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Total Domination Dot Critical and Dot Stable Graphs

A thesis

presented to

the faculty of the Department of Mathematics

East Tennessee State University

In partial fulfillment

of the requirements for the degree

Master of Science in Mathematical Sciences

by

Stephanie McMahon

May 2010

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Keywords: total domination, dot-critical, dot-stable, identify, dotting

ABSTRACT

Total Domination Dot Critical and Dot Stable Graphs

by

Stephanie McMahon

Two vertices are said to be identified if they are combined to form one vertex whose neighborhood is the union of their neighborhoods. A graph is total domination dot-critical if identifying any pair of adjacent vertices decreases the total domination number. On the other hand, a graph is total domination dot-stable if identifying any pair of adjacent vertices leaves the total domination number unchanged. Identifying any pair of vertices cannot increase the total domination number. Further we show it can decrease the total domination number by at most two. Among other results, we characterize total domination dot-critical trees with total domination number three and all total domination dot-stable graphs.

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DEDICATION

I would like to dedicate my thesis to the close friends and family who supported me through my education endeavors. I also want to extend a special thanks to my grandparents who have been supportive throughout my life and made it possible for me to even attempt getting an education. Last but not least I need to thank my 'Tennessee Mom', Daia Stager, for all of her love, support, and guidance.

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I would also like to thank my advisor Dr. Teresa Haynes for not only working with me, but for introducing me to graph theory. Her love for graph theory ignited my love for it as well. I would like to acknowledge the National Science Foundation GK-12 Program at East Tennessee State University for an excellent opportunity to grow professionally in so many ways. I would like to acknowledge Dr. Aimee Govett, Dr. Anant Godbole, Dr. Jeff Knisley and Dr. Debra Knisley for their guidance and contribution to my development as a mathematician.

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1 INTRODUCTION

The purpose of this thesis is to study the effect that identifying or dotting two vertices has on the total domination number. We also study which graphs have special properties with respect to this change. In Section 1.1, we introduce the graph theory terminology and notation used throughout this paper. In Section 1.2, we define the domination parameters of interest. In Section 1.3, we introduce the topic being studied as well as more topic-specific terminology and notation.

1.1 Graph Theory Terminology and Notation

A graph is a mathematical representation of a relationship. For this paper, we are only considering simple graphs therefore we define a graph with a simple graph definition. A *graph* $G = (V(G), E(G))$ consists of two sets: a nonempty finite set V of vertices and a finite set E of edges consisting of unordered pairs of distinct vertices from V. An edge between two vertices means that the vertices are related, as defined by the relationship being modeled. A graph is *connected* if for any two vertices in the graph, there is a path between them. The cardinality of $V(G)$, denoted n, is the *order* of G. The cardinality of $E(G)$, denoted m, is the *size* of G. A pair of vertices u and v are *adjacent* if $uv \in E$, that is, if uv is an edge of G. The *degree of* v, denoted $deg(v)$, is the number of vertices adjacent to v. A vertex of degree zero is called an *isolate*, and a vertex degree one is called an *endvertex* or a *leaf*. The vertex adjacent to a leaf is called the *support vertex* of the leaf. We will be using $G[S]$ to denote the subgraph of G induced by a set of vertices S. There are some common graphs that we will discuss throughout this thesis, for example, paths and stars. A *star* is a graph with a single center vertex which is adjacent to r other vertices, denoted $K_{1,r}$. We let P_n and C_n denote the *path* and *cycle*, respectively, on n vertices. Any graph which contains no cycle is called a *tree*. A *complete graph*, denoted K_n, is a graph on n vertices that has every possible edge present. In Figure 1, we give examples of graphs from these families.

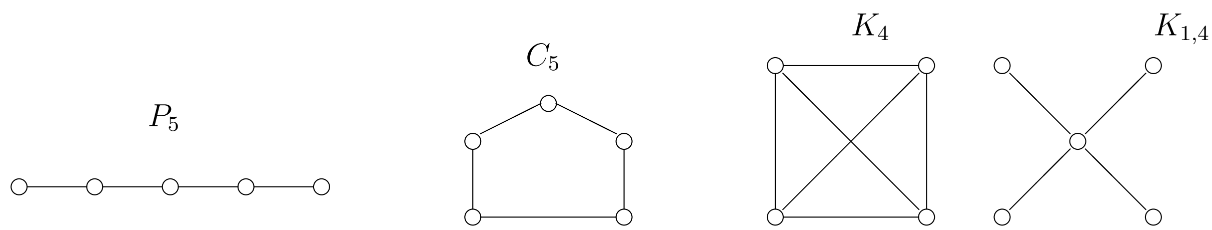


Figure 1: From left to right: path, cycle, complete graph, and a star.

For any vertex $v \in V(G)$, the *open neighborhood* of v is $N(v) = \{u \in V(G) | uv \in E(G)\}$ and the *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. Given any set $S \subset V(G)$ and $v \in S$, a vertex $u \in V(G)$ is a *S-private neighbor* of v if $N(u) \cap S = \{v\}$. The *private neighborhood* $pn(v, S)$ is the set of S-private neighbors of v. The *S-external private neighborhood* of v, denoted $epn(v, S)$, is the set of all S-private neighbors of v in $V(G) \setminus S$. A vertex w is a *common neighbor* of u and v when $w \in (N(u) \cap N(v))$. *Identifying* or *dotting* two vertices can be described as combining them to form one vertex whose neighborhood is the union of their neighborhoods. An edge uv can be *subdivided* by replacing the edge with a new vertex that is adjacent only to u and v. An *independent set of vertices (respectively, edges)* is a set of vertices (respectively, edges) of which no pair is adjacent.

1.2 Domination Parameters

A set S is called a *dominating set* if every vertex in $V \setminus S$ is adjacent to a vertex in S. The minimum cardinality of any dominating set of G is the *domination number* of G, denoted $\gamma(G)$. Similarly, a set S is a total dominating set (TDS) if every vertex in V is adjacent to a vertex in S. The minimum cardinality of any TDS of G is the *total domination number* of G, denoted $\gamma_t(G)$. A TDS of G with minimum cardinality is called a $\gamma_t(G)$-*set*.

1.3 Dotting and Total Domination

When considering graph domination parameters, it is of interest to study criticality with respect to changes in graphs. The effects on the total domination number of a graph by adding an edge, removing an edge, and removing a vertex have been studied. The operation of interest for this thesis is dotting. Burton and Sumner [2] studied effects of dotting on the domination number of a graph and published their work on domination dot-critical graphs in 2006. A graph is defined to be *domination dot-critical* if dotting any adjacent pair of vertices decreases the domination number.

For our work, we expand the same concept to total domination. We introduce more terminology. When dotting vertices a and b, the new vertex formed is denoted (ab). The graph $G.ab$ is the graph formed by dotting a and b.

For the graphs in Figure 2, $\gamma_t(G) = 4$, $\gamma_t(G.ab) = 4$, and $\gamma_t(G.bc) = 3$. Notice that the total domination number does not change when dotting a and b, but it decreases when dotting b and c.

Any pair of vertices in a graph can be dotted, but for most of this thesis we

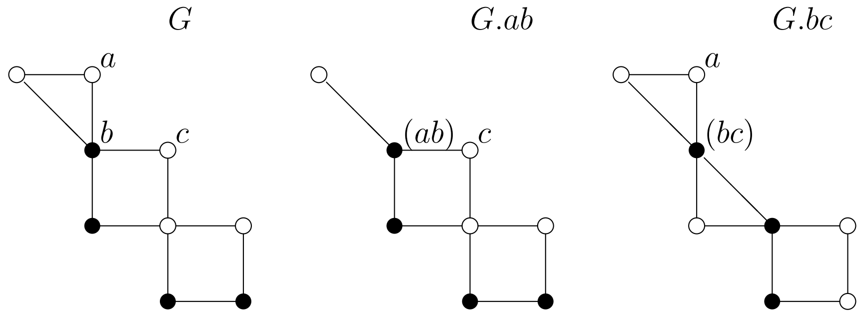


Figure 2: Dotting example in which the darkened vertices represent γ_t-sets of the respective graphs.

consider only adjacent vertices. A graph G is called *γ_t-dot-critical* if for every pair of *adjacent* vertices $a, b \in V(G)$, $\gamma_t(G.ab) < \gamma_t(G)$. If graph G is γ_t-dot-critical and $\gamma_t(G) = k$, then G is called *k_t-dot-critical*. Similarly, a graph G is called *γ_t-dot-stable* if for any pair of *adjacent* vertices $a, b \in V(G)$, $\gamma_t(G.ab) = \gamma_t(G)$. Again, if G is γ_t-dot-stable and $\gamma_t(G) = k$, then G is called *k_t-dot-stable*. If uv is an edge and $\gamma_t(G.uv) < \gamma_t(G)$, then uv is a *critical edge*.

2 LITERATURE REVIEW

2.1 Domination Dot-Critical Graphs

As previously mentioned, Burton and Sumner [2] studied the effects of dotting on the domination number. We present some additional terminology. A graph G is *edge-critical* with respect to the domination number if for every two non-adjacent vertices v and u, $\gamma(G+uv) < \gamma(G)$. A vertex v of G is *critical* if $\gamma(G-v) < \gamma(G)$. A graph G is *vertex-critical* if every vertex of G is critical. Burton and Sumner [2] denote the set of critical vertices of G by G'. A vertex v is called stable if $\gamma(G-v) = \gamma(G)$. A graph is *domination dot-critical* (hereafter, just dot-critical) if identifying any two adjacent vertices (i.e., contracting the edge comprising those vertices) results in a graph with smaller domination number. If identifying any two vertices of G causes the domination number to decrease, then G is *totally dot-critical*. A graph G is *k-edge-critical, k-vertex-critical, k-dot-critical*, or *totally-k-dot-critical*, when it has the indicated property and $\gamma(G) = k$. A graph is *critically dominated* if its set of critical vertices forms a dominating set. The *corona* of two graphs G and H, denoted GoH, is the graph formed from one copy of G and $|V(G)|$ copies of H where the i^{th} vertex of G is adjacent to every vertex in the i^{th} copy of H.

The majority of results they found relate to the criticality of vertices. Unless otherwise noted, all results in this section are from [2].

Lemma 2.1 *Let $a, b \in V(G)$ for a graph G. Then $\gamma(G.ab) < \gamma(G)$ if and only if either there exists a $\gamma(G)$-set S of G such that $a, b \in S$ or at least one of a or b is critical in G.*

Lemma 2.2 *If G is any graph with $\gamma(G) = k \geq 2$, then G is dot-critical (respectively totally dot-critical) if and only if every two adjacent non-critical vertices (respectively any two non-critical vertices) belong to a common $\gamma(G)$-set.*

A vertex in a graph G is *useable* if it belongs to some $\gamma(G)$-set. If every vertex of G is useable, then G is *vertex-useable.* The following lemma is used to show more dot-critical properties.

Lemma 2.3 *Let G be any graph, and $v \in G'$. Then each vertex in $N[v]$ is useable.*

Theorem 2.4 *For every graph G,*

1. *If G is dot-critical, then G is vertex-useable.*

2. *If G is critically dominated, then G is vertex-useable.*

The next result allows them to discuss only connected graphs.

Lemma 2.5 *The graph G is dot-critical (respectively totally dot-critical) if and only if each of its components is dot-critical (respectively totally dot-critical).*

The following lemma gives an interesting property for stability in G.

Lemma 2.6 *If $v, u \in V(G)$ for a graph G such that $N[v] = N[u]$, then $\gamma(G) = \gamma(G.vu)$.*

A graph G is *point-distinguishing* if every two distinct vertices have distinct closed neighborhoods. It follows from the next lemma that every dot-critical graph is point-distinguishing.

Theorem 2.7 *Every point-distinguishing, edge-critical graph is totally dot-critical.*

In one section of [2], they considered graphs with $\gamma(G) = 2$.

Lemma 2.8 *Let a, b and v be vertices of 2-dot-critical graph G such that in $\bar{G}$, v is adjacent to a, v is adjacent to b, and a is not adjacent to b, then*

1. *One of a, b is adjacent to an endvertex in $\bar{G}$*

2. *v is adjacent to an endvertex of $\bar{G}$.*

A graph G is said to be *spiked* if G is the corona of a connected graph H with a single vertex.

Theorem 2.9 *Let G be a graph on $n \geq 4$ vertices. Then G is 2-dot-critical if and only if $\bar{G}$ is not complete, but every component of $\bar{G}$ is spiked or a complete graph $K_m,\ m \geq 2$.*

The following is a characterization for the 2-dot-critical graphs.

Theorem 2.10 *The graph G is a totally 2-dot-critical graph on $n \geq 2$ vertices if and only if every component of $\bar{G}$ is spiked.*

Theorem 2.11 *A 2-dot-critical graph has no critical vertices if and only if it is complete multipartite with each part containing at least three vertices.*

Now we see the more general results for dot-critical graphs.

Lemma 2.12 *If G is a dot-critical and $N[v] \subseteq N[u]$, then $v \in G'$.*

Corollary 2.13 *Every end vertex of a dot-critical graph is a critical vertex.*

Next are some interesting results for graphs with $\gamma(G) = 3$.

Theorem 2.14 *A connected* 3*-dot-critical graph with* $G' = \emptyset$ *has a diameter of at most three.*

Theorem 2.15 *A connected totally* 3*-dot-critical graph with no critical vertices has a diameter of at most two.*

Burton and Sumner posed the open question: Is it true that for $k \geq 4$, there exists a totally k-dot-critical graph with no critical vertices? This was shown to have a positive answer by Chengye, Yuansheng, and Linlin in [6].

Theorem 2.16 *[6] There exists a totally k-dot-critical graph with no critical vertices for any* $k \geq 4$.

Theorem 2.17 *[6] A connected* 4*-dot-critical graph* G *with* $G' = \emptyset$ *has a diameter of at most five.*

Since every totally 4-dot-critical graph is 4-dot-critical, the corollary results.

Corollary 2.18 *[6] A connected totally* 4*-dot-critical graph* G *with* $G' = \emptyset$ *has a diameter of at most five.*

Burton and Sumner showed the following in [3]. Note that graph is γ-excellent if every vertex of the graph is contained in some minimum dominating set of the graph.

Theorem 2.19 *[3] Let T be a tree on $n \geq 4$ vertices. Then the following are equivalent:*

1. *T is dot-critical.*
2. *T is critically dominated.*
3. *T is γ-excellent.*

Nader [10] studied the restrictions on the diameter for dot-critical graphs and among other results gave the following.

Theorem 2.20 *[10] A connected k-dot-critical graph G with $G' = \emptyset$ has a diameter of at most 7 when $k = 5$ and $3k - 9$ when $k \geq 6$.*

In this section, we place no restriction on the vertices being dotted, that is, they may or may not be adjacent. We note that dotting two vertices of a graph with order $n \geq 3$ cannot increase the total domination number. As we have seen in Figure 2, the total domination number of a graph can remain the same or decrease. For our first result, we show that dotting vertices can decrease the total domination number by at most two, see Figure 3.

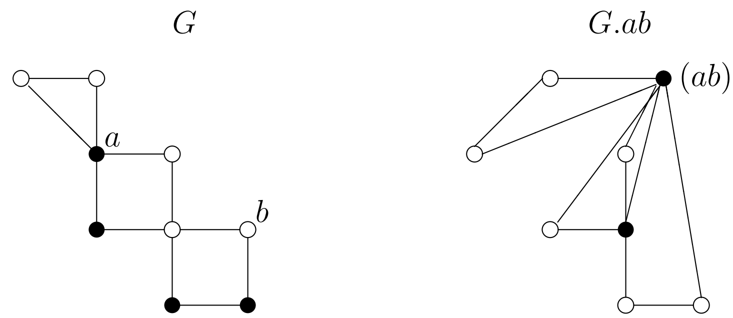


Figure 3: Example where $\gamma_t(G.ab) = \gamma_t(G) - 2$

Proposition 3.1 *Let G be a connected graph of order $n \geq 3$. For any two vertices a and b, $\gamma_t(G) - 2 \leq \gamma_t(G.ab) \leq \gamma_t(G)$.*

Proof Clearly dotting two vertices does not increase the total domination number so the upper bound holds. For the lower bound, let M be a $\gamma_t(G.ab)$-set. We consider two cases.

Case 1: $(ab) \in M$. Let $S = (M \backslash \{(ab)\}) \cup \{a, b\}$. If $N(a) \cap S \neq \emptyset$ and $N(b) \cap S \neq \emptyset$, then S is a TDS of G, implying that $\gamma_t(G) \leq |S| = |M| + 1 = \gamma_t(G.ab) + 1$. Without loss of generality, assume $N(a) \cap S = \emptyset$. Since M is a TDS of $G.ab$, we know that

(ab) must have at least one neighbor in M. It follows that $N(b) \cap S \neq \emptyset$. Since G has no isolates a must have a neighbor, say y, in $V \setminus S$. Thus $S \cup \{y\}$ is a TDS of G, and $\gamma_t(G) \leq |S| + 1 = |M| + 2 = \gamma_t(G.ab) + 2$.

Case 2: $(ab) \notin M$. Consider M in G. If M total dominates G, then $\gamma_t(G) \leq \gamma_t(G.ab)$. Assume that M does not total dominate G. Since M is a TDS of $G.ab$, without loss of generality, M total dominates $G - \{a\}$ but does not dominate a. Let $x \in N_G(a)$. Then $x \in V \setminus M$ has a neighbor in M. Therefore $S = M \cup \{x\}$ is a TDS of G, and hence $\gamma_t(G) \leq |M| + 1 = \gamma_t(G.ab) + 1$. $\square$

Considering the cases in the proof, we deduce the following corollary.

Corollary 3.2 *Let G be a connected graph with order $n \geq 3$.*
If a and b are adjacent vertices of G, then $\gamma_t(G) - 1 \ \leq \gamma_t(G.ab) \ \leq \gamma_t(G)$.

Proof It is only possible for $\gamma_t(G.ab) = \gamma_t(G) - 2$ in Case 1 of the proof of Proposition 3.1. Using the same notation, assume a and b are adjacent and $(ab) \in M$. Then $S = (M \setminus \{(ab)\}) \cup \{a, b\}$ is a TDS of G, and so $\gamma_t(G) \leq |S| = |M| + 1 = \gamma_t(G.ab) + 1$.
$\square$

4 TOTAL DOMINATION DOT CRITICAL GRAPHS

In this section, we restrict our attention to dotting only adjacent vertices.

4.1 Existence of γ_t-Dot-Critical Graphs

In this section, we show that k_t-dot-critical graphs exist for all values of $k \geq 3$.

Definition 4.1 *A spider is the graph formed by subdividing all edges of a star $K_{1,r}$ with $r \geq 1$. Similarly, a wounded spider is the graph formed by subdividing exactly $r-1$ edges of a star $K_{1,r}$ with $r \geq 2$.*

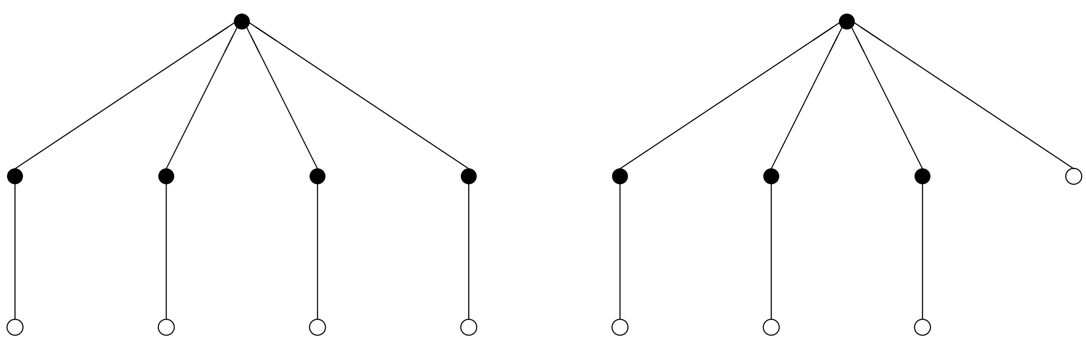


Figure 4: A spider and a wounded spider formed from a $K_{1,4}$

A specific example of a spider and wounded spider is in Figure 4. We now present our realizability results.

Proposition 4.2 *Let G be a connected graph of order n and $k \geq 3$ be an integer. There exists a k_t-dot-critical graph G for all values of k.*

Proof Let G be a spider with $k-1$ leaves. Then $\gamma_t(G) = k$. Dotting any pair of adjacent vertices results in a wounded spider with the total domination number $k-1$. $\square$

4.2 Characterization of 3_t-Dot-Critical Trees

Observation 4.3 *If v is a support vertex in graph G, then v is in every $\gamma_t(G)$-set.*

The previous observation helps us to consider leaves in γ_t-dot-critical graphs.

Lemma 4.4 *If G is a γ_t-dot-critical graph and u is a support vertex, then u is adjacent to exactly one leaf.*

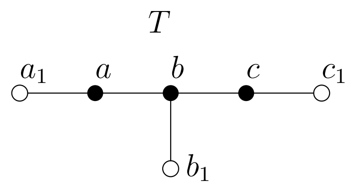
Proof Let G be a γ_t-dot-critical graph and u a support vertex with adjacent leaf set $\{v_1, ..., v_j\}$. By the definition of a support vertex, $j \geq 1$. Assume for the purpose of a contradiction that $j > 1$. Let M be a $\gamma_t(G.uv_1)$-set, then $|M| = \gamma_t(G) - 1$. Since (uv_1) is a support vertex, $(uv_1) \in M$. Moreover (uv_1) has a neighbor, say x, in M. But, since $N_G(v_1) = u$, $x \in N(u)$ thus $S = (M \setminus \{(uv_1)\}) \cup \{u\}$ is a TDS of G with cardinality $\gamma_t(G) - 1$, a contradiction. Thus $j = 1$ and u is adjacent to exactly one leaf. $\square$

Recall that P_n denotes the path on n vertices. Next we characterize 3_t-dot-critical trees.

Proposition 4.5 *A tree T is 3_t-dot-critical if and only if $T \cong P_5$.*

Proof $\Rightarrow$ Assume T is a 3_t-dot-critical tree. We wish to show that T is a P_5. Let S be a $\gamma_t(T)$-set. Notice since T is a tree that S induces a $P_3 = (a, b, c)$. Let

$A = N(a) \setminus S, B = N(b) \setminus S$ and $C = N(c) \setminus S$. Notice it follows from the definition of a tree that $A \cap B = \emptyset$, $B \cap C = \emptyset, C \cap A = \emptyset$ and $A \cup B \cup C$ is an independent set. By the minimality of S, A and C are nonempty. Thus $|A| \geq 1$ and $|C| \geq 1$. Since S dominates T, $\{a\} \cup A, \{b\} \cup B$, and $\{c\} \cup C$ partition $V(T)$ and each vertex of $A \cup B \cup C$ is a leaf. Lemma 4.4 implies that $|A| = |C| = 1$ and $|B| \leq 1$.



Assume for the purpose of a contradiction that $B = \{b'\}$. Then $T.bb'$ is a P_5 and $\gamma_t(P_5) = 3$, contradicting that T is 3_t-dot-critical. Therefore it follows $B = \emptyset$ and T is a P_5.

$\Leftarrow$ Clearly P_5 is a tree and $\gamma_t(P_5) = 3$. Notice that dotting any adjacent vertices forms a P_4 and $\gamma_t(P_4) = 2$. Thus P_5 is a 3_t-dot-critical tree.$\square$

5 TOTAL DOMINATION DOT STABLE GRAPHS

5.1 Existence of γ_t-Dot-Stable Graphs

We first show that no graph with odd total domination number is stable. Then we show that every even total domination number is achievable by a γ_t-dot-stable graph.

Observation 5.1 *[9] For $n \geq 3$, $\gamma_t(P_n) = \gamma_t(C_n) = \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{4} \rceil - \lfloor \frac{n}{4} \rfloor$.*

This observation can be rewritten for our purposes. For $n \geq 3$,

$$\begin{aligned}\gamma_t(P_n) = \gamma_t(C_n) &= \tfrac{n}{2} \text{ when } n \equiv 0 \pmod 4\\ &= \tfrac{n-1}{2} + 1 \text{ when } n \equiv 1,\ 3 \pmod 4\\ &= \tfrac{n}{2} + 1 \text{ when } n \equiv 2 \pmod 4.\end{aligned}$$

Proposition 5.2 *If G is γ_t-dot-stable, then $\gamma_t(G)$ is even.*

Proof We prove the contrapositive. Assume that $\gamma_t(G)$ is odd, and let S be a $\gamma_t(G)$-set. Since $\gamma_t(G)$ is odd, $G[S]$ has an odd component. The new set formed by dotting any two adjacent vertices, say x and y, in the odd component of S is a TDS for $G.xy$, so $\gamma_t(G.xy) < |S| = \gamma_t(G)$. Hence, G is not γ_t-dot-stable. □

Proposition 5.3 *Let $k \geq 2$ be an even integer. There exists a k_t-dot-stable graph G for all even values of k.*

Proof Let G be a P_{2k}. Since k is even $2k \equiv 0 \pmod 4$. Notice that when dotting any pair of adjacent vertices of P_{2k}, the resulting graph is a P_{2k-1}. By Observation 5.1, $\gamma_t(P_{2k}) = \gamma_t(P_{2k-1}) = k$. □

Next we show that γ_t-sets of γ_t-dot-stable graphs have a unique property.

Lemma 5.4 *Let G be a graph with $\gamma_t(G) \geq 4$. If G is γ_t-dot-stable with $\gamma_t(G)$-set S, then the induced subgraph $G[S]$ is a set of independent edges.*

Proof Let S be a $\gamma_t(G)$-set. Let x_1 and x_2 be adjacent vertices in S. Notice that because G is γ_t-dot-stable the set $(S \setminus \{x_1, x_2\}) \cup \{(x_1x_2)\}$ is not a TDS of $G.x_1x_2$. Since every vertex in $V \setminus S$ dominated by x_1 or x_2 is now dominated by (x_1x_2), the only possibility is that (x_1x_2) itself is not dominated. Thus in G, x_1 and x_2 have no neighbors in $S \setminus \{x_1, x_2\}$. Since S is a TDS of G and x_1x_2 is an arbitrary edge in $G[S]$, the result follows. $\square$

We are now ready to characterize the γ_t-dot-stable graphs.

Theorem 5.5 *Let G be a graph with $\gamma_t(G) \geq 4$. Graph G is γ_t-dot-stable if and only if for every $\gamma_t(G)$-set S the induced subgraph $G[S]$ is a set of independent edges.*

Proof ($\Rightarrow$) Assume G is $\gamma_t(G)$-dot-stable. Let S be a $\gamma_t(G)$-set. By Lemma 5.4 it follows that $G[S]$ is a set of independent edges.

($\Leftarrow$) Assume that every $\gamma_t(G)$-set induces a set of independent edges. Assume to the contrary that G is not stable. Thus there exists a critical edge ab. Let M be a $\gamma_t(G.ab)$-set, then $|M| = \gamma_t(G) - 1$. We consider two cases.

Case 1: $(ab) \in M$.

Since M is a TDS of $G.ab$, (ab) is adjacent to a vertex, say x, in M. Notice, in G, x is adjacent to a or b. Thus $M' = (M \setminus \{(ab)\}) \cup \{a, b\}$ is a TDS of G with cardinality $\gamma_t(G.ab) + 1 = \gamma_t(G)$. Hence, M' is a $\gamma_t(G)$-set such that $\{x, a, b\}$ induces a P_3 or a

K_3 in $G[M']$, contradicting that $G[M']$ is an independent set of edges.

Case 2: $(ab) \notin M$.

Since M is a TDS of $G.ab$, (ab) is adjacent to a vertex, say x, in M. Also x is adjacent to a vertex, say y, in M. Thus, without loss of generality, x is adjacent to a in G. It follows that $M' = M \cup \{a\}$ is a $\gamma_t(G)$-set for which $\{a, x, y\}$ induces a P_3 or a K_3 in $G[M']$, a contradiction.

Hence, there is no critical edge and G is γ_t-dot-stable. □

Observation 5.6 *For graph G with $\gamma_t(G) \geq 3$, G has a $\gamma_t(G)$-set containing no leaves.*

Lemma 5.4 and Observation 5.6 imply the following result.

Lemma 5.7 *If G is a γ_t-dot-stable graph with $\gamma_t(G) \geq 4$, then there exists a $\gamma_t(G)$-set S such that each vertex in S has a neighbor in $V \setminus S$.*

Definition 5.8 *Let G be a graph and $S \subset V(G)$. A vertex $x \in V \setminus S$ is component common if x is adjacent to two or more components of $G[S]$.*

Lemma 5.9 *Let G be a γ_t-dot-stable graph with $\gamma_t(G) \geq 4$ and γ_t-set S. If $u, v \in S$ are adjacent vertices and u is adjacent to a component common vertex, then $epn(v, S) \neq \emptyset$.*

Proof Without loss of generality, assume u is adjacent to a component common vertex, say x. Notice that $S' = (S \setminus \{v\}) \cup \{x\}$ is a $\gamma_t(G)$-set if $epn(v, S) = \emptyset$. Since, by definition, x is adjacent to some other component of $G[S]$, $G[S']$ does not induce a set of independent edges, a contradiction. Thus $epn(v, S) \neq \emptyset$. □

The corollary follows directly from Lemma 5.9.

Corollary 5.10 *Let G be a γ_t-dot-stable graph with γ_t-set S and $\gamma_t(G) \geq 4$. Let u_i and v_i be adjacent pairs in $G[S]$. If $[N(v_p) \cap (V \setminus S)] \subseteq [N(v_j) \cap (V \setminus S)]$, then $epn(u_p, S) \neq \emptyset$ and $epn(u_j, S) \neq \emptyset$.*

Lemma 5.11 *Let G be a γ_t-dot-stable graph with $\gamma_t(G)$-set S and $\gamma_t(G) \geq 4$. If u and v are adjacent vertices in S such that $epn(u, S) = \emptyset$ and $epn(v, S) = \emptyset$, then $A = N(u) \cap (V \setminus S) = N(v) \cap (V \setminus S)$. Furthermore, no vertex of A dominates A and so $|A| \geq 2$.*

Proof Lemma 5.9 implies that there are no component common vertices in $N(u) \cap N(v)$. Since $epn(u, S) = \emptyset$ and $epn(v, S) = \emptyset$, it follows that $A = N(u) \cap (V \setminus S) = N(v) \cap (V \setminus S)$. Suppose $x \in A$ dominates A. Since x is not component common, x is adjacent to some $y \in V \setminus S$. Since y is dominated by S, $(S - \{u, v\}) \cup \{x, y\}$ is a $\gamma_t(G)$-set which does not induce a set of independent edges, a contradiction. Therefore no vertex of A dominates A. We note that $|A| \geq 2$. $\square$

Theorem 5.12 *[9] For any graph G with no isolates, $\gamma_t(G) \leq \frac{2n}{3}$.*

We can now make slight improvement on the upper bound of Theorem 5.12 for the total domination number of γ_t-dot-stable graphs.

Proposition 5.13 *If G is a k_t-dot-stable graph of order n and $\gamma_t(G) \geq 4$, then $\gamma_t(G) = k \leq \frac{2(n-1)}{3}$.*

Proof Let G be a k_t-dot-stable graph, and by Lemma 5.6 choose S to be a $\gamma_t(G)$-set such that every vertex in S has a neighbor in $V \setminus S$. By Theorem 5.4, $G[S] = \frac{k}{2}K_2$. Label the vertices of S as u_i and v_i where u_i is adjacent to v_i in the i^{th} component of $G[S]$ and $1 \leq i \leq \frac{k}{2}$. To establish our bound we count the vertices of $V \setminus S$. If $u_i \in S$ and $epn(u_i, S) \neq \emptyset$, then we associate a unique vertex from $epn(u_i)$ with u_i. Let P be the set of external private neighbors of vertices in S. Moreover, if $epn(u_i, S) = \emptyset$ and $epn(v_i, S) = \emptyset$, then, by Lemma 5.11, $A_i = N(u_i) \cap (V \setminus S) = N(v_i) \cap (V \setminus S)$ and $|A_i| \geq 2$. Let $A = \bigcup_i A_i$. Note that A does not contain a component common vertex and $A \cap P = \emptyset$. Thus again we can count two unique vertices in A for each such u_iv_i component. Hence, the only vertices in S that we have not associated with a unique vertex in $V \setminus S$ are the ones with no external private neighbors and adjacent to a component common vertex. Let x be the number of such vertices in S, and let c be the number of component common vertices in $V \setminus S$. Now $c \geq 1$ for otherwise $|V \setminus S| \geq |S|$ implying that $\gamma_t(G) \leq \frac{n}{2}$ and we are finished. Moreover, Lemma 5.9 implies that $x \leq \frac{|S|}{2} = \frac{k}{2}$. Hence, $n = |S| + |V \setminus S| \geq k + k - x + c$. To minimize $2k - x + c$, we must maximize x and minimize c. Thus $n \geq 2k - \frac{k}{2} + 1$. Hence, $k \leq \frac{2(n-1)}{3}$ and the result follows. $\square$

5.3 Realizability of γ_t-Dot-Stable Graphs

In this section, we are able to use a minor observation and a family of subdivided stars to show the realizability of γ_t-dot-stable graphs.

Observation 5.14 *If a graph G is γ_t-dot-stable and zero or more leaves are appended to a vertex in $\gamma_t(G)$-set S, then G remains γ_t-dot-stable.*

Definition 5.15 *Let $\mathcal{F}$ be the family of subdivided stars formed by subdividing each edge of a star $K_{1,r}$ with $r \geq 2$ twice.*

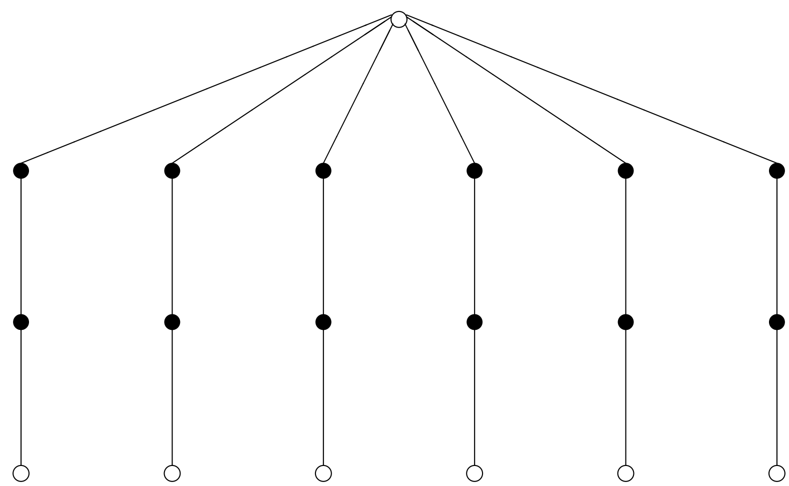


Figure 5: Example of a star with edges subdivided twice which is a γ_t-dot-stable graph where $n = \frac{3}{2}k + 1$

The following proposition illustrates the sharpness of Lemma 5.11.

Proposition 5.16 *Each graph in family $\mathcal{F}$ is a γ_t-dot-stable graph with order $n = \frac{3}{2}\gamma_t + 1$.*

Theorem 5.17 *Given any even integer k and integer n such that $4 \leq k \leq \frac{2(n-1)}{3}$, there exists a k_t-dot-stable graph of order n.*

Proof The result follows for $n = \frac{3}{2}k + 1$ by Observation 5.14. The value of n can be increased by appending leaves to any support vertex. $\square$

5.4 γ_t-Dot-Stable Trees

In this section, we specifically look at γ_t-dot-stable graphs that are trees.

Definition 5.18 *A tree T is in $\mathcal{H}_5$ if T is a caterpillar with spine code* $(1^+,\ 0^+,\ 0,\ 0^+,\ 1^+)$ *or in $\mathcal{H}_6$ if T is a caterpillar with spine code* $(1^+,\ 0^+,\ 0,\ 0,\ 0^+,\ 1^+)$*. This means T has a spine of length 5 or 6 with 1 or more legs on the endvertices of the spine, zero or more legs on the support vertices of the spine, and zero legs on the remaining vertices of the spine.*

A specific example of trees in $\mathcal{H}_5$ and $\mathcal{H}_6$ is given in Figure 6.

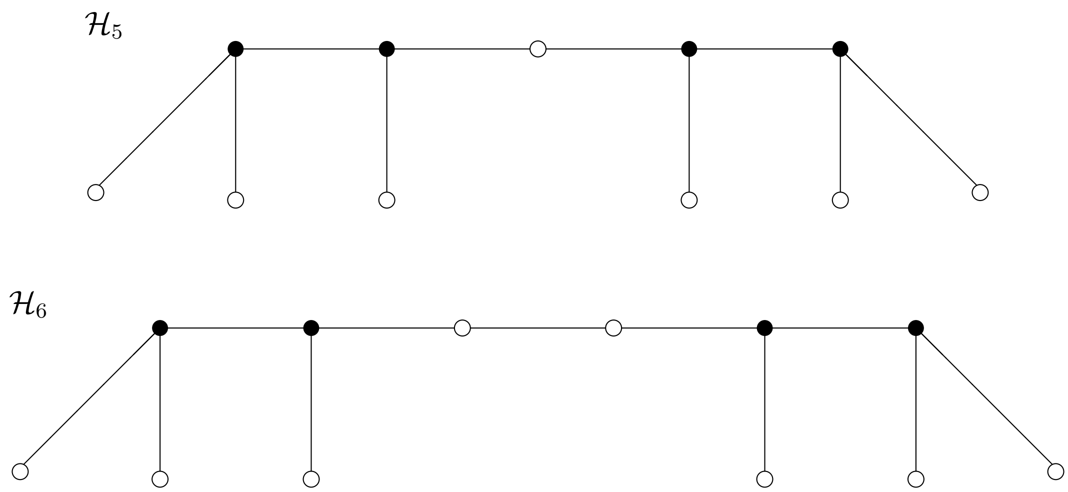


Figure 6: Examples of each caterpillar family

Theorem 5.19 *A graph T is a 4_t-dot-stable tree if and only if $T \in \mathcal{H}_5$ or $T \in \mathcal{H}_6$.*

Proof $\Rightarrow$ Let T be a 4_t-dot stable tree. Then $\gamma_t(T) = 4$ and $T[S] = 2K_2$ for any $\gamma_t(T)$-set S. Let $S = \{a, b, c, d\}$ where a is adjacent to b and c is adjacent to d. Since T is connected, we consider two cases.

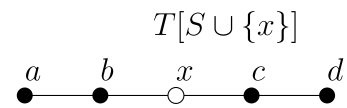


Figure 7: Case 1

Case 1: (See Figure 7) Either a or b have a common neighbor with c or d. Without loss of generality, b and c have a common neighbor x. Notice that all other neighbors of S are leaves since a common neighbor creates a cycle.

Claim: Both a and d each have at least one leaf. If neither has a leaf, then $\{b, x, c\}$ is a TDS contradicting that $\gamma_t(T) = 4$. If only one, say a, has a leaf, then $\{a, b, x, c\}$ is a γ_t-set that does not induce a $2K_2$. Thus both a and d have one or more leaves. Consider b and c, by Observation 5.14 each vertex can have zero or more leaves. Therefore, $T \in \mathcal{H}_5$.

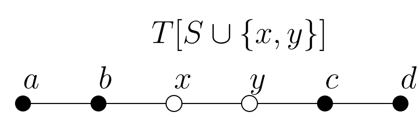


Figure 8: Case 2

Case 2: (See Figure 8) Assume there are no component common vertices. Then since G is connected, we may assume that b has a neighbor, say x, and c has a neighbor, say y, such that x is adjacent to y. The set $\{b, x, y, c\}$ induces a P_4, for otherwise a cycle is formed. As before, the remaining neighbors are leaves otherwise a cycle is created.

Claim: Each of a and d is adjacent to at least one leaf. If neither has a leaf neighbor, then $\{b, x, y, c\}$ is a $\gamma_t(T)$-set which does not induce a $2K_2$. If only one, say a, has a leaf, then $T \in \mathcal{H}_5$ and we are finished.

By Observation 5.6, b and c have zero or more leaves and $T \in \mathcal{H}_6$.

$\Leftarrow$ Every $T \in \mathcal{H}_5$ or $T \in \mathcal{H}_6$ is a P_7 or P_8 with leaves appended to a vertices of the γ_t-set, thus are $\gamma_t(T)$-dot-stable graphs. Notice that $\gamma_t(P_7) = \gamma_t(P_8) = 4$, therefore the result holds. $\square$

6 FAMILY $\mathcal{J}$ AND TOTAL DOMINATION DOT-SUPERCRITICAL GRAPHS

6.1 $\mathcal{J}_c$ Graphs

We found that there are some graphs that remain critical after a first pair of vertices are identified. That is, the total domination number decreases with dotting the first pair as well as the second. In order to better discuss these graphs, we define a particular family of graphs.

Definition 6.1 *A graph G that is γ_t-dot-critical and when any two pairs of adjacent vertices are dotted the new graph $G.ab$ is also γ_t-dot-critical, then $G \in \mathcal{J}_c$.*

Definition 6.2 *The Q_3, drawn below in Fig 9, is called a hypercube.*

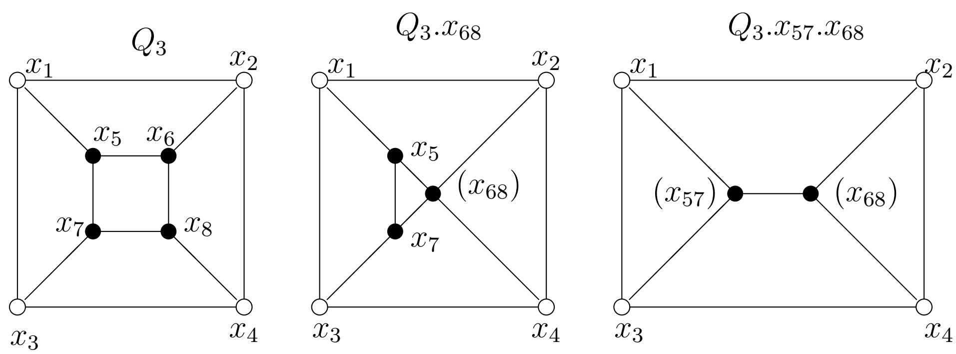


Figure 9: The cube $Q_3 \in \mathcal{J}_c$

The hypercube Q_3 is an example of a graph in $\mathcal{J}_c$. To illustrate this, we dot the necessary pairs of vertices in Fig. 9 and the Appendix. By symmetry the first pair of vertices is arbitrary. We need only consider dotting x_6 with x_8 followed by x_5 with x_7 or x_1 with x_2 or x_{68} with x_5 or x_{68} with x_2. Since dotting the remaining pairs is necessary but not constructive we demonstrate them in the appendix.

6.2 γ_t-Dot-Supercritical Graphs

Throughout most of this thesis, we considered only adjacent pairs of vertices. We now look at graphs that are critical with respect to *any* pair of vertices, adjacent or non-adjacent.

Definition 6.3 *A graph G is γ_t-dot-supercritical when for every pair of vertices $a, b \in V(G)$, $\gamma_t(G.ab) < \gamma_t(G)$.*

Proposition 6.4 *If a graph $G \in \mathcal{J}_c$, then G is also γ_t-dot supercritical.*

Proof Since G is γ_t-dot-critical, we need only consider any two nonadjacent vertices a and b in $V(G)$. In order to show G is γ_t-dot-supercritical, we will show that $\gamma_t(G.ab) < \gamma_t(G)$. We consider two cases.

Case 1: a and b have common neighbor, say x.

Since $G \in \mathcal{J}_c$, $\gamma_t(G.ax.b(ax)) = \gamma_t(G) - 2$. Let S be a $\gamma_t(G.ax.b(ax))$-set. If $((ax)b) \in S$, then $(S \setminus \{((ax)b)\}) \cup \{x, (ab)\}$ is a TDS of $G.ab$ with cardinality $\gamma_t(G) - 1$.

If $((ax)b) \notin S$, then $((ax)b)$ has a neighbor, say y, in S. If a or b is adjacent to y in G, the $S \cup \{(ab)\}$ is a TDS of G.ab with cardinality $\gamma_t(G) - 1$. If not, then x is adjacent to y and hence $S \cup \{x\}$ is a TDS of G.ab with cardinality $\gamma_t(G) - 1$.

In all cases, $\gamma_t(G.ab) < \gamma_t(G)$.

Case 2: a and b have no common neighbor.

In this case, we dot b with some neighbor b' and a with a neighbor a'. Since $G \in \mathcal{J}_c$, $\gamma_t(G.bb'.aa') = \gamma_t(G) - 2$. Let P be a $\gamma_t(G.bb'.aa')$-set.

Case 2a: $(aa'), (bb') \in P$

Since P is a TDS, (aa') and (bb') both have at least one neighbor in P. Since every vertex dominated by (aa') or (bb') in $G.aa'.bb'$ is dominated by $\{(ab), a', b'\}$ in $G.ab$ it follows that $P' = (P - \{(aa'), (bb')\}) \cup \{(ab), a', b'\}$ is a TDS of $G.ab$ and $|P'| = |P| + 1 = \gamma_t(G) - 1$. Thus $\gamma_t(G.ab) < \gamma_t(G)$.

Case 2b: One of $(aa'), (bb')$ is in P and the other is not.

Without loss of generality, say $(aa') \in P$ and $(bb') \notin P$. Then $P' = (P - \{(aa')\}) \cup \{(ab), a'\}$ is a TDS of $G.ab$ and $|P'| = |P|+1 = \gamma_t(G)-1$. Therefore $\gamma_t(G.ab) < \gamma_t(G)$.

Case 2c: $(aa'), (bb') \notin P$

If in $G.ab$, (ab) has a neighbor in P, then $P \cup \{(ab)\}$ is a TDS of $G.ab$ with cardinality $\gamma_t(G) - 1$ and we are finished.

Hence assume that (ab) has no neighbor in P. Thus a' and b' have neighbors in P and $P \cup \{a'\}$ is a TDS of $G.ab$ and again the result holds. $\square$

6.3 Existence of $\mathcal{J}_s$ Graphs

In this section, we consider a special family of γ_t-dot-stable graphs.

Definition 6.5 *A graph $G \in \mathcal{J}_s$ when G is γ_t-dot-stable and the graph formed by dotting any pair of* adjacent *vertices is also γ_t-dot-stable.*

Lemma 6.6 *For any even $\gamma_t(G) = k$, there exists a graph $G \in \mathcal{J}_s$.*

Proof Construct G as a path or a cycle of length $n \equiv 0 \pmod 4$ for $n > 4$. Notice then $\gamma_t(P_n) = \gamma_t(C_n) = \frac{n}{2}$. Clearly dotting any pair of vertices forms a P_{n-1} and C_{n-1} respectively. Repeating the process with the newly formed graph produces a P_{n-2} and a C_{n-2}. Since $\gamma_t(P_{n-2}) = \gamma_t(C_{n-2}) = \frac{n-2}{2} + 1 = \frac{n}{2}$, both paths and cycles on $n \equiv 0 \pmod 4$ vertices are in $\mathcal{J}_s$. $\square$

7 CONCLUSION

We found that identifying any pair of vertices cannot increase the total domination number. Further, we have shown it can decrease the total domination number by at most two. A graph is total domination dot-critical if identifying any pair of adjacent vertices decreases the total domination number. On the other hand, a graph is total domination dot-stable if identifying any pair of adjacent vertices leaves the total domination number unchanged. We have shown the existence for both of these types of graphs and presented our realizability results for γ_t-dot-stable graphs. We characterized total domination dot-critical trees with total domination number three and all total domination dot-stable graphs. A graph G that is γ_t-dot-critical and when any two pairs of adjacent vertices are dotted the new graph $G.ab$ is also γ_t-dot-critical, then $G \in \mathcal{J}_c$. While considering $\mathcal{J}_c$ graphs we showed that if a graph $G \in \mathcal{J}_c$, then G is also γ_t-dot supercritical.

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Here we illustrate that the cube $Q_3 \in \mathcal{J}_c$ by demonstrating the remaining pairs of vertices also decrease the total domination number.

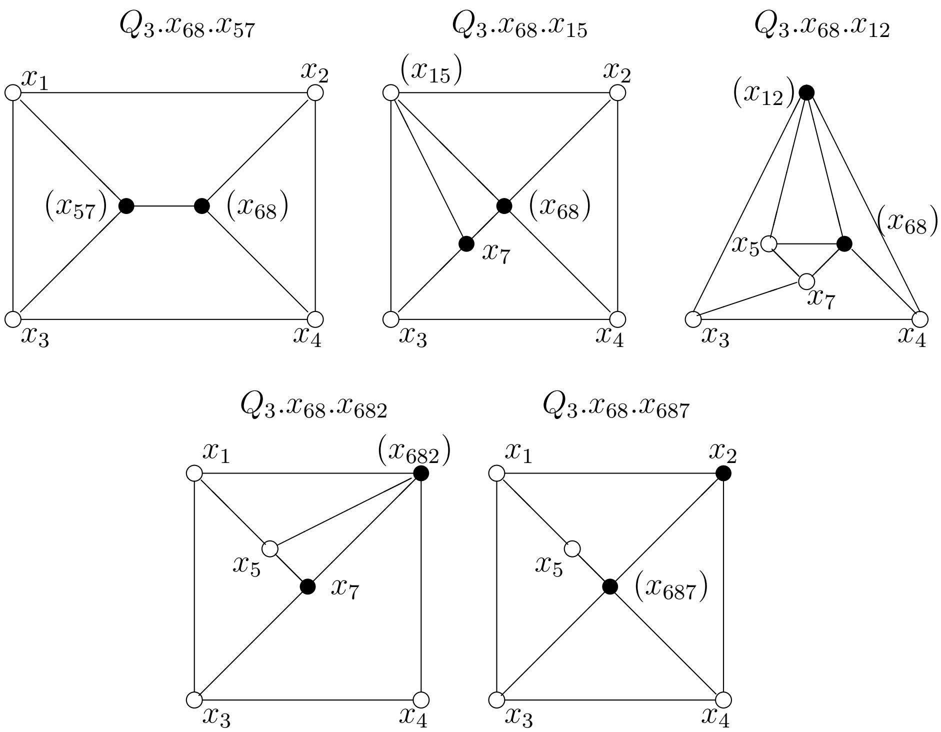

Figure 10: The remaining cases of vertex pairs are verified. The darkened vertices represent γ_t-sets of the respective graphs.

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