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Independent Domination in Complementary Prisms.

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Independent Domination in Complementary Prisms

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by

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ABSTRACT

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Let G be a graph and $\overline{G}$ be the complement of G. The complementary prism $G\overline{G}$ of G is the graph formed from the disjoint union of G and $\overline{G}$ by adding the edges of a perfect matching between the corresponding vertices of G and $\overline{G}$. For example, if G is a 5-cycle, then $G\overline{G}$ is the Petersen graph. In this paper we investigate independent domination in complementary prisms.

DEDICATION

I would like to dedicate this to my parents Everardo and Maria G. Góngora who have dedicated their lives to give me the opportunities I have today. Also, to my brother and sister, Homero and Sandra who have always been my number one fans. Finally to Bobbi J. Darretta, the crazy Italian who followed me here.

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1 INTRODUCTION

In [8] the authors study the domination and total domination numbers for complementary prisms. Here we study independent domination for complementary prisms. First, we will cover some basic definitions of graph theory followed by some standard notation that will be seen throughout the paper. Lastly, we will give some examples of complementary prisms.

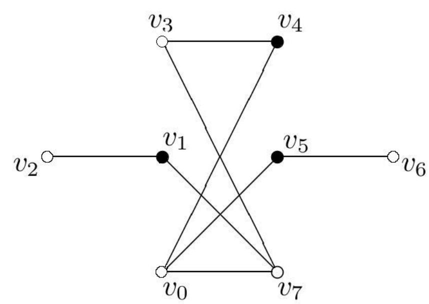

Figure 1: Example of an $i(G)$-set.

1.1 Basic Definitions

Let $G = (V, E)$ be a graph with vertex set V and edge set E. A *dominating set*, denoted DS, of G is a set S of vertices of G such that every vertex in $V \setminus S$ is adjacent to a vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a DS of G. A DS of G of cardinality $\gamma(G)$ is called a $\gamma(G)$-*set.*

If I is a nonempty subset of the vertex set $V(G)$ of G, then the subgraph $< I >$ of G induced by I is the graph having vertex set $V(I)$ and whose edge set consists of those edges of G incident with two elements of $V(I)$. A subset I of $V(G)$ is said to

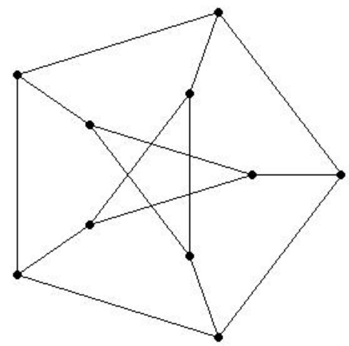

Figure 2: $G\overline{G}$ where $G = C_5$.

be *independent* if and only if the graph $< I >$ has no edges. If the set I dominates the graph G, then we call I an *independent dominating set*, abbreviated IDS.

The *independent domination number* of G, denoted by $i(G)$, is the minimum cardinality of an IDS. An IDS of G of cardinality $i(G)$ is called a $i(G)$-set. Since an IDS is a DS, we have $\gamma(G) \leq i(G)$ for all graphs G. In Figure 1, the set $I = \{v_1, v_4, v_5\}$ is an i(G)-set of the graph G.

The *complementary prism* $G\overline{G}$ of G is the graph formed from the disjoint union $G \cup \overline{G}$ of G and $\overline{G}$ by adding the edges of a perfect matching between the corresponding vertices (same label) of G and $\overline{G}$. As illustrated in Figure [2], the graph $C_5\overline{C}_5$ is the Petersen graph. Also, if $G = K_n$, the graph $K_n\overline{K}_n$ is the corona $K_n \circ K_1$, where the *corona*, $H \circ K_1$, of a graph H is the graph obtained from H by attaching a pendant edge to each vertex of H.

1.2 Notation

For notation and graph theory terminology we, in general, follow [5]. Specifically, let $G = (V, E)$ be a graph with vertex set V of order $n = |V|$ and edge set E of size $m = |E|$, and let v be a vertex in V. The *open neighborhood* of v is the set $N(v) = \{u \in V \mid uv \in E\}$, and the *closed neighborhood of* v is $N[v] = \{v\} \cup N(v)$. For a set $S \subseteq V$, its *open neighborhood* is the set $N(S) = \cup_{v \in S} N(v)$, and its *closed neighborhood* is the set $N[S] = N(S) \cup S$. If $X, Y \subseteq V$, then the set X is said to *independent dominate* the set Y if $Y \subseteq N[X]$, and X is independent. A vertex $w \in V$ is an *S-private neighbor of* $v \in S$ if $N[w] \cap S = \{v\}$, while the *S-private neighbor set of* v, denoted $\text{pn}[v, S]$, is the set of all S-private neighbors of v. An *open S-private neighborhood* is defined similarly for $N(w) \cap S = \{v\}$ and denoted $\text{pn}(v, S)$. The degree of a vertex v is $deg_G(v) = |N(v)|$. The *minimum degree* of G is $\delta(G) = \min\{deg_G(v) | v \in V(G)\}$. The *maximum degree* of G is $\Delta(G) = \max\{deg_G(v) | v \in V(G)\}$. A vertex of degree zero is an *isolated vertex*. A vertex of degree one is called a *leaf* or an *endvertex*, and its neighbor is called a *support vertex*. For any leaf vertex v and support vertex w, the edge vw is called a *pendant edge*.

Let $G\overline{G}$ be the complementary prism of a graph $G = (V, E)$. For notational convenience, we let $\overline{V} = V(\overline{G})$. Note that $V(G\overline{G}) = V \cup \overline{V}$. To simplify our discussion of complementary prisms, we say simply G and $\overline{G}$ to refer to the subgraph copies of G and $\overline{G}$, respectively, in $G\overline{G}$. Also, for a vertex v of G, we let $\overline{v}$ be the corresponding vertex in $\overline{G}$, and for a set $X \subseteq V$, we let $\overline{X}$ be the corresponding set of vertices in $\overline{V}$.

Since we study the independent domination number of complementary prisms, here are some examples of $i(G)$-sets in various complementary prisms, these sets are denoted by the darkened vertices in Figures 3 and 4.

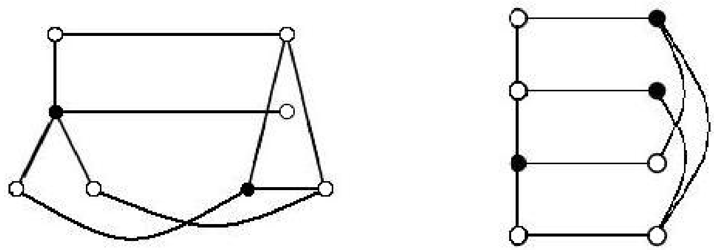

Figure 3: $G\overline{G}$ where $G = K_{3,1}$ (left) and $G = P_4$ (right).

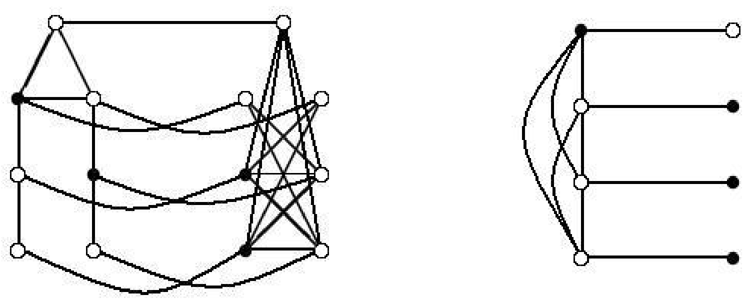

Figure 4: $G\overline{G}$ where $G = K_5$ (right).

2 LITERATURE REVIEW

This chapter consists of a review of the literature of past research on complementary prisms. In the first section, we investigate the relationship between complementary prisms and complementary products which were first introduced by Haynes, Henning, Slater and van der Merwe in [7]. In the following section, we will review previous parameters studied in complementary prisms from [7, 8].

2.1 The Complementary Product of Two Graphs

In [7] Haynes, Henning, Slater and van der Merwe introduced a generalized form of the Cartesian product of two graphs. Let G_1 and G_2 be graphs with $V(G_1) = \{u_1, u_2, ..., u_n\}$ and $V(G_2) = \{v_1, v_2, ..., v_p\}$. The *Cartesian product* of the graphs G_1 and G_2, symbolized by $G_1 \Box G_2$, is the graph formed from G_1 and G_2 in the following manner.

The graph $G_1 \Box G_2$ has np vertices. Each of these vertices has a label taken from $V(G_1) \times V(G_2)$. In $G_1 \Box G_2$, two vertices (u_i, v_j) and (u_r, v_s) are adjacent if and only if one of the following conditions hold:

(i.) $i = r$, and $v_j v_s \in E(G_2)$.

(ii.) $j = s$, and $u_i u_r \in E(G_1)$.

For each i, the induced subgraph on the vertices (u_i, v_j) for $1 \leq j \leq p$ is a copy of G_2, and for each j, the induced subgraph on the vertices (u_i, v_j) for $1 \leq i \leq n$ is a copy of G_1. Note that, $G_1 \Box G_2$ can either be viewed as the graph formed by taking each vertex of G_1, replacing it with a copy of G_2 and matching the corresponding vertices and taking each vertex of G_2, replacing it with a copy of G_1 and matching the corresponding vertices.

Also, the *complementary product* of two graphs is defined as follows: Let R be a subset of $V(G)$ and S be a subset of $V(H)$. The complementary product, denoted by $G(R)\Box H(S)$, is constructed as follows. The vertex set $V(G(R)\Box H(S))$ is $\{(u_i, v_j) : 1 \leq i \leq n, 1 \leq j \leq p\}$ where the edge $(u_i, v_j)(u_h, v_k)$ is in $E(G(R)\Box H(S))$ if one of the following conditions hold.

(i.) If $i = h$, $u_i \in R$, and $v_j v_k \in E(H)$, or if $i = h$, $u_i \notin R$ and $v_j v_k \notin E(H)$.

(ii.) If $j = k$, $v_j \in S$, and $u_i u_h \in E(G)$, or if $j = k$, $v_j \notin S$, and $u_i u_h \notin E(G)$.

In simpler terms, for each $u_i \in V(G)$, we replace u_i with a copy of H if u_i is in R and with a copy of its complement $\overline{H}$ if u_i is not in R, and for each $v_j \in V(H)$, we replace each v_j with a copy of G if $v_j \in S$ and a copy of $\overline{G}$ if $v_j \notin S$.

In the case where $R = V(G)$ and $S = V(H)$, the complementary product $G(R)\Box H(S)$ is written $G\Box H(S)$ and $G(R)\Box H$, respectively. In other words $G\Box H(S)$ is the graph obtained by replacing each vertex $v \in V(H)$ with a copy of G if $v \in S$ and by a copy of $\overline{G}$ if $v \notin S$, and replacing each u_i with a copy of H. In the extreme case where $R = V(G)$, and $S = V(H)$, the complementary product $G(V(G))\Box H(V(H)) = G\Box H$ is simply the same as the Cartesian product $G\Box H$. See Figure 5 for an illustration of $C_4(\{u_1,\ u_4\}) \Box C_3(\{v_3\})$. A complementary prism $G\overline{G}$ is the complementary product $G\Box K_2(S)$ with $|S| = 1$.

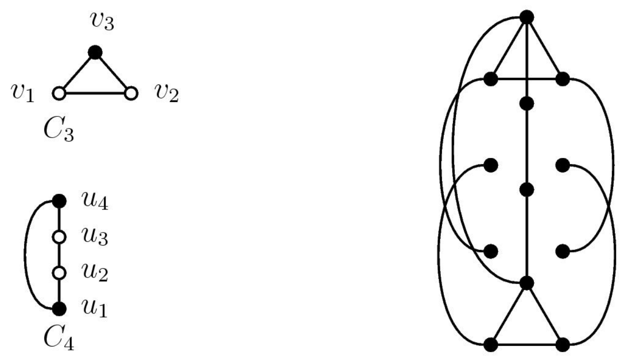

Figure 5: $C_4(\{u_1, u_4\}) \,\square\, C_3(\{v_3\})$

2.2 Domination and Total Domination in Complementary Prisms

In [8], Haynes, Henning and Van der Merwe studied domination in complementary prisms and acquired the following results. The first result we review is whenever G is a specific family of graphs.

Proposition 2.1 *[8]*

(a) *If $G = K_n$, then $\gamma(G\overline{G}) = n$.*

(b) *If $G = tK_2$, then $\gamma(G\overline{G}) = t + 1$.*

(c) *If $G = K_t \circ K_1$ and $t \geq 3$, then $\gamma(G\overline{G}) = \gamma(G) = t$.*

(d) *If $G = C_n$ and $n \geq 3$, then $\gamma(G\overline{G}) = \lceil (n + 4)/3 \rceil$.*

(e) *If $G = P_n$ and $n \geq 2$, then $\gamma(G\overline{G}) = \lceil (n + 3)/3 \rceil$.*

Similarly in [8] they found the total domination numbers for graphs of specific families.

Proposition 2.2 *[8]*

(a) *If $G = K_n$, then $\gamma_t(G\overline{G}) = n$.*

(b) *If $G = tK_2$, then $\gamma_t(G\overline{G}) = n = 2t$.*

(c) *If* $G = K_t \circ K_1$ *and* $t \geq 3$*, then* $\gamma_t(G\overline{G}) = \gamma_t(G) = t$*.*

(d) *If* $G \in \{C_n,\ P_n\}$ *with order* $n \geq 5$*, then*

$$\gamma_t(G\overline{G}) = \begin{cases} \gamma_t(G), & \textit{if } n \equiv 2 \pmod{4} \\ \gamma_t(G) + 2, & \textit{if } G = C_n, \textit{ and } n \equiv 0 \pmod{4} \\ \gamma_t(G) + 1, & \textit{otherwise.} \end{cases}$$

They also characterized graphs for which $\gamma(G\overline{G})$ and $\gamma_t(G\overline{G})$ are small.

Proposition 2.3 *[8] Let* G *be a graph of order* n*. Then,*

(a) $\gamma(G\overline{G}) = 1$ *if and only if* $G = K_1$*.*

(b) $\gamma(G\overline{G}) = 2$ *if and only if* $n \geq 2$ *and* G *has a support vertex that dominates* $V(G)$ *or* $\overline{G}$ *has a support vertex that dominates* $V(\overline{G})$*.*

Proposition 2.4 *[8] Let* G *be a graph of order* $n \geq 2$*, with* $|E(G)| = |E(\overline{G})|$*. Then*

(a) $\gamma_t(G\overline{G}) = 2$ *if and only if* $G = K_2$*.*

(b) $\gamma_t(G\overline{G}) = 3$ *if and only if* $n \geq 3$ *and* $G = K_3$ *or* G *has a support vertex that dominates* $V(G)$ *or* $\overline{G}$ *has a support vertex that dominates* $V(\overline{G})$*.*

As we continue through the results found in [8] we see that they were able to find a lower and upper bound for $\gamma(G\overline{G})$ and $\gamma_t(G\overline{G})$.

Proposition 2.5 *[8] For any graph* G*,* $\max\{\gamma(G), \gamma(\overline{G})\} \leq \gamma(G\overline{G}) \leq \gamma(G) + \gamma(\overline{G})$*.*

Proposition 2.6 *[8] If* G *and* $\overline{G}$ *are without isolates, then* $\max\{\gamma_t(G), \gamma_t(\overline{G})\} \leq \gamma_t(G\overline{G}) \leq \gamma_t(G) + \gamma_t(\overline{G})$*.*

In [8], they then characterized graphs G for which $\gamma(G\overline{G}) = max\{\gamma(G), \gamma(\overline{G})\}$ and $\gamma_t(G\overline{G}) = max\{\gamma_t(G), \gamma_t(\overline{G})\}$. First, let us define a packing. A set S of vertices of a graph G is a *packing* of G if $d_G(x, y) \geq 3$ for all pairs of distinct vertices x and y in S. The *packing number* $\rho(G)$ of G is the maximum cardinality of a packing set in G.

Proposition 2.7 *[8] A graph G satisfies $\gamma(G\overline{G}) = \gamma(G) \geq \gamma(\overline{G})$ if and only if G has an isolated vertex or there exists a packing P of G such that $|P| \geq 2$ and $\gamma(G \setminus P) = \gamma(G) - |P|$.*

Proposition 2.8 *[8] Let G be a graph such that neither G nor $\overline{G}$ has an isolated vertex. Then $\gamma_t(G\overline{G}) = \gamma_t(G) \geq \gamma_t(\overline{G})$ if and only if $G = n/2K_2$ or there exists an open packing $P = P_1 \cup P_2$ in G satisfying the following conditions:*

$|P| \geq 2$;

$P_1 \cap P_2 = \emptyset$;

if $P_1 \neq \emptyset$, then P_1 is a packing in G;

if $P_1 = \emptyset$, then $|P| \geq 3$ or $G[P] = \overline{K}_2$; and

$\gamma_t(G \setminus N[P_1] \setminus P_2) = \gamma_t(G) - 2|P_1| - |P_2|$.

3 SPECIFIC FAMILIES

To illustrate independent domination in complementary prisms, we determine $i(G\overline{G})$ for selected graphs G. A *subdivided star* $K^*_{1,t}$ is the graph obtained from the star $K_{1,t}$ by subdividing each edge exactly once. Let $K_{r,s}$ denote the complete bipartite graph with partite sets of cardinality r and s, and let P_n denote the path of order n.

We begin with a straightforward observation and a result from [5].

Observation 3.1 *For a path P_n,*

(a) $i(P_n) = \lceil n/3 \rceil$*, and*

(b) *there exists an $i(P_n)$-set I such that I does not contain one of the endvertices of P_n.*

Proposition 3.2 [8] *Let $G = P_n$. Then $\gamma(G\overline{G}) = \lceil n/3 \rceil + 1$.*

Proposition 3.3

(a) *If $G = P_n$ and $n \geq 2$, then $i(G\overline{G}) = \lceil n/3 \rceil + 1$.*

(b) *If $G = K_{r,s}$ where $2 \leq r \leq s$, then $i(G\overline{G}) = r + 1$.*

(c) *If G is a subdivided star $K^*_{1,t}$, then $i(G\overline{G}) = t + 1$.*

Proof. (a) Since $i(G) \geq \gamma(G)$ for any graph G, by Proposition 3.2 we have $\lceil n/3 \rceil + 1 = \gamma(G\overline{G}) \leq i(G\overline{G})$. Next, we will show that $\lceil n/3 \rceil + 1 \geq i(G\overline{G})$. Let the vertices of $G = P_n$ be labeled sequentially $v_0, v_1, \cdots, v_{n-1}$ such that v_0 and v_{n-1} are endvertices. Let I be an $i(G)$-set that does not contain v_0, (such a set exists and has cardinality $\lceil n/3 \rceil$ by Observation 3.1). This implies that $v_1 \in I$. Notice that $\overline{v}_0$ dominates $\overline{G} \setminus \{\overline{v}_1\}$.

Thus $I \cup \{\overline{v}_0\}$ is an IDS of $G\overline{G}$. Therefore, $i(G\overline{G}) \leq |I \cup \{\overline{v}_0\}| = |I| + 1 = \lceil n/3 \rceil + 1$, and hence $i(G\overline{G}) = \lceil n/3 \rceil + 1$.

(b) Let $G = K_{r,s}$, $2 \leq r \leq s$, with partite sets $R = \{u_1, u_2, \cdots, u_r\}$ and $S = \{v_1, v_2, \cdots, v_s\}$. To show the upper bound, we note that $\{\overline{u}_1, \overline{v}_1\} \cup R \setminus \{u_1\}$ is an IDS of $G\overline{G}$, so $i(G\overline{G}) \leq 2 + r - 1 = r + 1$. To show the lower bound, let L be an $i(G\overline{G})$-set. If $L \cap \overline{V} = \emptyset$, then $V \subseteq L$ to dominate $\overline{V}$, a contradiction because V is not independent. Hence, $|L \cap \overline{V}| \geq 1$. Since each of $\overline{R}$ and $\overline{S}$ induces a complete graph, $|L \cap \overline{V}| \leq 2$. If $L \cap \overline{V} = \{\overline{u}_i\}$, then $S \subseteq L$ to independent dominate $\overline{S}$. Therefore, $i(G\overline{G}) = |L| \geq |S| + 1 \geq |R| + 1 = r + 1$. If $L \cap \overline{V} = \{\overline{v}_i\}$, then $R \subseteq L$ to independent dominate $\overline{R}$. Again $i(G\overline{G}) = |L| \geq |R| + 1 = r + 1$. If $L \cap \overline{V} = \{\overline{v}_i, \overline{u}_j\}$, then to independent dominate $(R \cup S) \setminus \{u_i, v_j\}$, either $R \setminus \{u_i\} \subseteq L$ or $S \setminus \{v_j\} \subseteq L$. Since $r \leq s$, we have $i(G\overline{G}) = |L| \geq |\{\overline{u}_i, \overline{v}_j\} \cup R \setminus \{\overline{u}_i\}| = 2 + r - 1 = r + 1$. Hence $i(G\overline{G}) = r + 1$.

(c) Let $G = K^*_{1,t}, 2 \leq t$, be a subdivided star with center v_0. Label the leaves of G, v_i for $1 \leq i \leq t$, and let u_i be the support vertex of v_i. To show that $|L| \leq t+1$, we note that $\{\overline{v}_1, u_1, v_i \mid 2 \leq i \leq t\}$ is an IDS of $G\overline{G}$. Let L be an $i(G\overline{G})$-set. To independent dominate v_i, at least one of v_i, u_i, and $\overline{v}_i$ must be in L for $1 \leq i \leq t$. Moreover, to independent dominate $\overline{v}_0$, L includes at least one vertex from $\{\overline{v}_0, v_0, \overline{v}_i \mid 1 \leq i \leq t\}$. If either $\overline{v}_0$ or v_0 is in L, it follows that $|L| \geq t + 1$ as desired. Hence assume that $v_0 \notin L$ and $\overline{v}_0 \notin L$. Thus $\overline{v}_i \in L$ for some i to dominate $\overline{v}_0$. Then to independent dominate $\overline{u}_i$, L contains at least one vertex from $\{\overline{u}_i, u_i\}$ implying that $|L| \geq t + 1$. Hence, $i(G\overline{G}) = t + 1$. $\square$

4 LOWER BOUND

In this section, we present a lower bound on the independent domination number of the complementary prism $G\overline{G}$ in terms of the independent domination number of G, as well as a characterization of the graphs attaining this lower bound.

Theorem 4.1 *For any graph G, $max\{i(G), i(\overline{G})\} \leq i(G\overline{G})$.*

Proof. If $G = K_n$, then $G\overline{G}$ is the corona $K_n \circ K_1$. The result holds because $i(G) = n = i(G\overline{G})$. Thus, we may assume that neither G nor $\overline{G}$ is a complete graph. Let I be an $i(G\overline{G})$-set, and let $I_1 = I \cap V$ and $I_2 = I \cap \overline{V}$. Without loss of generality, let $i(G) = max\{i(G), i(\overline{G})\}$. If I_1 is an IDS of G, then we are finished. If not, then I_1 is independent but does not dominate G. Let $T \subseteq V$ be the set of vertices that are not dominated by I_1, and let T_1 be an IDS of T. Then $|T_1| \leq |T|$, and each vertex in T is dominated by exactly one vertex in I_2, so $|T_1| \leq |T| \leq |I_2|$. Moreover $I_1 \cup T_1$ is an IDS of G. Hence, $i(G) \leq |I_1 \cup T_1| \leq |I_1| + |I_2| = |I| = i(G\overline{G})$. Therefore, $max\{i(G), i(\overline{G})\} \leq i(G\overline{G})$. $\square$

Next we characterize the graphs attaining the lower bound of Theorem 4.1.

Theorem 4.2 *A graph G satisfies $i(G\overline{G}) = i(G) \geq i(\overline{G})$ if and only if G has an isolated vertex.*

Proof. First we consider the sufficiency. Assume that G has an isolated vertex v. Since $\overline{v}$ dominates $\overline{G}$, we have $i(G) \geq i(\overline{G}) = 1$. Let I be an $i(G)$-set. Then any isolated vertex of G must be in I, so $v \in I$. Then $A = I \setminus \{v\} \cup \{\overline{v}\}$ is an IDS of $G\overline{G}$. Thus $i(G\overline{G}) \leq |A| = i(G)$. By Theorem 4.1, $i(G) \leq i(G\overline{G})$ and so $i(G\overline{G}) = i(G)$.

Next we consider the necessity. Assume that $i(G\overline{G}) = i(G) \geq i(\overline{G})$, and let I be an $i(G\overline{G})$-set. If $I \subseteq V$, then $I = V$ since I must also dominate $\overline{G}$, hence V is an independent set and the result follows. Thus, we may assume that $I \cap V \neq \emptyset$ and $I \cap \overline{V} \neq \emptyset$. Let $I_1 = I \cap V$ and $I_2 = I \cap \overline{V}$. Since $|I_1| < i(G)$, I_1 does not dominate G. Let $X \subseteq V$ be the set that is not dominated by I_1. Hence for each $x \in X$, $\overline{x} \in I_2$. Since I_2 is independent, X induces a complete graph in G. Therefore, $I_1 \cup \{x\}$ for any $x \in X$ is an IDS of G. Thus $i(G) \leq |I_1| + 1 \leq |I_1| + |I_2| = i(G\overline{G})$. But, since $i(G) = i(G\overline{G})$, we have that $|I_2| = 1$, that is, $I_2 = \{\overline{x}\}$. Now I_2 dominates $\overline{V} \setminus \overline{I}_1$. Also since x is not dominated by I_1, it follows that $\overline{x}$ dominates $\overline{I}_1$. Hence, $deg_{\overline{G}}(\overline{x}) = n - 1$, implying that x is an isolated vertex in G. $\square$

5 UPPER BOUND

Let $\delta(G)$ (respectively, $\Delta(G)$) denote the minimum (respectively, maximum) degree of G. In this section we present upper bounds on the independent domination number of the complementary prism $G\overline{G}$.

Observation 5.1 [5] *For any graph G, $i(G) \leq \delta(\overline{G})$.*

Theorem 5.2 *Let G be a graph with order n and maximum degree $\Delta(G)$. Then $i(G\overline{G}) \leq 2(n-1) - max\{\Delta(G), \Delta(\overline{G})\}$, and this bound is sharp.*

Proof. Let G be a graph of order n. We know from Observation 5.1 that $i(G) \leq \delta(\overline{G})$. Since $\Delta(G\overline{G}) + \delta(\overline{G\overline{G}}) = 2n-1$, we have $\delta(\overline{G\overline{G}}) = 2n-1-\Delta(G\overline{G})$. Since $\Delta(G\overline{G}) = max\{\Delta G, \Delta\overline{G}\}+1$, we have $\delta(\overline{G\overline{G}}) = 2n-2-max\{\Delta(G), \Delta(\overline{G})\}$. Therefore $i(G\overline{G}) \leq 2(n-1) - max\{\Delta(G), \Delta(\overline{G})\}$.

Next we will show that the bound is sharp. Let $G = K_2 \cup (n-2)K_1$. Notice that $max\{\Delta(G), \Delta(\overline{G})\} = \Delta(\overline{G}) = n-1$. It is straightforward to show that $i(G\overline{G}) = n-1$. Thus $2(n-1)-max\{\Delta(G), \Delta(\overline{G})\} = 2n-2-\Delta(\overline{G}) = 2n-2-(n-1) = n-1 = i(G\overline{G})$.

$\square$

Lemma 5.3 *For any graph G, let I_1 be an $i(G)$-set and I_2 be an $i(\overline{G})$-set. Then $I_1 \cup I_2$ is not an IDS of $G\overline{G}$ if an only if there exists exactly one vertex $v \in I_1$ such that $\overline{v} \in I_2$.*

Proof. Let G be a graph, and let I_1 be an $i(G)$-set and I_2 be an $i(\overline{G})$-set. Assume that $I_1 \cup I_2$ is not an IDS of $G\overline{G}$. Clearly, $I_1 \cup I_2$ dominates $G\overline{G}$, so it must be the case that $I_1 \cup I_2$ is not independent. Thus assume, for a contradiction, that there exist two vertices $v_0, v_1 \in I_1$ such that $\overline{v}_0, \overline{v}_1 \in I_2$. Since I_1 is an $i(G)$-set, then $v_0v_1 \notin E(G)$. Thus $\overline{v}_0\overline{v}_1 \in E(\overline{G})$, which is a contradiction to the fact that I_2 is an $i(\overline{G})$-set, concluding the sufficiency. Next let v be a unique vertex, such that $v \in I_1$ and $\overline{v} \in I_2$, then clearly $I_1 \cup I_2$ is not an IDS of $G\overline{G}$. $\square$

In [8] the authors observed that for any graph G, $\gamma(G\overline{G}) \leq \gamma(G) + \gamma(\overline{G})$ and $\gamma_t(G\overline{G}) \leq \gamma_t(G)+\gamma_t(\overline{G})$. Yet for independent domination, the upper bound $i(G)+i(\overline{G})$ does not hold. In fact the difference $i(G\overline{G}) - (i(G) + i(\overline{G}))$ can be arbitrarily large as we show with the following theorem.

Theorem 5.4 *For a graph G and its complementary prism $G\overline{G}$, the difference $i(G\overline{G})-[i(G) + i(\overline{G})]$ can be arbitrarily large.*

Proof. Let $2 \leq t < r$ and $H_x^{r,t}$ be the graph formed by identifying t copies of K_{r+1} at a single vertex labeled x. For the i^{th} copy of $K_{r+1} \setminus \{x\}$ where $1 \leq i \leq t$, label the vertices $\{x_{i,1}, \cdots, x_{i,r}\}$. Now let $G = G^{r,t}$ be the graph formed from $H_x^{r,t} \cup H_y^{r,t}$ by adding the edge xy. See Figure 6 for an example.

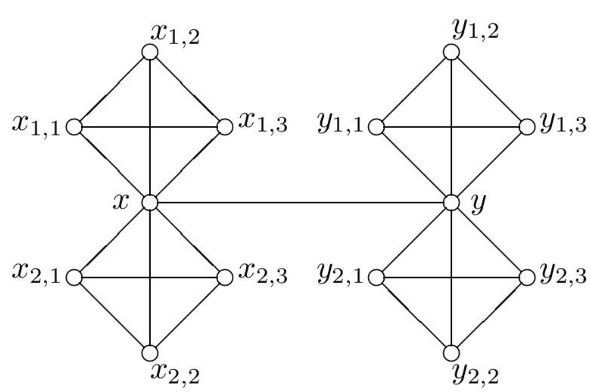

Figure 6: The Graph $G^{3,2}$

Note that $i(G) = t + 1$ and $i(\overline{G}) = 2$. We now proceed to show that $i(G\overline{G}) = 2t + 2$. Let $I = \{\overline{x}, \overline{y}, y_{i,1}, x_{i,1} \mid 1 \leq i \leq t\}$. Observe that I is an IDS of $G\overline{G}$ with $|I| = 2t+2$. Thus $i(G\overline{G}) \leq 2t+2$. It remains to show that $i(G\overline{G}) \geq 2t+2$. Let I be an arbitrary IDS of $G\overline{G}$. We note that since I is independent, $|I \cap \{x, y, \overline{x}, \overline{y}\}| \leq 2$. Without loss of generality, we consider the following cases:

Case 1. $I \cap \{x, y, \overline{x}, \overline{y}\} = \{\overline{x}, \overline{y}\}$. Observe that $\{\overline{x}, \overline{y}\}$ independent dominates $\overline{G} \cup \{x, y\}$ in $G\overline{G}$. Since $x, y \notin I$, to independent dominate $G \setminus \{x, y\}$, we need at least $2t$ additional vertices. Thus $i(G\overline{G}) \geq 2t + 2$.

Case 2. $I \cap \{x, y, \overline{x}, \overline{y}\} = \{\overline{x}, y\}$. Observe that $\{\overline{x}\}$ dominates $\{x, \overline{x}\} \cup (\overline{H}_y^{r,t} \setminus \{\overline{y}\})$, and $\{y\}$ dominates $H_y^{r,t} \cup \{x, \overline{y}\}$. Thus it remains to independent dominate $(H_x^{r,t} \setminus \{x\}) \cup (\overline{H}_x^{r,t} \setminus \{\overline{x}\})$. Since I is independent, no vertex of $\overline{H}_y^{r,t}$ is in I. Since $\overline{H}_x^{r,t} \setminus \{\overline{x}\}$ is a complete multipartite graph, and since $\overline{y} \notin I$, we need to select at least r vertices from one of the partite sets in order to independent dominate $\overline{H}_x^{r,t} \setminus \{\overline{x}\}$. This implies that to independent dominate $(H_x^{r,t} \setminus \{x\}) \cup (\overline{H}_x^{r,t} \setminus \{\overline{x}\})$, we need at least $r + t - 1$ vertices. Hence $i(G\overline{G}) \geq 2 + r + t - 1 = r + t + 1 \geq 2(t + 1)$, since $r \geq t + 1$.

Case 3. $I \cap \{x, y, \overline{x}, \overline{y}\} = \{x\}$. Clearly, $\{x\}$ dominates $H_x^{r,t} \cup \{y, \overline{x}\}$. In order to independent dominate $\overline{G} \setminus \{\overline{x}\}$ in $G\overline{G}$, we need at least r vertices from one of the partite sets in $\overline{H}_x^{r,t} \setminus \{\overline{x}\}$. It remains to independent dominate $H_y^{r,t} \setminus \{y\}$ in $G\overline{G}$. To do this, we need at least one vertex from each of the copies of K_r in $H_y^{r,t} \setminus \{y\}$. Thus, $i(G\overline{G}) \geq 1 + r + t \geq 2(t + 1)$, since $r \geq t + 1$.

Case 4. $I \cap \{x, y, \overline{x}, \overline{y}\} = \{\overline{x}\}$. Observe that $\{\overline{x}\}$ dominates $\{x, \overline{x}\} \cup \overline{H}_y^{r,t} \setminus \{\overline{y}\}$ in $G\overline{G}$. Moreover, since $y \notin I$, $\overline{x} \in I$, and no vertex of $\overline{H}_y^{r,t} \setminus \{\overline{y}\}$ is in I, we need at least t vertices in $H_y^{r,t}$ to independent dominate $H_y^{r,t}$. Also since $\overline{y} \notin I$, then we need at least r vertices of $\overline{H}_x^{r,t} \setminus \{\overline{x}\}$ to independent dominate $(\overline{H}_x^{r,t} \setminus \{\overline{x}\}) \cup \{\overline{y}\}$. It remains to independent dominate the vertices $H_x^{r,t} \setminus \{x\}$ that are not independent dominated by the vertices of $\overline{H}_x^{r,t}$. But to do this we need at least $t-1$ additional vertices. Thus, $i(G\overline{G}) \geq 1+t+r+t-1 > 2(t+1)$, since $r \geq t+1$.

Case 5. $I \cap \{x, y, \overline{x}, \overline{y}\} = \emptyset$. Then, to independent dominate $\{\overline{x}\}$ and $\{\overline{y}\}$ in $G\overline{G}$, we need to select at least one vertex from $\overline{H}_x^{r,t} \setminus \{\overline{x}\}$ and at least one vertex from $\overline{H}_y^{r,t} \setminus \{\overline{y}\}$, which contradicts the fact that I is independent. Thus, this case is not possible.

Therefore, in all cases $i(G\overline{G}) \geq 2t+2$. Hence, $i(G\overline{G}) = 2t+2$. Also $|i(G\overline{G}) - [i(G) + i(\overline{G})]| = 2t+2-(t+1)-2 = t-1$. Thus as $t \to \infty$, the difference $i(G\overline{G})-(i(G)+i(\overline{G}))$ is arbitrarily large. $\square$

Next we present another upper bound on $i(G\overline{G})$.

Theorem 5.5 *If G is a graph with minimum degree $\delta(G)$ and no isolated vertices, then $i(G\overline{G}) \leq i(G) + \delta(G)$, and this bound is sharp.*

Proof. Let G be a graph with no isolated vertices, and let $x \in V$ be a vertex of minimum degree $\delta(G)$. Let I be an $i(G)$-set.

Case 1. Assume $x \in I$, and let $X = N_G(x)$. If $pn_G[x, I] = \{x\}$, then let $\overline{S}$ be an IDS of $\langle \overline{X} \rangle$. Clearly, $|\overline{S}| \leq \delta(G)$ and $(I \setminus \{x\}) \cup \{\overline{x}\} \cup \overline{S}$ is an IDS of $G\overline{G}$. Thus, $i(G\overline{G}) \leq |I \setminus \{x\}| + |\overline{S}| + 1 \leq i(G) - 1 + \delta(G) + 1 = i(G) + \delta(G)$.

Suppose then that $pn_G(x, I) \neq \emptyset$, that is, x is necessary in I to dominate at least one vertex in $V \setminus I$. Let $A = pn_G(x, I)$, and let S be an IDS of $\langle A \rangle$. Then $S \subseteq A \subseteq X$. Note that $(I \setminus \{x\}) \cup S$ is an IDS of G. Now let $\overline{D}$ be an IDS of $\langle \overline{X} \setminus \overline{S} \rangle$, then

$(I \setminus \{x\}) \cup S \cup \overline{D} \cup \{\overline{x}\}$ is an IDS of $G\overline{G}$. Note that $|S| + |\overline{D}| \leq |X| = \delta(G)$. Hence, $i(G\overline{G}) \leq |I \setminus \{x\}| + |S| + |\overline{D}| + 1 \leq i(G) - 1 + \delta(G) + 1 = i(G) + \delta(G)$.

Case 2. Assume that $x \notin I$. Since I dominates G, x has at least one neighbor in I. Let $A = N_G(x) \cap I$ where $|A| = t \geq 1$, and let $B = N_G(x) \setminus A$, and $|B| = \delta(G) - t$. Let $\overline{S}$ be an IDS of $\overline{B}$. Note that $\overline{S} \cup \{\overline{x}\}$ is an IDS of $\overline{G} \setminus \overline{A}$ and $|\overline{S}| \leq |\overline{B}| = \delta(G) - t$. Moreover, $I \cup \{\overline{x}\} \cup \overline{S}$ is an IDS of $G\overline{G}$, and so $i(G\overline{G}) \leq |I| + 1 + |\overline{S}| \leq i(G) + 1 + \delta(G) - t \leq i(G) + \delta(G)$. We note that the bound $i(G\overline{G}) \leq i(G) + \delta(G)$ is sharp for the complete graph $G = K_n$, where $i(G\overline{G}) = n = 1 + (n-1) = i(G) + \delta(G)$. □

We note that strict inequality is possible for the upper bound of Theorem 5.5. Consider the graph $G = G^{r,t}$ from the proof of Theorem 5.4. Let $t < r + 1$. Notice that $i(G\overline{G}) = 2t + 2 < (t + 1) + r = i(G) + \delta(G)$.

Theorem 5.6 *If G is a graph with $\delta(G) = 1$ and $\delta(\overline{G}) \geq 1$, then $i(G\overline{G}) = i(G) + 1$.*

Proof. Let G be a graph where neither G nor $\overline{G}$ contains isolated vertices, and G has minimum degree $\delta(G) = 1$. By Theorem 5.5, $i(G\overline{G}) \leq i(G) + \delta(G) = i(G) + 1$. Without loss of generality, let $i(G) \geq i(\overline{G})$. Now from Theorem 4.1, we have that for any graph G, $i(G) \leq i(G\overline{G})$ and from Theorem 4.2, we know $i(G) = i(G\overline{G})$ if and only if $i(G)$ contains an isolated vertex. Thus, $i(G) + 1 \leq i(G\overline{G})$. Therefore, $i(G\overline{G}) = i(G) + 1$. □

We note that the converse of Theorem 5.6 is not necessarily true. Consider, for example, the graph $G = C_5$. Then G is self-complementary with no isolated vertices and has $i(G) = 2$. Also, $G\overline{G}$ is the Petersen graph with $i(G\overline{G}) = 3 = i(G) + 1$, but $\delta(G) = 2$.

6 CONCLUSION

In this thesis, we studied independent domination in complementary prisms as well as the relationship between independent domination and other parameters. In Chapter 1, we introduced some basic definitions for thesis. The chapter concludes with the definition of complementary prisms and some examples. In Chapter 2, we review the some of the existing literature on complementary prisms.

Chapter 3 begins with our results. We found the independent domination number of $G\overline{G}$ whenever G is a path, a subdivided star, and when G is a complete bipartite graph. In Chapter 4, we discovered that the lower bound found in [8] for $\gamma(G\overline{G})$ and $\gamma_t(G\overline{G})$ also held for independent domination. We were also able to characterize the lower bound.

In Chapter 5, we give various upper bounds to our parameter. When we attempted to mimic the results found in [8], we ran into a problem, yielding an interesting result. We conclude with some open questions:

- Does there exist and upper bound we can characterize?
- Is $G^{3,2}$ the smallest graph that poses a contradiction to the upper bound $i(G\overline{G}) \leq i(G) + i(G)$?
- If $\delta(G) = 1$ is replaced with $\delta(G) = 2$ in Theorem 5.6, will $i(G\overline{G}) = i(G) + 2$?

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