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Finding Edge and Vertex Induced Cycles within Circulants.

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Finding Edge and Vertex Induced Cycles Within Circulants

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by

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ABSTRACT

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Let H be a graph. G is a subgraph of H if $V(G) \subseteq V(H)$ and $E(G) \subseteq E(H)$. The subgraphs of H can be used to determine whether H is planar, a line graph, and to give information about the chromatic number. In a recent work by Beeler and Jamison [3], it was shown that it is difficult to obtain an automorphic decomposition of a triangle- free graph. As many of their examples involve circulant graphs, it is of particular interest to find triangle-free subgraphs within circulants. As a cycle with at least four vertices is a canonical example of a triangle-free subgraph, we concentrate our efforts on these. In this thesis, we will state necessary and sufficient conditions for the existence of edge induced and vertex induced cycles within circulants.

DEDICATION

First, I would like to dedicate this thesis to my Heavenly Father. Next, to my late father Tennyson W. Wooten Sr., who was very instrumental in my interest in mathematics. Finally, my mother Marie, my brothers Tennyson Jr. and Micah, and my sister Leanna. Thank you for all your love and encouragement throughout my academic career.

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1 INTRODUCTION

The purpose of this thesis is to find edge and vertex induced subgraphs (particularly cycles) within circulant graphs. In Section 1.1, basic graph theoretical definitions are introduced that pertain to this paper. In Section 1.2, we include background information from modern algebra and number theory that will be used throughout. For Section 1.3, we define the circulant graph, its properties, and we also state the problem we are analyzing. For Section 1.4, we define labellings of graphs. Finally, in Section 1.5, we show a preview of major results and theorems that will be proven in Chapter 4.

1.1 Basic Definitions of Graph Theory

A *graph* G consists of a *vertex set* $V(G)$, an *edge set* $E(G)$, and a relation that associates one edge with two vertices, called the *endpoints*. We will assume that all the graphs presented throughout this paper are *simple*, i.e., there are no *loops* (edges whose endpoints are equal) or *multiple edges* (edges that have the same endpoints) [17]. All graphs presented here are finite unless otherwise noted. In Figure 1, we have an example of a graph G.

The *order* of G, denoted as $n(G) = |V(G)|$, is the number of vertices in G. The *size* of G, denoted as $e(G) = |E(G)|$, is the number of edges in G. For example, in Figure 1, $n(G) = 8$ and $e(G) = 11$. The *complement* of G, denoted as $\overline{G}$, has a vertex set $V(G)$, and edges defined by $v_1v_2 \in E(\overline{G})$ if and only if $v_1v_2 \notin E(G)$. A vertex v is *incident* to an edge e if v is an endpoint of e. The *degree* of a vertex v is the number of edges incident to v, denoted as $deg_G(v)$ [17]. The *maximum degree* of G, denoted

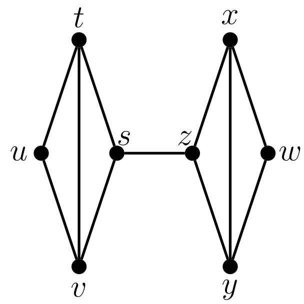


Figure 1: A Graph G

as $\Delta(G) = \max\{deg_G(v)\}$, is the largest number of edges incident to a vertex v in G. The *minimum degree* of G, denoted as $\delta(G) = \min\{deg_G(v)\}$, is the smallest number of edges incident to a vertex v in G. A graph G is *d-regular* when $\Delta(G) = \delta(G) = d$ [7]. Therefore, in Figure 1, $\Delta(G) = 3$ and $\delta(G) = 2$ but it is not a regular graph.

Two vertices $v, w \in V(G)$ are *adjacent* if and only if v and w are endpoints of an edge e [17]. A $v - w$ *walk* denoted as W, is a sequence of vertices such that consecutive vertices in this sequence are adjacent. In other words, $W : v = w_0, w_1, w_2, \ldots, w_j = w$ where $j \geq 0$ and w_k and w_{k+1} are adjacent for $k = 0, 1, 2, \ldots, j - 1$. In any walk W, the vertices and edges can be traversed more than once [7]. This is the case for a $t - s$ walk in Figure 2 with $W : t, u, v, t, s, v, s$.

A $v - w$ *path* in G is a walk where the vertices and edges are traversed only once. A graph G is *connected* if there is a $v - w$ path for all $v, w \in V(G)$. In particular, a path on n vertices is denoted P_n. We will assume that all graphs in this thesis are connected. A *cycle* is a $v - w$ path with $v = w$. A cycle on n vertices is denoted as C_n [7]. In Figure 3, we have an example of a $t - s$ path P_4 on the left and a C_4 cycle on the right. A graph is *complete* when all the vertices are mutually adjacent.

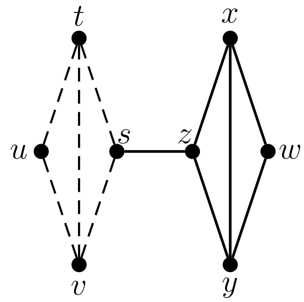


Figure 2: A Walk W in Graph G

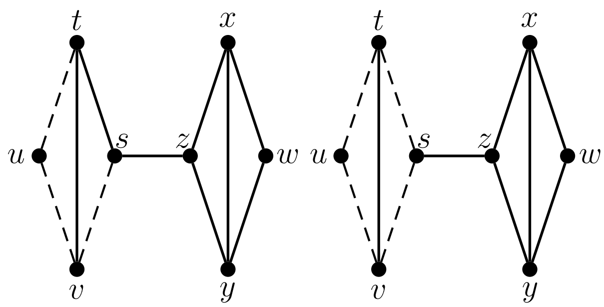


Figure 3: A Path and a Cycle on G

A complete graph on n vertices is denoted as K_n. A *tree* denoted as T is an acyclic graph, i.e., it has no cycles [7]. So in Figure 1, G is a connected graph.

An *isomorphism* from a graph G_1 to another graph G_2 is a bijection $\alpha : V(G_1) \to V(G_2)$ such that $vw \in E(G_1)$ if and only if $\alpha(v)\alpha(w) \in E(G_2)$. If such a bijection exists, we can say that G_1 is isomorphic to G_2 and which is denoted as $G_1 \cong G_2$ [17].

A graph H has a *subgraph* G if every vertex in G is also a vertex from H and every edge of G is also an edge of H, i.e., $V(G) \subseteq V(H)$ and $E(G) \subseteq E(H)$. A *maximum clique* is the largest complete subgraph in H. The order of the maximum clique is denoted as $\omega(G)$ [17]. Referring back to Figure 1, the largest complete subgraph is K_3, i.e., $\omega(H) = 3$. This particular subgraph is also a triangle, i.e., $K_3 \cong C_3$. So a *triangle-free* graph contains no subgraph isomorphic to C_3 [17].

An *edge induced subgraph* of G induced by $A \subseteq E(G)$ denoted as $\langle A \rangle$ consists of all the vertices that are incident with at least one edge in A. A *vertex induced subgraph* of G induced by $B \subseteq V(G)$ is denoted as G_B. We define G_B as follows:

$$V(G_B) = B \text{ and}$$

$$E(G_B) = \{b_1b_2 : b_1, b_2 \in B, b_1b_2 \in E(G)\} \text{ [4].}$$

This means G_B has vertices from G with all their corresponding edges [10]. We have an example in Figure 4, of an edge induced subgraph and a vertex induced subgraph. On the far left, we have our original graph G. In the center, we have an edge induced subgraph of G with $A \subseteq E(G)$ such that $A = \{vu, ut, ts\}$. On the far right, we have a vertex induced subgraph of G with $B \subseteq V(G)$ such that $B = \{s, t, v, x, y, z\}$ with $V(G_B) = B$ and $E(G_B) = \{st, tv, sv, sz, zx, yz, xy\}$.

We have another example of edge and vertex induced subgraphs in Figure 5 using a graph called a *wheel* denoted as $W_{1,n}$ on the first line, i.e., a cycle with an appended universal vertex [17]. On the second line towards the left, we have an edge induced subgraph P_3. On the right of this same line, we have a vertex induced subgraph C_3. Notice that in both of these subgraphs they consist of the same vertices. However, one is an edge induced P_3, the other is a vertex induced C_3. On the third line, we have a P_3 subgraph which is both an edge and vertex induced subgraph. Our goal is to find edge and vertex induced cycles within any circulant. After defining edge and vertex induced subgraphs, we now can prove the following proposition:

Proposition 1.1 *[5] If G is a vertex induced subgraph of H, then G is an edge*

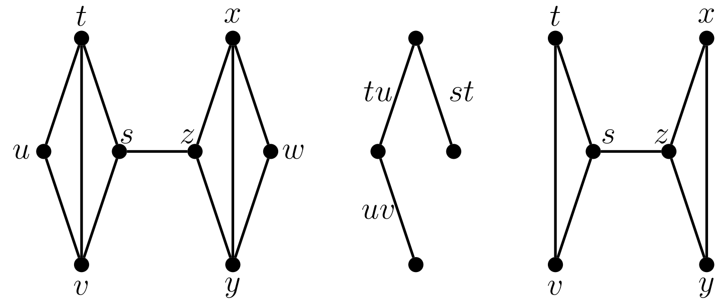


Figure 4: Edge and Vertex Induced Subgraphs of G

induced subgraph of H.

Proof. Let H_A be a vertex induced subgraph of H such that H_A is isomorphic to G. Since H has a vertex induced subgraph, there exists an edge induced subgraph $\langle L \rangle$ that is isomorphic to G. So we have that $L \subseteq E(H)$. Therefore, $L = E(H_A)$. In conclusion, we have that $\langle L \rangle$ is isomorphic to G. ■

The converse of Proposition 1.1 is false as we have seen using the wheel $W_{1,4}$ in Figure 5. Remember that we had an edge induced subgraph P_3 and a vertex induced subgraph C_3. Both of these subgraphs had the same vertices from $W_{1,4}$, but these subgraphs are not isomorphic to each other. Thus, if G is an edge induced subgraph of H, then G is not always a vertex induced subgraph of H.

We define a *coloring* of graph G as an assignment of colors to the vertices. A *proper coloring* of G is an assignment of colors to the vertices of G such that adjacent vertices receive different colors [17]. The *chromatic number* of a graph G is the minimum amount of colors needed for a proper coloring of a graph. This is denoted as $\chi(G)$ [10]. We will use the graph from Figure 1, to show that for this particular graph $\chi(G) = 3$, in Figure 6 with the colors labeled as 1, 2, and 3.

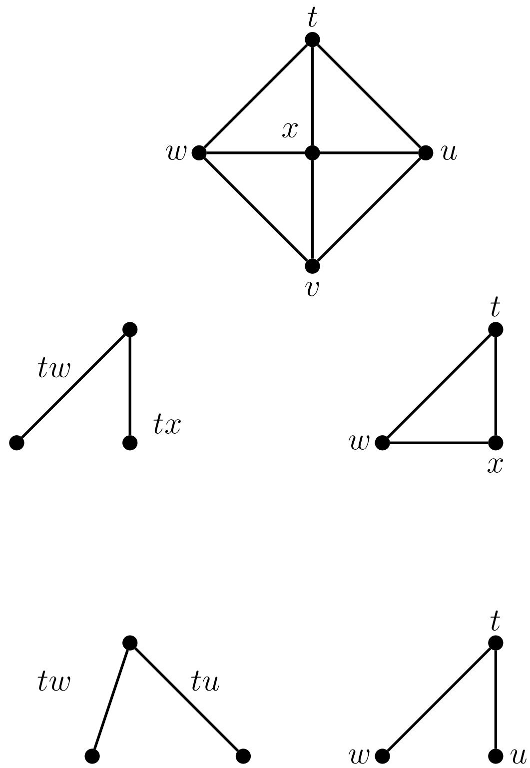


Figure 5: Induced Subgraphs of $W_{1,4}$

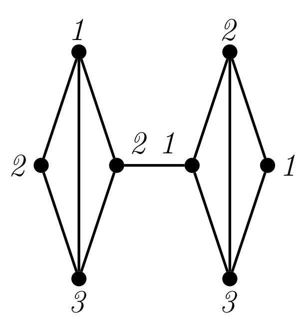


Figure 6: A Coloring of G

1.2 Modern Algebra and Number Theory

In this section, we will give brief background information from modern algebra and number theory. Let Γ be a non empty set. A *group* $(\Gamma, *)$ with a binary operation $*$, satisfies the following properties below [8]:

i) Closure: If $c, d \in \Gamma$, then $c * d \in \Gamma$.

ii) Associativity: For all $c, d, f \in \Gamma, (c * d) * f = c * (d * f)$.

iii) Identity: For all $c \in \Gamma$ there exist an identity $e \in \Gamma$ such that $c * e = e * c = c$.

iv) Inverse: For every $c \in \Gamma$ there exist an element $d \in \Gamma$ such that $c * d = d * c = e$.

The *order* of a group is the number of elements in $(\Gamma, *)$, denoted as $|\Gamma|$. In a group Γ, if there exist $h \in \Gamma$ such that $h = h^{-1}$, then h is called an *involution*. All groups have at least one involution, namely the identity. If for all $c, d \in \Gamma$, $c * d = d * c$, then Γ is *Abelian*. A group Γ is *cyclic* if there exists $c \in \Gamma$, such that $\Gamma = \{c^m : m \in \mathbb{Z}\}$. This element c is a generator of Γ, denoted as $\Gamma =< c >$. In this thesis, we will be using $(\mathbb{Z}_n, +)$ as a cyclic group of order n. The involutions of this group are $\frac{n}{2}$ and 0 [8]. These involutions will be important in the notation listed below and on labellings in Section 1.4. We will define the following below:

i) The set of positive integers as $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$;

ii) Integers modulo n, $\mathbb{Z}_n = \{0, 1, 2, \dots, n-1\}$;

iii) "Positive" integers modulo n: $\mathbb{Z}_n^+ = \{x \in \mathbb{Z}_n : 0 < x < \frac{n}{2}\}$;

iv) $\mathbb{Z}_n^* = \{x \in \mathbb{Z}_n : 0 < x \leq \frac{n}{2}\}$[2];

v) Modular Absolute Value:

$$|a|_n = \begin{cases} a & \text{if } 0 \le a \le \frac{n}{2} \\ n-a & \text{if } \frac{n}{2} < a < n. \end{cases}$$

Let $a, b \in \mathbb{Z}$ with $b > 0$. We have that b is a *divisor* of a if there exists $c \in \mathbb{Z}$ such that $a = bc$. This is denoted as $b|a$. The *greatest common divisor* of a and b is the largest d such that $d|a$ and $d|b$. This is denoted as $gcd(a, b)$. We have that a and b are *relatively prime* if $gcd(a, b) = 1$ [8].

Proposition 1.2 *[8] We have that* $gcd(a_1, a_2, a_3, \ldots, a_n) = 1$ *if and only if there exists* $x_1, x_2, x_3, \ldots, x_n \in \mathbb{Z}$ *such that* $x_1a_1 + x_2a_2 + x_3a_3 + \cdots + x_na_n = 1$.

In addition, Proposition 1.2 extends gcd to any number of variables.

1.3 Circulant Graphs

In this section, we define a circulant and state its properties. Let $n \in \mathbb{Z}^+$ and $S \subseteq \mathbb{Z}_n^*$ be given. A *circulant*, denoted as $C_n(S)$, is an undirected graph with $V(C_n(S)) = \mathbb{Z}_n$ and $E(C_n(S))$ defined as follows:

$$E(C_n(S)) = \{vw : v, w \in \mathbb{Z}_n \quad \text{and} \quad |v - w|_n \in S\}.$$

The elements in S are called *lengths*. So we have that $vw \in E(C_n(S))$ is of length a if $|v - w|_n = a \in S$ [11]. An example of a circulant is shown in Figure 7 as $C_6(1, 2)$.

Heuberger [11] presented the following properties of a circulant below without proof. For completeness, we include the proofs.

Proposition 1.3 *[11] Let* $n \in \mathbb{Z}$ *with* $n \ge 3$ *and* $S = \{a_1, a_2, a_3, \ldots, a_m\} \subseteq \mathbb{Z}_n^*$ *be given. We have the following properties:*

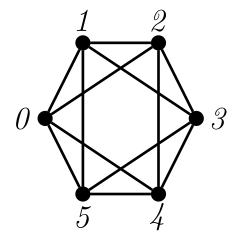


Figure 7: $C_6(1,2)$

i) $C_n(S)$ *is connected if and only if* $gcd(a_1, a_2, a_3, \ldots, a_m, n) = 1$*;*

ii) $C_n(S)$ *with* $gcd(a_1, a_2, a_3, \ldots, a_m, n) = d$ *is isomorphic to* d *disjoint copies of* $C_{\frac{n}{d}}(\frac{a_1}{d}, \frac{a_2}{d}, \frac{a_3}{d}, \ldots, \frac{a_m}{d})$*;*

iii) If $gcd(a_i, n) = 1$ *for some* $i = 1, 2, 3, \ldots, n$ *and* $a_i^{-1} \equiv b \pmod{n}$*, then* $C_n(a_1, a_2, a_3, \ldots, a_m)$ *is isomorphic to* $C_n(a_1b, a_2b, a_3b, \ldots, a_mb)$*;*

iv) $C_n(S)$ *is r-regular where:*

$$r = \begin{cases} 2m & \text{if } S \subset \mathbb{Z}_n^+ \\ 2m-1 & \text{if otherwise}; \end{cases}$$

v) $C_n(a)$ *is isomorphic to* C_n *if and only if* $gcd(a, n) = 1$*;*

vi) $C_n(1, 2, 3, \ldots, \lfloor \frac{n}{2} \rfloor)$ *is isomorphic to* K_n*;*

vii) $\overline{C_n(S)}$ *is isomorphic to* $C_n(\mathbb{Z}_n^* - S)$*.*

Proof.

i) Assume that $C_n(S)$ is connected. We must show that $\gcd(a_1, \ldots, a_m, n) = 1$. Since $C_n(S)$ is a connected graph, there is a path between any two vertices.

Without loss of generality, we must show that there is a path between 0 and 1. Thus there is some combination of edge lengths that sum to one modulo n. This implies that there exist $s_1, \dots, s_k \in \mathbb{Z}$ such that $s_1a_1 + \cdots + s_ka_k \equiv 1 \pmod{n}$.

$$s_1a_1 + \cdots + s_ka_k \equiv 1 \pmod{n}$$

$$\iff s_1a_1 + \cdots + s_ka_k = qn + 1 \text{ for some integer } q$$

$$\iff s_1a_1 + \cdots + s_ka_k - qn = 1$$

$$\iff \gcd(a_1, \dots, a_k, n) = 1 \quad \text{by Proposition 1.2} \quad .$$

If $\gcd(a_1, \dots, a_k, n) = 1$ then there exists $s_1, \dots, s_k$ such that

$$s_1a_1 + \cdots + s_ka_k = 1 \pmod{n}.$$

Thus you can make step size one. Hence $C_n(S)$ is a connected graph.

ii) Assume $C_n(S)$ has a $\gcd(a_1, a_2, a_3, \dots, a_m, n) = d$. We want to show that $C_n(S)$ is isomorphic to d disjoint copies of $C_{\frac{n}{d}}(\frac{a_1}{d}, \frac{a_2}{d}, \frac{a_3}{d}, \dots, \frac{a_m}{d})$. Let $j \in V(C_n(S))$. Let C_j be a connected component containing j. Therefore, the elements of C_j form $j + id$ with an integer i. We have that $d|n$ by assumption. This means that there are d connected components consisting of $\frac{n}{d}$ vertices. Therefore,

$$s_1\frac{a_1}{d} + s_2\frac{a_2}{d} + s_3\frac{a_3}{d} + \cdots + s_m\frac{a_m}{d} \equiv 1 \pmod{n}.$$

Next, we need to show that C_j is isomorphic to $C_{\frac{n}{d}}(\frac{a_1}{d}, \frac{a_2}{d}, \frac{a_3}{d}, \dots, \frac{a_m}{d})$. Let β be a function where $\beta : V(C_j) \to V(C_{\frac{n}{d}}(\frac{a_1}{d}, \frac{a_2}{d}, \frac{a_3}{d}, \dots, \frac{a_m}{d}))$ such that for all $s \in V(C_j)$ we have that $\beta(s) = \frac{s}{d} \pmod{\frac{n}{d}}$. So, if $vw \in E(C_n(S))$, then

$|v-w|_n = a_i \in S$. Hence,

$$\beta(vw) = |\beta(v)-\beta(w)|_{\frac{n}{d}} = |\frac{v}{d} - \frac{w}{d}|_{\frac{n}{d}} = \frac{a_i}{d}.$$

Therefore, $\beta(vw) \in C_{\frac{n}{d}}(\frac{a_1}{d}, \frac{a_2}{d}, \frac{a_3}{d}, \ldots, \frac{a_m}{d})$. So, $C_j \cong C_{\frac{n}{d}}(\frac{a_1}{d}, \frac{a_2}{d}, \frac{a_3}{d}, \ldots, \frac{a_m}{d})$.

iii) Assume that $gcd(a_i, n) = 1$ with $j \in N$ and $a_i^{-1} \equiv b \pmod{n}$. We need to show that $C_n(a_1, a_2, a_3, \ldots, a_m) \cong C_n(a_1b, a_2b, a_3b, \ldots, a_mb)$. Suppose $x_1x_2 \in E(C_n(S))$, such that $|x_1 - x_2|_n = a_j$ with $a_j \in S$. Define $s_n^* = \{a_1b, a_2b, a_3b, \ldots, a_mb\}$ and note the following below:

$$\beta(x_1x_2) = |\beta(x_1) - \beta(x_2)|_n = |a_j^{-1}x_1 - a_j^{-1}x_2|_n$$

$$= a_j^{-1}|x_1 - x_2| \in s^*.$$

Therefore,

$$C_n(a_1, a_2, a_3, \ldots, a_m) \cong C_n(a_1b, a_2b, a_3b, \ldots, a_mb).$$

iv) We need to show that $C_n(S)$ is $r-regular$ with the following conditions below:

$$r = \begin{cases} 2m & \text{if } S \subset \mathbb{Z}_n^+ \\ 2m-1 & \text{otherwise.} \end{cases}$$

Let $x_1 \in V(C_n(S))$ and $a_j \in S$. So x_1 is adjacent to all the vertices of the form $x_1 + a_j \pmod{n}$ or $x_1 - a_j \pmod{n}$. If $a_j \neq \frac{n}{2} \pmod{n}$, then these vertices are distinct. In other words, there are two of these for each difference. So, $deg(x_i) = 2m$. Since x_i was chosen arbitrarily $C_n(S)$ is $2m$-regular. If one these differences is an involution, $x_i + \frac{n}{2} = x_i - \frac{n}{2}$. This means that x_1 is adjacent to one less vertex. Therefore,

$$r = \begin{cases} 2m & \text{if } S \subset \mathbb{Z}_n^+ \\ 2m-1 & \text{otherwise.} \end{cases}$$

v) Assume that $C_n(a) \cong C_n$. We must show that $gcd(a, n) = 1$. We have that C_n is a 2-regular connected graph. We have that $C_n(a)$ is 2-regular if and only if $a \neq \frac{n}{2}$ and $\gcd(a, n) = 1$. Assume that $\gcd(a, n) = 1$. We must show that $C_n(a) \cong C_n$. From (i) and (ii), we know that $C_n(S)$ with $\gcd(a, n) = 1 \in S$. A connected graph is 2-regular if and only if it is a cycle [17]. So from (iv) $C_n(a)$ is a 2-regular graph. So the length $a \in S$ with $a = 1$. Therefore, if $C_n(a)$, then $C_n(1) \cong C_n$.

vi) We must show that $C_n(1, 2, 3, \ldots, \lfloor \frac{n}{2} \rfloor)$ is isomorphic to K_n. Since we are using simple graphs, every complete graph K_n is a $(n-1)$-regular graph. Using (iv) $C_n(S)$ is r-regular where:

$$r = \begin{cases} 2m & \text{if } S \subset \mathbb{Z}_n^+ \\ 2m-1 & otherwise. \end{cases}$$

If $\lfloor \frac{n}{2} \rfloor = \frac{n}{2}$ then $r = 2(\frac{n}{2}) - 1 = n - 1$. If $\lfloor \frac{n}{2} \rfloor \neq \frac{n}{2}$ implies $\lfloor \frac{n}{2} \rfloor = \frac{n-1}{2} = r = 2(\frac{n-1}{2}) = n - 1$. Hence, $C_n(1, 2, 3, \ldots, \lfloor \frac{n}{2} \rfloor) \cong K_n$.

vii) We must show that $\overline{C_n(S)}$ is isomorphic to $C_n(\mathbb{Z}_n^* - S)$. We have that $vw \in E(C_n(S))$ if and only if $|v - w|_n \in S$. Therefore,

$$|v - w|_n \notin \mathbb{Z}^* - S$$

$$\iff vw \notin E(C_n(\mathbb{Z}^* - S)).$$

Hence, $\overline{C_n(S)}$ is isomorphic to $C_n(\mathbb{Z}_n^* - S)$.

■

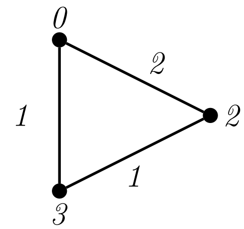


Figure 8: Example of Labelling with $f : C_3 \to \mathbb{Z}_4$

1.4 Labellings

In this section, we will go over labellings that pertain to the purpose of this thesis. A $\mathbb{Z}_n$-labelling of a graph G is an injective function f that maps $V(G)$ to $\mathbb{Z}_n$, i.e., $f : V(G) \to \mathbb{Z}_n$. Each of these vertex labels induces an edge label as follows: for all $ab \in E(G)$, $f'(ab) = |f(a) - f(b)|_n$. In Figure 8, we have an example of a labelling $f : C_3 \to \mathbb{Z}_4$. So the edge labels are obtained by taking the difference of two vertices modulo 4.

A $\mathbb{Z}_n$-labelling f of $G = (V, E)$ is a $\mathbb{Z}_n$-*valuation* if and only if the edge induced labelling f' is injective and if the involution $\frac{n}{2}$ is excluded from the edge label [4]. Labellings were introduced in Rosa's paper [15] which described a $\mathbb{Z}_n$-valuation on a graph of size n as a ρ-valuation if and only if $n = 2r+1$ and β-valuation if and only if f is positive. We have that f is positive if and only if $V \to \mathbb{Z}_n^+ \cup \{0\}$ [4]. So we have Rosa's β-valuations were popularized by Golomb [9] which he called *graceful*. The reason that $f : V(G) \to \mathbb{Z}_n$ must be injective is to prevent degeneracy of a subgraph. If the labelling of the vertices is not injective, G will collapse as shown in Figure 9. Next, we will look at a cycle that is induced within a circulant. So for G to be an

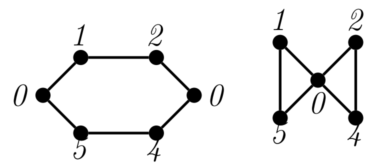


Figure 9: Consequence of a Vertex Labelling not Being Injective

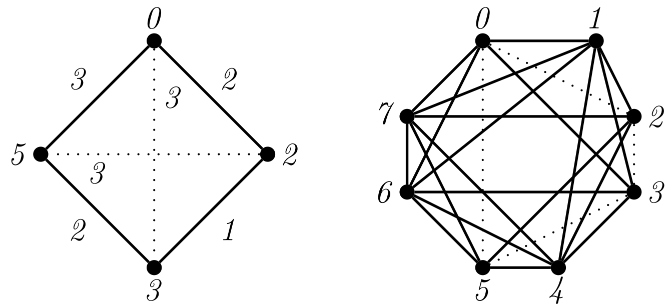


Figure 10: Edge Induced C_4 within $C_8(1, 2, 3)$

edge induced subgraph of $H = C_n(S)$ we need $f(E(G)) \subseteq S$. The concept of induced edge labels can be extended to all pairs of vertices in the obvious way, which we call co-edges [5].

We define a set of *co-edge* labels, $f(E(\overline{G}))$

$$f(E(\overline{G})) = \{|f(x) - f(y)|_n : xy \notin E(G)\}.$$

For G to be a vertex induced subgraph of $C_n(S)$, none of these co-edges can be in the difference set S, i.e., $f(E(\overline{G})) \cap S = \emptyset$ [5]. Referring to Figure 10, we have an example of an edge induced C_4 within $C_8(1, 2, 3)$.

1.5 Preview

In this section, we are introducing five major ideas that are essential to the problem that we are analyzing. The first idea mentions the necessary and sufficient conditions to have a vertex induced subgraph within a circulant. The other four ideas will be proven in Chapter 4, which deal with the major results obtained in this thesis. Also in both of these chapters, an explanation will be given of each one. In Chapter 3, we have two major theorems that come from the results of the tables that are listed. They show the conditions for an edge and a vertex induced $C_4 \in C_n(a, b)$.

1) Let $S \subseteq \mathbb{Z}_n^+$ be given. A circulant, $C_n(S)$, contains a vertex induced K_{k+1} if and only if there exists $\{a_1, \ldots, a_k\} \subseteq S$ such that:

$$|a_i - a_j|_n \in S \quad \text{for all} \quad 1 \leq i \leq j \leq k$$

[2].

2) Suppose that $C_n(S)$ is a d-regular circulant where $d \geq 2$. If C_m is a vertex induced subgraph of $C_n(S)$, then $m \leq \frac{nd}{2(d-1)}$.

3) Let $|S| \geq 2$. If C_m is a vertex induced subgraph of $C_n(S)$, then $m \leq \frac{n}{2} + 1$.

4) There exists an edge induced C_m if and only if there exists $\ell_i \in \{0, 1\}$ and $x_i \in S$ for $i = 1, 2, 3, \ldots, m$ such that:

   i) $\sum_{i=1}^{m} (-1)^{\ell_i} x_i \equiv 0 \pmod{n}$ and

   ii) $\sum_{i=1}^{j} (-1)^{\ell_i} x_i \not\equiv \sum_{i=1}^{k} (-1)^{\ell_i} x_i \pmod{n}$ for $j \neq k$.

5) There exists a vertex induced C_m if and only if (4) holds and:

$$\left| \sum_{i=1}^{k-1} (-1)^{\ell_i} x_i - \sum_{i=1}^{j-1} (-1) x_i \right|_n \notin S$$

for all $k, j \in \{0, 1, 2, \ldots, m\}$ such that $|k - j|_n \geq 2$.

For (1) we have the largest possible size of a vertex induced subgraph that can be within any circulant. This is obtained by looking at the largest clique that is possible.

We have that (2) gives us the largest size possible for a vertex induced cycle within any circulant. This is the upper bound of a vertex induced cycle. Notice that as d becomes larger, k approaches $\frac{n}{2}$.

Using the information from (2), we obtain another bound in (3). This idea shows that we can not have long cycles within any circulant. We can however achieve equality, i.e., a vertex induced C_m for $m = \frac{n}{2} + 1$ within a circulant. For example, $B = \{0, 1, 3, 5\}$ will induce a C_4 in $C_6(1, 2)$.

For (4) and(5) we show necessary and sufficient conditions of having edge and vertex induced cycles within circulants in general. For the edge induced cycles, two conditions must be present: $\sum_{i=1}^{m}(-1)^{\ell_i}x_i \equiv 0 \pmod{n}$, i.e., we should have a cycle and $\sum_{i=1}^{j}(-1)^{\ell_i}x_i \neq \sum_{i=1}^{k}(-1)^{\ell_i}x_i \pmod{n}$ for $j \neq k$ i.e, to prevent degeneracy. For vertex induced cycles, the conditions for an edge induced cycle must be present and none of the co-edges from that cycle can be in the difference set S.

In conclusion, these five ideas will later become theorems. They are essential in solving the problem of finding edge and vertex induced cycles within circulants. The first three ideas give bounds on finding induced cycles within circulants. The last two theorems give us the necessary and sufficient conditions for edge and vertex induced cycles to exist in any circulant. In Chapters 2 - 5 we will do the following: We will give more background information on this major problem, a specific example of a cycle (C_4) within a two difference circulant i.e., $C_n(a, b)$, proofs of these ideas presented here and open problems that will be analyzed in the future.

2 LITERATURE REVIEW

In this chapter, we review the literature that relates to finding edge and vertex induced cycles within circulants. In Section 2.1, we will give more detailed information about labellings, valuations, and decompositions. We will use this information to understand how to find subgraphs within circulants. In Section 2.2, we will look at the previous work that leads to our problem. In Section 2.3, we will have a review of work that shows the importance of finding subgraphs within circulants. In Section 2.4, we review the work of finding cliques.

2.1 Labels, Valuations, and Decompositions

A *labelling* is an injection between the vertices v_i and $f(v_i) \in \mathbb{N}$. This induces an edge labelling $|f(v_i) - f(v_j)|$ on an edge $v_i v_j$ [15]. A *decomposition* $\mathscr{D}$ of a graph H by a graph G is a partition of $E(H)$ such that the subgraph induced by each part of the partition, $\mathscr{P}$, is isomorphic to G [2]. Labellings were developed by Rosa when he was working on Ringel's conjecture.

Conjecture 2.1 *(Ringel [17]) Given any tree, T, of size q, the edge set of K_{2q+1} can be partitioned into isomorphic copies of T.*

Rosa showed this is possible, provided there exists a β-valuation on T [15]. Also there is another paper titled "How to Number a Graph" in which Golumb calls Rosa's β-valuations graceful labellings, which is the popular term today [9].

Our problem is a bit easier than that of decompositions. A decomposition partitions H into several copies of G. We need to find a only single copy of G. Nonetheless, labelling methods are valuable to our problem. A paper by Beeler and Jamison, "Valuations, Rulers, and Cyclic Decompositions of Graphs" [4], deals with graphs that can host a decomposition by a graph G. This particular paper extends graceful to arbitrary cyclic groups.

2.2 Where the Problem Comes From

The subject of automorphic decompositions gives us the problem of finding induced cycles within circulant. Beeler, working under the supervision of Jamison, described a specific kind of decomposition, namely automorphic [3]. In this paper, many of their examples come from circulants [3]. Furthermore, he shows that it is difficult to obtain an automorphic decomposition in a triangle-free host.

Theorem 2.2 *[3] For the existence of an automorphism G-decomposition of H the following conditions are necessary:*

i) For all $A \subseteq V(H)$ such that H_A is triangle-free we require that $n(G) \geq \Delta(H_A)$;

ii) If H is a triangle-free, we require that $\chi(H) \geq n(G) \geq \Delta(H)$;

iii) If H is a triangle-free and not an odd cycle, we require that $\chi(H) = n(G) = \Delta(H)$ and $G = P_2$.

2.3 Importance of Finding Subgraphs

In this section, we deal with the subjects of forbidden subgraphs, which relates to finding edge and vertex induced subgraphs within circulants. Theorem 2.3 gives necessary and sufficient conditions on the existence of planar graphs consisting of

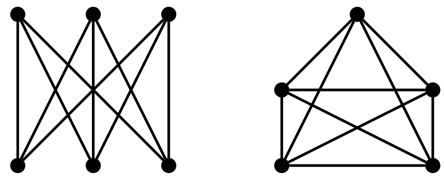


Figure 11: Graphs $K_{3,3}$ and K_5

edge induced subgraphs. A *planar graph* is when G can be drawn in the plane where no two edges are crossing each other. In a graph G, a *subdivision* of $v_1v_2 \in E(G)$ by replacing v_1v_2 with a path v_1, v_3, v_2 from a new vertex v_3 [17].

Theorem 2.3 *(Kuratowski [12]) G is planar if and only if it does not contain a subdivision of $K_{3,3}$ or K_5 (seen in Figure 11) as an edge induced subgraph*

In Theorem 2.4, we have necessary and sufficient conditions that must be valid to have a vertex induced subgraph. A graph G is a *line graph*, denoted as $L(G)$, has a vertex for each edge in G. Two vertices in $L(G)$ are adjacent if and only if they share a common endpoint in G [17]. In Figure 12, these are the nine subgraphs that will not be vertex induced within line graphs.

Theorem 2.4 *(Beineke [6]) A simple graph G is a line graph if and only if G does not have any of the nine graphs in Figure 12 as a vertex induced subgraph.*

In the paper "Circulants and the chromatic index of Steiner Triple Systems", the authors give the colorability of $C_n(a, b, a+b)$ [13]. Colorability is related to subgraphs if $G \subseteq H$, then $\chi(G) \leq \chi(H)$ [17]. There are two distinct elements of the difference

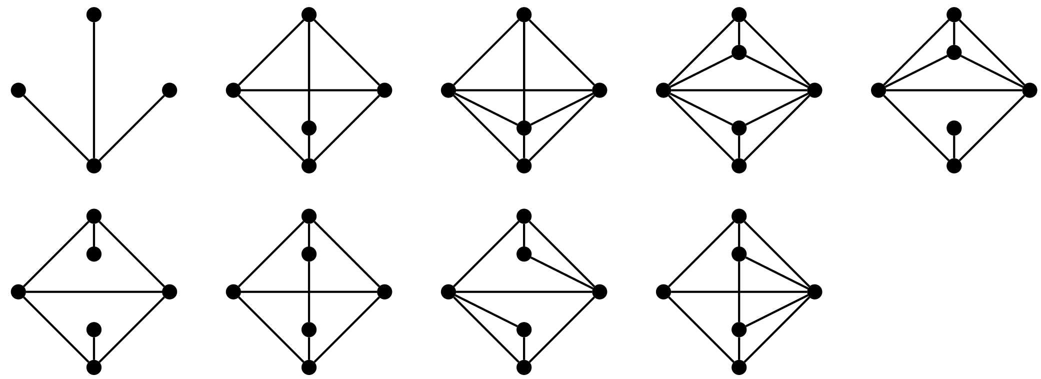


Figure 12: Graphs That Are Not Vertex Induced Subgraphs of Line Graphs

set S that we analyze, which are a and b. Mainly in this paper "On planarity and colorability of circulant graphs", Heuberger looks at the planarity and colorability of $C_n(a,b)$ [11]. The following propositions below come from Rosa's [13] and Heuberger's [11] papers dealing with chromatic numbers of circulants.

Proposition 2.5 *[11] $G = C_n(a,b)$ be a connected circulant with $|a|_n \neq |b|_n$. Then:*

i) $\chi(G) = 2$ *if and only if a and b are odd and n is even.*

ii) $\chi(G) = 4$ $3 \nmid n$, $n \neq 5$ *and* $b \equiv \pm 2a \pmod{n}$ *or* $a \equiv \pm 2b \pmod{n}$.

iii) $\chi(G) = 4$ *if* $n = 13$ *and* $b \equiv \pm 5a \pmod{13}$ *or* $a \equiv \pm 5b \pmod{13}$.

iv) $\chi(G) = 5$ *if* $n = 5$.

v) $\chi(G) = 3$ *in all other cases.*

Proposition 2.6 *[13] Let $G = C_n(a, b, a+b)$ be a connected $6-regular$ circulant where $n \geq 7$ and $a, b,$ and $a+b$ are pairwise distinct positive integers. Then we have the following:*

i) $\chi(G) = 7$ *if and only if* $n = 7$.

ii) $\chi(G) = 6$ *if and only if* $C_n(a, b, a+b) \cong C_{11}(1, 2, 3)$.

iii) $\chi(G) = 5$ *if and only if* $C_n(a, b, a+b) \cong C_n(1, 2, 3)$ *and* $n \neq 7, 11$ *is not by 4 or* G *is isomorphic to one of these circulants* $C_{13}(1, 3, 4)$, $C_{17}(1, 3, 4)$,$C_{18}(1, 3, 4)$, $C_{19}(1, 7, 8)$,$C_{25}(1, 3, 4)$,$C_{26}(1, 7, 8)$,$C_{33}(1, 6, 7)$,$C_{37}(1, 10, 11)$.

iv) $\chi(G) = 3$ *if and only if* $3 \mid n$ *and* $3 \nmid a, b, a+b$.

v) $\chi(G) = 4$ *in all other cases.*

In his dissertation, Beeler gives a general upper bound for the chromatic number in the following proposition [2].

Proposition 2.7 *[2] Let* $C_{kn}(S)$ *be a circulant such that for all* $a \in S$, $k \nmid a$ *and* n *is sufficiently large. Then we have* $\chi(C_{kn}(S)) \leq k$.

He also shows that this bound is tight in certain circumstances. The chromatic number of any circulant will be at most k.

2.4 Finding Cliques

In this section, we go over work done with finding cliques. In Theorems 2.8 and 2.9, we give a general upper bound and lower bound on the chromatic number when observing cliques.

Theorem 2.8 *(Turan [16]) If* G *is of order* n *with no* k*-clique, then:*

$$e(G) \leq \frac{n^2(k-2)}{2(k-1)}.$$

Turan's Theorem is very important in what we are trying to accomplish in finding edge and vertex induced subgraphs within circulants that are triangle-free. This relates the clique number to the order and size of G. A *Turan graph* is a graph that has the maximum number of edges of any graph of order n which contains a complete subgraph K_k. Therefore, it is very important not to have any K_3 subgraphs within any circulant [1, 14].

Theorem 2.9 *[2] Let $S \subseteq \mathbb{Z}_n^+$ be given. A circulant, $C_n(S)$, contains a vertex induced K_{k+1} if and only if there exists $\{a_1, a_2, a_3, \ldots, a_k\} \subseteq S$ such that:*

$$|a_i - a_j|_n \in S \quad \text{for all} \quad 1 \leq i \leq j \leq k.$$

Proof. Let $H = C_n(S)$ and suppose that H contains K_{k+1} as a vertex induced subgraph. Let $A \subseteq V(H)$ be such that $H_A \cong K_{k+1}$. By the cyclic nature of the circulant, we may assume that $0 \in A$. Let $A = \{0, v_1, \ldots, v_k\}$. Since 0 is adjacent to all of the v_i, it follows that $|v_i|_n \in S$ for all i. We claim that $\{|v_1|_n, \ldots, |v_k|_n\}$ is the required set of differences. Since $\{0, v_1, \ldots, v_k\}$ are vertices of a circulant and mutually adjacent, it follows that $|v_i - v_j|_n \in S$ for all i and j.

Conversely, assume that $\{a_1, \ldots, a_k\} \subseteq S$ is such that $|a_i - a_j|_n \in S$ for all $1 \leq i < j \leq k$. In order to show that $C_n(S)$ contains K_{k+1}, we need to find $k+1$ distinct mutually adjacent vertices. We claim that $\{0, a_1, \ldots, a_k\}$ is the required set of vertices. Clearly, 0 is adjacent to every vertex in the set as $a_i \in S$ for all i. Further, $|a_i - a_j|_n \in S$ for all $1 \leq i < j \leq k$ by definition of S. Thus, all of the vertices in $\{0, a_1, a_2, \ldots, a_k\}$ are mutually adjacent. As such, we have constructed the required clique. ■

For any $C_n(S)$ has a vertex induced complete subgraph if and only if the elements of the difference set S have lengths that are distinct and/or equal to each other. Producing a vertex induced subgraph within any circulant is more difficult than developing an edge induced one (not with cliques). We are more interested in necessary and sufficient conditions in finding vertex induced cycles within circulants.

3 C_4 WITHIN $C_n(S)$

In this chapter, we have specific examples of a C_4 within a two difference circulant, i.e, $C_n(a, b)$ where $a \not\equiv \pm b \pmod{n}$. Tables 1 - 4 will be set up in the following fashion. In the first column are the edge labels of C_4. The second column contains the actual figures of C_4, with vertex labels. Each edge label goes in a clockwise direction giving the vertex labels. The horizontal and vertical dashed lines are co-edges of C_4, i.e., these edges are not in C_4 but they are in $\overline{C}_4$. Column three gives the conditions for C_4 to be an edge induced subgraph. The top line of the third column gives the condition necessary to prevent degeneracy. The bottom line in the third column represents the condition necessary for the completion of that particular cycle. Also in this same column, the initials N.C.N. stand for "No Completion Necessary". The last column gives additional conditions for C_4 to be a vertex induced subgraph. In order for this C_4 to be vertex induced none of these co-edges can be in the difference set S. The top line in the fourth column represents the horizontal co-edge condition and the bottom line gives the vertical co-edge condition. Tables 5 - 7 do not have any figures because they have similar properties to those above. These will be discussed in more detail later.

Here are the following two theorems for edge and vertex induced C_4 being within $C_n(a, b)$.

Theorem 3.1 *We have that C_4 is an edge induced subgraph of $C_n(a, b)$ if and only if one of the following holds:*

i) $4a \equiv 0 \pmod{n}$ *and* $2a \not\equiv 0$*;*

ii) $3a + b \equiv 0 \pmod{n}$ *and* $2a \neq 0$*;*

iii) $3a + b \equiv 0 \pmod{n}$ *and* $a \neq -b$*;*

iv) $3a - b \equiv 0 \pmod{n}$ *and* $2a \neq 0$*;*

v) $3a - b \equiv 0 \pmod{n}$ *and* $a \neq b$*;*

vi) $2a + 2b \equiv 0 \pmod{n}$ *and* $a \neq -b$*;*

vii) $2a - 2b \equiv 0 \pmod{n}$ *and* $a \neq b$*;*

viii) Note it is always possible to achieve edge induced C_4 *via cancellation.*

Proof. Refer to Tables 1 - 4; specifically column three. ∎

Note it is always possible to achieve edge induced C_4 using cancellation through arithmetic. This is the condition in Theorem 3.1 (viii). However, our real interest is vertex induced cycles. Hence, it is necessary to list the other more restrictive conditions for achieving on edged induced C_4. These may lead to alternate vertex induced cycles, as shown in the next theorem.

Theorem 3.2 *We have that* C_4 *is a vertex induced subgraph of* $C_n(a, b)$ *if and only if one of the following holds:*

i) Theorem 3.1 (i) holds and $2a \notin S$*;*

ii) Theorem 3.1 (ii) holds and $2a \notin S$*;*

iii) Theorem 3.1 (iii) holds and $a + b \notin S$*;*

iv) Theorem 3.1 (iv) holds and $2a \notin S$ *or* *v) Theorem 3.1 (v) holds and* $a - b \notin S$*;*

vi) Theorem 3.1(vi) holds and $a + b \notin S$*;*

vii) Theorem 3.1 (vii) holds and $a - b \notin S$*;*

viii) Theorem 3.1 (viii) holds and $a + b \notin S$ *and* $a - b \notin S$*.*

Proof. Refer to Tables 1 - 4; specifically column four. ■

Note that provided $a \pm b \notin S$, then cancellation will yield vertex induced C_4.

The conditions in Theorems 3.1 and 3.2 are given in terms of a and positive values. These theorems are given in terms of a because it is possible to reverse roles of a and b, i.e., the condition $4a \equiv 0 \pmod{n}$ can be interpreted as $4b \equiv 0 \pmod{n}$. Conditions are listed in terms of positive values because $a \in S$ implies $-a \in S$ and $-4a \equiv 0 \pmod{n}$ if and only if $4a \equiv 0 \pmod{n}$. For completeness, these are listed in Tables 5 - 7.

Table 1: $4a \equiv 0 \pmod{n}$ and $3a + b \equiv 0 \pmod{n}$

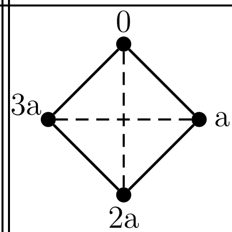
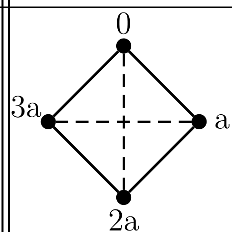
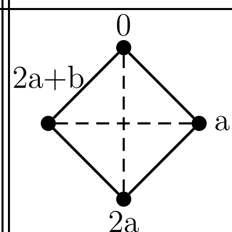
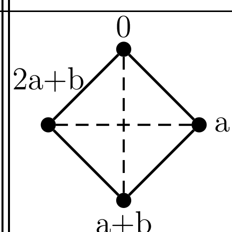
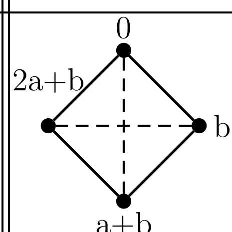
$E(C_4)$	C_4	Edge Cond.	Vertex Cond.
a, a, a, a		$2a \neq 0$ $4a \equiv 0$	$2a \notin S$ $2a \notin S$
a, a, a, b		$2a \neq 0$ $3a + b \equiv 0$	$2a \notin S$ $2a \notin S$
a, a, b, a		$a \neq -b$,$2a \neq 0$ $3a + b \equiv 0$	$a + b \notin S$ $2a \notin S$
a, b, a, a		$a \neq -b$ $3a + b \equiv 0$	$a + b \notin S$ $a + b \notin S$
b, a, a, a		$2a \neq 0$, $a \neq -b$ $3a + b \equiv 0$	$2a \notin S$ $a + b \notin S$

Table 2: $3a - b \equiv 0 \pmod{n}$

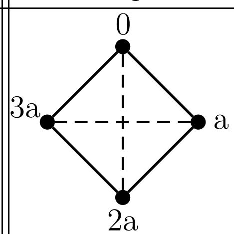
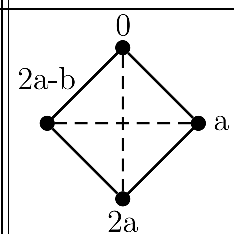
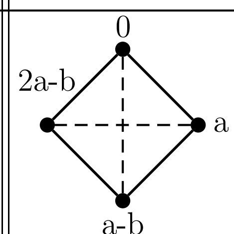
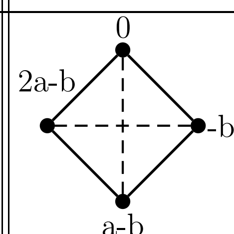
$E(C_4)$	C_4	Edge Cond.	Vertex Cond.
$a, a, a, -b$		$2a \neq 0$ $3a - b \equiv 0$	$2a \notin S$ $2a \notin S$
$a, a, -b, a$		$2a \neq 0$,$a \neq b$ $3a - b \equiv 0$	$a - b \notin S$ $2a \notin S$
$a, -b, a, a$		$a \neq b$ $3a - b \equiv 0$	$a - b \notin S$ $a - b \notin S$
$-b, a, a, a$		$a \neq b$, $2a \neq 0$ $3a - b \equiv 0$	$2a \notin S$ $a - b \notin S$

Table 3: $2a + 2b \equiv 0 \pmod{n}$

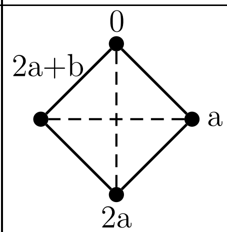
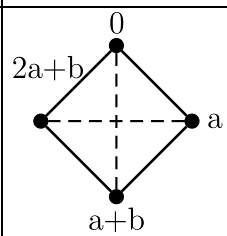
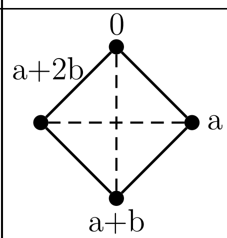
$E(C_4)$	C_4	Edge Cond.	Vertex Cond.
a, a, b, b		$a \neq -b$,$2a \neq 0$ $2a + 2b \equiv 0$	$a + b \notin S$ $2a \notin S$
a, b, a, b		$a \neq -b$ $2a + 2b \equiv 0$	$a + b \notin S$ $a + b \notin S$
a, b, b, a		$2b \neq 0$,$a \neq -b$ $2a + 2b \equiv 0$	$2b \notin S$ $a + b \notin S$

Table 4: $2a - 2b \equiv 0 \pmod{n}$ and Cancellation

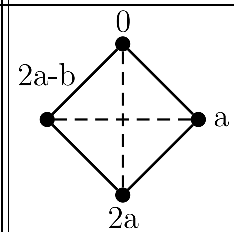
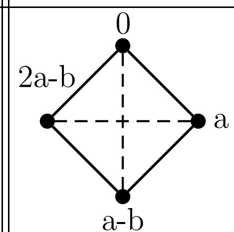
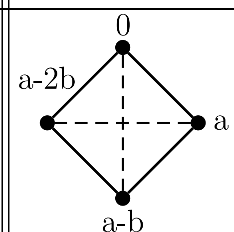
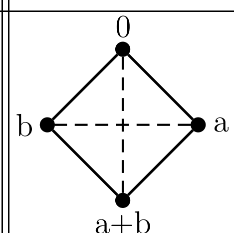
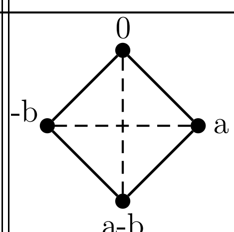
$E(C_4)$	C_4	Edge Cond.	Vertex Cond.
$a, a, -b, -b$		$a \neq b$,$2a \neq 0$ $2a - 2b \equiv 0$	$a + b \notin S$ $2a \notin S$
$a, -b, a, -b$		$a \neq b$ $2a - 2b \equiv 0$	$a - b \notin S$ $a - b \notin S$
$a, -b, -b, a$		$a \neq b$,$2b \neq 0$ $2a + 2b$	$-2b \notin S$ $a - b \notin S$
$a, b, -a, -b$		$a \neq \pm b$ N.C.N.	$a - b \notin S$ $a + b \notin S$
$a, -b, -a, b$		$a \neq \pm b$ N.C.N.	$a + b \notin S$ $a - b \notin S$

Table 5: Negation of Tables 1 - 4

$E(C_4)$	Notes	Edge	Vertex
$-a,-a,-a,-a$	Negate (a,a,a,a)	$-2a \neq 0$ $-4a \equiv 0$	$-2a \notin S$ $-2a \notin S$
$-a,-a,-a,-b$	Negate (a,a,a,b)	$-2a \neq 0$ $-3a-b \equiv 0$	$-2a \notin S$ $-2a \notin S$
$-a,-a,-b,-a$	Negate (a,a,b,a)	$-2a \neq 0$,$-a \neq b$ $-3a-b \equiv 0$	$-a-b \neq S$ $-2a \neq S$
$-a,-b,-a,-a$	Negate (a,b,a,a)	$-a \neq b$ $-3a-b \equiv 0$	$-a-b \notin S$ $-a-b \notin S$
$-b,-a,-a,-a$	Negate (b,a,a,a)	$-a \neq b$, $-2a \neq 0$ $-3a-b \equiv 0$	$-2a \notin S$ $-a-b \notin S$
$-a,-a,-a,b$	Negate (a,a,a,-b)	$-2a \neq 0$ $-3a+b \equiv 0$	$-2a \notin S$ $-2a \notin S$
$-a,-a,b,-a$	Negate (a,a,-b,a)	$-2a \neq 0$,$-a \neq -b$ $-3a+b \equiv 0$	$-a+b \notin S$ $2a \notin S$
$-a,b,-a,-a$	Negate (a,-b,a,a)	$a \neq b$ $-3a+b \equiv 0$	$-a+b \neq S$ $-a+b \neq S$
$b,-a,-a,-a$	Negate (-b,a,a,a)	$-a \neq -b$,$-2a \neq 0$ $-3a+b \equiv 0$	$-2a \notin S$ $-a+b \notin S$
$-a,-a,-b,-b$	Negate (a,a,b,b)	$2a \neq 0$, $a \neq -b$ $-2a-2b \equiv 0$	$-a-b \notin S$ $-2a \notin S$
$-a,-b,-a,-b$	Negate (a,b,a,b)	$-a \neq b$ $-2a-2b \equiv 0$	$-a-b \notin S$ $-a-b \notin S$
$-a,-b,-b,-a$	Negate (a,b,b,a)	$-a \neq b$,$-2b \neq 0$ $-2a-2b \equiv 0$	$-2b \notin S$ $-a-b \notin S$
$-a,-a,b,b$	Negate (a,a,-b,-b)	$-a \neq -b$,$-2a \neq 0$ $-2a+2b \equiv 0$	$-a+b \neq S$ $-2a \neq S$
$-a,b,-a,b$	Negate (a,-b,a,-b)	$-a \neq -b$ $-2a+2b \equiv 0$	$-a+b \notin S$ $-a+b \notin S$
$-a,b,b,-a$	Negate (a,-b,-b,a)	$2b \neq 0$,$-a \neq -b$ $-2a+2b \equiv 0$	$-2b \notin S$ $a-b \notin S$
$-a,-b,a,b$	Negate (a,b,-a,-b)	$-a \neq \pm b$ N.C.N.	$-a+b \notin S$ $-a-b \notin S$
$-a,b,a,-b$	Negate (a,-b,-a,b)	$-a \neq \pm b$ N.C.N.	$-a-b \notin S$ $-a+b \notin S$

Table 6: Role Reversal of Tables 1 - 4

$E(C_4)$	Notes	Edge	Vertex
b, b, b, b	Reverse (a,a,a,a)	$2b \neq 0$ $4b \equiv 0$	$2b \notin S$ $2b \notin S$
b, b, b, a	Reverse (a,a,a,b)	$2b \neq 0$ $a + 3b \equiv 0$	$2b \notin S$ $2b \notin S$
b, b, a, b	Reverse (a,a,b,a)	$2b \neq 0, -a \neq b$ $a + 3b \equiv 0$	$a + b \neq S$ $2b \neq S$
b, a, b, b	Reverse (a,b,a,a)	$-a \neq b$ $a + 3b \equiv 0$	$a + b \notin S$ $a + b \notin S$
a, b, b, b	Reverse (b,a,a,a)	$-a \neq b,\ 2b \neq 0$ $a + 3b \equiv 0$	$2b \notin S$ $a + b \notin S$
$b, b, b, -a$	Reverse (a,a,a,-b)	$2b \neq 0$ $-a + 3b \equiv 0$	$2b \notin S$ $2b \notin S$
$b, b, -a, b$	Reverse (a,a,-b,a)	$2b \neq 0, a \neq b$ $-a + 3b \equiv 0$	$-a + b \notin S$ $2b \notin S$
$b, -a, b, b$	Reverse (a,-b,a,a)	$a \neq b$ $-a + 3b \equiv 0$	$-a + b \neq S$ $-a + b \neq S$
$-a, b, b, b$	Reverse(-b,a,a,a)	$a \neq b, 2b \neq 0$ $-a + 3b \equiv 0$	$2b \notin S$ $-a + b \notin S$
b, b, a, a	Reverse(a,a,b,b)	$2b \neq 0,\ -a \neq b$ $2a + 2b \equiv 0$	$a + b \notin S$ $2b \notin S$
b, a, b, a	Reverse (a,b,a,b)	$-a \neq b$ $2a + 2b \equiv 0$	$a + b \notin S$ $a + b \notin S$
b, a, a, b	Reverse (a,b,b,a)	$-a \neq b, 2a \neq 0$ $2a + 2b \equiv 0$	$2a \notin S$ $a + b \notin S$
$b, b, -a, -a$	Reverse (a,a,-b,-b)	$a \neq b, 2b \neq 0$ $-2a + 2b \equiv 0$	$-a + b \neq S$ $2b \neq S$
$b, -a, b, -a$	Reverse(a,-b,a,-b)	$a \neq b$ $-2a + 2b \equiv 0$	$-a + b \notin S$ $-a + b \notin S$
$b, -a, -a, b$	Reverse (a,-b,-b,a)	$-2a \neq 0, a \neq b$ $-2a + 2b \equiv 0$	$-2a \notin S$ $-a + b \notin S$
$b, a, -b, -a$	Reverse (a,b,-a,-b)	$\pm a \neq b$ N.C.N.	$-a + b \notin S$ $a + b \notin S$
$b, -a, -b, a$	Reverse (a,-b,-a,b)	$\pm a \neq b$ N.C.N.	$a + b \notin S$ $-a + b \notin S$

Table 7: Negation of Table 6

$E(C_4)$	Notes	Edge	Vertex
$-b,-b,-b,-b$	Neg./Rev. (a,a,a,a)	$-2b \neq 0$ $-4b \equiv 0$	$-2b \notin S$ $-2b \notin S$
$-b,-b,-b,-a$	Neg./Rev. (a,a,a,b)	$-2b \neq 0$ $-a-3b \equiv 0$	$-2b \notin S$ $-2b \notin S$
$-b,-b,-a,-b$	Neg./Rev. (a,a,b,a)	$-2b \neq 0$,$a \neq -b$ $-a-3b \equiv 0$	$-a-b \neq S$ $-2b \neq S$
$-b,-a,-b,-b$	Neg./Rev. (a,b,a,a)	$a \neq -b$ $-a-3b \equiv 0$	$-a-b \notin S$ $-a-b \notin S$
$-a,-b,-b,-b$	Neg./Rev. (b,a,a,a)	$a \neq -b$, $-2b \neq 0$ $-a-3b \equiv 0$	$-2b \notin S$ $-a-b \notin S$
$-b,-b,-b,a$	Neg./Rev. (a,a,a,-b)	$-2b \neq 0$ $a-3b \equiv 0$	$-2b \notin S$ $-2b \notin S$
$-b,-b,a,-b$	Neg./Rev. (a,a,-b,a)	$-2b \neq 0$,$-a \neq -b$ $a-3b \equiv 0$	$a-b \notin S$ $-2b \notin S$
$-b,a,-b,-b$	Neg./Rev. (a,-b,a,a)	$-a \neq -b$ $a-3b \equiv 0$	$a-b \neq S$ $a-b \neq S$
$a,-b,-b,-b$	Neg./Rev.(-b,a,a,a)	$-a \neq -b$,$-2b \neq 0$ $a-3b \equiv 0$	$-2b \notin S$ $a-b \notin S$
$-b,-b,-a,-a$	Neg./Rev.(a,a,b,b)	$-2b \neq 0$, $a \neq -b$ $-2a-2b \equiv 0$	$-a-b \notin S$ $-2b \notin S$
$-b,-a,-b,-a$	Neg./Rev. (a,b,a,b)	$a \neq -b$ $-2a-2b \equiv 0$	$-a-b \notin S$ $-a-b \notin S$
$-b,-a,-a,-b$	Neg./Rev. (a,b,b,a)	$a \neq -b$,$-2a \neq 0$ $-2a-2b \equiv 0$	$-2a \notin S$ $-a-b \notin S$
$-b,-b,a,a$	Neg./Rev. (a,a,-b,-b)	$-a \neq -b$,$-2b \neq 0$ $2a-2b \equiv 0$	$a-b \neq S$ $-2b \neq S$
$-b,a,-b,a$	Neg./Rev.(a,-b,a,-b)	$-a \neq -b$ $2a-2b \equiv 0$	$a-b \notin S$ $a-b \notin S$
$-b,a,a,-b$	Neg./Rev. (a,-b,-b,a)	$2a \neq 0$,$-a \neq -b$ $2a-2b \equiv 0$	$2a \notin S$ $a-b \notin S$
$-b,-a,b,a$	Neg./Rev. (a,b,-a,-b)	$\pm a \neq -b$ N.C.N.	$a-b \notin S$ $-a-b \notin S$
$-b,a,b,-a$	Neg./Rev. (a,-b,-a,b)	$\pm a \neq -b$ N.C.N.	$-a-b \notin S$ $a-b \notin S$

4 MAJOR RESULTS

This chapter consists of the major results we have proven about necessary and sufficient conditions in finding induced cycles within circulants. There are four theorems presented in this chapter. Explanations and proofs will be given for each one.

In the following theorem, we have an upper bound on the size of the largest vertex induced cycle within any circulant.

Theorem 4.1 *Suppose that $C_n(S)$ is a d-regular circulant where $d \geq 2$. If C_m is a vertex induced subgraph of $C_n(S)$, then $m \leq \frac{nd}{2(d-1)}$.*

Proof. A cycle on m vertices has m edges and is 2-regular. However, $C_n(S)$ is d-regular by hypothesis. This means that each vertex in the cycle has $d-2$ incident edges that are not in the cycle. As this cycle is vertex induced, these edges may not be incident with two vertices in the cycle. Hence we have $m + (d-2)m = (d-1)m$ edges that are incident with at least one vertex in the cycle. However, a d-regular graph on n vertices has $nd/2$ edges [17]. Thus the number of edges not incident with the cycle is given by:

$$\frac{nd}{2} - (d-1)m \geq 0.$$

Solving for m yields:

$$m \leq \frac{nd}{2(d-1)}.$$

■

For this theorem, we proved that long vertex induced cycles can not exist within circulants. This is another bound on how large a cycle can be within any circulant.

Theorem 4.2 *Let $|S| \geq 2$. If C_m is a vertex induced subgraph of $C_n(S)$, then $m \leq \frac{n}{2} + 1$.*

Proof. We note that the more differences in S, the more difficult it is to achieve a vertex induced cycle. Without loss of generality, assume that $|S| = 2$. Suppose that for all $a \in S$ we have that $gcd(a, n) = d \geq 2$. Thus, by Proposition 1.3, we have that the edges of length a form a cycle of length $n/d \leq n/2$. Any edge of length b would then make a shorter cycle. Thus, we can not have a vertex induced cycle of length $k > n/2 + 1$. We may then assume that $\gcd(a, n) = 1$. Therefore, our circulant is isomorphic to $C_n(1, c)$ where $c = a^{-1}b$ by Proposition 1.3. The only edge induced cycle of length n is chorded by the edges of length c. Take any edge induced cycle of length m where $n/2 + 1 < m < n$. Such a cycle must contain at least two non-adjacent vertices that are of distance 1 or c apart. Hence, this cycle cannot be vertex induced. ■

Theorem 4.3 *There exists an edge induced C_m if and only if there exists $\ell_i \in \{0, 1\}$ and $x_i \in S$ for $i = 1, 2, 3, \ldots, m$ such that:*

i) $\sum_{i=1}^{m} (-1)^{\ell_i} x_i \equiv 0 \pmod{n}$ *and*

ii) $\sum_{i=1}^{j} (-1)^{\ell_i} x_i \not\equiv \sum_{i=1}^{k} (-1)^{\ell_i} x_i \pmod{n}$ *for $j \neq k$.*

Proof. Assume that C_m is an edge induced subgraph of $C_n(S)$. We must show that there exists $\ell_i \in \{0, 1\}$ and $x_i \in S$ for $i = 1, 2, 3, \ldots, m$ such that we have the following:

i) completion of C_m and

ii) prevention of degeneracy.

So, there exists $A \subseteq E(C_n(S))$ such that $\langle A \rangle \cong C_m$. For each edge $v_iw_i \in A$, there must be a difference in S. In other words, either $v_i - w_i \equiv x_i \pmod{n}$ or $v_i - w_i \equiv n - x_i \pmod{n}$ where $x_i \in S$. Define:

$$\ell_i = \begin{cases} 0 & \text{if } v_i - w_i \equiv x_i \pmod{n} \\ 1 & \text{if } v_i - w_i \equiv n - x_i \pmod{n}. \end{cases}$$

For completion of C_m we must have the following:

$$\sum_{i=1}^{m} (-1)^{\ell_i} x_i \equiv 0 \pmod{n}.$$

Notice that the jth vertex in the cycle is mapped to $\sum_{i=1}^{j} (-1)^{\ell_i} x_i$ in $C_n(S)$. There is an injective relationship between each vertex from C_m within $C_n(S)$. Hence, we have that:

$$\sum_{i=1}^{j} (-1)^{\ell_i} x_i \not\equiv \sum_{i=1}^{k} (-1)^{\ell_i} x_i \pmod{n} \quad \text{for} \quad j \neq k.$$

Assume to the contrary that there exists $\ell_i \in \{0, 1\}$ and $x_i \in S$ for $i = 1, 2, 3, \ldots, m$ such that:

$$\sum_{i=1}^{m} (-1)^{\ell_i} x_i \equiv 0 \pmod{n} \quad \text{and}$$

$$\sum_{i=1}^{j} (-1)^{\ell_i} x_i \not\equiv \sum_{i=1}^{k} (-1)^{\ell_i} x_i \pmod{n} \quad \text{for} \quad j \neq k.$$

Define $A = \{v_j v_{j+1} : j = 0, 1, \ldots, m\}$ where $v_0 = 0$ and for $j = 1, \ldots, m$:

$$v_j = \sum_{i=1}^{j} (-1)^{\ell_i} x_i.$$

We have that $|v_{j+1} - v_j|_n = x_{j+1} \in S$. So $A \subseteq E(C_n(S))$. Also, since $v_0 = v_m = 0$, $\langle A \rangle$ is cyclic. Since $v_j \neq v_k$ there is an injective relationship with each vertex in $C_n(S)$. Hence $\langle A \rangle \cong C_m$. ■

Theorem 4.4 *There exists a vertex induced C_m if and only if Theorem 4.3 holds and:*

$$\left|\sum_{i=1}^{k-1}(-1)^{\ell_i}x_i - \sum_{i=1}^{j-1}(-1)x_i\right|_n \notin S$$

for all $k, j \in \{0, 1, \ldots, m\}$ such that $|k-j|_n \geq 2$.

Proof. Assume that there is a vertex induced C_m in $C_n(S)$. There must be an edge induced cycle by Proposition 1.1. Therefore, Theorem 4.3 holds. Each vertex satisfies:

$$v_j = \sum_{i=1}^{j}(-1)^{\ell_i}x_i.$$

Since C_m is a vertex induced subgraph, we must have that $|v_k - v_j|_n \notin S$ for non-adjacent vertices v_k and v_j. The non-adjacent vertices in C_m have indices that must satisfy $|k-j|_m \geq 2$. Therefore, we have:

$$\left|\sum_{i=1}^{k}(-1)^{\ell_i}x_i - \sum_{i=1}^{j}(-1)^{\ell_i}x_i\right|_n \notin S$$

for all $k, j \in \{1, \ldots, m\}$ such that $|k-j|_m \geq 2$.

Suppose that Theorem 4.3 holds and

$$\left|\sum_{i=1}^{k}(-1)^{\ell_i}x_i - \sum_{i=1}^{j}(-1)^{\ell_i}x_i\right|_n \notin S$$

for all $k, j \in \{1, \ldots, m\}$ such that $|k-j|_k \geq 2$. Since Theorem 4.3 holds, it follows that there exists an edge induced C_m, with vertex set:

$$V(C_m) = \left\{ v_j = \sum_{i=1}^{j}(-1)^{\ell_i}x_i : j = 1, \ldots, m \right\}.$$

To show that this is a vertex induced C_m, we must have that $|v_k - v_j|_n \notin S$ for all non-adjacent vertices in the cycle. Notice that vertices that are not adjacent in C_m

these indices must satisfy $|k-j|_m \geq 2$. We have that

$$\left|\sum_{i=1}^{k}(-1)^{\ell_i}x_i - \sum_{i=1}^{j}(-1)^{\ell_i}x_i\right|_n \notin S$$

for all $k, j \in \{1, \ldots, m\}$ such that $|k-j|_m \geq 2$, it follows that this cycle is vertex induced. ■

With these major results of the first two theorems, we were able to apply bounds on how large C_m can be within $C_n(S)$. Notice that these bounds deal with vertex induced cycles. Since we have vertex induced cycles there are also edge induced cycles as shown by Proposition 1.1. The last two theorems give the conditions of edge and vertex induced cycles in general. All four of these theorems give us the conditions of how large C_m can be and their appearance. In the last chapter, we will summarize everything up to this point and present some more open problems that may be studied in the future.

5 CONCLUSION

This thesis dealt with finding edge and vertex induced subgraphs within circulants. The subgraphs we used were cycles. There are several results that have been discovered on the necessary and sufficient conditions in finding cycles within any circulant.

We established that if there is a vertex induced subgraph, then we have an edge induced subgraph. The converse of this proposition is not true because you can have an edge induced subgraph that is not a vertex induced subgraph.

We analyzed C_4 within any $C_n(a, b)$. There were several cases that were presented. For each case, we had tables that consist of the following possibilities when $4a \equiv 0 \pmod{n}$, $3a + b \equiv 0 \pmod{n}$, $3a - b \equiv 0 \pmod{n}$, $2a + 2b \equiv 0 \pmod{n}$, $2a - 2b \equiv 0 \pmod{n}$ and cancellation through arithmetic. The last three tables gave necessary and sufficient conditions without figures when the roles of a and b were reversed and the edge labels were multiplied by -1. Tables 1-7 allowed us to classify necessary and sufficient conditions for $C_4 \in C_n(a, b)$.

We also placed bounds on the size and length of a vertex induced cycle within any circulant. We looked at the necessary and sufficient conditions for finding edge and vertex induced cycles within any circulant.

In conclusion, we have proven the major results from the preview section that pertain to our problem. They are as follows:

- **Theorem 2.9** *[2] Let $S \subseteq \mathbb{Z}_n^+$ be given. A circulant, $C_n(S)$, contains a vertex* *induced K_{k+1} if and only if there exists $\{a_1, a_2, a_3, \ldots, a_k\} \subseteq S$ such that:*

$$|a_i - a_j|_n \in S \quad \text{for all} \quad 1 \leq i \leq j \leq k.$$

- **Theorem 4.1** *Suppose that $C_n(S)$ is a d-regular circulant where $d \geq 2$. If C_m is a vertex induced subgraph of $C_n(S)$, then $m \leq \frac{nd}{2(d-1)}$.*

- **Theorem 4.2** *Let $|S| \geq 2$. If C_m is a vertex induced subgraph of $C_n(S)$, then $m \leq \frac{n}{2} + 1$.*

- **Theorem 4.3** *There exists an edge induced C_m if and only if there exists $\ell_i \in \{0,1\}$ and $x_i \in S$ for $i = 1, 2, 3, \ldots, m$ such that:*

  i) $\sum_{i=1}^{m}(-1)^{\ell_i}x_i \equiv 0 \pmod{n}$ and

  ii) $\sum_{i=1}^{j}(-1)^{\ell_i}x_i \neq \sum_{i=1}^{k}(-1)^{\ell_i}x_i \pmod{n}$ for $j \neq k$.

- **Theorem 4.4** *There exists a vertex induced C_m if and only if Theorem 4.3 holds and:*

$$\left|\sum_{i=1}^{k-1}(-1)^{\ell_i}x_i - \sum_{i=1}^{j-1}(-1)x_i\right|_n \notin S$$

  for all $k, j \in \{0, 1, \ldots, m\}$ such that $|k-j|_n \geq 2$.

However, there are still more open problems to be studied. Using cycles as subgraphs, we would like to explore C_5, C_6, $\ldots$,C_m with $m \geq 5$ within $C_n(a,b)$. Also, it would be interesting to see these cycles within larger difference sets. The other subgraphs that would be of interest are cycle-related graphs, like kites, coronas, and cycles glued together in Figure 13. For the cycles glued together we are looking at two situations: (1) when they are connected at one vertex and (2) when they share at least one edge. Also we would like to answer the following questions:

i.) Are C_k and C_{k+t} vertex induced subgraphs in $C_n(S)$ but not $C_{k+1}, \ldots, C_{k+t-1}$?

ii.) What is the structure of $\mathscr{C}(G)$ with $G = C_n(S)$ where,

$$\mathscr{C}(G) = \{k : C_k \text{ is a vertex induced subgraph of } G\}?$$

In conclusion, we assert that the study of subgraphs within circulants warrants further research. We hope that this thesis serves as a valuable resource for those pursuing such research.

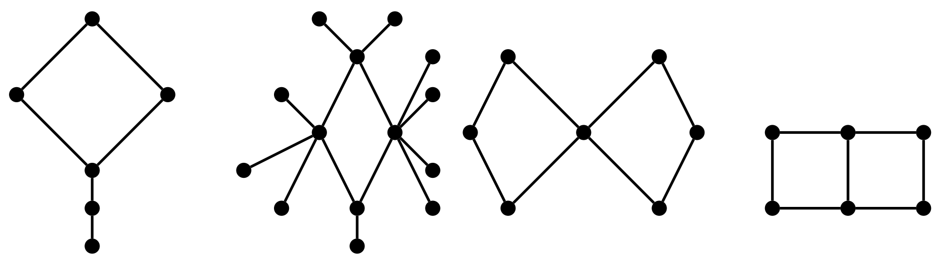


Figure 13: A Kite, Corona, and Cycles Glued Together

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