By Theorem 2.5, it follows that a decomposition exists.

Case 13: If $m \equiv 3 \pmod 4$ and $n \equiv 0 \pmod 4$, then $mn \equiv 0 \pmod 2$. Therefore, by Theorem 2.5, a decomposition exists.

Case 14: If $m \equiv 3 \pmod 4$ and $n \equiv 1 \pmod 4$, where $m, n \geq 2$, then $D_{m,n}$=$D_{3,5}$ $\cup$ $D_{m-3,5}$ $\cup$ $D_{m,n-5}$ $\cup$ $D_{m-3,n-5}$. Now cover each smaller graph with Y. For $D_{3,5}$, the partite sets are $\{0_1, 1_1, 2_1\}$ and $\{0_2, 1_2, 2_2, 3_2, 4_2\}$. One possible way of covering $D_{3,5}$ so that the padding is minimal is: $\{[0_1, 0_2, 1_1, 1_2]_Y, [1_1, 3_2, 2_1, 4_2]_Y, [2_1, 1_2, 0_1, 2_2]_Y, [2_2, 2_1, 0_2, 1_1]_Y, [4_2, 0_1, 2_2, 1_1]_Y, [0_2, 2_1, 4_2, 0_1]_Y, [1_2, 0_1, 3_2, 2_1]_Y, [3_2, 0_1, 1_2, 1_1]_Y\}$ and $|A(P)|$=$\{(0_1, 1_2),(1_2, 0_1)\}$. The partite sets for $D_{m-3,5}$ are $\{3_1, 4_1, \ldots, (m-1)_1\}$ and $\{0_2, 1_2, 2_2, 3_2, 4_2\}$. For $D_{m,n-5}$, the partite sets are $\{0_1, 1_1, 2_1\}$ and $\{5_2, 6_2, \ldots, (n-1)_2\}$. Finally, for $D_{m-3,n-5}$, the partite sets are $\{3_1, 4_1, \ldots, (m-1)_1\}$ and $\{5_2, 6_2, \ldots, (n-1)_2\}$. Since $m \equiv 3 \pmod 4$ $n \equiv$ and $1 \pmod 4$ and $m, n \geq 2$, then $m-3 \equiv 0 \pmod 2$ and $n-5 \equiv 0 \pmod 2$. Therefore, $D_{m-3,5}$, $D_{m,n-5}$, and $D_{m-3,n-5}$ can be decomposed by Theorem 2.5. Thus, the padding, P, satisfies $|A(P)|$=2.

Case 15: If $m \equiv 3 \pmod 4$ and $n \equiv 2 \pmod 4$, then $m \equiv n \equiv 0 \pmod 2$. Therefore, by Theorem 2.5, a decomposition exists.

Case 16: If $m \equiv n \equiv 3 \pmod 4$, where $m, n \geq 2$, then $D_{m,n}$=$D_{3,3}$ $\cup$ $D_{m-3,3}$ $\cup$ $D_{m,n-3}$ $\cup$ $D_{m-3,n-3}$. Now cover each smaller graph with Y. For $D_{3,3}$, the partite sets are $\{0_1, 1_1, 2_1\}$ and $\{0_2, 1_2, 2_2\}$. One possible way of covering $D_{3,3}$ so that the padding is minimal is: $\{[2_1, 0_2, 0_1, 1_2]_Y, [0_1, 2_2, 1_1, 0_2]_Y, [1_1, 0_2, 2_1, 2_2]_Y, [1_2, 2_1, 2_2, 1_1]_Y, [2_2, 1_1, 1_2, 0_1]_Y\}$ and $|A(P)|$=$\{(1_1, 2_2),(2_2, 1_1)\}$. The partite sets for $D_{m-3,3}$ are $\{3_1, 4_1, \ldots, (m-1)_1\}$ and $\{0_2, 1_2, 2_2\}$. For $D_{m,n-3}$, the partite sets are $\{0_1, 1_1,$

$2_1\}$ and $\{3_2, 4_2, \ldots, (n-1)_2\}$. Finally, for $D_{m-3,n-3}$, the partite sets are $\{3_1, 4_1, \ldots, (m-1)_1\}$ and $\{3_2, 4_2, \ldots, (n-1)_2\}$. Since $m \equiv 3 \pmod 4$, $n \equiv 3 \pmod 4$ and $m, n \geq 2$, then $m-3 \equiv 0 \pmod 2$ and $n-3 \equiv 0 \pmod 2$. Therefore, $D_{m-3,3}$, $D_{m,n-3}$, and $D_{m-3,n-3}$ can be decomposed by Theorem 2.5. Thus, the padding, P, satisfies $|A(P)|$=2.*Q.E.D.*

Thus, each case for Y satisfies the conditions of the theorem. Therefore, theorems 2.1-2.8 provide the necessary and sufficient conditions for maximal packings and minimal coverings of the complete bipartite graph with C_4, X, and Y. Thus, packings and coverings of $D_{m,n}$ have been completed. Now we will pack and cover the complete directed graphs.

3 COVERING D_v WITH THE C_4, X, AND Y ORIENTATIONS OF A 4−CYCLE

If $\{d_1, d_2, \ldots, d_n\}$ is a covering of D_v with copies of d, then we define the digraph P_v with arc set $A(P) = \bigcup_{i=1}^{n} A(g_i) \setminus A(G)$, as the padding of the covering. Therefore a minimal covering of D_v minimizes $|A(P)|$. We will now give necessary and sufficient conditions for minimizing $|A(P)|$ with a minimal covering of D_v for the various orientations of the 4−cycle. We need to only consider graphs where $v \geq 4$. In this chapter we denote $V(D_v)$ as {0,1,…, v-1}.

Theorem 3.1 *A minimal C_4 covering of D_v satisfies:*

1. $|A(P)|$=0 *if* $v \equiv 0$ *or* 1 (*mod* 4), $v \neq 4$,
2. $|A(P)|$=4 *if* v=4,
3. P=D_2 *if* $v \equiv 2$ *or* 3 (*mod* 4).

Proof. Since $|A(C_4)|$=4, it is necessary that the padding P satisfy $|A(D_v)| + |A(P)| \equiv 0$ (mod 4). Therefore, for $v \equiv 0$ or 1 (mod 4) we have $|A(P)| \geq 0$. For $v \equiv$ 2 or 3 (mod 4), we need $|A(P)| \geq 2$. Since a decomposition does not exist for D_4, it is necessary that $|A(P)| \geq 4$ when v= 4.

A C_4-decomposition of D_v exists if and only if $v \equiv 0$ or 1 (mod 4), $v \neq 4$. [12].

If $v = 4$, then we have a covering of D_4 with copies of C_4: $\{[0,3,2,1]_C, [0,2,1,3]_C, [0,3,1,2]_C, [0,1,2,3]_C\}$. Then the padding is $A(P)$=$\{(0,3),(1,2),(2,1),(3,0)\}$ and $|A(P)|$=4.

For $v \equiv 2$ (mod 4), we know D_v=$D_{v-6} \cup D_{v-6,6} \cup D_6$. The vertex set for D_{v-6} is {0, 1, …, $(v-5)$}. The partite sets for $D_{v-6,6}$ are {0, 1, …, $(v-5)$} and {0, 1, …, 5}.

For D_6, the vertex set is $\{0, 1, \ldots, 5\}$. Since $v-6 \equiv 0 \pmod 4$, D_{v-6} decomposes with the exception of v=10. By Theorem 2.1, $D_{v-6,6}$ can be decomposed. If $v = 6$, then we have a covering of D_6 with copies of C_4: $\{[0,1,2,3]_C, [0,2,4,1]_C, [0,3,1,4]_C, [0,4,2,5]_C, [0,5,3,2]_C, [1,3,5,2]_C, [1,3,4,5]_C, [1,5,4,3]_C\}$. The padding is then $A(P)=\{(1,3),(3,1)\}$. Thus $|A(P)| = 2$.

If $v = 10$, then we have a covering of D_{10} with copies of C_4: $\{[0,1,3,4]_C, [0,2,1,5]_C, [0,3,1,6]_C, [0,4,1,7]_C, [0,5,1,8]_C$ $[0,6,1,9]_C, [0,7,8,1]_C, [0,9,1,2]_C, [1,4,2,7]_C, [2,3,5,9]_C, [2,4,7,5]_C, [2,5,3,8]_C, [2,6,3,7]_C, [2,8,9,3]_C, [2,9,4,6]_C, [3,6,4,8]_C, [3,9,5,7]_C, [4,5,8,7]_C, [4,9,6,5]_C, [5,6,9,8]_C, [6,7,9,8]_C, [6,8,9,7]_C\}$. The padding is then $A(P)=\{(8,9),(9,8)\}$ and $|A(P)|$=2.

For $v \equiv 3 \pmod 4$, we know D_v=$D_{v-7} \cup D_{v-7,7} \cup D_7$. The vertex set for D_{v-7} is $\{0, 1, \ldots, (v-6)\}$. The partite sets for $D_{v-7,7}$ are $\{0, 1, \ldots, (v-6)\}$ and $\{0, 1, \ldots, 6\}$. For D_7, the vertex set is $\{0, 1, \ldots, 6\}$. Since $v - 7 \equiv 0 \pmod 4$, D_{v-7} can be decomposed. By Theorem 2.1, $D_{v-7,7}$ can be decomposed. If $v = 7$, then we have a covering of D_7 with copies of C_4: $\{[0,1,2,3]_C, [0,2,4,6]_C, [0,3,4,5]_C, [0,4,3,2]_C, [0,5,2,1]_C, [0,6,1,4]_C, [1,3,5,4]_C, [1,5,6,3]_C, [1,6,2,5]_C, [2,6,5,3]_C, [3,6,4,2]_C\}$. The padding is then $A(P)=\{(2,3),(3,2)\}$. Then $|A(P)| = 2$. *Q.E.D.*

Thus, each case for C_4 satisfies the conditions of the theorem.

Theorem 3.2 1. *$|A(P)|$=0 if $v \equiv 0$ or 1 (mod 4), $v \neq 5$,*

2. *$|A(P)|$=4 if $v = 5$,*

3. *P=D_2 if $v \equiv 2$ or 3 (mod 4).*

Proof. There necessary conditions follow similar to those in Theorem 3.1.

An X- decomposition of D_v exists if and only if $v \equiv 0$ or 1 (mod 4), $v \neq 5$[6].

If $v = 5$, then we have a covering of D_5 with copies of X: $\{[3,0,1,4]_X, [4,0,2,3]_X, [2,0,3,1]_X, [1,0,4,2]_X, [4,1,3,2]_X, [3,2,4,1]_X\}$. The padding is then $A(P)=\{(3,1),(3,2),(4,1),(4,2)\}$ and $|A(P)|=4$.

For $v \equiv 2 \pmod 4$, we know $D_v = D_{v-6} \cup D_{v-6,6} \cup D_6$. The vertex set for D_{v-6} is $\{0, 1, \ldots, (v-5)\}$. The partite sets for $D_{v-6,6}$ are $\{0, 1, \ldots, (v-5)\}$ and $\{0, 1, \ldots, 5\}$. For D_6, the vertex set is $\{0, 1, \ldots, 5\}$. Since $v - 6 \equiv 0 \pmod 4$, D_{v-6} decomposes except when v=10. By Theorem 2.3, $D_{v-6,6}$ can be decomposed. If v=6, then we have a covering of D_6 with copies of X: $\{[5,0,1,2]_X, [4,0,2,3]_X, [2,0,3,4]_X, [1,0,4,5]_X, [3,0,5,1]_X, [4,1,3,2]_X, [5,3,1,4]_X, [2,1,3,5]_X\}$. Then the padding is $A(P)=\{(1,3),(3,1)\}$. Then $|A(P)|=2$.

If $v = 10$, then we have a covering of D_{10} with copies of X: $\{[3,0,1,2]_X, [6,0,4,5]_X, [9,0,7,8]_X, [8,0,2,4]_X, [7,0,3,5]_X, [2,0,5,7]_X, [5,0,6,9]_X, [1,0,8,3]_X, [4,0,9,1]_X, [2,1,4,9]_X, [6,1,5,2]_X, [5,1,6,8]_X, [3,1,7,6]_X, [7,1,8,9]_X, [8,1,9,5]_X, [9,2,5,3]_X, [8,2,6,7]_X, [4,2,8,6]_X, [6,3,9,4]_X, [7,2,3,4]_X, [9,4,3,7]_X, [5,4,9,6]_X, [4,7,3,8]_X\}$. Then the padding is $A(P)=\{(4,9), (9,4)\}$ and $|A(P)|=2$.

For $v \equiv 3 \pmod 4$, we know $D_v = D_{v-7} \cup D_{v-7,7} \cup D_7$. The vertex set for D_{v-7} is $\{0, 1, \ldots, (v-6)\}$. The partite sets for $D_{v-7,7}$ are $\{0, 1, \ldots, (v-6)\}$ and $\{0, 1, \ldots, 6\}$. For D_7, the vertex set is $\{0, 1, \ldots, 6\}$. Since $v - 7 \equiv 0 \pmod 4$, D_{v-7} can be decomposed. By Theorem 2.3, $D_{v-7,7}$ can be decomposed.If $v = 7$, then we have a covering of D_7 with copies of X: $\{[3,0,1,2]_X, [6,0,2,3]_X, [3,1,0,4]_X, [3,5,0,6]_X, [2,5,1,6]_X, [4,5,2,0]_X, [1,5,2,4]_X, [0,5,4,3]_X, [6,4,2,5]_X, [5,6,1,3]_X, [4,6,2,1]_X\}$. The padding is then $A(P)=\{(2,5),(5,2)\}$ and $|A(P)|=2$. *Q.E.D*

Thus, each case for X satisfies the conditions of the theorem.

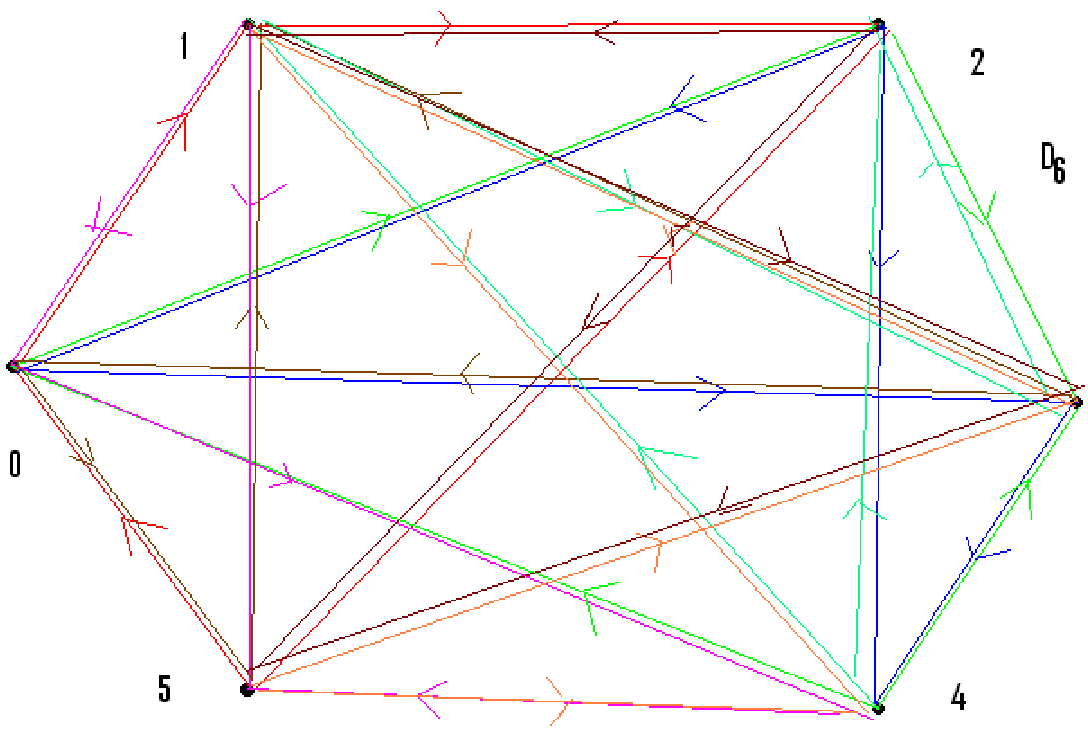

Figure 7: Covering D_6 with Copies of X

Theorem 3.3 *1. $|A(P)|= 0$ if $v \equiv 0$ or 1 $(mod\ 4)$, $v \notin \{4, 5\}$,*

2. $|A(P)|$=4 if $v \in \{4, 5\}$,

3. P=D_2 if $v \equiv 2$ or 3 $(mod\ 4)$.

Proof. The necessary conditions follow similar to those of Theorem 3.1.

A Y-decomposition of D_v exists if and only if $v \equiv 0$ or 1 (mod 4), $v \notin [4, 5]$. [6]

If $v = 4$, then we have a covering of D_4 with copies of Y: {$[0,1,2,3]_Y$, $[2,3,0,1]_Y$, $[2,1,3,0]_Y$, $[3,1,2,0]_Y$}. The padding is then $A(P)$=$\{(0, 3), (0, 2), (2, 0), (3, 0)\}$ and $|A(P)|$=4.

If $v = 5$, then we have a covering of D_5 with copies of Y: $\{[3,0,1,2]_Y$, $[4,0,2,1]_Y$, $[2,1,4,0]_Y$, $[4,2,3,1]_Y$, $[3,4,0,1]_Y$, $[2,4,3,0]_Y\}$. Then the padding is $A(P)=\{(2,0),(2,1),(4,0),(4,1)\}$ and $|A(P)|=4$.

For $v \equiv 2 \pmod 4$, we know $D_v=D_{v-6} \cup D_{v-6,6} \cup D_6$. The vertex set for D_{v-6} is $\{0, 1, \ldots, (v-5)\}$. The partite sets for $D_{v-6,6}$ are $\{0, 1, \ldots, (v-5)\}$ and $\{0, 1, \ldots, 5\}$. For D_6, the vertex set is $\{0, 1, \ldots, 5\}$. Since $v-6 \equiv 0 \pmod 4$, D_{v-6} decomposes except when $v=10$. By Theorem 2.5, $D_{v-6,6}$ can be decomposed. If $v = 6$, then we have a covering of D_6 with copies of Y: $\{[3,0,1,2]_Y$, $[4,0,2,5]_Y$, $[2,0,5,3]_Y$, $[0,3,1,4]_Y$, $[5,1,2,4]_Y$, $[1,4,3,5]_Y$, $[1,2,4,3]_Y$, $[2,1,0,5]_Y\}$. The padding is then $A(P)=\{(1,2),(2,1)\}$ and $|A(P)|=2$.

If $v = 10$, then we have a covering of D_{10} with copies of Y: $\{[3,0,1,2]_Y$, $[6,3,4,5]_Y$, $[7,1,2,8]_Y$, $[8,4,5,9]_Y$, $[2,5,6,3]_Y$, $[6,1,0,9]_Y$, $[0,4,3,7]_Y$, $[8,1,3,5]_Y$, $[3,5,8,1]_Y$, $[1,7,4,9]_Y$, $[1,4,0,5]_Y$, $[7,5,1,9]_Y$, $[7,0,8,2]_Y$, $[4,1,6,2]_Y$, $[0,5,2,6]_Y$, $[5,4,7,9]_Y$, $[0,2,9,3]_Y$, $[6,0,9,4]_Y$, $[2,4,6,7]_Y$, $[9,3,8,6]_Y$, $[8,3,7,6]_Y$, $[9,2,0,8]_Y$, $[4,8,7,5]_Y\}$. The padding is then $A(P)=\{(4,5),(5,4)\}$and $|A(P)|=2$.

For $v \equiv 3 \pmod 4$, we know $D_v=D_{v-7} \cup D_{v-7,7} \cup D_7$. The vertex set for D_{v-7} is $\{0, 1, \ldots, (v-6)\}$. The partite sets for $D_{v-7,7}$ are $\{0, 1, \ldots, (v-6)\}$ and $\{0, 1, \ldots, 6\}$. For D_7, the vertex set is $\{0, 1, \ldots, 6\}$. Since $v-7 \equiv 0 \pmod 4$, D_{v-7} can be decomposed . By Theorem 2.5, $D_{v-7,7}$ can be decomposed.If $v = 7$, then we have a covering of D_7 with copies of Y: $\{[3,0,1,2]_Y$, $[5,0,2,4]_Y$, $[6,0,2,3]_Y$, $[1,0,4,3]_Y$, $[4,0,5,1]_Y$, $[2,0,6,5]_Y$, $[6,3,1,5]_Y$, $[5,2,6,3]_Y$, $[1,6,4,2]_Y$, $[1,3,5,4]_Y$, $[4,6,1,3]_Y\}$. The padding is then $A(P)=\{(1,3),(3,1)\}$ and $|A(P)|=2$. *Q.E.D.*

Thus, each case for Y satisfies the conditions of the theorem. Therefore, Theorems 3.1-3.3 provide the necessary and sufficient conditions for maximal packings and minimal coverings of the complete directed graph with C_4, X, and Y. Now we will pack and cover the complete digraph on v vertices and a hole of size w.

4 PACKING AND COVERING $D(v, w)$ WITH THE C_4, X, AND Y ORIENTATIONS OF THE 4−CYCLE

With the completion of packings and coverings of the complete bipartite graph and the complete directed graph, we can pack and cover complete directed graphs with holes. The complete directed graph on v vertices with a hole of size w is a graph with v vertices that are mutually adjacent and a set of w vertices that are adjacent to the v vertices and incident to the w vertices. We will begin with the maximal C_4-packings.

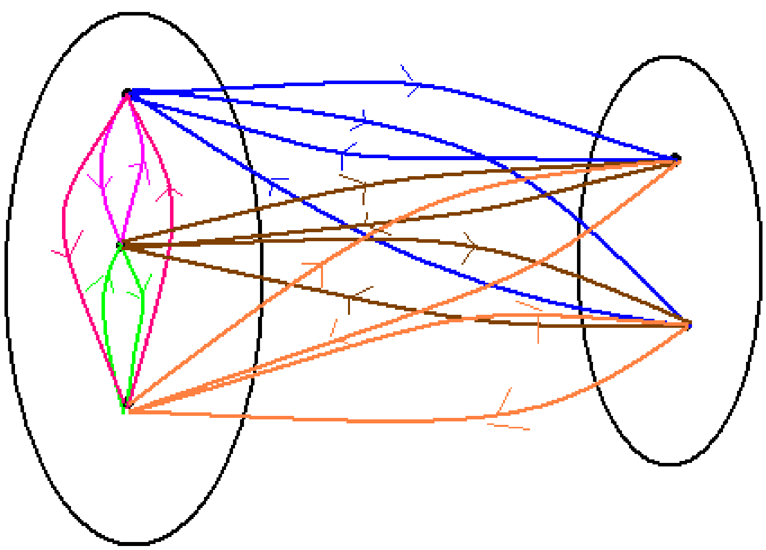

Figure 8: Example of $D(v, w)$

Theorem 4.1 [5] *A C_4-decomposition of D_v with a hole of size w exists if and only if $\{v\ (mod\ 4),\ w\ (mod\ 4)\} \subset \{0,1\}$ or $\{v\ (mod\ 4),\ w\ (mod\ 4)\} \subset \{2,3\}$ and $v - w > 3$.*

Theorem 4.2 [5] *An optimal packing of D_v with copies of C_4 and leave L satisfies:*

1. *$L = \emptyset$ if $v \equiv 0$ or $1\ (mod\ 4)$, $v \neq 4$,*
2. *$|A(L)| = 4$ if $v = 4$,*
3. *$L = D_2$ if $v \equiv 2$ or $3\ (mod\ 4)$.*

Theorem 4.3 *A maximal C_4 packing of $D(v, w)$ satisfies:*

1. *$|A(L)| = 0$ if $\{v\ (mod\ 4),\ w\ (mod\ 4)\} \subset \{0,1\}$ or $\{v\ (mod\ 4),\ w\ (mod\ 4)\} \subset \{2,3\}$, $v - w > 3$,*
2. *$|A(L)| = 2$, otherwise.*

Proof. Since $|A(C_4)| = 4$, it is necessary that the leave, L, satisfy $|A(L)| \equiv |A(D(v,w))| (\text{mod } 4)$. The necessary conditions on $|A(L)|$ follow.

Case 1: If $v \equiv w \equiv 0$ (mod 4), then $v \equiv w \equiv 0$ (mod 2). Therefore, by Theorem 4.1, a decomposition exists.

Case 2: If $v \equiv 0$ (mod 4) and $w \equiv 1$ (mod 4), then $\{v$ (mod 4), w (mod 4)$\} \subset$ {0,1}. Thus, by Theorem 4.1, a decomposition exists.

Case 3: If $v \equiv 0$ (mod 4) and $w \equiv 2$ (mod 4), then we know the number of arcs is $v(v-1) - w(w-1) \equiv 2$ (mod 4). Then $D(v, w) = D_{v-w} \cup D_{v-w,w}$. Note that for D_{v-w}, the vertex set is $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$. For $D_{v-w,w}$, the partite sets are $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$ and $\{(0_2, 1_2, \ldots, (w-1)_2\}$. Since $v \equiv 0$ (mod 4) and $w \equiv 2$ (mod 4), $D_{v-w,w}$ can be decomposed by Theorem 2.1. D_{v-w} can be packed with leave $L = D_2$ by Theorem 4.2. Thus $|A(L)| = 2$.

Case 4: If $v \equiv 0 \pmod 4$ and $w \equiv 3 \pmod 4$, then we know the number of arcs is $v(v-1) - w(w-1) \equiv 2 \pmod 4$. Then $D(v,w)$=$D_{v-w} \cup D_{v-w,w}$. Note that for D_{v-w}, the vertex set is $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$. For $D_{v-w,w}$, the partite sets are $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$ and $\{(0_2, 1_2, \ldots, (w-1)_2\}$. It follows from Theorem 2.2, that $D_{v-w,w}$ can be packed and $|A(L)|$=2. Thus D_{v-w} can be decomposed by Theorem 4.1.

Case 5: If $v \equiv 1 \pmod 4$ and $w \equiv 0 \pmod 4$, then $\{v \pmod 4, w \pmod 4\} \subset \{0,1\}$. Therefore, by Theorem 4.1, a decomposition exists.

Case 6: If $v \equiv w \equiv 1 \pmod 4$, then $\{v \pmod 4, w \pmod 4\} \subset \{0,1\}$. Thus, by Theorem 4.1, a decomposition exists.

Case 7: If $v \equiv 1 \pmod 4$ and $w \equiv 2 \pmod 4$, then we know the number of arcs is $v(v-1) - w(w-1) \equiv 2 \pmod 4$. Then $D(v,w)$=$D_{v-w} \cup D_{v-w,w}$. Note that for D_{v-w}, the vertex set is $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$. For $D_{v-w,w}$, the partite sets are $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$ and $\{(0_2, 1_2, \ldots, (w-1)_2\}$. Since $v \equiv 1 \pmod 4$ and $w \equiv 2 \pmod 4$, then $D_{v-w,w}$ can be decomposed by Theorem 2.1. D_{v-w} can be packed with leave L=D_2 by Theorem 4.2. Thus $|A(L)|$=2.

Case 8: If $v \equiv 1 \pmod 4$ and $w \equiv 3 \pmod 4$, then we know the number of arcs is $v(v-1) - w(w-1) \equiv 2 \pmod 4$. Then $D(v,w)$=$D_{v-w} \cup D_{v-w,w}$. Note that for D_{v-w}, the vertex set is $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$. For $D_{v-w,w}$, the partite sets are $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$ and $\{(0_2, 1_2, \ldots, (w-1)_2\}$. Note that since, $v \equiv 1 \pmod 4$ and $w \equiv 3 \pmod 4$, then $D_{v-w,w}$ can be decomposed by Theorem 2.1. D_{v-w} can be packed with leave L=D_2 by Theorem 4.2. Thus $|A(L)|$=2.

Case 9: If $v \equiv 2 \pmod 4$ and $w \equiv 0 \pmod 4$, then we know the number of arcs is $v(v-1) - w(w-1) \equiv 2 \pmod 4$. Then $D(v,w)$=$D_{v-w} \cup D_{v-w,w}$. Note that for D_{v-w}, the vertex set is $\{(0_1,\ 1_1,\ \ldots,\ (v-w-1)_1)\}$. For $D_{v-w,w}$, the partite sets are $\{(0_1,\ 1_1,\ \ldots,\ (v-w-1)_1)\}$ and $\{(0_2,\ 1_2,\ \ldots,\ (w-1)_2\}$. Note that since, $v \equiv 2$ (mod 4) and $w \equiv 0 \pmod 4$, then $D_{v-w,w}$ can be decomposed by Theorem 2.1. D_{v-w} can be packed with leave L=D_2 by Theorem 4.2. Then $|A(L)|$=2.

Case 10: If $v \equiv 2 \pmod 4$ and $w \equiv 1 \pmod 4$, then we know the number of arcs is $v(v-1) - w(w-1) \equiv 2 \pmod 4$. Then $D(v,w)$=$D_{v-w} \cup D_{v-w,w}$. Note that for D_{v-w}, the vertex set is $\{(0_1,\ 1_1,\ \ldots,\ (v-w-1)_1)\}$. For $D_{v-w,w}$, the partite sets are $\{(0_1,\ 1_1,\ \ldots,\ (v-w-1)_1)\}$ and $\{(0_2,\ 1_2,\ \ldots,\ (w-1)_2\}$. By Theorem 2.2, $D_{v-w,w}$ can be packed and $|A(L)|$=2. D_{v-w} can be decomposed by Theorem 4.1.

Case 11: If $v \equiv w \equiv 2 \pmod 4$, then $\{v \pmod 4,\ w \pmod 4\} \subset \{2,3\}$. Therefore, by Theorem 4.1, a decomposition exists.

Case 12: If $v \equiv 2 \pmod 4$ and $w \equiv 3 \pmod 4$, then $\{v \pmod 4,\ w \pmod 4\} \subset \{2,3\}$. Thus, by Theorem 4.1, a decomposition exists.

Case 13: If $v \equiv 3 \pmod 4$ and $w \equiv 0 \pmod 4$, then the number of arcs is $v(v-1) - w(w-1) \equiv 2 \pmod 4$. Thus $D(v,w)$=$D_{v-w} \cup D_{v-w,w}$. Note that for D_{v-w}, the vertex set is $\{(0_1,\ 1_1,\ \ldots,\ (v-w-1)_1)\}$. For $D_{v-w,w}$, the partite sets are $\{(0_1,\ 1_1,\ \ldots,\ (v-w-1)_1)\}$ and $\{(0_2,\ 1_2,\ \ldots,\ (w-1)_2\}$. Since $v \equiv 3 \pmod 4$ and $w \equiv 0 \pmod 4$, then $D_{v-w,w}$ can be decomposed by Theorem 2.1. D_{v-w} can be packed with leave L=D_2 by Theorem 4.2. Thus $|A(L)|$=2.

Case 14: If $v \equiv 3 \pmod 4$ and $w \equiv 1 \pmod 4$, then we know the number of arcs is $v(v-1) - w(w-1) \equiv 2 \pmod 4$. Then $D(v,w)$=$D_{v-w} \cup D_{v-w,w}$. Note that for

D_{v-w}, the vertex set is $\{(0_1,\ 1_1,\ \ldots,\ (v-w-1)_1)\}$. For $D_{v-w,w}$, the partite sets are $\{(0_1,\ 1_1,\ \ldots,\ (v-w-1)_1)\}$ and $\{(0_2,\ 1_2,\ \ldots,\ (w-1)_2\}$. Since $v \equiv 3 \pmod 4$ and $w \equiv 0 \pmod 4$, then $D_{v-w,w}$ can be decomposed by Theorem 2.1. D_{v-w} can be packed with leave L=D_2 by Theorem 4.2. Thus $|A(L)|$=2.

Case 15: If $v \equiv 3 \pmod 4$ and $w \equiv 2 \pmod 4$, then $\{v \pmod 4,\ w \pmod 4\} \subset \{2{,}3\}$. Therefore, by Theorem 4.1, a decomposition exists.

Case 16: If $v \equiv w \equiv 3 \pmod 4$, then $\{v \pmod 4,\ w \pmod 4\} \subset \{2{,}3\}$. Then, by Theorem 4.1, a decomposition exists. *Q.E.D*

Thus, each case for C_4 satisfies the conditions of the theorem.

Theorem 4.4 [5] *An X decomposition of D_v with a hole of size w exists if and only if $\{v\ (mod\ 4),\ w\ (mod\ 4)\} \subset \{0,1\}$ or $\{v\ (mod\ 4),\ w\ (mod\ 4)\} \subset \{2,3\}$ and $v-w \neq 3$ in the case of $v \equiv 2\ (mod\ 4)$ and $w \equiv 3\ (mod\ 4)$.*

Theorem 4.5 [5] *An optimal packing of D_v with copies of X and leave L satisfies:*

1. *L=$\emptyset$ if $v \equiv 0$ or $1\ (mod\ 4)$, $v \neq 5$,*
2. *$|A(L)|$=4 if $v = 5$,*
3. *L=D_2 if $v \equiv 2$ or $3\ (mod\ 4)$.*

Theorem 4.6 *A maximal X packing of $D(v, w)$ satisfies:*

1. *L=$\emptyset$ if $\{v\ (mod\ 4),\ w\ (mod\ 4)\} \subset \{0,1\}$ or $\{v\ (mod\ 4),\ w\ (mod\ 4)\} \subset \{2,3\}$,*
2. *$|A(L)|$=2, otherwise.*

Proof. The necessary conditions follow similarly to that of Theorem 4.3.

Case 1: If $v \equiv w \equiv 0 \pmod 4$, then $v \equiv w \equiv 0 \pmod 2$. Thus, by Theorem 4.4, a decomposition exists.

Case 2: If $v \equiv 0 \pmod 4$ and $w \equiv 1 \pmod 4$, then $\{v \pmod 4, w \pmod 4\} \subset \{0,1\}$. Therefore, by Theorem 4.4, a decomposition exists.

Case 3: If $v \equiv 0 \pmod 4$ and $w \equiv 2 \pmod 4$, then we know the number of arcs is $v(v-1) - w(w-1) \equiv 2 \pmod 4$. Then $D(v,w)$=$D_{v-w} \cup D_{v-w,w}$. Note that for D_{v-w}, the vertex set is $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$. For $D_{v-w,w}$, the partite sets are $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$ and $\{(0_2, 1_2, \ldots, (w-1)_2\}$. Since $v \equiv 0 \pmod 4$ and $w \equiv 2 \pmod 4$, then $D_{v-w,w}$ can be decomposed by Theorem 2.3. Then D_{v-w} can be packed with leave L=D_2 by Theorem 4.5. Thus $|A(L)|$=2.

Case 4: If $v \equiv 0 \pmod 4$ and $w \equiv 3 \pmod 4$, then we know the number of arcs is $v(v-1) - w(w-1) \equiv 2 \pmod 4$. Then $D(v,w)$=$D_{v-w} \cup D_{v-w,w}$. Note that for D_{v-w}, the vertex set is $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$. For $D_{v-w,w}$, the partite sets are $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$ and $\{(0_2, 1_2, \ldots, (w-1)_2\}$. By Theorem 2.4, $D_{v-w,w}$ can be packed and $|A(L)|$=2. It follows that D_{v-w} can be decomposed by Theorem 4.5. Thus $|A(L)|$=2.

Case 5: If $v \equiv 1 \pmod 4$ and $w \equiv 0 \pmod 4$, then $\{v \pmod 4, w \pmod 4\} \subset \{0,1\}$. Therefore, by Theorem 4.4 a decomposition exists.

Case 6: If $v \equiv w \equiv 1 \pmod 4$, then $\{v \pmod 4, w \pmod 4\} \subset \{0,1\}$. Therefore, by Theorem 4.4 a decomposition exists.

Case 7: If $v \equiv 1 \pmod 4$ and $w \equiv 2 \pmod 4$, then we know the number of arcs is $v(v-1) - w(w-1) \equiv 2 \pmod 4$. Then $D(v,w)$=$D_{v-w-2} \cup D_{v-w-2,w+2} \cup D_{2,w}$. Note that the vertex set for D_{v-w-2} is $\{0_1, 1_1, \ldots, (v-w-3)_1\}$. For $D_{v-w-2,w+2}$, the partite sets are $\{0_1, 1_1, \ldots, (v-w-3)_1\}$ and $\{0_2, 1_2, \ldots, (v-w+1)_2\}$. For $D_{2,w}$, the partite sets are $\{(v-2)_1, (v-1)_1\}$ and $\{0_2, 1_2, \ldots, (w-1)_2\}$. By Theorem 4.5,

D_{v-w-2} can be decomposed. $D_{v-w-2,w+2}$ can be decomposed by Theorem 2.3. $D_{2,w}$ can be decomposed by Theorem 2.3. Therefore $|A(L)|$=2.

Case 8: If $v \equiv 1 \pmod 4$ and $w \equiv 3 \pmod 4$, then we know the number of arcs is $v(v-1) - w(w-1) \equiv 2 \pmod 4$. Then $D(v,w)$=$D_{v-w-2} \cup D_{v-w-2,w+2} \cup D_{2,w}$. Note that the vertex set for D_{v-w-2} is $\{0_1, 1_1, \ldots, (v-w-3)_1\}$. For $D_{v-w-2,w+2}$, the partite sets are $\{0_1, 1_1, \ldots, (v-w-3)_1\}$ and $\{0_2, 1_2, \ldots, (v-w+1)_2\}$. For $D_{2,w}$, the partite sets are $\{(v-2)_1, (v-1)_1\}$ and $\{0_2, 1_2, \ldots, (w-1)_2\}$. Now note by Theorem 4.5, D_{v-w-2} can be decomposed. $D_{v-w-2,w+2}$ by Theorem 2.3. $D(w+2,2)$ can be packed by case 7 with L=D_2. Therefore $|A(L)|$=2.

Case 9: If $v \equiv 2 \pmod 4$ and $w \equiv 0 \pmod 4$, then we know the number of arcs is $v(v-1) - w(w-1) \equiv 2 \pmod 4$. Then $D(v,w)$=$D_{v-w} \cup D_{v-w,w}$. Note that for D_{v-w}, the vertex set is $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$. For $D_{v-w,w}$, the partite sets are $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$ and $\{(0_2, 1_2, \ldots, (w-1)_2\}$. Since $v \equiv 2 \pmod 4$ and $w \equiv 0 \pmod 4$, then $D_{v-w,w}$ can be decomposed by Theorem 2.3. D_{v-w} can be packed with leave L=D_2 by Theorem 4.5. Then $|A(L)|$=2.

Case 10: If $v \equiv 2 \pmod 4$ and $w \equiv 1 \pmod 4$, then we know the number of arcs is $v(v-1) - w(w-1) \equiv 2 \pmod 4$. Then $D(v,w)$=$D_{v-w} \cup D_{v-w,w}$. Note that for D_{v-w}, the vertex set is $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$. For $D_{v-w,w}$, the partite sets are $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$ and $\{(0_2, 1_2, \ldots, (w-1)_2\}$. By Theorem 2.4, $D_{v-w,w}$ can be packed and $|A(L)|$=2. Thus D_{v-w} can be decomposed by Theorem 4.5. Then $|A(L)|$=2.

Case 11: If $v \equiv w \equiv 2 \pmod 4$, then $\{v \pmod 4, w \pmod 4\} \subset \{2,3\}$. Therefore, by Theorem 4.4, a decomposition exists.

Case 12: If $v \equiv 2 \pmod 4$ and $w \equiv 3 \pmod 4$, then $\{v \pmod 4, w \pmod 4\} \subset \{2,3\}$. Thus, by Theorem 4.4, a decomposition exists.

Case 13: If $v \equiv 3 \pmod 4$ and $w \equiv 0 \pmod 4$, then we know the number of arcs is $v(v-1) - w(w-1) \equiv 2 \pmod 4$. Then $D(v,w)$=$D_{v-w} \cup D_{v-w,w}$. Note that for D_{v-w}, the vertex set is $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$. For $D_{v-w,w}$, the partite sets are $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$ and $\{(0_2, 1_2, \ldots, (w-1)_2\}$. Since $v \equiv 3 \pmod 4$ and $w \equiv 0 \pmod 4$, then $D_{v-w,w}$ can be decomposed by Theorem 2.3. Thus D_{v-w} can be packed with leave L=D_2 by Theorem 4.5. Then $|A(L)|$=2.

Case 14: If $v \equiv 3 \pmod 4$ and $w \equiv 1 \pmod 4$, then we know the number of arcs is $v(v-1) - w(w-1) \equiv 2 \pmod 4$. Then $D(v,w)$=$D_{v-w-3} \cup D_{v-w-3,3} \cup D_{v-w-3,w}$ $\cup$ $D(w+3,w)$=$D_{v-w-3} \cup D_{v-w-3,w+3} \cup D(w+3,w)$. Note that the vertex set for D_{v-w-3} is $\{0_1, 1_1, \ldots, (v-w-4)_1\}$. For $D_{v-w-3,w+3}$, the partite sets are $\{0_1, 1_1, \ldots, (v-w-4)_1\}$ and $\{0_2, 1_2, \ldots, (v-w+2)_2\}$. For $D(w+3,w)$, the partite sets are $\{0_2, 1_2, \ldots, (w+2)_2\}$ and $\{0_2, 1_2, \ldots, (w-1)_2\}$. D_{v-w-3} can be packed by Theorem 4.4. By Theorem 2.3, $D_{v-w-3,w+3}$ can be decomposed. $D(w+3,w)$ can be decomposed by Theorem 4.5. Therefore $|A(L)|$=2.

Case 15: If $v \equiv 3 \pmod 4$ and $w \equiv 2 \pmod 4$, then $\{v \pmod 4, w \pmod 4\} \subset \{2,3\}$. Therefore, by Theorem 4.4 a decomposition exists.

Case 16: If $v \equiv w \equiv 3 \pmod 4$, then $\{v \pmod 4, w \pmod 4\} \subset \{2,3\}$. Therefore, by Theorem 4.4 a decomposition exists.*Q.E.D*

Thus, each case for X satisfies the conditions of the theorem.

Theorem 4.7 [5] *A Y decomposition of D_v with a hole of size w exists if and only if $\{v\ (mod 4),\ w\ (mod 4)\} \subset \{0,1\}$ or $\{v\ (mod 4),\ w\ (mod 4)\} \subset \{2,3\}$ and $v - w \neq 3$.*

Theorem 4.8 [5] *An optimal packing of D_v with copies of Y and leave L satisfies:*

1. $L=\emptyset$ *if* $v \equiv 0$ *or* 1 $(mod\ 4)$, $v \notin \{4, 5\}$,

2. $|A(L)|=4$ *if* $v \in \{4, 5\}$,

3. $|A(L)|=2$ *if* $v \equiv 2$ *or* 3 $(mod\ 4)$.

Theorem 4.9 *A maximal Y packing of* $D(v, w)$ *satisfies:*

1. $|A(L)|=0$ *if* $\{v\ (mod\ 4),\ w\ (mod\ 4)\} \subset \{0, 1\}$ *or* $\{v\ (mod\ 4),\ w\ (mod\ 4)\} \subset \{2, 3\}$,

2. $|A(L)|=2$, *otherwise.*

Proof. The necessary conditions follow similarly to those of Theorem 4.3.

Case 1: If $v \equiv w \equiv 0$ (mod 4), then $v \equiv w \equiv 0$ (mod 2). Therefore, by Theorem 4.7 a decomposition exists.

Case 2: If $v \equiv 0$ (mod 4) and $w \equiv 1$ (mod 4), then $\{v$ (mod 4), w (mod 4)$\} \subset$ {0,1}. Therefore, by Theorem 4.7 a decomposition exists.

Case 3: If $v \equiv 0$ (mod 4) and $w \equiv 2$ (mod 4), then we know the number of arcs is $v(v-1) - w(w-1) \equiv 2$ (mod 4). Then $D(v, w)$=$D_{v-w} \cup D_{v-w,w}$. Note that for D_{v-w}, the vertex set is $\{(0_1,\ 1_1,\ \ldots,\ (v-w-1)_1)\}$. For $D_{v-w,w}$, the partite sets are $\{(0_1,\ 1_1,\ \ldots,\ (v-w-1)_1)\}$ and $\{(0_2,\ 1_2,\ \ldots,\ (w-1)_2\}$. Note that since , $v \equiv 0$ (mod 4) and $w \equiv$ and 2 (mod 4), then $D_{v-w,w}$ can be decomposed by Theorem 2.5. D_{v-w} can be packed with leave L=D_2 by Theorem 4.8. Then $|A(L)|$=2.

Case 4: If $v \equiv 0$ (mod 4) and $w \equiv 3$ (mod 4), then we know the number of arcs is $v(v-1) - w(w-1) \equiv 2$ (mod 4). Then $D(v, w)$=$D_{v-w} \cup D_{v-w,w}$. Note that for D_{v-w}, the vertex set is $\{(0_1,\ 1_1,\ \ldots,\ (v-w-1)_1)\}$. For $D_{v-w,w}$, the partite sets are

$\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$ and $\{(0_2, 1_2, \ldots, (w-1)_2\}$.By Theorem 2.6, $D_{v-w,w}$ can be packed and $|A(L)|$=2. D_{v-w} can be decomposed by Theorem 4.8.

Case 5: If $v \equiv 1 \pmod 4$ and $w \equiv 0 \pmod 4$, then $\{v \pmod 4, w \pmod 4\} \subset \{0,1\}$. Therefore, by Theorem 4.7 a decomposition exists.

Case 6: If $v \equiv w \equiv 1 \pmod 4$, then $\{v \pmod 4, w \pmod 4\} \subset \{0,1\}$. Therefore, by Theorem 4.7 a decomposition exists.

Case 7: If $v \equiv 1 \pmod 4$ and $w \equiv 2 \pmod 4$, then we know the number of arcs is $v(v-1) - w(w-1) \equiv 2 \pmod 4$. Then $D(v,w)$=$D_{v-w-2} \cup D_{v-w-2,2} \cup D_{v-w-2,w}$=$D_{v-w-2} \cup D_{v-w-2,w+2} \cup D_{2,w} \cup D_2$. Note that the vertex set for D_{v-w-2} is $\{0_1, 1_1, \ldots, (v-w-3)_1\}$. For $D_{v-w-2,w+2}$, the partite sets are $\{0_1, 1_1, \ldots, (v-w-3)_1\}$ and $\{0_2, 1_2, \ldots, (v-w+1)_2\}$. For $D_{2,w}$, the partite sets are $\{(v-2)_1, (v-1)_1\}$ and $\{0_2, 1_2, \ldots, (w-1)_2\}$. By Theorem 4.8, D_{v-w-2} can be decomposed. $D_{v-w-2,w+2}$ can be decomposed by Theorem 2.5. $D_{2,w}$ can be decomposed by Theorem 2.5. Therefore $|A(L)|$=2.

Case 8: If $v \equiv 1 \pmod 4$ and $w \equiv 3 \pmod 4$, then we know the number of arcs is $v(v-1) - w(w-1) \equiv 2 \pmod 4$. Then $D(v,w)$=$D_{v-w-2} \cup D_{v-w-2,2} \cup D_{v-w-2,w}$=$D_{v-w-2} \cup D_{v-w-2,w+2} \cup D_{2,w} \cup D_2$. Note that the vertex set for D_{v-w-2} is $\{0_1, 1_1, \ldots, (v-w-3)_1\}$. For $D_{v-w-2,w+2}$, the partite sets are $\{0_1, 1_1, \ldots, (v-w-3)_1\}$ and $\{0_2, 1_2, \ldots, (v-w+1)_2\}$. For $D_{2,w}$, the partite sets are $\{(v-2)_1, (v-1)_1\}$ and $\{0_2, 1_2, \ldots, (w-1)_2\}$. Now note by Theorem 4.8, D_{v-w-2} can be decomposed. $D_{v-w-2,w+2}$ can be decomposed by Theorem 2.5. $D(w+2,2)$ can be packed by Case 7 with L=D_2. Therefore $|A(L)|$=2.

Case 9: If $v \equiv 2 \pmod 4$ and $w \equiv 0 \pmod 4$, then we know the number of arcs is $v(v-1) - w(w-1) \equiv 2 \pmod 4$. Then $D(v,w)$=$D_{v-w} \cup D_{v-w,w}$. Note that for D_{v-w}, the vertex set is $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$. For $D_{v-w,w}$, the partite sets are $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$ and $\{(0_2, 1_2, \ldots, (w-1)_2\}$. Since $v \equiv 2 \pmod 4$ and $w \equiv 0 \pmod 4$, then $D_{v-w,w}$ can be decomposed by Theorem 2.5. D_{v-w} can be packed with leave L=D_2 by Theorem 4.8. Then $|A(L)|$=2.

Case 10: If $v \equiv 2 \pmod 4$ and $w \equiv 1 \pmod 4$, then we know the number of arcs is $v(v-1) - w(w-1) \equiv 2 \pmod 4$. Then $D(v,w)$=$D_{v-w} \cup D_{v-w,w}$. Note that for D_{v-w}, the vertex set is $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$. For $D_{v-w,w}$, the partite sets are $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$ and $\{(0_2, 1_2, \ldots, (w-1)_2\}$. By Theorem 2.6, $D_{v-w,w}$ can be packed and $|A(L)|$=2. D_{v-w} can be decomposed by Theorem 4.8.

Case 11: If $v \equiv w \equiv 2 \pmod 4$, then $\{v \pmod 4, w \pmod 4\} \subset \{2,3\}$. Therefore, by Theorem 4.7 a decomposition exists.

Case 12: If $v \equiv 2 \pmod 4$ and $w \equiv 3 \pmod 4$, then $\{v \pmod 4, w \pmod 4\} \subset \{2,3\}$. Thus, by Theorem 4.7 a decomposition exists.

Case 13: If $v \equiv 3 \pmod 4$ and $w \equiv 0 \pmod 4$, then we know the number of arcs is $v(v-1) - w(w-1) \equiv 2 \pmod 4$. Then $D(v,w)$=$D_{v-w} \cup D_{v-w,w}$. Note that for D_{v-w}, the partite sets are $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$. For $D_{v-w,w}$, the partite sets are $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$ and $\{(0_2, 1_2, \ldots, (w-1)_2\}$. Since $v \equiv 3 \pmod 4$ $w \equiv 0 \pmod 4$, $D_{v-w,w}$ can be decomposed by Theorem 2.5. D_{v-w} can be packed with leave L=D_2 by Theorem 4.8. Then $|A(L)|$=2.

Case 14: If $v \equiv 3 \pmod 4$ and $w \equiv 1 \pmod 4$, then we know the number of arcs is $v(v-1) - w(w-1) \equiv 2 \pmod 4$. Then $D(v,w)$=$D_{v-w-2} \cup D_{v-w-2,2} \cup$

$D_{v-w-2,w}$=$D_{v-w-2} \cup D_{v-w-2,w+2} \cup D(w+2,w)$. Note that the vertex set for D_{v-w-2} is $\{0, 1, \ldots, (v-w-3)\}$. For $D_{v-w-2,w+2}$, the partite sets are $\{0_1, 1_1, \ldots, (v-w-3)_1\}$ and $\{0_2, 1_2, \ldots, (v-w+1)_2\}$. For $D_{2,w}$, the partite sets are $\{(v-2)_1, (v-1)_1\}$ and $\{0_2, 1_2, \ldots, (w-1)_2\}$. D_{v-w-2} can be decomposed by Theorem 4.8. By Theorem 2.5, $D_{v-w-2,w+2}$ can be decomposed. $D(w+2,w)$=$D_{2,w} \cup D_2$. $D_{2,w}$ can be decomposed by Theorem 4.7. Therefore $|A(L)|$=2.

Case 15: If $v \equiv 3 \pmod 4$ and $w \equiv 2 \pmod 4$, then $\{v \pmod 4, w \pmod 4\} \subset$ {2,3}. Therefore, by Theorem 4.7 a decomposition exists.

Case 16: If $v \equiv w \equiv 3 \pmod 4$, then $\{v \pmod 4, w \pmod 4\} \subset$ {2,3}. Therefore, by Theorem 4.7 a decomposition exists. *Q.E.D*

Thus, each case for Y satisfies the conditions of the theorem. Now we want to cover $D(v,w)$ with C_4.

Theorem 4.10 *A minimal C_4-covering of $D(v,w)$ satisfies:*

1. $|A(L)|$=0 *if* $\{v \ (mod\ 4),\ w \ (mod\ 4)\} \subset \{0,1\}$ *or* $\{v \ (mod\ 4),\ w \ (mod\ 4)\} \subset \{2,3\}$, *and* $v-w>3$,

2. $|A(L)|$=2, *otherwise.*

Proof. The necessary conditions follow similar to those of Theorem 2.9.

Case 1: If $v \equiv w \equiv 0 \pmod 4$, then $v \equiv w \equiv 0 \pmod 2$. Therefore, by Theorem 4.1, a decomposition exists.

Case 2: If $v \equiv 0 \pmod 4$ and $w \equiv 1 \pmod 4$, then $\{v \pmod 4, w \pmod 4\} \subset$ {0,1}. Therefore, by Theorem 4.1, a decomposition exists.

Case 3: If $v \equiv 0 \pmod 4$ and $w \equiv 2 \pmod 4$, then we know the number of arcs is $v(v-1) - w(w-1) \equiv 2 \pmod 4$. Then $D(v,w)$=$D_{v-w} \cup D_{v-w,w}$. Note that for

D_{v-w}, the vertex set is $\{(0_1,\ 1_1,\ \ldots,\ (v-w-1)_1)\}$. For $D_{v-w,w}$, the partite sets are $\{(0_1,\ 1_1,\ \ldots,\ (v-w-1)_1)\}$ and $\{(0_2,\ 1_2,\ \ldots,\ (w-1)_2\}$. Since , $v \equiv 0 \pmod 4$ and $w \equiv 2 \pmod 4$, then $D_{v-w,w}$ can be decomposed by Theorem 2.1. D_{v-w} can be covered with padding P=D_2 by Theorem 3.2. Thus $|A(P)|$=2.

Case 4: If $v \equiv 0 \pmod 4$ and $w \equiv 3 \pmod 4$, then we know the number of arcs is $v(v-1) - w(w-1) \equiv 2 \pmod 4$. Then $D(v,w)$=$D_{v-w} \cup D_{v-w,w}$. Note that for D_{v-w}, the vertex set is $\{(0_1,\ 1_1,\ \ldots,\ (v-w-1)_1)\}$. For $D_{v-w,w}$, the partite sets are $\{(0_1,\ 1_1,\ \ldots,\ (v-w-1)_1)\}$ and $\{(0_2,\ 1_2,\ \ldots,\ (w-1)_2\}$. Since, $v \equiv 0 \pmod 4$ and $w \equiv 3 \pmod 4$, then $D_{v-w,w}$ can be decomposed by Theorem 2.1. D_{v-w} can be covered with padding P=D_2 by Theorem 3.2. Thus $|A(P)|$=2.

Case 5: If $v \equiv 1 \pmod 4$ and $w \equiv 0 \pmod 4$, then $\{v \pmod 4,\ w \pmod 4\} \subset \{0,1\}$. Therefore, by Theorem 4.1 a decomposition exists.

Case 6: If $v \equiv w \equiv 1 \pmod 4$, then $\{v \pmod 4,\ w \pmod 4\} \subset \{0,1\}$. Thus, by Theorem 4.1 a decomposition exists.

Case 7: If $v \equiv 1 \pmod 4$ and $w \equiv 2 \pmod 4$, then we know the number of arcs is $v(v-1) - w(w-1) \equiv 2 \pmod 4$. Then $D(v,w)$=$D_{v-w} \cup D_{v-w,w}$. Note that for D_{v-w}, the partite sets are $\{(0_1,\ 1_1,\ \ldots,\ (v-w-1)_1)\}$. For $D_{v-w,w}$, the partite sets are $\{(0_1,\ 1_1,\ \ldots,\ (v-w-1)_1)\}$ and $\{(0_2,\ 1_2,\ \ldots,\ (w-1)_2\}$. Since $v \equiv 1 \pmod 4$ and $w \equiv 2 \pmod 4$, then $D_{v-w,w}$ can be decomposed by Theorem 2.1. D_{v-w} can be covered with padding P=D_2 by Theorem 3.2. Thus $|A(P)|$=2.

Case 8: If $v \equiv 1 \pmod 4$ and $w \equiv 3 \pmod 4$, then we know the number of arcs is $v(v-1) - w(w-1) \equiv 2 \pmod 4$. Then $D(v,w)$=$D_{v-w} \cup D_{v-w,w}$. Note that for D_{v-w}, the vertex set is $\{(0_1,\ 1_1,\ \ldots,\ (v-w-1)_1)\}$. For $D_{v-w,w}$, the partite sets are

$\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$ and $\{(0_2, 1_2, \ldots, (w-1)_2\}$. Note that since, $v \equiv 1 \pmod 4$ and $w \equiv 3 \pmod 4$, then $D_{v-w,w}$ can be decomposed by Theorem 2.1. D_{v-w} can be covered with padding P=D_2 by Theorem 3.2. Then $|A(P)|$=2.

Case 9: If $v \equiv 2 \pmod 4$ and $w \equiv 0 \pmod 4$, then we know the number of arcs is $v(v-1) - w(w-1) \equiv 2 \pmod 4$. Then $D(v,w)$=$D_{v-w} \cup D_{v-w,w}$. Note that for D_{v-w}, the vertex set is $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$. For $D_{v-w,w}$, the partite sets are $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$ and $\{(0_2, 1_2, \ldots, (w-1)_2\}$. Since $v \equiv 2 \pmod 4$ and $w \equiv 0 \pmod 4$, then $D_{v-w,w}$ can be decomposed by Theorem 2.1. D_{v-w} can be covered with padding P=D_2 by Theorem 3.2. Thus $|A(P)|$=2.

Case 10: If $v \equiv 2 \pmod 4$ and $w \equiv 1 \pmod 4$, then we know the number of arcs is $v(v-1) - w(w-1) \equiv 2 \pmod 4$. Then $D(v,w)$=$D_{v-w} \cup D_{v-w,w}$. Note that for D_{v-w}, the vertex set is $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$. For $D_{v-w,w}$, the partite sets are $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$ and $\{(0_2, 1_2, \ldots, (w-1)_2\}$. Since $v \equiv 2 \pmod 4$ and $w \equiv 1 \pmod 4$, then $D_{v-w,w}$ can be decomposed by Theorem 2.1. D_{v-w} can be covered with padding P=D_2 by Theorem 3.2. Thus $|A(P)|$=2.

Case 11: If $v \equiv w \equiv 2 \pmod 4$, then $\{v \pmod 4, w \pmod 4\} \subset \{2,3\}$. Therefore, by Theorem 4.1, a decomposition exists.

Case 12: If $v \equiv 2 \pmod 4$ and $w \equiv 3 \pmod 4$, then $\{v \pmod 4, w \pmod 4\} \subset \{2,3\}$. Thus, by Theorem 4.1, a decomposition exists.

Case 13: If $v \equiv 3 \pmod 4$ and $w \equiv 0 \pmod 4$, then we know the number of arcs is $v(v-1) - w(w-1) \equiv 2 \pmod 4$. Then $D(v,w)$=$D_{v-w} \cup D_{v-w,w}$. Note that for D_{v-w}, the vertex set is $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$. For $D_{v-w,w}$, the partite sets are $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$ and $\{(0_2, 1_2, \ldots, (w-1)_2\}$. Since $v \equiv 3 \pmod 4$ and $w \equiv$

0 (mod 4), then $D_{v-w,w}$ can be decomposed by Theorem 2.1. D_{v-w} can be covered with padding P=D_2 by Theorem 3.2. Thus $|A(P)|$=2.

Case 14: If $v \equiv 3 \pmod 4$ and $w \equiv 1 \pmod 4$, then we know the number of arcs is $v(v-1) - w(w-1) \equiv 2 \pmod 4$. Then $D(v,w)$=$D_{v-w} \cup D_{v-w,w}$. Note that for D_{v-w}, the vertex set is $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$. For $D_{v-w,w}$, the partite sets are $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$ and $\{(0_2, 1_2, \ldots, (w-1)_2\}$. Since $v \equiv 3 \pmod 4$ and $w \equiv 1 \pmod 4$, then $D_{v-w,w}$ can be decomposed by Theorem 2.1. D_{v-w} can be covered with padding P=D_2 by Theorem 3.2. Thus $|A(P)|$=2.

Case 15: If $v \equiv 3 \pmod 4$ and $w \equiv 2 \pmod 4$, then $\{v \pmod 4, w \pmod 4\} \subset \{2,3\}$. Therefore, by Theorem 4.1, a decomposition exists.

Case 16: If $v \equiv w \equiv 3 \pmod 4$, then $\{v \pmod 4, w \pmod 4\} \subset \{2,3\}$. Thus, by Theorem 4.1, a decomposition exists. *Q.E.D*

Thus, each case for C_4 satisfies the conditions of the theorem.

Theorem 4.11 *A minimal X covering of $D(v,w)$ satisfies:*

1. $|A(L)|$=0 *if* $\{v \ (mod\ 4),\ w \ (mod\ 4)\} \subset \{0,1\}$ *or* $\{v \ (mod\ 4),\ w \ (mod\ 4)\} \subset \{2,3\}$,

2. $|A(L)|$=2, *otherwise.*

Proof. The necessary conditions follow similar to those of Theorem 2.9.

Case 1: If $v \equiv w \equiv 0 \pmod 4$, then $v \equiv w \equiv 0 \pmod 2$. Therefore, by Theorem 4.4 a decomposition exists.

Case 2: If $v \equiv 0 \pmod 4$ and $w \equiv 1 \pmod 4$, then $\{v \pmod 4, w \pmod 4\} \subset \{0,1\}$. Therefore, by Theorem 4.4 a decomposition exists.

Case 3: If $v \equiv 0 \pmod 4$ and $w \equiv 2 \pmod 4$, then we know the number of arcs is $v(v-1) - w(w-1) \equiv 2 \pmod 4$. Then $D(v,w)$=$D_{v-w} \cup D_{v-w,w}$. Note that for D_{v-w}, the vertex set is $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$. For $D_{v-w,w}$, the partite sets are $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$ and $\{(0_2, 1_2, \ldots, (w-1)_2\}$. Since $v \equiv 0 \pmod 4$ and $w \equiv 2 \pmod 4$, then $D_{v-w,w}$ can be decomposed by Theorem 2.3. D_{v-w} can be covered with padding P=D_2 by Theorem 3.1. Thus $|A(P)|$=2.

Case 4: If $v \equiv 0 \pmod 4$ and $w \equiv 3 \pmod 4$, then we know the number of arcs is $v(v-1) - w(w-1) \equiv 2 \pmod 4$. Then $D(v,w)$=$D_{v-w} \cup D_{v-w,w}$. Note that for D_{v-w}, the vertex set is $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$. For $D_{v-w,w}$, the partite sets are $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$ and $\{(0_2, 1_2, \ldots, (w-1)_2\}$. Since $v \equiv 0 \pmod 4$ and $w \equiv 3 \pmod 4$, $D_{v-w,w}$ can be packed by Theorem 2.9 with padding P=D_2. D_{v-w} can be decomposed by Theorem 4.5. Thus $|A(P)|$=2.

Case 5: If $v \equiv 1 \pmod 4$ and $w \equiv 0 \pmod 4$, then $\{v \pmod 4, w \pmod 4\} \subset \{0,1\}$. Therefore, by Theorem 4.4, a decomposition exists.

Case 6: If $v \equiv w \equiv 1 \pmod 4$, then $\{v \pmod 4, w \pmod 4\} \subset \{0,1\}$. Thus, by Theorem 4.4, a decomposition exists.

Case 7: If $v \equiv 1 \pmod 4$ and $w \equiv 2 \pmod 4$, then we know the number of arcs is $v(v-1) - w(w-1) \equiv 2 \pmod 4$. Then $D(v,w)$=$D_{v-w-2} \cup D_{v-w-2,2} \cup D_{v-w-2,w}$=$D_{v-w-2} \cup D_{v-w-2,w+2} \cup D_{2,w} \cup D_2$. Note that the vertex set for D_{v-w-2} is $\{0_1, 1_1, \ldots, (v-w-3)_1\}$. For $D_{v-w-2,w+2}$, the partite sets are $\{0_1, 1_1, \ldots, (v-w-3)_1\}$ and $\{0_2, 1_2, \ldots, (v-w+1)_2\}$. For $D_{2,w}$, the partite sets are $\{(v-2)_1, (v-1)_1\}$ and $\{0_2, 1_2, \ldots, (w-1)_2\}$. Now note by Theorem 4.5, D_{v-w-2} can be decomposed. $D_{v-w-2,w+2}$ by Theorem 2.3. $D_{2,w}$ can be decomposed by Theorem 2.3.

When covering D_2 the padding is D_2. Therefore $|A(P)|$=2.

Case 8: If $v \equiv 1 \pmod 4$ and $w \equiv 3 \pmod 4$, then we know the number of arcs is $v(v-1) - w(w-1) \equiv 2 \pmod 4$. Then $D(v,w)$=$D_{v-w-2} \cup D_{v-w-2,2} \cup D_{v-w-2,w}$=$D_{v-w-2} \cup D_{v-w-2,w+2} \cup D_{2,w} \cup D_2$. Note that the vertex set for D_{v-w-2} is $\{0_1,\ 1_1,\ \ldots,\ (v-w-3)_1\}$. For $D_{v-w-2,w+2}$, the partite sets are $\{0_1,\ 1_1,\ \ldots,\ (v-w-3)_1\}$ and $\{0_2,\ 1_2,\ \ldots,\ (v-w+1)_2\}$. For $D_{2,w}$, the partite sets are $\{(v-2)_1,\ (v-1)_1\}$ and $\{0_2,\ 1_2,\ \ldots,\ (w-1)_2\}$. By Theorem 4.5, D_{v-w-2} can be decomposed. $D_{v-w-2,w+2}$ can be decomposed by Theorem 2.3. $D(w+2,2)$ can be covered by case 7 with P=D_2. Therefore $|A(P)|$=2.

Case 9: If $v \equiv 2 \pmod 4$ and $w \equiv 0 \pmod 4$, then we know the number of arcs is $v(v-1) - w(w-1) \equiv 2 \pmod 4$. Then $D(v,w)$=$D_{v-w} \cup D_{v-w,w}$. Note that for D_{v-w}, the vertex set is $\{(0_1,\ 1_1,\ \ldots,\ (v-w-1)_1)\}$. For $D_{v-w,w}$, the partite sets are $\{(0_1,\ 1_1,\ \ldots,\ (v-w-1)_1)\}$ and $\{(0_2,\ 1_2,\ \ldots,\ (w-1)_2\}$. Since $v \equiv 0 \pmod 4$ and $w \equiv 2 \pmod 4$, $D_{v-w,w}$ can be decomposed by Theorem 2.3. D_{v-w} can be covered with padding P=D_2 by Theorem 3.1. Thus $|A(P)|$=2.

Case 10: If $v \equiv 2 \pmod 4$ and $w \equiv 1 \pmod 4$, then we know the number of arcs is $v(v-1) - w(w-1) \equiv 2 \pmod 4$. Then $D(v,w)$=$D_{v-w} \cup D_{v-w,w}$. Note that for D_{v-w}, the vertex set is $\{(0_1,\ 1_1,\ \ldots,\ (v-w-1)_1)\}$. For $D_{v-w,w}$, the partite sets are $\{(0_1,\ 1_1,\ \ldots,\ (v-w-1)_1)\}$ and $\{(0_2,\ 1_2,\ \ldots,\ (w-1)_2\}$. Since $v \equiv 2 \pmod 4$ and $w \equiv 1 \pmod 4$, then $D_{v-w,w}$ can be covered by Theorem 2.9 with padding P=D_2. D_{v-w} can be decomposed by Theorem 4.5. Thus $|A(P)|$=2.

Case 11: If $v \equiv w \equiv 2 \pmod 4$, then $\{v \pmod 4,\ w \pmod 4\} \subset \{2,3\}$. Therefore, by Theorem 4.4, a decomposition exists.

Case 12: If $v \equiv 2 \pmod 4$ and $w \equiv 3 \pmod 4$, then $\{v \pmod 4, w \pmod 4\} \subset \{2,3\}$. Thus, by Theorem 4.4, a decomposition exists.

Case 13: If $v \equiv 3 \pmod 4$ and $w \equiv 0 \pmod 4$, then we know the number of arcs is $v(v-1) - w(w-1) \equiv 2 \pmod 4$. Then $D(v,w)$=$D_{v-w} \cup D_{v-w,w}$. Note that for D_{v-w}, the vertex set is $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$. For $D_{v-w,w}$, the partite sets are $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$ and $\{(0_2, 1_2, \ldots, (w-1)_2\}$. Since $v \equiv 3 \pmod 4$ and $w \equiv 0 \pmod 4$, then $D_{v-w,w}$ can be decomposed by Theorem 2.3. D_{v-w} can be covered with padding P=D_2 by Theorem 3.1. Thus $|A(P)|$=2.

Case 14: If $v \equiv 3 \pmod 4$ and $w \equiv 1 \pmod 4$, then we know the number of arcs is $v(v-1) - w(w-1) \equiv 2 \pmod 4$. Then $D(v,w)$=$D_{v-w-3} \cup D_{v-w-3,3} \cup D_{v-w-3,w}$=$D_{v-w-3} \cup D_{v-w-3,w+3} \cup D(w+3,w)$. Note that the vertex set for D_{v-w-3} is $\{0_1, 1_1, \ldots, (v-w-4)_1\}$. For $D_{v-w-3,w+3}$, the partite sets are $\{0_1, 1_1, \ldots, (v-w-4)_1\}$ and $\{0_2, 1_2, \ldots, (v-w+2)_2\}$. For $D(w+3,w)$, the partite sets are $\{0_2, 1_2, \ldots, (w+2)_2\}$ and $\{0_2, 1_2, \ldots, (w-1)_2\}$. D_{v-w-3} can be covered by Theorem 3.2. By Theorem 2.3, $D_{v-w-3,w+3}$ can be decomposed. $D(w+3,w)$ can be decomposed by Theorem 4.7. Therefore $|A(P)|$=2.

Case 15: If $v \equiv 3 \pmod 4$ and $w \equiv 2 \pmod 4$, then $\{v \pmod 4, w \pmod 4\} \subset \{2,3\}$. Therefore, by Theorem 4.4, a decomposition exists.

Case 16: If $v \equiv w \equiv 3 \pmod 4$, then $\{v \pmod 4, w \pmod 4\} \subset \{2,3\}$. Thus, by Theorem 4.4, a decomposition exists. *Q.E.D*

Thus, each case for X satisfies the conditions of the theorem.

Theorem 4.12 *A minimal Y covering of $D(v,w)$ satisfies:*

1. $|A(L)|$=0 *if* $\{v \ (mod\ 4),\ w \ (mod\ 4)\} \subset \{0,1\}$ *or* $\{v \ (mod\ 4),\ w \ (mod\ 4)\} \subset$

$\{2, 3\}$,

2. $|A(L)|$=2, *otherwise.*

Proof. The necessary conditions follow as in Theorem 2.9.

Case 1: If $v \equiv w \equiv 0 \pmod 4$, then $v \equiv w \equiv 0 \pmod 2$. Therefore, by Theorem 4.7 a decomposition exists.

Case 2: If $v \equiv 0 \pmod 4$ and $w \equiv 1 \pmod 4$, then $\{v \pmod 4, w \pmod 4\} \subset \{0,1\}$. Therefore, by Theorem 4.7 a decomposition exists.

Case 3: If $v \equiv 0 \pmod 4$ and $w \equiv 2 \pmod 4$, then we know the number of arcs is $v(v-1) - w(w-1) \equiv 2 \pmod 4$. Then $D(v, w)$=$D_{v-w} \cup D_{v-w,w}$. Note that for D_{v-w}, the vertex set is $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$. For $D_{v-w,w}$, the partite sets are $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$ and $\{(0_2, 1_2, \ldots, (w-1)_2\}$. Since $v \equiv 0 \pmod 4$ and $w \equiv 2 \pmod 4$, then $D_{v-w,w}$ can be decomposed by Theorem 2.5. D_{v-w} can be covered with padding P=D_2 by Theorem 3.3. Thus $|A(P)|$=2.

Case 4: If $v \equiv 0 \pmod 4$ and $w \equiv 3 \pmod 4$, then we know the number of arcs is $v(v-1) - w(w-1) \equiv 2 \pmod 4$. Then $D(v, w)$=$D_{v-w} \cup D_{v-w,w}$. Note that for D_{v-w}, the vertex set is $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$. For $D_{v-w,w}$, the partite sets are $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$ and $\{(0_2, 1_2, \ldots, (w-1)_2\}$. Since $v \equiv 0 \pmod 4$ and $w \equiv 3 \pmod 4$, then $D_{v-w,w}$ can be covered by Theorem 2.11 with padding P=D_2. D_{v-w} can be decomposed by Theorem 4.8. Thus $|A(P)|$=2.

Case 5: If $v \equiv 1 \pmod 4$ and $w \equiv 0 \pmod 4$, then $\{v \pmod 4, w \pmod 4\} \subset \{0,1\}$. Therefore, by Theorem 4.7, a decomposition exists.

Case 6: If $v \equiv w \equiv 1 \pmod 4$, then $\{v \pmod 4, w \pmod 4\} \subset \{0,1\}$. Thus, by Theorem 4.7, a decomposition exists.

Case 7: If $v \equiv 1 \pmod 4$ and $w \equiv 2 \pmod 4$, then we know the number of arcs is $v(v-1) - w(w-1) \equiv 2 \pmod 4$. Then $D(v,w)$=$D_{v-w-2} \cup D_{v-w-2,2} \cup D_{v-w-2,w}$=$D_{v-w-2} \cup D_{v-w-2,w+2} \cup D_{2,w} \cup D_2$. Note that the vertex set for D_{v-w-2} is $\{0_1, 1_1, \ldots, (v-w-3)_1\}$. For $D_{v-w-2,w+2}$, the partite sets are $\{0_1, 1_1, \ldots, (v-w-3)_1\}$ and $\{0_2, 1_2, \ldots, (v-w+1)_2\}$. For $D_{2,w}$, the partite sets are $\{(v-2)_1, (v-1)_1\}$ and $\{0_2, 1_2, \ldots, (w-1)_2\}$. By Theorem 3.3, D_{v-w-2} can be decomposed. $D_{v-w-2,w+2}$ by Theorem 2.5. $D_{2,w}$ can be decomposed by Theorem 2.5. When covering D_2 the padding is D_2. Therefore $|A(P)|$=2.

Case 8: If $v \equiv 1 \pmod 4$ and $w \equiv 3 \pmod 4$, then we know the number of arcs is $v(v-1) - w(w-1) \equiv 2 \pmod 4$. The $D(v,w)$=$D_{v-w} \cup D_{v-w,w}$. Note that for D_{v-w}, the vertex set is $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$. For $D_{v-w,w}$, the partite sets are $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$ and $\{(0_2, 1_2, \ldots, (w-1)_2\}$. Since , $v \equiv 2 \pmod 4$ and $w \equiv 1 \pmod 4$, then $D_{v-w,w}$ can be covered by Theorem 2.11 with padding P=D_2. D_{v-w} can be decomposed by Theorem 4.8. Thus $|A(P)|$=2.

Case 9: If $v \equiv 2 \pmod 4$ and $w \equiv 0 \pmod 4$, then we know the number of arcs is $v(v-1) - w(w-1) \equiv 2 \pmod 4$. Then $D(v,w)$=$D_{v-w} \cup D_{v-w,w}$. Note that for D_{v-w}, the vertex set is $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$. For $D_{v-w,w}$, the partite sets are $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$ and $\{(0_2, 1_2, \ldots, (w-1)_2\}$. Since $v \equiv 2 \pmod 4$ and $w \equiv 0 \pmod 4$, then $D_{v-w,w}$ can be decomposed by Theorem 2.5. D_{v-w} can be covered with padding P=D_2 by Theorem 3.3. Thus $|A(P)|$=2.

Case 10: If $v \equiv 2 \pmod 4$ and $w \equiv 1 \pmod 4$, then we know the number of arcs is $v(v-1) - w(w-1) \equiv 2 \pmod 4$. Then $D(v,w)$=$D_{v-w} \cup D_{v-w,w}$. Note that for D_{v-w}, the vertex set is $\{(0_1, 1_1, \ldots, (v-w-1)_1)\}$. For $D_{v-w,w}$, the partite sets are

$\{(0_1,\ 1_1,\ \ldots,\ (v-w-1)_1)\}$ and $\{(0_2,\ 1_2,\ \ldots,\ (w-1)_2\}$. Since , $v \equiv 2 \pmod 4$ and $w \equiv 1 \pmod 4$, then $D_{v-w,w}$ can be covered by Theorem 2.11 with padding P=D_2. D_{v-w} can be decomposed by Theorem 4.8. Thus $|A(P)|$=2.

Case 11: If $v \equiv w \equiv 2 \pmod 4$, then $\{v \pmod 4,\ w \pmod 4\} \subset \{2,3\}$. Therefore, by Theorem 4.7, a decomposition exists.

Case 12: If $v \equiv 2 \pmod 4$ and $w \equiv 3 \pmod 4$, then $\{v \pmod 4,\ w \pmod 4\} \subset \{2,3\}$. Thus, by Theorem 4.7, a decomposition exists.

Case 13: If $v \equiv 3 \pmod 4$ and $w \equiv 0 \pmod 4$, then we know the number of arcs is $v(v-1) - w(w-1) \equiv 2 \pmod 4$. Then $D(v,w)$=$D_{v-w} \cup D_{v-w,w}$. Note that for D_{v-w}, the vertex set is $\{(0_1,\ 1_1,\ \ldots,\ (v-w-1)_1)\}$. For $D_{v-w,w}$, the partite sets are $\{(0_1,\ 1_1,\ \ldots,\ (v-w-1)_1)\}$ and $\{(0_2,\ 1_2,\ \ldots,\ (w-1)_2\}$. Since $v \equiv 3 \pmod 4$ and $w \equiv 0 \pmod 4$, then $D_{v-w,w}$ can be decomposed by Theorem 2.5. D_{v-w} can be covered with padding P=D_2 by Theorem 3.3. Thus $|A(P)|$=2.

Case 14: If $v \equiv 3 \pmod 4$ and $w \equiv 1 \pmod 4$, then we know the number of arcs is $v(v-1) - w(w-1) \equiv 2 \pmod 4$. Then $D(v,w)$=$D_{v-w-2} \cup D_{v-w-2,2} \cup D_{v-w-2,w}$=$D_{v-w-2} \cup D_{v-w-2,w+2} \cup D(w+2,w)$. Note that the vertex set for D_{v-w-2} is $\{0_1,\ 1_1,\ \ldots,\ (v-w-3)_1\}$. For $D_{v-w-2,w+2}$, the partite sets are $\{0_1,\ 1_1,\ \ldots,\ (v-w-3)_1\}$ and $\{0_2,\ 1_2,\ \ldots,\ (v-w+1)_2\}$. For $D_{2,w}$, the partite sets are $\{(v-2)_1,\ (v-1)_1\}$ and $\{0_2,\ 1_2,\ \ldots,\ (w-1)_2\}$. D_{v-w-2} can be decomposed by Theorem 4.8. By Theorem 2.5, $D_{v-w-2,w+2}$ can be decomposed. $D(w+2,w)$=$D_{2,w} \cup D_2$. $D_{2,w}$ can be decomposed by Theorem 4.7. When covering D_2 the padding is D_2. Therefore $|A(P)|$=2.

Case 15: If $v \equiv 3 \pmod 4$ and $w \equiv 2 \pmod 4$, then $\{v \pmod 4, w \pmod 4\} \subset \{2{,}3\}$. Therefore, by Theorem 4.7, a decomposition exists.

Case 16: If $v \equiv w \equiv 3 \pmod 4$, then $\{v \pmod 4, w \pmod 4\} \subset \{2{,}3\}$. Thus, by Theorem 4.7, a decomposition exists. *Q.E.D*

Thus, each case for Y satisfies the conditions of the theorem. Therefore, Theorems 4.1-4.12 provide the necessary and sufficient conditions for maximal packings and minimal coverings of the complete directed graph on v vertices with holes of size w with C_4, X, and Y.

5 CONCLUSION

Within this paper we have considered C_4, X, and Y orientations of the 4-cycle. For C_4 and Y packings and coverings of complete bipartite graphs, the leave or padding of the graph was of size 0 or 2. If m and n were both even or both odd, then the size of the leave or padding was 0 or 2. For X, we have found that if m was even and n was odd or m was odd and n was even then the leave and padding was of size 4. In packing and covering D_v with C_4 our leave or padding was of size 0 or 2 unless v=4. If v=4 then the size of the leave or padding was 4. For X, our leave or padding was of size 0 or 2 unless v=5. If v=5 then the size of the leave or padding was 4. For Y, our leave or padding was of size 0 or 2 unless v=4 or 5. If v=4 or 5, then the size of the leave or padding was 4. When we packed and covered $D(v, w)$, if a decomposition existed then the leave or padding was zero. If a decomposition did not exist, then the leave or padding was of size 2. We have shown the necessary and sufficient conditions for finding the maximal padding and minimal covering of the complete bipartite graph, $D_{m,n}$, the complete directed graph, D_v, and the complete directed graph on v vertices with a hole of size w, through several proofs.

BIBLIOGRAPHY

[1] A. Brouwer, Optimal Packings of $K_4's$, *Journal of Combinatorial Theory, Series A* **26** (3) (1979), 278-297.

[2] F. Buckley, A Friendly Introduction to Graph Theory, *Pearson Education, Inc.* (2003), 261-262.

[3] M. Fort and G. Hedlund, Minimal Coverings of Pairs by Triples, *Pacific Journal of Mathematics* **8** (1958), 709-719.

[4] R. Gardner, Optimal Packings and Coverings of the Complete Directed Graph with 3-Circuits and with Transitive Triples, *Congressus Numerantium*, **127** (1997) 161-170.

[5] R. Gardner, C. Huff, and J. Kennedy, Decompositions and Packings of Digraphs with Orientations of a 4-cycle, *Ars Combinatoria*, **54** (2000), 269-282.

[6] F. Harary, W. Wallis, and K. Heinrich, Decompositions of complete symmetric digraphs into the four oriented quadrilaterals, *Combinatorial Mathematics* in Lecture Notes in Mathematics **686** (1978), 165-173.

[7] S. Hung and N. Mendelsohn, Directed Triple Systems, *Journal of Combinatorial Theory Series A*, **14** (1973), 310-318.

[8] J. Kennedy, Two Perfect Maximum Packings and Minimum Coverings of K_n with Hexagons, Ph.D. dissertation, Auburn University, U.S.A. 1995.

[9] E. Mendelsohn and A. Rosa, Embedding maximal packings of triples, *Congressus Numerantium* **40** (1983), 235-247.

[10] N. Mendelsohn, A Natural Generalization of Steiner Triple Systems, *Computers in Number Theory*, eds. A. O. Atkin and B. Birch, Academic Press, London, (1971) 323-338.

[11] J. Schönheim, On Maximal Systems of k-Tuples, *Studia Sci. Math. Hungarica* (1966), 363-368.

[12] J. Schönheim, Partition of the edges of the directed complete graph in 4-cycles, *Discrete Math.* **11** (1975), 67-70.

[13] J. Schönheim and A. Bialostocki, Packing and Covering of the Complete Graph with 4-Cycles, *Canadian Mathematics Bulletin* **18** (5) (1975), 703-708.

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