5-2011

Universal Hypergraphs.

Michael Deren
East Tennessee State University

Universal Hypergraphs

A thesis

presented to

the faculty of the Department of Mathematics & Statistics

East Tennessee State University

In partial fulfillment

of the requirements for the degree

Master of Science in Mathematical Sciences

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by

Michael R. Deren

May 2011

Anant Godbole, Ph.D., Chair

Robert Beeler, Ph.D.

Robert Gardner, Ph.D.

Keywords: graph theory, universal graphs, universal hypergraphs

ABSTRACT

Universal Hypergraphs

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In this thesis, we study universal hypergraphs. What are these? Let us start with defining a universal graph as a graph on n vertices that contains each of the many possible graphs of a smaller size $k < n$ as an induced subgraph. A *hypergraph* is a discrete structure on n vertices in which edges can be of any size, unlike graphs, where the edge size is always two. If all edges are of size three, then the hypergraph is said to be 3-uniform. If a 3-uniform hypergraph can have edges colored one of a colors, then it is called a 3-uniform hypergraph with a colors. Analogously with universal graphs, a universal, induced, 3-uniform, k-hypergraph, with a possible edge colors is then defined to be a 3-uniform a-colored hypergraph on n vertices that contains each of the many possible 3-uniform a-colored hypergraphs on k vertices, $k < n$. In this thesis, we study conditions for the existence of a such a universal hypergraph, and address the question of how large n must be, given a fixed k, so that hypergraphs on n vertices are universal with high probability. This extends the work of Alon, [2] who studied the case of $a = 2$, and that too for graphs (not hypergraphs).

ACKNOWLEDGMENTS

I would like to acknowledge all of my friends, family, and teachers that have encouraged me to complete my degree and work on my thesis. Firstly, I would like to acknowledge my thanks for Dr. Godbole in his relentless work in making this thesis a reality. I would also like to thank my advisory committee, Dr. Beeler and Dr. Gardner for their support and encouragement. Finally, I would like to thank all of my college professors that made this adventure fun.

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1 INTRODUCTION

One might wonder how long a string of letters, numbers, or characters would need to be in order to include any random string of letters, numbers, or characters of size k as a substring of (not necessarily consecutive) letters, numbers, or characters. Next, a person may wonder how big a picture would need to be in order to include all smaller pictures of size $k \times k$, as submatrices with not necessarily consecutive rows and columns.

Examples of minimal omnisequences are given by Abraham et al., [1] and minimal omnimosiacs are given by Banks et al., [4]. Eroglu [8] has studied minimal omnisculptues, which are 3-dimensional analogs. This thesis considers different 3-dimensional "super structures" which contain all structures that we call 3-uniform k-hypergraphs.

We will make some basic definitions on the structures that we will be working with. First, let us start with the basic definition of a graph's key components, vertices and edges. *Vertices* are a collection of points, often drawn in the plane. A graph consists of n vertices for some $n = 1, 2, \ldots$. Some pairs of vertices are connected by *edges*. There may be multiple edges from a vertex x to vertex y. However, we exclude the possiblilty of an edge existing from a vertex a to itself. Thus, our graphs may have multiple edges but no loops. Due to this reason, they are not *simple*.

Graphs are very useful in the fact that vertices may represent locations on a map, or computers in a network. We may use edges for representing roads or lines between the vertices in the previous examples. Families of graphs are often defined by certain properties of the number of edges or vertices that they have. For example, the family of graphs known as trees are simple *connected* graphs that contain n vertices and $n-1$ edges and lack a cycle.

Another aspect of graphs that we study is the *degree* of the vertices of a graph. The degree of a vertex simply is the number of edges that are incident to a vertex. We will use the following example, Figure 1, in order to illustrate a basic, simple graph, G.

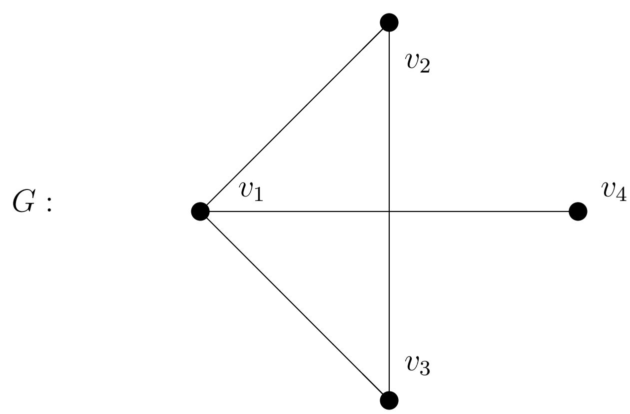

Figure 1: Vertices and Edges for a Simple Graph

Looking at Figure 1, we first notice that G has four vertices, namely those in the vertex set $V(G) = \{v_1,\ v_2, v_3, v_4\}$. G also has four edges, with edge set $E(G) = \{v_1v_2,\ v_1v_3,\ v_1v_4,\ v_2v_3\}$. We finally examine the degree of each vertex, and notice that the degree of the vertices is 3, 2, 2, 1 for v_1, v_2, v_3, and v_4, respectively.

In this thesis, we will study the case where we allow multiple edges between two vertices. These types of graphs are also known as *weighted graphs*. This is a much more practical application of graph theory, as it is typical for there to be more than one route between two locations on a map, or edges of equal or different weights or lengths. In this thesis, we think of these multiple edges being coded by different colors. For example, an edge of weight zero may be represented by the color white, an edge of weight one by the color black, and perhaps an edge of weight two may be colored by blue or red.

One of the easiest way to study graphs is to look at what is known as the *adjacency matrix* of a graph. An adjacency matrix is the representation of a graph using a matrix. We label the columns and rows of the matrix with the listed vertices, and the entries of the matrix are typically 0 or 1, with 0 in the (i, j) position representing the lack of an edge between vertices v_i and v_j, and 1 representing the presence of an edge. Edge labels may also reflect the weight of an edge, or the color of an edge, or the number of edges between the vertices. Let us look at a small example of the adjacency matrix for G, the graph in Figure 1.

$$\begin{array}{cc} G & \begin{matrix} v_1 & v_2 & v_3 & v_4 \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}.$$

It is common, however, to omit the headings of the matrix, and just list the entries. We assume the headings or labels to be listed in order from smallest to largest, or in alphabetical order. It may be necessary to include the headings if one has labeled the vertices as colors for example. Omitting labels would make our matrix for G appear as:

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Clearly, the adjacency matrix is symmetric and the entries on the diagonal (0, 0), (1, 1), etcetera are zero since G has no loops.

We say that two graphs H and I are *isomorphic* if there exists a bijection f from $V(H)$ to $V(I)$ such that $(u, v) \in E(H)$ if and only if $(f(u), f(v)) \in E(I)$. We look at the two graphs H and I below as an example. In Figure 2, the bijection takes v_1, v_2, v_3, v_4 to v_1, v_4, v_3, v_2.

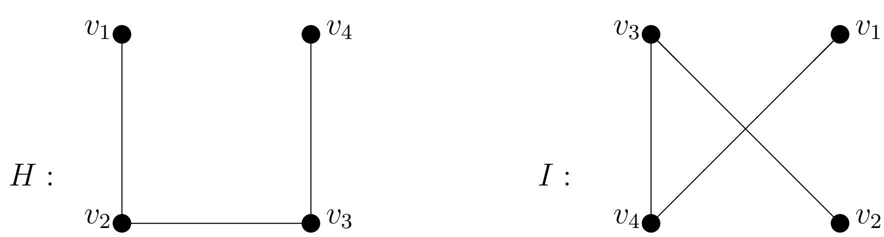

Figure 2: Graphs H and I are Isomorphic

The next concept that is crucial in this thesis is the idea of a subgraph. A *subgraph* is a smaller piece of a larger graph. All elements in a subgraph are elements of the original graph, which includes vertices and edges.

An *induced subgraph* is a subgraph of a larger graph where a set of vertices is deleted, along with their incident edges. (Remember that an edge is incident to a vertex if the edge is touching the vertex.)

Another example of a subgraph is a spanning subgraph. A *spanning subgraph* is a subgraph where all of the vertices are present however the subgraph lacks some edges. Let us look at some examples of subgraphs of G, the graph from Figure 1.

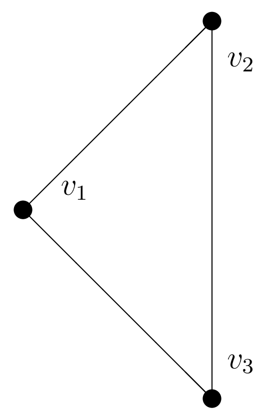

Figure 3: An Induced Subgraph of G

As we can see in Figure 3, we have deleted from G vertex v_4 and all of the incident edges. The graph we are left with is known as an *induced subgraph* of G. Induced subgraphs are an essential definition that we need to understand in order to consider universal graphs.

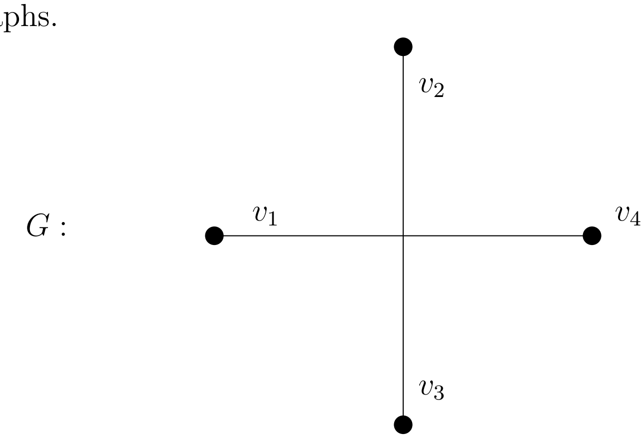

Figure 4: A Spanning Subgraph of G

As we can see in Figure 4, all of the vertices in G are present, however we are missing some of the edges. This is an example of a *spanning subgraph*.

2 EXAMPLES

2.1 Universal Graphs

A *Universal Graph* is a graph on n vertices, $\{v_1, v_2, \ldots, v_n\}$, such that every graph on k vertices occurs as an induced subgraph on the vertices $\{v_{i_1}, v_{i_2}, \ldots, v_{ik}\}$ where $i_1 < i_2 < \cdots < i_k$. We give the reader an example of a universal graph, and then we will carry the thesis over to the idea of and an example of a universal hypergraph.

It is easiest to visualize the universal graph by drawing the graph, and then encoding it by an adjacency matrix that represents the prospective universal graph, and finally locating all of the subgraphs on k vertices. It is important to note that there exist two distinct cases, one where isomorphisms are allowed and a second where we do not allow isomorphisms. We shall give an example of both cases. However, this thesis will primarily focus on the case where isomorphisms are not allowed, hence the restriction $i_1 < i_2 < \cdots < i_k$. above.

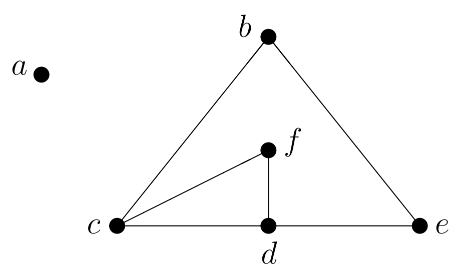

Figure 5: Example of a Universal Graph

The adjacency matrix of the alleged universal graph in Figure 5 looks like:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

One can find the empty graph on three vertices by looking in Rows $1, 2, 6$ and the corresponding columns. It is important to note that the rows and columns need not be next to one another, however when you take the intersection of them you obtain the required matrix.

Another example that we will look at is the one in Figure 6.

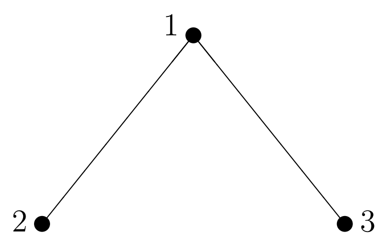

Figure 6: Example of One Subgraph that a Universal Graph Must Contain

We note that it is important to remove the labels on the subgraph and replace them with new labels, such as $1, 2, 3, \ldots$ and relabel the graph in a counterclockwise fashion. This prevents the use of isomorphisms. As the figure shows, we have an edge between the smallest number and both the middle and largest number, however no edge exists from the middle number to the largest. The subgraph has an adjacency matrix given in Figure 7. If we look at the adjacency matrix of the prospective universal graph, we can find the subgraph's matrix on rows and columns $2, 3,$ and 5.

An important subgraph to consider is the subgraph in Figure 8.

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Figure 7: The Adjacency Matrix of Figure 6

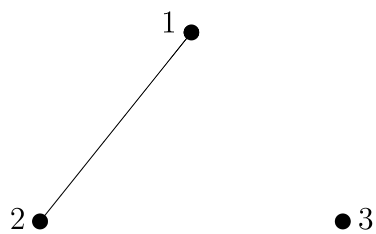

Figure 8: Another Example of a Subgraph of the Universal Graph

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The matrix above is the adjacency matrix for the subgraph as shown in Figure 7.

This is a unique subgraph because it can be found only on vertices b, e, f and is verified by the adjacency matrix. The reader can usually just look at the universal graph and verify that each subgraph exists. However this example does not look correct in its representation. When we look at the adjacency matrix of the subgraph with the correct counterclockwise labeling, we find that the subgraph placement is indeed correct.

For completeness, we have included where you can find all possible subgraphs within the universal graph. These subgraphs are labeled as $A, B, \ldots H$ in Table 1. It is important to look not only at the picture, but also at the adjacency matrices.

We also give an example of a universal graph where we allow isomorphisms within the subgraphs on k vertices. By allowing isomorphisms, our n goes from six to five,

Table 1: Universal Graph Subgraph Locations

Subgraph	Position
A	a, b, f
B	b, e, f
C	a, b, c
D	c, e, f
E	b, c, e
F	b, c, d
G	b, d, e
H	c, d, f

as seen in Figure 9.

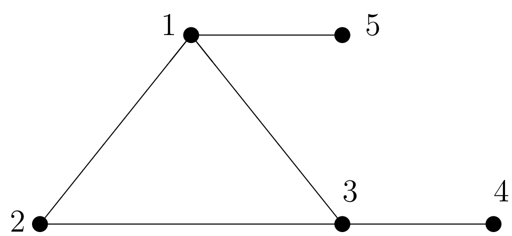

Figure 9: A Universal Graph with Isomorphic Subgraphs

Firstly, we note that the empty subgraph on three vertices is obtained through vertices 2, 4, 5, and the complete subgraph is shown in vertices 1, 2, 3. This takes care of two of the subgraphs, however we are still looking for the remaining six.

We give an example of how one subgraph will cover for a number of subgraphs, in this case three in total, due to isomorphisms. If we look at the induced subgraph on vertices 1, 3, 5, we have the situation in Figure 10.

We note that, due to isomorphisms, we can relabel the subgraph on any fashion that we please. Accordingly, Figure 11 shows us how three possible configurations

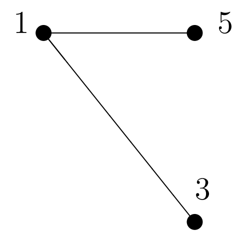

Figure 10: A Subgraph of the Universal Graph

are formed.

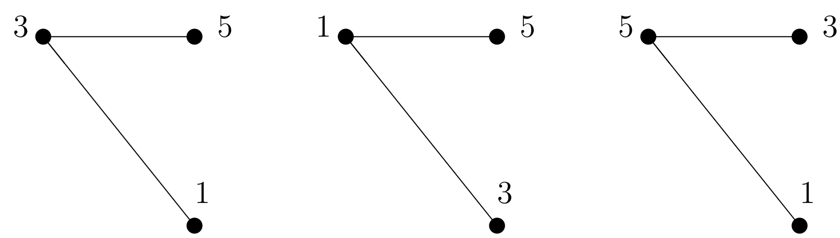

Figure 11: Possible Subgraph Isomorphisms

As we can see, the vertex of degree two is a different label in all three possible subgraphs. Without the allowance of isomorphisms, we would have to locate three distinct subgraphs within the prospective universal graph. By allowing isomorphisms, one can decrease the amount of work that needs to be done. If one is interested, the other class of subgraphs, the class with only one edge on the three vertices can be located on our universal graph in vertices 3, 4, and 5.

2.2 Universal Hypergraphs

A *hypergraph* is different from a graph in that the edge set of a hypergraph is composed of a collection of sets of vertices of arbitrary size. When all sets are of size two, we obtain a regular graph. With that being said, we need at least three vertices from the vertex set to complete one hyperedge. In this thesis, we focus on 3-regular hypergraphs where each hyperedge is created with three vertices from a graph. We can thus identify each edge with a triangle. For example, we can have $V = \{1, 2, 3, 4\}$ and $H = \{\{1, 2, 4\}, \{1, 2, 3\}, \{2, 3, 4\}, \{1, 3, 4\}\}$, which gives us the tetrahedron hypergraph. We illustrate this 3-uniform hypergraph on 4 vertices in Figure 12.

Note that this hypergraph has $2^4 = 16$ subhypergraphs, as we choose each of the four hyperedges (or not). In general, there are $2^{\binom{k}{3}}$ subhypergraphs on k vertices, and $a^{\binom{k}{3}}$ subhypergraphs if edges are colored one of a colors.

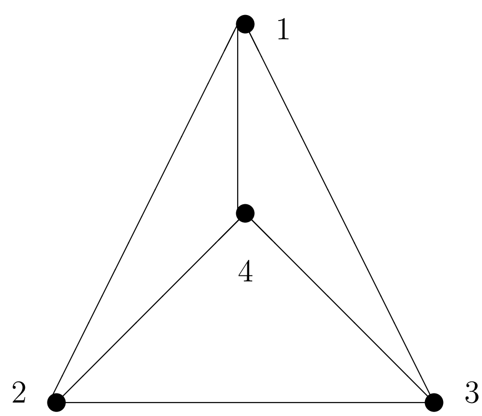

Figure 12: Tetrahedron Hypergraph

How do we draw the adjacency matrix of these subhypergraphs? Let us illustrate for $a = 2$. First we list all of the the zeroes that occur in the matrices due to conditions such as the (i, i, j) position being zero, and we use this as a base to generate the 16 possible tetrahedrons, with the edges either present or absent. If the edges are present, we include a 1 in the matrix, and a 0 is filled in if the edge is absent. We also note that if an edge is present, a 1 is included in six of the positions due to symmetry. For example, the hyperedge {2, 3, 4} will be found in the second column, of the third row, of the fourth matrix, with this labeling throughout. We also point out, due to symmetry, 1's will also be placed in locations (2, 4, 3), (4, 2, 3), (3, 2, 4), (3, 4, 2), (4, 3, 2) , for the (2, 3, 4) edge.

A *universal hypergraph* is a hypergraph on n vertices such that every possible hypergraph on k vertices is present as an induced subgraph on vertices $1 \leq n_1 < n_2 < \cdots < n_k \leq n$. In our quest for a universal hypergraph, we first listed all of the induced zeros in the adjacency matrices which are present because the three regular hypergraph must consist of edges of the form (x, y, z) where x, y, z are distinct.

Figure 13 shows the induced zeros on the Tetrahedron Hypergraph Matrices.

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & & \\ 0 & & 0 & \\ 0 & & & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & & \\ 0 & 0 & 0 & 0 \\ & 0 & 0 & \\ & 0 & & 0 \end{bmatrix} \begin{bmatrix} 0 & & 0 & \\ & 0 & 0 & \\ 0 & 0 & 0 & 0 \\ & & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & & & 0 \\ & 0 & & 0 \\ & & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Figure 13: Induced Zeroes on the Tetrahedron

We include a small example, in Figure 14. This shows the following subtetrahedron, which is missing the back and left hyperedge, hyperedges {1, 2, 4} and {1, 2, 3}.

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Figure 14: Subtetrahderon

To create a universal hypergraph, we first must list all of the possible 3-regular hypergraphs existing on four vertices. They are shown in Figures 15, 16, and 17.

To make a universal hypergraph with $n = 8$, $k = 4$, $a = 2$ we first looked at all of the positions where we could insert our subgraphs. We will begin with a 8 by 8 matrix, and we will have eight of these. We put the zeroes in the places were no edge can exist as in Figure 18.

a

$$\begin{bmatrix} 0&0&0&0\\ 0&0&1&1\\ 0&1&0&1\\ 0&1&1&0 \end{bmatrix} \begin{bmatrix} 0&0&1&1\\ 0&0&0&0\\ 1&0&0&1\\ 1&0&1&0 \end{bmatrix} \begin{bmatrix} 0&1&0&1\\ 1&0&0&1\\ 0&0&0&0\\ 1&1&0&0 \end{bmatrix} \begin{bmatrix} 0&1&1&0\\ 1&0&1&0\\ 1&1&0&0\\ 0&0&0&0 \end{bmatrix}$$

b

$$\begin{bmatrix} 0&0&0&0\\ 0&0&1&1\\ 0&1&0&1\\ 0&1&1&0 \end{bmatrix} \begin{bmatrix} 0&0&1&1\\ 0&0&0&0\\ 1&0&0&0\\ 1&0&0&0 \end{bmatrix} \begin{bmatrix} 0&1&0&1\\ 1&0&0&0\\ 0&0&0&0\\ 1&0&0&0 \end{bmatrix} \begin{bmatrix} 0&1&1&0\\ 1&0&0&0\\ 1&0&0&0\\ 0&0&0&0 \end{bmatrix}$$

c

$$\begin{bmatrix} 0&0&0&0\\ 0&0&1&1\\ 0&1&0&0\\ 0&1&0&0 \end{bmatrix} \begin{bmatrix} 0&0&1&1\\ 0&0&0&0\\ 1&0&0&1\\ 1&0&1&0 \end{bmatrix} \begin{bmatrix} 0&1&0&0\\ 1&0&0&1\\ 0&0&0&0\\ 0&1&0&0 \end{bmatrix} \begin{bmatrix} 0&1&0&0\\ 1&0&1&0\\ 0&1&0&0\\ 0&0&0&0 \end{bmatrix}$$

d

$$\begin{bmatrix} 0&0&0&0\\ 0&0&1&0\\ 0&1&0&1\\ 0&0&1&0 \end{bmatrix} \begin{bmatrix} 0&0&1&0\\ 0&0&0&0\\ 1&0&0&1\\ 0&0&1&0 \end{bmatrix} \begin{bmatrix} 0&1&0&1\\ 1&0&0&1\\ 0&0&0&0\\ 1&1&0&0 \end{bmatrix} \begin{bmatrix} 0&0&1&0\\ 0&0&1&0\\ 1&1&0&0\\ 0&0&0&0 \end{bmatrix}$$

e

$$\begin{bmatrix} 0&0&0&0\\ 0&0&0&1\\ 0&0&0&1\\ 0&1&1&0 \end{bmatrix} \begin{bmatrix} 0&0&0&1\\ 0&0&0&0\\ 0&0&0&1\\ 1&0&1&0 \end{bmatrix} \begin{bmatrix} 0&0&0&1\\ 0&0&0&1\\ 0&0&0&0\\ 1&1&0&0 \end{bmatrix} \begin{bmatrix} 0&1&1&0\\ 1&0&1&0\\ 1&1&0&0\\ 0&0&0&0 \end{bmatrix}$$

Figure 15: The First Five Subtetrahedrons

f

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

g

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

h

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

i

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

j

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

k

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Figure 16: The Next Six Subtetrahedrons

l

$$\begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix} \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&1&0 \end{bmatrix} \begin{bmatrix} 0&0&0&0\\0&0&0&1\\0&0&0&0\\0&1&0&0 \end{bmatrix} \begin{bmatrix} 0&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&0 \end{bmatrix}$$

m

$$\begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&1&0 \end{bmatrix} \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix} \begin{bmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0 \end{bmatrix} \begin{bmatrix} 0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0 \end{bmatrix}$$

n

$$\begin{bmatrix} 0&0&0&0\\0&0&0&1\\0&0&0&0\\0&1&0&0 \end{bmatrix} \begin{bmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0 \end{bmatrix} \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix} \begin{bmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}$$

o

$$\begin{bmatrix} 0&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&0 \end{bmatrix} \begin{bmatrix} 0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0 \end{bmatrix} \begin{bmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix} \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}$$

p

$$\begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix} \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix} \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix} \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}$$

Figure 17: The Last Five Subtetrahedrons

$$
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & & & & & & \\
0 & & 0 & & & & & \\
0 & & & 0 & & & & \\
0 & & & & 0 & & & \\
0 & & & & & 0 & & \\
0 & & & & & & 0 & \\
0 & & & & & & & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & & & & & & \\
 & 0 & & 0 & & & & \\
 & 0 & & & 0 & & & \\
 & 0 & & & & 0 & & \\
 & 0 & & & & & 0 & \\
 & 0 & & & & & & 0
\end{bmatrix}
\begin{bmatrix}
0 & & 0 & & & & & \\
 & 0 & 0 & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & 0 & 0 & & & & \\
 & & 0 & & 0 & & & \\
 & & 0 & & & 0 & & \\
 & & 0 & & & & 0 & \\
 & & 0 & & & & & 0
\end{bmatrix}
$$

$$
\begin{bmatrix}
0 & & & 0 & & & & \\
 & 0 & & 0 & & & & \\
 & & 0 & 0 & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & 0 & 0 & & & \\
 & & & 0 & & 0 & & \\
 & & & 0 & & & 0 & \\
 & & & 0 & & & & 0
\end{bmatrix}
\begin{bmatrix}
0 & & & & 0 & & & \\
 & 0 & & & 0 & & & \\
 & & 0 & & 0 & & & \\
 & & & 0 & 0 & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & 0 & 0 & & \\
 & & & & 0 & & 0 & \\
 & & & & 0 & & & 0
\end{bmatrix}
\begin{bmatrix}
0 & & & & & 0 & & \\
 & 0 & & & & 0 & & \\
 & & 0 & & & 0 & & \\
 & & & 0 & & 0 & & \\
 & & & & 0 & 0 & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & 0 & 0 & \\
 & & & & & 0 & & 0
\end{bmatrix}
$$

$$
\begin{bmatrix}
0 & & & & & & 0 & \\
 & 0 & & & & & 0 & \\
 & & 0 & & & & 0 & \\
 & & & 0 & & & 0 & \\
 & & & & 0 & & 0 & \\
 & & & & & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & & & & & & & 0 \\
 & 0 & & & & & & 0 \\
 & & 0 & & & & & 0 \\
 & & & 0 & & & & 0 \\
 & & & & 0 & & & 0 \\
 & & & & & 0 & & 0 \\
 & & & & & & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

Figure 18: Induced Zeroes of the Universal Graph

The next step was to insert all of the possible subgraphs into the 8 by 8 matrix, one by one. We will show how and where we inserted the tetrahedron in the example before. Remember it is missing hyperedges (1, 2, 4) and (1, 2, 3). We select rows (2, 3, 6, 8), columns (2, 3, 6, 8) and matrices (2, 3, 6, 8) to place this example. This is shown in Figure 19.

$$
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & & & & & & \\
0 & & 0 & & & & & \\
0 & & & 0 & & & & \\
0 & & & & 0 & & & \\
0 & & & & & 0 & & \\
0 & & & & & & 0 & \\
0 & & & & & & & 0
\end{bmatrix}
\begin{bmatrix}
0 & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & & & 0 & & 0 \\
 & 0 & & 0 & & & & \\
 & 0 & & & 0 & & & \\
 & 0 & 0 & & & 0 & & 1 \\
 & 0 & & & & & 0 & \\
 & 0 & 0 & & & 1 & & 0
\end{bmatrix}
\begin{bmatrix}
0 & & 0 & & & & & \\
 & 0 & 0 & & & 0 & & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & 0 & 0 & & & & \\
 & & 0 & & 0 & & & \\
 & 0 & 0 & & & 0 & & 1 \\
 & & 0 & & & & 0 & \\
 & 0 & 0 & & & 1 & & 0
\end{bmatrix}
$$

$$
\begin{bmatrix}
0 & & & 0 & & & & \\
 & 0 & & 0 & & & & \\
 & & 0 & 0 & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & 0 & 0 & & & \\
 & & & 0 & & 0 & & \\
 & & & 0 & & & 0 & \\
 & & & 0 & & & & 0
\end{bmatrix}
\begin{bmatrix}
0 & & & & 0 & & & \\
 & 0 & & & 0 & & & \\
 & & 0 & & 0 & & & \\
 & & & 0 & 0 & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & 0 & 0 & & \\
 & & & & 0 & & 0 & \\
 & & & & 0 & & & 0
\end{bmatrix}
\begin{bmatrix}
0 & & & & & 0 & & \\
 & 0 & 0 & & & 0 & & 1 \\
 & 0 & 0 & & & 0 & & 1 \\
 & & & 0 & & 0 & & \\
 & & & & 0 & 0 & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & 0 & 0 & \\
 & 1 & 1 & & & 0 & & 0
\end{bmatrix}
$$

$$
\begin{bmatrix}
0 & & & & & & 0 & \\
 & 0 & & & & & 0 & \\
 & & 0 & & & & 0 & \\
 & & & 0 & & & 0 & \\
 & & & & 0 & & 0 & \\
 & & & & & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & & & & & & & 0 \\
 & 0 & 0 & & & 1 & & 0 \\
 & 0 & 0 & & & 1 & & 0 \\
 & & & 0 & & & & 0 \\
 & & & & 0 & & & 0 \\
 & 1 & 1 & & & 0 & & 0 \\
 & & & & & & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

Figure 19: Placement Within the Universal Graph

We have placed tetrahedron h into our example on rows (2, 3, 6, 8) columns (2, 3, 6, 8) and in matrices (2, 3, 6, 8). We continued to place all the adjacency matrices of the tetrahedron exhaustively. With this in mind, we shall show the following adjacency matrix, exhibited in Figure 20, does give a universal hypergraph on 8 vertices.

$$\begin{bmatrix} 0&0&0&0&0&0&0&0\\ 0&0&1&1&0&1&1&1\\ 0&1&0&1&0&0&0&0\\ 0&1&1&0&0&1&0&0\\ 0&0&0&0&0&1&1&0\\ 0&1&0&1&1&0&0&1\\ 0&1&0&0&1&0&0&0\\ 0&1&0&0&0&1&0&0 \end{bmatrix} \begin{bmatrix} 0&0&1&1&0&0&1&1\\ 0&0&0&0&0&0&0&0\\ 1&0&0&1&1&0&0&0\\ 1&0&1&0&0&&&\\ 0&0&1&0&0&0&0&1\\ 0&0&0&&0&0&0&1\\ 1&0&0&&0&0&0&\\ 1&0&0&&1&1&&0 \end{bmatrix} \begin{bmatrix} 0&1&0&1&0&1&0&0\\ 1&0&0&1&1&0&0&0\\ 0&0&0&0&0&0&0&0\\ 1&1&0&0&1&0&&\\ 0&1&0&1&0&1&&0\\ 1&0&0&0&1&0&0&1\\ 0&0&0&&&0&0&\\ 0&0&0&&0&1&&0 \end{bmatrix}$$

$$\begin{bmatrix} 0&1&1&0&0&1&0&0\\ 1&0&1&0&0&&&\\ 1&1&0&0&1&0&&\\ 0&0&0&0&0&0&0&0\\ 0&0&1&0&0&1&0&\\ 1&&0&0&1&0&0&0\\ 0&&&0&0&0&0&\\ 0&&&0&&0&&0 \end{bmatrix} \begin{bmatrix} 0&0&0&0&0&1&1&0\\ 0&0&1&0&0&0&0&1\\ 0&1&0&1&0&1&&0\\ 0&0&1&0&0&1&0&\\ 0&0&0&0&0&0&0&0\\ 1&0&1&1&0&0&1&1\\ 1&0&&0&0&1&0&1\\ 0&1&0&&0&1&1&0 \end{bmatrix} \begin{bmatrix} 0&1&1&1&1&0&0&1\\ 1&0&0&&0&0&0&1\\ 1&0&0&0&1&0&0&1\\ 1&&0&0&1&0&0&0\\ 1&0&1&1&0&0&1&1\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 1&1&1&0&1&0&0&0 \end{bmatrix}$$

$$\begin{bmatrix} 0&1&0&0&1&0&0&0\\ 1&0&0&&0&0&0&\\ 0&0&0&&&0&0&\\ 0&&&0&0&0&0&\\ 1&0&&0&0&1&0&1\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&&&&1&0&0&0 \end{bmatrix} \begin{bmatrix} 0&1&0&0&0&1&0&0\\ 1&0&0&&1&1&&0\\ 0&0&0&&0&1&&0\\ 0&&&0&&0&&0\\ 0&1&0&&0&1&1&0\\ 1&1&1&0&1&0&0&0\\ 0&&&&1&0&0&0\\ 0&0&0&0&0&0&0&0 \end{bmatrix}$$

Figure 20: Universal Hypergraph on 8 Vertices

As we can see, there exists a signifigant amount of blank spaces within the adjacency matrix. This is because we can possibly create a universal hypergraph on a smaller amount of vertices namely $n = 7$. It can be shown that $n = 6$ is impossible. However, for an example, it was best to create a universal hypergraph on 8 vertices. One can fill the blank entries in with either one's or zero's. Note if you were to place a 1 in position (6, 7, 2), the 1 would be induced on (7, 6, 2), (6, 2, 7), (7, 2, 6), (2, 6, 7), and (2, 7, 6) as well. In all, there exist 60 blank spaces on the 8 matrices, so this correlates to 10 hyperedges. Table 2 shows where the subgraphs have been placed.

Table 2: Tetrahedron Subgraph Locations

Subgraph	Position
a	1, 2, 3, 4
b	5, 6, 7, 8
c	2, 3, 4, 5
d	3, 4, 5, 6
e	1, 4, 5, 6
f	1, 2, 5, 6
g	1, 2, 5, 8
h	2, 3, 6, 8
i	4, 5, 6, 7
j	1, 4, 6, 8
k	1, 2, 3, 7
l	2, 5, 6, 7
m	1, 4, 5, 7
n	1, 6, 7, 8
o	1, 3, 6, 7
p	1, 3, 5, 8

3 BOUNDS

3.1 Pigeonhole Bound on n

Lemma 3.1 *Let n be the least integer for which an induced 3-uniform k-hypergraph with a colors exists. Then*

$$n \geq \frac{k}{e} a^{\frac{(k-1)(k-2)}{6}}.$$

Proof:

There are $\binom{n}{k}$ locations where an induced hypergraph on k vertices can occur. Assume these all give different hypergraphs. If $\binom{n}{k} < a^{\binom{k}{3}}$, then the number of locations would be smaller than the number of a- colored 3-uniform hypergraphs on k vertices. Thus $\binom{n}{k} \geq a^{\binom{k}{3}}$. Since

$$\binom{n}{k} \leq \left(\frac{ne}{k}\right)^k,$$

we have

$$\left(\frac{ne}{k}\right)^k \geq a^{\frac{k(k-1)(k-2)}{6}},$$

or

$$n \geq \frac{k}{e} a^{\frac{(k-1)(k-2)}{6}},$$

as desired. □

3.2 Universal Hypergraph Bounds

Theorem 3.2 *The smallest value of* n *such that a hypergraph has a high probability of becoming universal is*

$$\frac{k}{e}a^{\frac{(k-1)(k-2)}{6}} \leq n \leq \frac{k}{e}a^{\frac{(k-1)(k-2)}{6}}(1+\epsilon).$$

where ϵ *is fixed yet positive. We will prove a series of lemmas before beginning the proof of Theorem 3.2.*

We have already shown the lower bound in Lemma 3.1, so now we begin on the upper bound. In order to obtain bounds on how large our n needs to be in order for the graph in question to be universal, we define a function $\phi(r)$. This will be related to our later use of Suen's Inequality. Let, for $3 \leq r \leq k-1$,

$$\phi(r) = \binom{k}{r}\binom{n}{k-r}a^{\binom{r}{3}}.$$

Next, we take this function and look at $\phi(r+1)$. The ratio of two consecutive values of the function, which we will call $\pi(r)$, will determine the "activity" of the graph. So, let

$$\pi(r) = \frac{\phi(r+1)}{\phi(r)}.$$

If $\pi(r) > 1$ we conclude that the function is increasing, which will further our analysis. Notice that,

$$\pi(r) = \frac{\binom{k}{r+1}\binom{n}{k-r-1}a^{\binom{r+1}{3}}}{\binom{k}{r}\binom{n}{k-r}a^{\binom{r}{3}}}$$

which reduces to

$$\pi(r) = \frac{(k-r)^2 a^{\frac{(r)(r-1)}{2}}}{(r+1)(n-k+r+1)}.$$

The behavior of π is studied in the next lemma.

Lemma 3.3

$$\frac{\pi'(r)}{\pi(r)} \geq 0, \text{thus } \pi\text{(r) is increasing}$$

Proof: Note

$$\frac{\pi'(r)}{\pi(r)} = \frac{-2}{k-r} + \frac{2r-1}{2}\ln(a) - \frac{1}{r+1} - \frac{1}{n-k+r+1}.$$

So we need to show that

$$\frac{2r-1}{2}\ln(a) \geq \frac{2}{k-r} + \frac{1}{r+1} + \frac{1}{n-k+r+1}.$$

Since we know that our a has to be at least 2, we can replace the $\ln(a)$ with $\ln(2)$ which we know is approximately 0.69. Note that our last term, $\frac{1}{n-k+r+1} \leq \epsilon$ where we can assume that $\epsilon \leq 0.09$. Since we know that $\frac{2r-1}{2} > \frac{2r-2}{2}$ we also make this slight change. We thus need to achieve the following equation,

$$\frac{2r-2}{2}(0.6) \geq \frac{2}{k-r} + \frac{1}{r+1}$$

or the weaker

$$(r-1)(0.6) \geq \frac{2}{k-r} + \frac{2}{r+1}.$$

We then split the inequality into two pieces, $(r-1)(0.3) \geq \frac{2}{r+1}$ and $(r-1)(0.3) \geq \frac{1}{k-r}$. Looking at the first piece,

$$(r-1)(0.3) \geq \frac{2}{r+1}$$

if and only if

$$r^2 - 1 \geq \frac{2}{.3}.$$

This leads us to

$$r^2 \geq \frac{23}{3},$$

and since we know that r is at least 3, we have solved the first half of the inequality.

Next we look at

$$(r-1)(0.3) \geq \frac{2}{k-r},$$

which holds if

$$(r-1)(k-r) \geq \frac{20}{3}.$$

As well as if

$$\alpha(r) := kr - k + r - r^2 \geq \frac{20}{3}.$$

We then take the derivative of α with respect to r in order to conclude where the minimum of the function is achieved. Setting the derivative of the function equal to zero yields

$$k + 1 - 2r = 0.$$

Therefore

$$r = \frac{k}{2}$$

is a critical point. We then check both endpoints, $r = 3$ and $r = k - 1$ and we verify that $2(k-3) \geq \frac{20}{3}$ and $(k-2)(1) \geq \frac{20}{3}$ since k is sufficiently large. At $r = \frac{k}{2}$,

$$\alpha\left(\frac{k}{2}\right) = \frac{k^2}{4} - \frac{k}{2} \geq \frac{20}{3}$$

if k is sufficiently large. Thus, the inequality holds for all r, hence $\pi(r)$ is increasing.

□

The next two lemmas enable us to identify the maximum value of ϕ.

Lemma 3.4

$$\frac{\phi(4)}{\phi(3)} \leq 1.$$

Proof:

$$\begin{aligned}
\frac{\phi(4)}{\phi(3)} &= \frac{(k-3)^2}{4(n-k+4)} a^{\frac{1}{2}(6)} \\
&= \frac{(k^2-6k+9)}{4n-4k+16} a^3 \\
&\leq \frac{(k^2-6k+9)}{\frac{4ka^{\frac{k^2-3k+2}{6}}}{e} - 4k + 16} a^3 \\
&\leq 1 \text{ for k large enough. } \square
\end{aligned}$$

Lemma 3.5

$$\frac{\phi(k-1)}{\phi(k-2)} \geq 1.$$

Proof:

$$\begin{aligned}
\frac{\phi(k-1)}{\phi(k-2)} &= \frac{(k-r)^2}{(r+1)(n-k+r+1)} a^{\frac{r(r-1)}{2}} \\
&= \frac{4}{n(k-1)} a^{\frac{(k-2)(k-1)}{2}}.
\end{aligned}$$

Since

$$n \leq \frac{4a^{\frac{k^2}{2}}}{k},$$

it follows that

$$\frac{\phi(k-1)}{\phi(k-2)} \geq 1. \ \square$$

For the next lemma we are interested in calculating the endpoints of the function, $\phi(r)$. In this lemma, we introduce a new "generic" function $o(1)$, where $o(1) \to 0$ as $k \to \infty$.

Lemma 3.6

$$\phi(3) \geq \phi(k-1).$$

Proof:

$$\begin{aligned}
\binom{k}{3}\binom{n}{k-3}a^{\binom{3}{3}} &\geq \binom{k}{k-1}\binom{n}{1}a^{\binom{k-1}{3}} \text{ if} \\
\frac{k^3}{6}\binom{n}{k-3}a &\geq kna^{\binom{k-1}{3}}(1+o(1)) \text{ if and only if} \\
\frac{\binom{n}{k-3}}{n} &\geq \frac{ka^{\binom{k-1}{3}}}{a^{\frac{k^3}{6}}}(1+o(1)).
\end{aligned}$$

We know that $\binom{n}{k-3} \leq (\frac{ne}{k-3})^{k-3}$, so the above holds, ignoring the $(1+o(1))$ terms, which vary from one line to the other, if

$$\frac{\frac{(ne)^{k-3}}{(k-3)^{k-3}}}{n} \geq \frac{ka^{\binom{k-1}{3}}}{a^{\frac{k^3}{6}}}, \text{ or if} \tag{1}$$

$$n^{k-4} \geq \frac{6a^{\frac{(k-1)(k-2)(k-3)}{6}}(k-3)^k}{ak^2e^{(k-3)}(k-3)^3}. \tag{2}$$

Multiply the right side of the inequality found in equation 2 by $\frac{k^k}{k^k}$. Then notice that

$$\left(\frac{k-3}{k}\right)^k = \left(1-\frac{3}{k}\right)^k \leq e^{-3}$$

So, the above is true if

$$n^{k-4} \geq \frac{6a^{\frac{(k-1)(k-2)(k-3)}{6}}(k)^{k-2}}{ae^k(k-3)^3}.$$

By taking the $(k-4)$th root of each side, we are left with

$$n \geq \frac{k^{\frac{k-2}{k-4}}}{e} a^{\frac{(k-1)(k-2)(k-3)}{6(k-4)}} (1 + o(1))$$

or

$$n \geq \frac{k}{e} a^{\frac{(k-1)(k-2)}{6}} (1 + o(1)).$$

We know from previous results that n should be

$$n \geq \frac{k}{e} a^{\frac{(k-1)(k-2)}{6}}$$

so by replacing n with our result, we conclude that $\phi(3) \geq \phi(k-1)$ if n is only slightly larger. Thus $\phi(x)$ is first decreasing and then increasing and thus by Lemma 3.5, $\phi(3)$ is the maximum of $\phi(r)$ for $3 \leq r \leq k-1$. $\square$

Due to the previous two lemmas, we conclude that the ratio, $\pi(x)$ is below one for $r \leq r_0$ and $\pi(x) \geq 1$ for $r > r_0$.

3.3 Suen's Inequality

In this section, we shall use a probability model where we take a $n \times n \times n$ array, insert the induced zeros, divide the result into a simplex and independently place one of a colors into each position of the simplex with probability $\frac{1}{a}$. This will validate whether our graph turns into a universal graph and is missing zero k-subhypergraphs.

First we note that $P(N \text{ is not a universal hypergraph}) = P(X \geq 1)$ where X is the number of missing k subhypergraphs. Next, by Markov's Inequality this is at most $E(X)$ which is equal to $E(\sum_{j=1}^{a^{\binom{k}{3}}} I_j)$, where $I_j = 1$ if and only if the jth hypergraph is missing ($I_j = 0$ otherwise). If this sum goes to 0, we conclude that we indeed do have a universal hypergraph with high probabilty.

Next, we have that this sum is equal to $\sum_{j=1}^{a^{\binom{k}{3}}} P(\cap_{l=1}^{\binom{n}{k}})E_{jl}$, where E_{jl} is the event that the jth hypergraph is not present in location l . This sum is equivalent to $\sum_{j=1}^{a^{\binom{k}{3}}} P(Z_j = 0)$ where Z_j is the number of locations at which j occurs.

In a paper written by Janson, [10], we find the following inequality:

$$\sum P(Z_j = 0) \leq \sum_{j=1}^{a^{\binom{k}{3}}} e^{-\lambda_j + \Delta_j e^{2\delta_j}}$$

where

$$\lambda_j = E\left(\sum_{l=1}^{\binom{n}{k}} I_l\right)$$

thus λ_j is the expected number of occurences of subhypergraph j. Note that

$$\Delta_j = \sum_{l} \sum_{l' \sim l} P(j \text{ occurs in both locations } l \text{ and } l'),$$

$$\Delta_j \leq \sum_{l} \sum_{l' \sim l} P(K_{k,3} \text{ occurs in both locations } l \text{ and } l') \text{ and}$$

$$\delta_j = \sup_l \sum_{l' \sim l} P(I_{l'} = 1).$$

Our goal is to make the right hand side of this inequality go to zero. If we can prove this, then we know that our graph is universal.

Lemma 3.7 *For any j,*

$$\lambda_j = \frac{\binom{n}{k}}{a^{\binom{k}{3}}}.$$

Proof:

$$\begin{aligned}
\lambda_j &= E(Z_j) \\
&= E(\sum_{l=1}^{\binom{n}{k}} I_l) \\
&= \binom{n}{k} P(I_l = 1) \\
&= \binom{n}{k} \frac{1}{a^{\binom{k}{3}}}. \quad \square
\end{aligned}$$

Before we state Lemma 3.7, we define a new symbol, $K_{k,3}$ which is the complete 3-uniform hypergraph on k vertices.

Lemma 3.8 *For any j,*

$$\Delta_j \leq \lambda * \frac{k^4}{6} * \frac{\binom{n}{k-3}}{a^{\binom{k}{3}-1}}$$

Proof:

$$\begin{aligned}
\Delta_j &= \sum_l \sum_{l' \sim l} P(j \text{ occurs in both locations } l \text{ and } l') \\
&\leq \sum_l \sum_{l' \sim l} P(K_{k,3} \text{ occurs in both locations } l \text{ and } l') \\
&\leq \binom{n}{k} \sum_{r=3}^{k-1} \binom{k}{r} \binom{n}{k-r} \frac{1}{a^{\binom{k}{3}+\binom{k}{3}-\binom{r}{3}}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\binom{n}{k}}{a^{\frac{k(k-1)(k-2)}{3}}} \sum \binom{k}{r}\binom{n}{k-r} a^{\binom{r}{3}} \\
&= \frac{\binom{n}{k}}{a^{2\binom{k}{3}}} \sum \phi(r) \\
&\leq \frac{k\binom{n}{k}}{a^{2\binom{k}{3}}} \phi(3) \\
&\leq \frac{k\binom{n}{k}}{a^{2\binom{k}{3}}} \binom{k}{3}\binom{n}{k-3} a \\
&\leq \frac{k^4\binom{n}{k}\binom{n}{k-3}}{6a^{2\binom{k}{3}-1}} \\
&= \lambda * \frac{k^4}{6} * \frac{\binom{n}{k-3}}{a^{\binom{k}{3}-1}}. \ \square
\end{aligned}
$$

Lemma 3.9 *For any j,*

$$\delta_j \leq \lambda \frac{k^6}{3n^3}.$$

Proof:

$$
\begin{aligned}
\delta_j &= \sum_{l' \sim l} P(I_{l'} = 1) \\
&\leq \sum_{r=3}^{k-1} \frac{\binom{k}{r}\binom{n-k}{k-r}}{a^{\binom{k}{3}}} \\
&\leq \sum_{r=3} \frac{\binom{k}{r}\binom{n-k}{k-r}}{a^{\binom{k}{3}}} * \frac{\binom{n}{k}}{\binom{n}{k}} \\
&\leq \frac{\lambda\binom{k}{3}\binom{n-3}{k-3}}{\binom{n}{k}} \\
&\leq \lambda \frac{k^3}{6} \frac{\binom{n}{k-3}}{\binom{n}{k}} \\
&\leq \lambda \frac{k^6}{3n^3}. \ \square
\end{aligned}
$$

The next step is to let $n = \frac{ka^{\frac{(k-1)(k-2)}{6}}}{e}(1+\epsilon)$ where $\epsilon > 0$ is arbitrarily small but fixed. We also know that for this choice of n

$$
\begin{aligned}
\lambda &= \frac{\binom{n}{k}}{a^{\binom{k}{3}}} \\
&\leq (\frac{ne}{k})^k * \frac{1}{a^{\binom{k}{3}}} \\
&= \frac{a^{\frac{k(k-1)(k-2)}{6}}}{a^{\binom{k}{3}}} * (1+\epsilon)^k \\
&= (1+\epsilon)^k.
\end{aligned}
$$

Lemma 3.10

$$e^{2\delta_j} \leq e^{\frac{2\lambda k^6}{3n^3}} \leq 2$$

Proof:

$$
\begin{aligned}
e^{\frac{2\lambda k^6}{3n^3}} &= e^{\frac{2(1+\epsilon)^k k^6}{3n^3}} \\
&\leq \exp\left\{\frac{2(1+\epsilon)^k(k^6)}{\frac{3k^3 a^{\frac{(k-1)(k-2)}{2}}}{e^3}}\right\} \\
&\leq \exp\left\{\frac{2e^3}{3}\frac{(1+\epsilon)^k k^3}{a^{\frac{(k-1)(k-2)}{2}}}\right\} \\
&\leq 2 \quad \square
\end{aligned}
$$

We now begin the proof of Theorem 3.2. By Suen's Inequality, we have that the $P(\text{Not universal}) \leq \mathrm{a}^{\binom{\mathrm{k}}{3}} \exp\left\{-\lambda + \frac{\lambda \mathrm{ak}^4}{3} * \frac{\binom{\mathrm{n}}{\mathrm{k}-3}}{\mathrm{a}^{\binom{\mathrm{k}}{3}}}\right\}$ So let us look at the last part of the equation, from after the second lambda. We have

$$\frac{ak^4}{3}\frac{\binom{n}{k-3}}{a^{\binom{k}{3}}} \leq \frac{ak^4}{3}(\frac{ne}{k-3})^{(k-3)} * \frac{1}{a^{\binom{k}{3}}}$$

So we plug in our value of n where

$$
\begin{aligned}
n &= \frac{k}{e} a^{\frac{(k-1)(k-2)}{6}} (1+\epsilon). \\
\frac{ne}{k-3} &= \frac{k}{k-3} a^{\frac{(k-1)(k-2}{6}} (1+\epsilon), \text{ so} \\
\frac{ak^4 \binom{n}{k-3}}{3a^{\binom{k}{3}}} &\leq \frac{ak^4}{3} \left(\frac{k}{k-3} a^{\frac{(k-1)(k-2)}{6}} (1+\epsilon) \right)^{k-3} \frac{1}{a^{\frac{k(k-1)(k-2)}{6}}} \\
&= \frac{ak^4}{3} \left(\frac{1}{1-\frac{3}{k}} \right)^{k-3} (1+\epsilon)^{k-3} * \frac{1}{a^{\frac{k(k-1)(k-2)}{6} - \frac{(k-1)(k-2)(k-3)}{6}}} \\
&= \frac{ak^4}{3} \left(\frac{1}{1-\frac{3}{k}} \right)^{k-3} (1+\epsilon)^{k-3} * \frac{1}{a^{\frac{(k-1)(k-2)}{6}(k-(k-3))}} \\
&\leq \frac{ak^4}{3} \left(\frac{1}{1-\frac{3}{k}} \right)^{k} (1+\epsilon)^{k} * \frac{1}{a^{\frac{(k-1)(k-2)}{2}}} \\
&\leq \frac{ak^4}{3} e^3 (1+\epsilon)^k * \frac{1}{a^{\frac{(k-1)(k-2)}{2}}}.
\end{aligned}
$$

We can conclude this is at most $\frac{1}{2}$. So we are finally we are left with

$$
\begin{aligned}
P(\text{Not universal}) &\leq a^{\binom{k}{3}} e^{-\lambda + \frac{\lambda}{2}} \\
&\leq a^{\binom{k}{3}} e^{\frac{-\lambda}{2}}.
\end{aligned}
$$

If we let $\epsilon = \frac{1}{2}$ and recall that $\lambda = (1+\epsilon)^k$. We then have that

$$
\begin{aligned}
P(\text{Not universal}) &\leq a^{\binom{k}{3}} e^{-(\frac{3}{2})^k * (\frac{1}{2})} \\
&\leq a^{\frac{k^3}{6}} e^{-(\frac{3}{2})^k * (\frac{1}{2})} \\
&\leq e^{\ln(a)\frac{k^3}{6} - (\frac{3}{2})^k * \frac{1}{2}} \\
&\to 0.
\end{aligned}
$$

We know that $\epsilon > 0$ however it is fixed. So regardless of ϵ

$$
\begin{aligned}
P(\text{Not universal}) &\leq a^{\frac{k^3}{6}} e^{-(1+\epsilon)^k * \frac{1}{2}} \\
&\leq e^{\frac{ln(a)k^3}{6} - (1+\epsilon)^k * \frac{1}{2}} \\
&\to 0. \quad \square
\end{aligned}
$$

This completes the proof of Theorem 3.2.

4 CONCLUSION

First, we were able to construct both a universal graph and a 3-regular universal hypergraph. Through a number of lemmas we were able to prove that if $\frac{k}{e}a^{\frac{(k-1)(k-2)}{6}} \leq n \leq \frac{k}{e}a^{\frac{(k-1)(k-2)}{6}}(1+\epsilon)$, then the probablity that a universal hypergraph exists is high. This thesis has made progress in exploring the existence of a universal hypergraph.

There is much more to be studied. For example, we wish to prove an analogous result on a 4-regular hypergraph, 5-regular hypergraph, and all the way up to a d-regular hypergraph. This may be straightforward or challenging; it remains to be seen which will be the case.

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VITA

MICHAEL R. DEREN

Education:	B.S. Mathematics, East Tennessee State University, Johnson City, Tennessee, 2009
	M.S. Mathematical Sciences, East Tennessee State University, Johnson City, Tennessee, 2011
Professional Experience:	Graduate Assistant, East Tennessee State University, Johnson City, Tennessee, 2009–2011