12-2012

Cost Effective Domination in Graphs

Tabitha Lynn McCoy
East Tennessee State University

Cost Effective Domination in Graphs

A thesis

presented to

the faculty of the Department of Mathematics

East Tennessee State University

In partial fulfillment

of the requirements for the degree

Master of Science in Mathematical Sciences

by

Tabitha McCoy

December 2012

Teresa W. Haynes, Ph.D., Chair

Robert A. Beeler, Ph.D.

Debra Knisley, Ph.D.

Keywords: cost effective domination, cost effective domination number

ABSTRACT

Cost Effective Domination in Graphs

by

Tabitha McCoy

A set S of vertices in a graph $G = (V, E)$ is a dominating set if every vertex in $V \setminus S$ is adjacent to at least one vertex in S. A vertex v in a dominating set S is said to be *cost effective* if it is adjacent to at least as many vertices in $V \setminus S$ as it is in S. A dominating set S is cost effective if every vertex in S is cost effective. The minimum cardinality of a cost effective dominating set of G is the cost effective domination number of G. In addition to some preliminary results for general graphs, we give lower and upper bounds on the cost effective domination number of trees in terms of their domination number and characterize the trees that achieve the upper bound. We show that every value of the cost effective domination number between these bounds is realizable.

DEDICATION

I'm dedicating this thesis to my grandfather, Mack Tuggle. He was a true inspiration to me and all those around him of how to live a faithful life in God. He taught me to seek God in all I do, and to never go a single day without praying. It is also he who taught me how to love the ones in my life. I am blessed to have had such a precious man to call my papaw.

ACKNOWLEDGMENTS

First and foremost, I want to thank Dr. Teresa Haynes for all of her encouragement and support throughout this process. The love she has for Graph Theory radiates from her, and without her this thesis would not have been possible. She believed in me when I did not believe in myself, and for that I will always be grateful. I also would like to thank Dr. Stephen Hedetniemi for all of his insight and feedback; it truly was invaluable. I'm also thankful to my committee members, Dr. Robert Beeler and Dr. Debra Knisley, for helping me along the way and during the revision process.

I also can't go without thanking one of my previous teachers, Mrs. Lisa Henley, for instilling the love of math in me. Through her enthusiasm for math, she inspired me to follow my heart and pursue a major in mathematics. Also, to my fellow graduate students that have grown to become my friends instead of just classmates, thank you for making this journey such a positive one.

I also want to thank my wonderful family for all of their love and support throughout my entire life, and especially during my college career. They are my biggest cheerleaders and have always been there for me. If it was not for them, I would not be the woman I am today. And to my boyfriend Ryan, you are the best. Thank you for being you.

CONTENTS

LIST OF FIGURES

1 INTRODUCTION

1.1 Introduction to Graph Theory

A *graph* G is a nonempty set $V(G)$ of objects called *vertices* together with a (possibly empty) set $E(G)$ of 2-element subsets of $V(G)$ called *edges*. To indicate that a graph G has *vertex set* $V(G)$ and *edge set* $E(G)$, we write $G = (V, E)$. We consider simple, finite graphs, that is, graphs with no loops or multiple edges. Each edge $\{u, v\}$ of G is typically denoted by uv or vu, and u and v are called *adjacent vertices*. Two adjacent vertices are called *neighbors* of each other. The *degree of a vertex* v in a graph G is the number of vertices in G adjacent to v. A vertex v is said to be *even* or *odd*, according to whether its degree in G is even or odd. Also, two edges are called *adjacent edges* if uv and vw are distinct edges in G. The vertex u and the edge uv are said to be *incident* to each other.

The number of vertices in a graph G is the *order* of G, and the number of edges is the *size* of G. We let $|V(G)| = n$ and $|E(G)| = m$. A graph of order 1 is called a *trivial graph*, and a graph of order 2 or more is called a *nontrivial graph*. A graph of size 0 is called an *empty graph*. A *nonempty graph* has one or more edges. The *complete graph* of order n, denoted K_n, is the graph for which every two distinct vertices are adjacent. Thus, K_n has size $n(n-1)/2$. The *path* on $n \geq 1$ vertices, denoted P_n, is a graph of order n and size $n-1$. The *length* of a path is the number of edges it contains. A graph G is connected if for every pair of vertices in $V(G)$, there exists a path between them. The *cycle* on n vertices, denoted C_n, is a closed path, P_n, and has order n and size n. The length of a cycle is the number of edges it contains. An *acyclic graph* has no cycles. A *tree* is a connected acyclic graph. A graph G is *bipartite* if $V(G)$ can be partitioned into two independent sets. A *complete bipartite graph* is a bipartite graph with partitions V_1 and V_2 such that every vertex in V_1 is adjacent to every vertex in V_2. If $|V_1| = s$ and $|V_2| = t$, then the complete bipartite graph is denoted $K_{s,t}$ and has order $s+t$ and size st. We note that trees are bipartite. A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A graph G is *regular* if the all vertices of G have the same degree, say r. Such graphs are called *r-regular.* A 3-regular graph is also called a *cubic graph.* The *cartesian product* of two graphs G_1 and G_2, commonly denoted by $G_1 \,\square\, G_2$, has vertex set

$$V(G) = V(G_1) \times V(G_2)$$

and two distinct vertices (u, v) and (x, y) of $G_1 \,\square\, G_2$ are adjacent if either

$$(1)\ u = x \ \text{ and } \ vy \in E(G_2) \ \text{ or } \ (2)\ v = y \ \text{ and } \ ux \in E(G_1).$$

Figure 1 gives examples of the graphs K_4, C_5, $P_3 \,\square\, P_4$ and $K_{2,3}$.

For a graph $G = (V, E)$, the *open neighborhood* of a vertex $u \in V$ is the set $N(u) = \{v \mid uv \in E\}$, and the *closed neighborhood* of u is the set $N[u] = N(u) \cup \{u\}$. The *open neighborhood* of a set $S \subseteq V$ is the set $N(S) = \bigcup_{u \in S} N(u)$, and the *closed neighborhood* of a set S is the set $N[S] = N(S) \cup S$. A set S of vertices is *independent* if no two vertices in S are adjacent and is a *dominating set* if $N[S] = V$, that is, every vertex in $V \setminus S$ is adjacent to at least one vertex in S. The *domination number* $\gamma(G)$ of a graph G equals the minimum cardinality of a dominating set in G (see Figure 2 for examples where the darkened vertices represent $\gamma(G)$-sets), while the *upper domination number* $\Gamma(G)$ equals the maximum cardinality of a minimal dominating

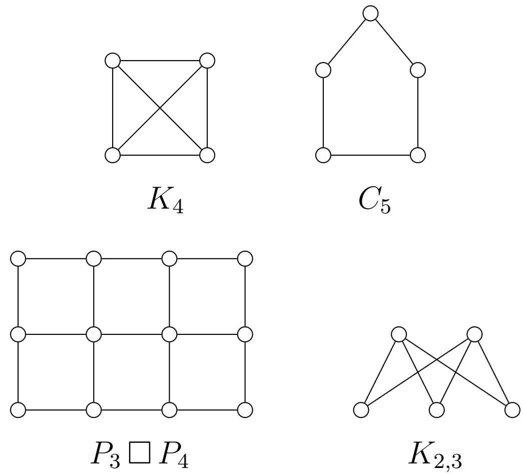


Figure 1: Graphs K_4, C_5, $P_3 \,\square\, P_4$ and $K_{2,3}$.

set in G. A dominating set of cardinality $\gamma(G)$ is called a $\gamma(G)$-*set*. For more details on domination, the reader is referred to *Fundamentals of Domination in Graphs* by Haynes, Hedetniemi, and Henning [13]. The *vertex independence number* $\beta_0(G)$ equals the maximum cardinality of an independent set in G, while the *independent domination number* $i(G)$ equals the minimum cardinality of a maximal independent set in G. The following inequalities are well-known in domination theory.

Proposition 1.1 *For any graph* G, $\gamma(G) \leq i(G) \leq \beta_0(G) \leq \Gamma(G)$.

We go into more depth with inequality chain from Proposition 1.1 in the Preliminary Results section of this thesis.

1.2 Cost Effective Domination

Motivated by the studies of unfriendly partitions and satisfactory partitions (for example, see [1, 2, 7, 8, 9, 19, 20]), cost effective domination was introduced in [10].

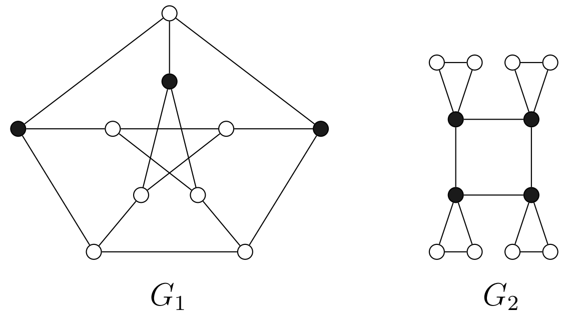
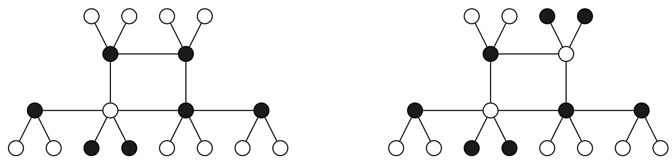


Figure 2: Graphs with $\gamma(G_1) = 3$ and $\gamma(G_2) = 4$.

A vertex v in a set S is said to be *cost effective* if it is adjacent to at least as many vertices in $V \setminus S$ as it is in S, and v is *very cost effective* if it is adjacent to more vertices in $V \setminus S$ than to vertices in S. A set S is *(very) cost effective* if every vertex in S is (very) cost effective. A set S is a *(very) cost effective dominating set* if S is both (very) cost effective and a dominating set.

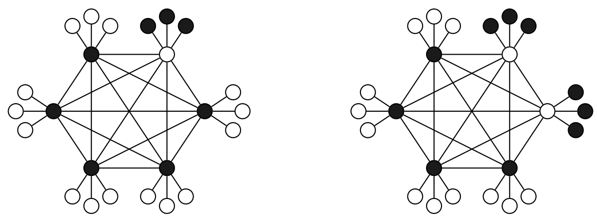
Definition 1.2 *The cost effective domination number $\gamma_{ce}(G)$ of a graph G equals the minimum cardinality of a cost effective dominating set in G. The upper cost effective domination number $\Gamma_{ce}(G)$ equals the maximum cardinality of a minimal dominating set that is cost effective in G. A cost effective dominating set of G with cardinality $\gamma_{ce}(G)$ is called a $\gamma_{ce}(G)$-set. The very cost effective domination number $\gamma_{vce}(G)$ and the upper very cost effective domination number $\Gamma_{vce}(G)$ are defined similarly.*

For examples, consider the graphs G in Figures 3(a) and 4(a) where the darkened vertices represent $\gamma_{ce}(G)$-sets and Figures 3(b) and 4(b) where the darkened vertices represent $\gamma_{vce}(G)$-sets.



(a) (b)

Figure 3: Graph with $\gamma_{c\epsilon}(G) = 7$ and $\gamma_{vc\epsilon}(G) = 8$.



(a) (b)

Figure 4: Graph with $\gamma_{c\epsilon}(G) = 8$ and $\gamma_{vc\epsilon}(G) = 10$.

It should be pointed out that while the property of being a dominating set is *superhereditary*, that is, every superset of a dominating set is also a dominating set, the property of being a cost effective dominating set is not superhereditary. This explains why the definition of the upper cost effective domination number does not include the word "minimal" as it does in the definition of the upper domination number. Without the word *minimal* in the definition of $\Gamma(G)$, the value of $\Gamma(G)$ would equal $n = |V|$ for all graphs.

In terms of application, we assume that maintaining edges in a network has an associated cost, and thus they should be used effectively. We assume that an edge between a vertex in a set S and a vertex in $V \setminus S$ is being used effectively, while an edge between two vertices in S is not necessarily being used cost effectively. Thus, a vertex is considered to be cost effective if at least as many edges incident to it are being used cost effectively as not being used cost effectively.

Another way of viewing the application is to consider a company, where the set S represents the employees and $V \setminus S$ represents the customers. Certainly the company would want to have only employees that add to its profits. Suppose the company offers a service to both its employees and its customers. Let the edges inside S represent services between employees (internal costs) and let edges between S and $V \setminus S$ represent income from paying customers. If the company allows employees to use the services it offers for free or at a discounted price, then to ensure that each employee $v \in S$ is profitable for the company it would be necessary for v to have at least as many neighbors in $V \setminus S$ as in S, that is, S needs to be a cost effective set. In this thesis, we study bounds on the cost effective domination number of graphs.

2 BACKGROUND AND TERMINOLOGY

2.1 Unfriendly Partitions

Cost effective domination is derived from the study of unfriendly partitions of graphs, as follows. Let C be a two-coloring of the vertices of a graph G, $C : V \rightarrow \{Red, Blue\}$. For every vertex $u \in V$, define $B(u) = \{v \in N(u), C(v) = Blue\}$ and $R(u) = \{v \in N(u), C(v) = Red\}$. Similarly, define $B(V) = \{v \in V, C(v) = Blue\}$ and $R(V) = \{v \in V, C(v) = Red\}$. A two-coloring produces a bipartition of V, $\pi = \{B(V), R(V)\}$. Given such a bipartition π, we say that an edge $uv \in E$ is *bicolored* if $C(u) \neq C(v)$. A bipartition π is called an *unfriendly partition* if every vertex $u \in B(V)$ has at least as many neighbors in $R(V)$ as it does in $B(V)$, and every vertex $v \in R(V)$ has at least as many neighbors in $B(V)$ as it does in $R(V)$. That is, if $C(u) = Blue$, then $|B(u)| \leq |R(u)|$, and if $C(u) = Red$, then $|R(u)| \leq |B(u)|$. These types of partitions were defined and studied by Borodin and Koshtochka [3], Aharoni, Milner and Prikry [1], and Shelah and Milner [20], who called these *unfriendly partitions*. They observed the following, a simple proof of which we provide here.

Theorem 2.1 [10] *Every finite connected graph G of order $n \geq 2$ has an unfriendly partition.*

Proof. Let $\pi = \{B(V), R(V)\}$ be any bipartition of $V(G)$ having the property that the number of bicolored edges is a maximum. Assume to the contrary that π is not an unfriendly partition. Then there must exist a vertex, say $v \in R(V)$, without loss of generality, having more Red neighbors than Blue neighbors. In this case, moving

v to $B(V)$ will increase the number of bicolored edges, contradicting the assumption that π has a maximum number of bicolored edges. $\square$

Unfriendly partitions have shown up indirectly in several other lines of research. In [4, 5] the concept of α-domination in graphs is defined and studied. A set $S \subseteq V$ of vertices in a graph $G = (V, E)$ is called an *α-dominating set* if for every vertex $v \in V \setminus S$, $|N(v) \cap S|/|N[v]| \geq \alpha$, where $0 \leq \alpha < 1$. In the case where $\alpha \geq 1/2$, every vertex in $V \setminus S$ meets the *unfriendly condition* in that it has at least as many neighbors in S as it has in $V \setminus S$. However, no unfriendly condition is imposed on the vertices in S.

Similarly, in [6, 11, 12, 14, 16, 18] global offensive alliances in graphs are defined and studied. A set $S \subseteq V$ of vertices is called a *global offensive alliance* if for every vertex $v \in V \setminus S$, $|N(v) \cap S| \geq |N[v] \cap (V \setminus S)|$. As with α-domination, if S is a global offensive alliance, then every vertex $v \in V \setminus S$ satisfies the unfriendly condition, in that it has at least as many neighbors in S as it has in $V \setminus S$ if you count the vertex v as one of its own neighbors. But no unfriendly condition is imposed on the vertices in S.

A partition that is in some sense dual to an unfriendly partition is a bipartition $\pi = \{B(V), R(V)\}$ called a *satisfactory partition* such that every vertex $u \in B(V)$ has at least as many neighbors in $B(V)$ as it does in $R(V)$, and every vertex $u \in R(V)$ has at least as many neighbors in $R(V)$ as it has in $B(V)$. That is, if $C(u) = Blue$, then $|B(u)| \geq |R(u)|$, and if $C(u) = Red$, then $|R(u)| \geq |B(u)|$. Satisfactory partitions have been studied in [7, 8, 9] and [19]. However, unlike unfriendly partitions, not every graph has a satisfactory partition. In fact, it is an NP-complete problem to determine if an arbitrary graph has a satisfactory partition [2].

2.2 Differentials in Graphs

The related concept of differentials in graphs was studied in [17], where the following game was considered for any arbitrary graph $G = (V, E)$. Assume you are allowed to buy as many tokens as you like, say k tokens, at the cost of \$1 each. You then place your tokens on some subset k vertices of V. For each vertex of G which is adjacent to a vertex with a token on it, but has no token on itself, you receive \$1. Note that you do not receive any credit for the vertices on which you place a token. Your objective it to maximize your profit, that is, the total value received minus the cost of the tokens bought. $B(X)$ is defined as the set of vertices in $V \setminus X$ that have a neighbor in a set X. Based on this game, the *differential* of a set X is defined to be $\partial(X) = |B(X)| - |X|$, and the *differential of a graph* to equal the $\max\{\partial(X)\}$ for any subset X of V.

In [17], it was shown that for any graph G,

$$n - 2\gamma(G) \leq \partial(G) \leq n - \gamma(G) - 1, \text{ and}$$

$$\Delta(G) - 1 \leq \partial(G).$$

The following realizability result was also given.

Theorem 2.2 [17] *For any triple (a, b, c) of positive integers such that $a \leq b \leq c$ and $c - 2a \leq b \leq c - a - 1$, there exists a tree T having order $n = c, \gamma(T) = a$, and $\partial(T) = b$.*

A *subdivision* of an edge uv is obtained by removing edge uv, adding a new vertex w, and adding the new egdes uw and wv. A *wounded spider* is the graph formed by subdividing at most $t-1$ of the edges of a star $K_{1,t}$ for $t \geq 0$. The following gives a characterization of trees that achieve the upper bound for $\partial(T)$, while the characterization of the trees T for which $\partial(T) = n - 2\gamma(T)$ is still being determined.

Theorem 2.3 [17] *A tree T has $\partial(T) = n - \gamma(T) - 1$ if and only if T is a nontrivial wounded spider.*

Also in [17], the trees having $\partial(T) = \Delta(T) - 1$ are characterized. For a rooted tree T, let T_u denote the subtree of T induced by u and its descendents.

A family $\mathcal{T}$ of trees is defined in [17] as follows. A tree T is in $\mathcal{T}$ if T is a tree rooted at a vertex v of maximum degree $\Delta(T)$ and one of the following properties holds:

1. v is adjacent to exactly one leaf x and for each $u \in N(v) \setminus \{x\}$, $T_u \in \{P_2, P_3\}$, where u is an endvertex of T_u, or

2. There exist two vertices $x, y \in N(v)$ such that $T_x \in \{P_1, P_2\}$ and $T_y \in \{P_1, P_2\}$. And, for each $u \in N(v) \setminus \{x, y\}$, the subtree $T_u \in \{P_1, P_2, P_3, P_4, P_5\}$ where u is the center of T_u or u is a leaf of $T_u = P_3$.

We conclude this section with the following theorem.

Theorem 2.4 [17] *A tree T has $\partial(T) = \Delta(T) - 1$ if and only if $T \in \mathcal{T}$.*

3 PRELIMINARY RESULTS

This section will begin with some preliminary results that build to the main results of this thesis.

Observation 3.1 *Every independent dominating set S in an isolate-free graph G is a very cost effective dominating set.*

Corollary 3.2 *For any isolate-free graph G,*

$$\gamma(G) \leq \gamma_{c\epsilon}(G) \leq \gamma_{vc\epsilon}(G) \leq i(G) \leq \beta_0(G) \leq \Gamma_{vce}(G) \leq \Gamma_{ce}(G) \leq \Gamma(G).$$

It is known [13] that $\beta_0(G) = \Gamma(G)$ for bipartite graphs so, from Corollary 3.2, we have that $\beta_0(G) = \Gamma_{vce}(G) = \Gamma_{ce}(G) = \Gamma(G)$ for bipartite graphs. On the other hand, in this section we will see that all combinations of the inequalities in the chain $\gamma(G) \leq \gamma_{c\epsilon}(G) \leq \gamma_{vc\epsilon}(G) \leq i(G)$ are possible, even when restricted to trees. We also give necessary conditions for a graph G to have $\gamma(G) = \gamma_{c\epsilon}(G)$, and for a graph G to have $\gamma_{c\epsilon}(G) = \gamma_{vc\epsilon}(G)$. In Section 4, we show that $\gamma_{c\epsilon}(T) \leq 2\gamma(T) - 3$ for trees T with $\gamma(T) \geq 3$, and characterize the trees achieving this bound. Then we show that, for trees T, every value of the cost effective domination number between $\gamma(T)$ and $2\gamma(T) - 3$ is realizable.

We first give some additional terminology. For a graph G and a subset $S \subseteq V$, we denote the subgraph induced by S as $G[S] = (S, E \cap (S \times S))$. An *$S$-external private neighbor* of a vertex $v \in S$ is a vertex $u \in V \setminus S$ which is adjacent to v but to no other vertex of S. The set of all S-external private neighbors of $v \in S$ is called the *S-external private neighbor set* of v and is denoted $\text{epn}(v, S)$. A vertex of degree one is called a *leaf* (or *endvertex*), and its neighbor is a *support vertex*. The *double star* $S_{r,s}$ is the tree with exactly two adjacent non-leaf vertices, one of which is adjacent to r leaves and the other to s leaves. The *corona* of graphs G and H, denoted $G \circ H$, is the graph formed from one copy of G and $|V(G)|$ copies of H, where the i^{th} vertex in $V(G)$ is adjacent to every vertex in the i^{th} copy of H.

The inequalities in Corollary 3.2 raise the following interesting questions: *Which graphs have a cost effective γ-set, that is, for which graphs G, is $\gamma(G) = \gamma_{ce}(G)$? For which graphs G is $\gamma_{ce}(G) = \gamma_{vce}(G)$?*

Note that if G is a cycle or a path P_k for $k \geq 5$, then $\gamma(G) = \gamma_{ce}(G) = \gamma_{vce}(G) = i(G)$. The graphs in Figure 5(a) and 5(b), where the darkened vertices represent $\gamma_{ce}(G)$-sets, have $\gamma_{ce}(G) > \gamma(G)$.

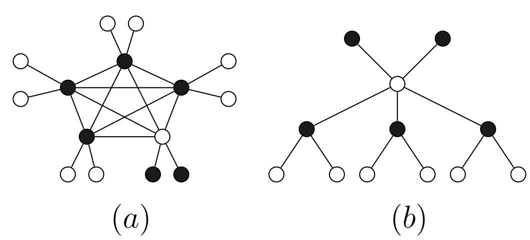


Figure 5: Graphs that do not have cost effective γ-sets.

Observation 3.3 [15] *Let S be a cost effective set of G. If every vertex in S has odd degree, then S is a very cost effective set of G.*

Corollary 3.4 *If G has a $\gamma_{ce}(G)$-set that consists of only odd vertices, then $\gamma_{ce}(G) = \gamma_{vce}(G)$.*

Corollary 3.5 *If every vertex of G has odd degree, then $\gamma_{ce}(G) = \gamma_{vce}(G)$.*

Note that in particular, $\gamma_{c\epsilon}(G) = \gamma_{vc\epsilon}(G)$ for cubic graphs.

Theorem 3.6 *If G has maximum degree $\Delta(G) \leq 4$, then $\gamma(G) = \gamma_{c\epsilon}(G)$.*

Proof. Among all $\gamma(G)$-sets, select S to be one with the minimum number of edges in $G[S]$. If S is cost effective, we are finished. Hence, assume to the contrary that there exists a vertex, say x, that is not cost effective. Therefore, $|N(x) \cap S| > |N(x) \cap (V \setminus S)|$. Thus, x has at least one neighbor in S. By the minimality of S, x has at least one external private neighbor, say x', with respect to S. But since $\Delta(G) \leq 4$ and $|N(x) \cap S| > |N(x) \cap (V \setminus S)|$, it follows that $N(x) \cap (V \setminus S) = \{x'\}$. But then $S' = (S \setminus \{x\}) \cup \{x'\}$ is a $\gamma(G)$-set with fewer edges in $G[S']$ than in $G[S]$, contradicting our choice of S. Hence, S is cost effective. $\Box$

Notice that the tree T in Figure 5(b) has maximum degree $\Delta(T) = 5$ and $\gamma(T) < \gamma_{ce}(T)$, and thus, the bound $\Delta(G) \leq 4$ in Theorem 3.6 is best possible.

From Theorem 3.6, we have the following,

Corollary 3.7 *If G is a grid graph $P_m \Box P_n$, a cylinder $C_m \Box P_n$, or a torus $C_m \Box C_n$, then $\gamma(G) = \gamma_{ce}(G)$.*

From Observation 3.3 and Theorem 3.6, we have the following:

Corollary 3.8 *If G is a cubic graph, then $\gamma(G) = \gamma_{c\epsilon}(G) = \gamma_{vc\epsilon}(G)$.*

Theorem 3.9 *If $\gamma(G) \leq 3$, then $\gamma(G) = \gamma_{c\epsilon}(G)$.*

Proof. Clearly, if $\gamma(G) = 1$, then $\gamma(G) = \gamma_{c\epsilon}(G)$, so assume that $2 \leq \gamma(G) \leq 3$. Among all $\gamma(G)$-sets, select S to be one with the minimum number of edges in $G[S]$.

If S is cost effective, then we are finished. Thus, assume that G is not cost effective. Then there exists a vertex $x \in S$, such that $|N(x) \cap S| > |N(x) \cap (V \setminus S)|$. Hence, x has at least one neighbor in S. By the minimality of S, x has at least one external private neighbor, say x'. Hence, $|N(x) \cap S| \geq 2$, implying that $|S| = 3$ and x has two neighbors in S and $|N(x) \cap (V \setminus S)| = 1$, that is, $N(x) \cap (V \setminus S) = \{x'\}$. But then $(S \setminus \{x\}) \cup \{x'\}$ is a $\gamma(G)$-set with fewer edges in its induced subgraph than in $G[S]$, contradicting our choice of S. Hence, S is cost effective. □

Notice that the tree T in Figure 5(b) has $\gamma(T) = 4$, but $\gamma_{ce}(T) = 5$, so the bound $\gamma(G) \leq 3$ in Theorem 3.9 is best possible. We conclude this section by showing that all eight combinations of the inequalities $\gamma(G) \leq \gamma_{c\epsilon}(G) \leq \gamma_{vc\epsilon}(G) \leq i(G)$ from Corollary 3.2 are possible, even when restricted to trees. For this purpose, let $K_{1,3}^x$ be the star with center x and leaves x_1, x_2, and x_3. Let T_x^j be the corona $K_{1,3}^x \circ \overline{K}_j$. For the following, T_i satisfies inequality i.

1. $\gamma(T) < \gamma_{c\epsilon}(T) < \gamma_{vc\epsilon}(T) < i(T)$.

   Let T_1 be the tree obtained from $T_x^2 \cup T_y^2$ by adding a new leaf vertex adjacent to x and an edge between x_1 and y_1. See Figure 6(a).

2. $\gamma(T) < \gamma_{c\epsilon}(T) < \gamma_{vc\epsilon}(T) = i(T)$.

   Let T_2 be the tree obtained from $T_x^2 \cup T_y^2$ by adding the edge xy.

3. $\gamma(T) < \gamma_{c\epsilon}(T) = \gamma_{vc\epsilon}(T) < i(T)$.

   Let T_3 be the tree obtained from $T_x^2 \cup T_y^2$ by adding the edge x_1y_1 and removing the two leaves adjacent to x_1. See Figure 6(b).

4. $\gamma(T) < \gamma_{c\epsilon}(T) = \gamma_{vc\epsilon}(T) = i(T)$.

   Let T_4 be the tree T_x^2.

5. $\gamma(T) = \gamma_{c\epsilon}(T) < \gamma_{vc\epsilon}(T) < i(T)$.

   Let T_5 be the corona $P_6 \circ \overline{K}_2$.

6. $\gamma(T) = \gamma_{c\epsilon}(T) < \gamma_{vc\epsilon}(T) = i(T)$.

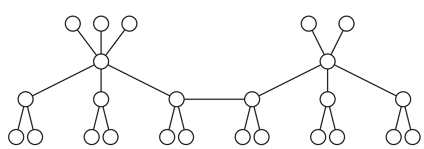
   Let T_6 be the tree T_x^3.

7. $\gamma(T) = \gamma_{c\epsilon}(T) = \gamma_{vc\epsilon}(T) < i(T)$.

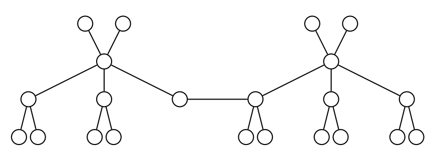
   Let T_7 be the double star $S_{r,s}$ where $2 \leq r \leq s$.

8. $\gamma(T) = \gamma_{c\epsilon}(T) = \gamma_{vc\epsilon}(T) = i(T)$.

   Let T_8 be the corona $T' \circ K_1$ of any tree T'.



(a)



(b)

Figure 6: Trees T_1 and T_3.

4 MAIN RESULTS

In this section, we determine an upper bound on the cost effective domination number of trees and characterize the trees obtaining this bound. We also show that every value of $\gamma_{c\epsilon}(T)$ between the upper and lower bounds of Theorem 4.1 is realizable.

Theorem 4.1 *If T is a tree with $\gamma(T) \geq 3$, then $\gamma(T) \leq \gamma_{c\epsilon}(T) \leq 2\gamma(T) - 3$, and these bounds are sharp.*

Proof. The lower bound is direct from Corollary 3.2. Let S be a $\gamma(T)$-set. If S is cost effective, then we are finished. Thus, assume that S is not cost effective and let $U = \{u_1, u_2, ..., u_k\}$ be the vertices of S that are not cost effective with respect to S. Let $s_i = |N(u_i) \cap S|$ and $o_i = |N(u_i) \cap (V \setminus S)|$, for $1 \leq i \leq k$. Thus for each $u_i \in U$, $s_i \geq o_i + 1$. Let $U' \subseteq V \setminus S$ be the vertices in $V \setminus S$ whose only neighbors in S are in U. Note that since each u_i is not cost effective, u_i has a neighbor in S, that is, $s_i \geq 1$. Hence, the minimality of S implies that u_i has at least one external private neighbor with respect to S in U'. Thus, $|U'| \geq \sum_{i=1}^{k} |\text{epn}(u_i, S)| \geq k$.

We first prove a claim:

Claim A $\sum_{i=1}^{k} s_i \leq \gamma(T) + k - 2$.

Proof. We establish the bound on the degree sum in $T[S]$ by considering the possible edges of $T[S]$ incident to a vertex in U. If both endvertices of an edge are in U, then we say the edge is a Type-1 edge, while if one endvertex is in U and the other is in $S \setminus U$, we say the edge is of Type-2. Thus, each Type-1 edge adds 2 to the degree sum in $T[S]$, and each Type-2 edge adds 1. Let t_i be the number of Type-i edges.

Note that if a pair of vertices in U are connected by a path in $T[U]$, then they have no common neighbor in $S \setminus U$, for otherwise a cycle is formed. Let $T[U]$ have c components. Since T is a tree, $t_1 = k - c$, and there are at most $c - 1$ pairs of vertices in U having a common neighbor in $S \setminus U$. By the Pigeonhole Principle, there are at least $t_2 - |S \setminus U|$ pairs of vertices in U having a common neighbor in $S \setminus U$. Thus, $t_2 - |S \setminus U| \leq c - 1$.

Hence, $\sum_{i=1}^{k} s_i = 2t_1 + t_2 \leq 2(k-c) + |S \setminus U| + c - 1 = 2k - 2c + \gamma(T) - k + c - 1 = \gamma(T) + k - c - 1 \leq \gamma(T) + k - 2$. □

Since $s_i \geq o_i + 1$ for each i, $1 \leq i \leq k$, by Claim A, we have $\sum_{i=1}^{k} o_i \leq \sum_{i=1}^{k}(s_i - 1) \leq \gamma(T) + k - 2 - k = \gamma(T) - 2$. Hence, $|U'| \leq \gamma(T) - 2$.

Next, we give an algorithm to recursively build a cost effective dominating set S_k from a $\gamma(T)$-set S. As before, let $U = \{u_1, u_2, \ldots, u_k\}$ be the subset of vertices in S that are not cost effective, and let U' be the set of vertices in $V \setminus S$ whose only neighbors in S are in U.

begin

let $S_0 = S$.

for $i = 1$ **to** k **do**

  if u_i is cost effective in S_{i-1}

    then let $S_i = S_{i-1}$

    else if $\text{epn}(u_i, S_{i-1}) = \emptyset$

      then let $S_i = S_{i-1} \setminus \{u_i\}$

      else let $S_i = (S_{i-1} \setminus \{u_i\}) \cup \text{epn}(u_i, S_{i-1})$

      fi

fi

end

We next prove that the algorithm produces a cost effective dominating set with cardinality at most $2\gamma(T) - 3$.

Claim B *The algorithm terminates with a cost effective dominating set, namely S_k, and $|S_k| \leq 2\gamma(T) - 3$.*

Proof. By definition the set $S = S_0$ is a dominating set and the vertices of $S \setminus \{u_1, u_2, ..., u_k\}$ are cost effective in S. We define the loop invariant: for $1 \leq i \leq k$, the set S_i is a dominating set and all of the vertices in $S_i \setminus \{u_{i+1}, \ldots, u_k\}$ are cost effective in S_i.

To see that S_i is a dominating set, we note that S_{i-1} is a dominating set, so if u_i is cost effective and $S_i = S_{i-1}$, clearly, S_i is a dominating set. If u_i is not cost effective in the set S_{i-1}, then u_i has at least one neighbor in S_{i-1}, implying that u_i is dominated by S_i. Moreover, the external private neighbors of u_i with respect to S_{i-1} are added to form S_i, so S_i is a dominating set.

To see that the set $S_i \setminus \{u_{i+1}, \ldots, u_k\}$ is cost effective, note if u_i is not cost effective in S_{i-1}, then $S_i = S_{i-1} \cup \text{epn}(u_i, S_{i-1})$. Let $X = \text{epn}(u_i, S_{i-1})$. Since T is a tree and each vertex in X is adjacent to u_i, X is an independent set. Moreover, since each vertex $x \in X$ is a private neighbor of u_i, x has no neighbors in $S_{i-1} \setminus \{u_i\}$. In other words, X is independent in $T[S_i]$ and so the vertices of X are cost effective with respect to S_i. Hence, the vertices that are not cost effective in S_i are the at the most the ones that are not cost effective in $S_{i-1} \setminus \{u_i\}$. On iteration k, the algorithm terminates with the cost effective dominating set S_k.

It remains to be shown that $|S_k| \leq 2\gamma(T) - 3$. To do this we count the maximum possible vertices being added to form the set S_k. Since U' consists of the vertices whose only neighbors in S are in U, we have that $\text{epn}(u_i, S) \subseteq U'$ for $1 \leq i \leq k$.

Consider the construction of set S_k. At iteration i, if u_i is cost effective in S_{i-1}, then we let $S_i = S_{i-1}$. Since $u_i \in U$, it is not cost effective in S so we have $|\text{epn}(u_i, S)| \geq 1$. Hence, for our counting purposes, letting $S_i = S_{i-1}$ is essentially the same as removing u_i and replacing it with a vertex from $\text{epn}(u_i, S) \subseteq U'$.

If u_i is not cost effective in S_{i-1}, then we remove u_i and add the set $\text{epn}(u_i, S_{i-1})$ to form S_i. To show that at most $|U'|$ vertices are added to S to form S_k, it suffices to show that $\text{epn}(u_i, S_{i-1}) \subseteq U'$. To see this, suppose to the contrary that $x \in \text{epn}(u_i, S_{i-1})$ and $x \notin U'$. By the definition of U', it follows that x has a neighbor in $S \setminus U$. Since $S \setminus U \subseteq S_{i-1}$, x has a neighbor in $S_{i-1} \setminus U$. But $u_i \in U$, contradicting that $x \in \text{epn}(u_i, S_{i-1})$. Hence, $\text{epn}(u_i, S_{i-1}) \subseteq U'$, and so we may conclude that every vertex added to form S_k is in the set U'.

It follows that to form S_k from our original set S, we add at most $|U'|$ vertices, while for the purposes of our count, we "remove" $|U| = k$ vertices. Since $|U'| \leq \gamma(T) - 2$, we have $|S_k| \leq |S| - |U| + |U'| \leq \gamma(T) - k + \gamma(T) - 2 = 2\gamma(T) - k - 2 \leq 2\gamma(T) - 3$ for $k \geq 1$. □

By Claim B, $\gamma_{ce}(T) \leq |S_k| \leq 2\gamma(T) - 3$, as desired. We conclude this proof by showing the bounds are sharp. The corona $T \circ K_1$ of any tree T achieves the lower bound. Let T be the corona $K_{1,t} \circ \overline{K}_{t-1}$. Then $\gamma(T) = t + 1$ and $\gamma_{ce}(T) = 2t - 1 = 2\gamma(T) - 3$, obtaining the upper bound. □

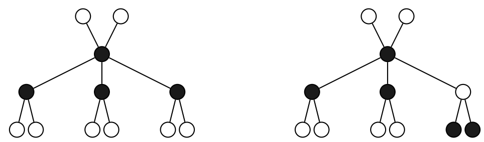


Figure 7: Tree T that achieves the upper bound of Theorem 4.1.

Note that the upper bound on the cost effective domination number of Theorem 4.1 does not hold for the very cost effective domination number of trees. For a counterexample, consider the tree $T = K_{1,t} \circ \overline{K}_t$ for which $\gamma(T) = \gamma_{c\epsilon}(T) = t + 1$ and $\gamma_{vc\epsilon}(T) = 2t > 2t - 2 = 2(t + 1) - 3 = 2\gamma(T) - 3$.

Next we characterize the trees obtaining the upper bound of Theorem 4.1. For this purpose, we define the family $\mathcal{F}$ of trees T_t, which are obtained from the star $K_{1,t}$, with center x and leaves $x_1, x_2, ..., x_t$ as follows. Add exactly $t - 1$ new vertices adjacent to x, and for $1 \leq i \leq t$, add at least $t - 1$ new vertices adjacent to x_i. Note that the corona $K_{1,t} \circ \overline{K}_{t-1} \in \mathcal{F}$.

Theorem 4.2 *A tree T with $\gamma(T) \geq 3$ has $\gamma_{c\epsilon}(T) = 2\gamma(T) - 3$ if and only if $T \in \mathcal{F}$.*

Proof. Let $T_t \in \mathcal{F}$. Then $\gamma(T_t) = t+1$, while $\gamma_{c\epsilon}(T) = t+t-1 = 2t-1 = 2\gamma(T_t)-3$.

Next assume that $\gamma_{c\epsilon}(T) = 2\gamma(T)-3$. Let S_k be a cost effective dominating set of T formed by the algorithm in the proof of Theorem 4.1. Then, $2\gamma(T) - 3 = \gamma_{c\epsilon}(T) \leq |S_k| \leq 2\gamma(T) - k - 2 \leq 2\gamma(T) - 3$. Since we have equality throughout, it follows that $2\gamma(T)-3 = 2\gamma(T)-k-2$, implying that for the set S_k, we have that $k = 1$. Thus, from our algorithm, we deduce that T has a $\gamma(T)$-set S with exactly one vertex, say u_1, that is not cost effective in S. Furthermore, $S_k = S_1 = (S \setminus \{u_1\}) \cup \text{epn}(u_1, S)$. Since $\gamma_{c\epsilon}(T) = 2\gamma(T)-3 \leq |S_k| = |S|-1+|\text{epn}(u_1, S)| = \gamma(T)-1+|\text{epn}(u_1, S)| \leq 2\gamma(T)-3$, we have that $|\mathrm{epn}(u_1, S)| = \gamma(T) - 2$. Moreover, since u_1 is not cost effective with respect to S, u_1 has exactly $\gamma(T) - 1$ neighbors in S. Since T is a tree, the induced subgraph $T[S]$ is the star $K_{1,\gamma(T)-1}$ with center u_1 and every vertex in $V \setminus S$ is a leaf in T. To see that $T \in \mathcal{F}$, we need to show that each vertex in $S \setminus \{u_1\}$ has at least $\gamma(T)-2$ leaf neighbors in $V \setminus S$. Suppose to the contrary that $x \in S \setminus \{u_1\}$ and x has at most $\gamma(T)-3$ leaf neighbors in $V \setminus S$. Then $(S \setminus \{x\}) \cup \mathrm{epn}(x, S)$ is a cost effective dominating set of T with cardinality $|S \setminus \{x\}| + |\mathrm{epn}(x, S)| \leq \gamma(T) - 1 + \gamma(T) - 3 < 2\gamma(T) - 3 = \gamma_{ce}(T)$, a contradiction. Thus, u_1 has exactly $\gamma(T) - 2$ leaf neighbors, and every vertex in $S \setminus \{u_1\}$ has at least $\gamma(T) - 2$ leaf neighbors, and so $T \in \mathcal{F}$. □

We conclude by showing that all values between the lower and upper bounds of Theorem 4.1 are realizable. Let $K^v_{1,t}$ be the star with center v and leaves $v_1, ..., v_t$.

Theorem 4.3 *Given positive integers a and b such that $4 \leq a \leq b \leq 2\gamma(T)-3$, there exists a tree T having $\gamma(T) = a$ and $\gamma_{ce}(T) = b$.*

Proof. To construct a tree T having $\gamma(T) = a$ and $\gamma_{ce}(T) = b$, we begin with the tree $(K^x_{1,a-2} \circ \overline{K}_{a-2}) \cup K^y_{1,b-a+1}$ and add the edge xy. Then, T has a support vertices.

We show that $\gamma(T) = a$ and $\gamma_{ce}(T) = b$. First note that since the set of support vertices of T is a dominating set, $\gamma(T) \leq a$, and since every leaf or its support must be in any $\gamma(T)$-set, we have $\gamma(T) \geq a$. Hence, $\gamma(T) = a$.

Let $S = \{x, x_1, ..., x_{a-2}, y_1, ..., y_{b-a+1}\}$. To see that S is a dominating set, note that every vertex in S is dominated by S. Assume $v \in V \setminus S$. Then, v is either a leaf adjacent to x_i or x, or $v = y$ and is dominated by y_j, for some i, j. Hence, S is a dominating set. To see that S is cost effective, note that y_i is independent in $T[S]$, so each y_i, $1 \leq i \leq b - a + 1$, is cost effective with respect to S. Moreover,

$|N(x_i) \cap S| = 1$ and $|N(x_i) \cap (V \setminus S)| = a - 2 \geq 2$, so x_i, for $1 \leq i \leq a-2$, is cost effective. Finally, $|N(x) \cap S| = a-2 < a-1 = |N(x) \cap (V \setminus S)|$, so x is cost effective. Hence, S is cost effective, and so $\gamma_{c\epsilon}(T) \leq |S| = 1 + a - 2 + b - a + 1 = b$.

Now, let S^* be a $\gamma_{c\epsilon}(T)$-set. To dominate T, each leaf or its support vertex must be in S^*. We show that at least one of the support vertices is not in S^*. Assume to the contrary that S^* contains all the support vertices of T. That is, $\{x, x_1, ..., x_{a-2}, y\} \subseteq S^*$. But then $|N(x) \cap (V \setminus S^*)| = a - 2 < |N(x) \cap S^*| = a - 1$, contradicting that S^* is a cost effective set. Hence, at least one support vertex, say w, of T is not in S^*, implying that S^* contains the leaves adjacent to w. Let l_w be the number of leaves adjacent to w. Recall that T has a support vertices, so $a - 1 + l_w \leq |S^*| = \gamma_{c\epsilon}(T) \leq b$. Thus, $l_w \leq b - a + 1$. Since $b \leq 2a - 3$, we have that $b - a + 1 \leq 2a - 3 - a + 1 = a - 2$. Now each support vertex of T is adjacent to either $a - 2$ or $b - a + 1$ leaves and $b - a + 1 \leq a - 2$, so we conclude that each support vertex is adjacent to at least $b - a + 1$ leaves. In particular, $l_w \geq b - a + 1$, and so, $l_w = b - a + 1$. Hence, $\gamma_{c\epsilon}(T) = |S^*| \geq a - 1 + l_w = a - 1 + b - a + 1 = b$. Therefore, $\gamma_{c\epsilon}(T) = b$. □

For an example, consider the tree T in Figure 8(a) where the darkened vertices represent a $\gamma(T)$-set and Figure 8(b) where the darkened vertices represent a $\gamma_{c\epsilon}(T)$-set.

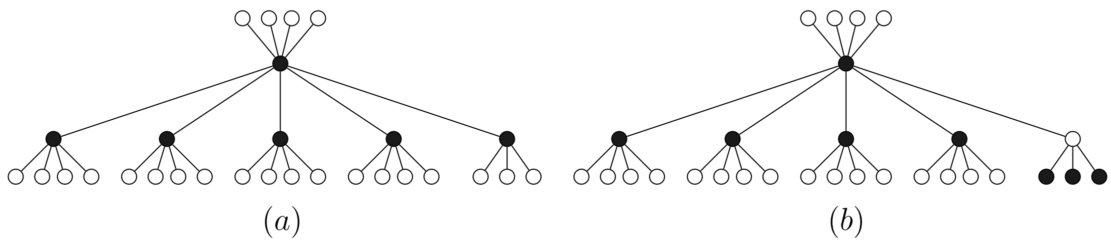


Figure 8: Tree T with $\gamma(T) = a = 6$ and $\gamma_{c\epsilon}(T) = b = 8$.

5 CONCLUDING REMARKS

We have determined an upper bound on the cost effective domination number of trees and characterized the trees obtaining the bound. We also showed that every value of $\gamma_{ce}(T)$ between the upper and lower bounds is realizable. We conclude with some open problems suggested by this work:

1. Characterize the trees T for which $\gamma(T) = \gamma_{c\epsilon}(T)$.

2. Characterize the trees T for which $\gamma_{c\epsilon}(T) = \gamma_{vc\epsilon}(T)$.

3. Characterize the trees T for which $\gamma_{vc\epsilon}(T) = i(T)$.

4. We have seen that the upper bound of $2\gamma(T)-3$ on the cost effective domination number of trees does not hold for the very cost effective domination number. Is there a bound on $\gamma_{vc\epsilon}(T)$ in terms of $\gamma(T)$ for trees T?

5. Although $2\gamma(T)-3$ is an upper bound on the cost effective domination number for trees, we have not been able to prove or disprove that it is a bound for the cost effective domination number of general graphs. Prove or disprove: For any graph G, $\gamma_{c\epsilon}(G) \leq 2\gamma(G) - 3$.

6. Investigate bounds on the upper parameters $\Gamma_{c\epsilon}(G)$ and $\Gamma_{vc\epsilon}(G)$.

BIBLIOGRAPHY

[1] R. Aharoni, E. C. Milner and K. Prikry. Unfriendly partitions of a graph. *J. Combin. Theory, Ser. B* 50(1):1-10, 1990.

[2] C. Bazgan, Z. Tuza and D. Vanderpooten. The satisfactory partition problem. *Discrete Appl. Math.* 154:1236-1245, 2006.

[3] A. V. Borodin and A. V. Kostochka. On an upper bound of a graph's chromatic number, depending on the graph's degree and density. *J. Combin. Theory Ser. B* 23:247-250, 1977.

[4] F. Dahme, D. Rautenbach and L. Volkmann. Some remarks on α-domination. *Discuss. Math. Graph Theory*, 24:423-430, 2004.

[5] J. E. Dunbar, D. G. Hoffman, R. C. Laskar and L. R. Markus. α-domination. *Discrete Math.*, 211:11-26, 2000.

[6] O. Favaron, G. H. Fricke, W. Goddard, S. M. Hedetniemi, S. T. Hedetniemi, P. Kristiansen, R. C. Laskar and R. D. Skaggs. Offensive alliances in graphs. *Discuss. Math. Graph Theory*, 24:263-275, 2004.

[7] M. U. Gerber and D. Kobler. Partitioning a graph to satisfy all vertices. *Tech. Rept. Swiss Federal Inst. Tech.*, Lausanne, 1998.

[8] M. U. Gerber and D. Kobler. Algorithmic approach to the satisfactory graph partitioning problem. *European J. Oper. Res.* 125(2):283-291, 2000.

[9] M. U. Gerber and D. Kobler. Classes of graphs that can be partitioned to satisfy all their vertices. *Australas. J. Combin.* 29:201-214, 2004.

[10] T. W. Haynes, S. M. Hedetniemi, S. T. Hedetniemi, T. L. McCoy, and I. Vasylieva, Cost effective domination in graphs. In press in *Congr. Numer.*

[11] T. W. Haynes, S. T. Hedetniemi and M. A. Henning. Global defensive alliances. *Proc.* 17^{th} *Int. Symp. Comput. Inform. Sci., I, ISCIS XVII (Orlando, FL, 2002)*, 303-307, CRC Press, 2002.

[12] T. W. Haynes, S. T. Hedetniemi and M. A. Henning. Global defensive alliances in graphs. *Electron. J. Combin.*, 10:Research Paper 47, 13 pp, 2003. http://www.combinatorics.org/ojs/index.php/eljc/article/view/v10i1r47/pdf

[13] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Fundamentals of Domination in Graphs*. Marcel Dekker, Inc., New York (1998).

[14] S. M. Hedetniemi, S. T. Hedetniemi and P. Kristiansen. Alliances in graphs. *J. Combin. Math. Combin. Comput.*, 48:157-177, 2004.

[15] S. M. Hedetniemi, S. T. Hedetniemi and A. A. McRae, Cost effective sets in graphs. Manuscript, August, 2012. hedet@clemson.edu

[16] P. Kristiansen, S. M. Hedetniemi and S. T. Hedetniemi. Introduction to alliances in graphs. *Proc.* 17^{th} *Int. Symp. Comput. Inform. Sci., ISCIS XVII (Orlando, FL, 2002)*, 308-312, CRC Press, 2002.

[17] J. L. Mashburn, T. W. Haynes, S. M. Hedetniemi, S. T. Hedetniemi and P. J. Slater. Differentials in Graphs. *Utilitas Mathematica*, 69:43-54, 2006.

[18] J. A. Rodriguez-Velazquez and J. M. Sigarreta. Global alliances in planar graphs. *AKCE Internat. J. Graphs Combin.*, 4(1):83-98, 2007.

[19] K. H. Shafique and R. D. Dutton. On satisfactory partitioning of graphs. *Congr. Numer.* 154:183-194, 2002.

[20] S. Shelah and E. C. Milner. Graphs with no unfriendly partitions. In A. Baker, B. Bollobás and A. Hajnal (eds.), *A Tribute to Paul Erdös.* Cambridge University Press, Cambridge, 373-384, 1990.

VITA

TABITHA MCCOY

Education:	B.S. Mathematics, University of Virginia's College at Wise, Wise, Virginia 2011 M.S. Mathematics, East Tennessee State University Johnson City, Tennessee 2013
Professional Experience:	National Science Foundation GK-12 Fellow East Tennessee State University Johnson City, Tennessee (2012-2013)
Publications:	T. W. Haynes, S. M. Hedetniemi, S. T. Hedetniemi, T. L. McCoy, and I. Vasylieva, Cost effective domination in graphs. In press in *Congr. Numer.* T. W. Haynes, S. T. Hedetniemi, T. L. McCoy, Cost effective domination. Submitted for publication, August 2012.
Honor Societies:	Kappa Mu Epsilon Darden Society