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Decompositions, Packings, and Coverings of Complete Directed Gaphs with a 3-Circuit and a Pendent Arc.

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Decompositions, Packings, and Coverings of Complete Directed Graphs with a 3-Circuit and a Pendent Arc

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by

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ABSTRACT

Decompositions, Packings, and Coverings of Complete Directed Graphs with a 3-Circuit and a Pendent Arc

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In the study of Graph theory, there are eight orientations of the complete graph on three vertices with a pendant edge, $K_3 \cup \{e\}$. Two of these are the 3-circuit with a pendant arc and the other six are transitive triples with a pendant arc. Necessary and sufficient conditions are given for decompositions, packings, and coverings of the complete digraph with the two 3-circuit with a pendant arc orientations.

DEDICATION

In the blessed memory of my late Mum, Winifred Naboh. May her soul rest in peace.

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Thank you.

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1 INTRODUCTION

Graph theory is an interesting area in the study of combinatorial mathematics. In this area of mathematics, we model objects as a set of points (vertices or nodes) and the relation between them as edges (arcs).

For a clear view and understanding of this thesis, we start by giving a list of definitions. A *graph* G is a finite nonempty set of objects called *vertices* (the singular is vertex) together with a (possibly empty) set of unordered pairs of distinct vertices of G called *edges*. The vertex set of G is denoted by $V(G)$, while the edge set is denoted by $E(G)$. A graph G is called *simple* if no two edges are equal as sets. In other words, a graph G is *simple* if at most one edge connects any two vertices (nodes). In the field of graph theory a *complete graph* is a simple graph where an edge connects every pair of vertices. In other words, a graph G is said to be *complete* if every two *vertices* (nodes) are *adjacent*. With a non-empty graph G, we can generate a directed graph D by assigning a direction (or by orienting) each edge of G. D is called the *orientation* of G. A directed graph D is thus a finite non-empty set of points called vertices, together with a set of ordered pairs of distinct vertices of D, called *arcs* or *directed edges*. If $a = [x, y]$ is an arc of a digraph D, then a is said to *join* x to y and a is *incident to* y and a is *incident from* x, while y is *incident from* a and x is *incident to* a. In graph theory, we say that x and y are adjacent. A *complete digraph* D_v of v vertices can be obtained from a complete graph G by replacing each edge of a complete graph with two arcs of opposite orientation as in Figure 1. Clearly, we see that in a complete digraph each pair of vertices are connected. In a directed graph, we define the *out-degree*, $od(u)$, of vertex u in D as the number of vertices of D that are *adjacent from* u, i.e., $od(u) = |N_o(u)|$ where the open neighborhood $N_o(u) = \{x \in V(D)/x$ is adjacent from $u\}$. The *in-degree*, $id(u)$ of vertex u in D refers to the number of vertices of D that are adjacent from u, i.e., $id(u) = |N_i(u)|$ where $N_i(u) = \{x \in V(D)/x$ is adjacent to u $\}$. By *total degree* of vertex u, we shall mean $od(u) + id(u)$.

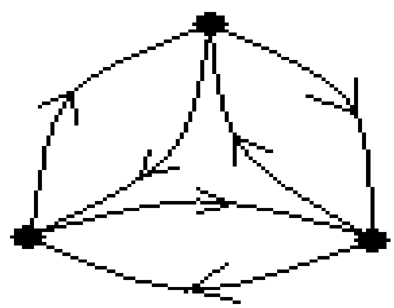

Figure 1: A Complete Digraph on 3 Vertices.

A *decomposition* of a *simple graph* with *isomorphic* copies of graph g is a set $\{g_1, g_2, ..., g_n\}$ where $g_i \cong g$ and $V(g_i) \subset V(G)$ for all i and $E(g_i) \cap E(g_j) = \emptyset$ for $i \neq j$ and the union over all g_i's gives the graph G. The g_i's are called blocks of the decomposition while $V(G)$ is the *vertex set* of G and $E(G)$ is the *edge set.* By replacing the *edge set* by *arc set* in the above definition, a similar definition can be obtained for the *decomposition* of *digraphs.* A graph (digraph) decomposition into isomorphic copies of a graph (respectively digraph) on three vertices is equivalent to a *triple system.* A K_3 decomposition of a complete graph on v vertices, K_v, is called a *Steiner Triple system*, $STS(v)$, which is known to exist if and only if $v \equiv 1$ or 3 (mod 6) [16].

In general, when a complete graph (digraph) is decomposed into graphs (respectively digraphs) on 3 vertices, the resulting structure is called a *triple system*. There are two orientations of K_3, namely the $3-$ *circuit* and the *transitive triple*. The following directed graphs below are the two orientations of a K_3. These can be labelled as C_3 for a 3-circuit and T for the transitive triple.

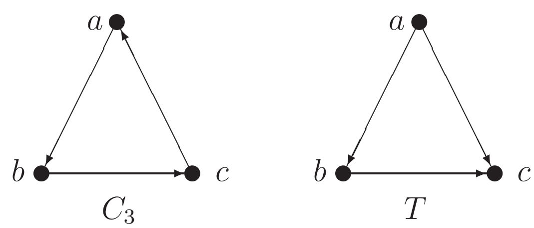

Figure 2: 3-Circuit and Transitive Triple

A decomposition of a complete digraph, denoted D_v into isomorphic copies of the 3-circuit is equivalent to a *Mendelson Triple System* of order v, denoted $MTS(v)$ and it exists if and only if $v \equiv 0$ or 1 (mod 3), $v \neq 6$ [14].

A *directed triple system* is equivalent to a transitive triple T (see Figure 2) decomposition of D_v and exists if and only if $v \equiv 0$ or 1 (mod 3) [9]. Also of relevance to our results are decompositions of K_v into copies of K_3 with a pendant edge (the graph L of Figure 4). Such decompositions exist if and only if $v \equiv 0$ or 1 (mod 8) [1]. For example, D_3 can be decomposed into two copies of the 3-circuit as shown on Figure 3.

Giving a $3-$*circuit* orientation to the K_3 subgraph in the L and the two different orientations on the pendent arc is the concentration of this work. We will decompose complete digraphs with these two orientations. This is illustrated by Figure 4. Firstly,

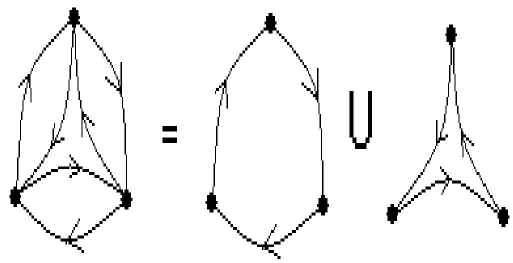

Figure 3: Decomposition of D_3 into 2 Copies of 3-Circuit.

m_1 has a vertex with out-degree equal to 2 and in-degree 1, two others with out-degree 1 and in-degree 1 and the last one with out-degree 0 and in-degree 1. Secondly, m_2 has a vertex with out-degree equal to 1 and in-degree 2 , two others with out-degree 1 and in-degree 1 and the last one with out-degree 1 and in-degree 0.

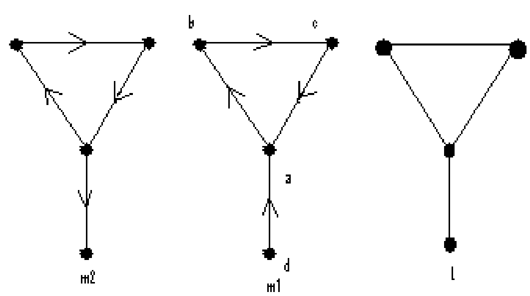

Figure 4: m_2, m_1 and a Lollipop L.

Mendelsohn, 1971, proved that a complete directed graph, D_v admits a decomposition into isomorphic copies of 3-circuit if and only if $v \equiv 0$ or 1 (mod 3), $v \neq 6$ [14]. In this work, we shall state and prove a similar results, viz.: a complete digraph admits a decomposition into isomorphic copies of 3-circuit with a pendent arc if and only if $v \equiv 0$ or 1 (mod 4). In this case, the 3-circuit with a pendent arc is either m_1 or m_2. Sufficiency is established in this case by using a well known method called *the difference method*, which involves direct constructions.

If a decomposition of D_v does not exist, then one question to address is, "*can we efficiently remove isomorphic copies of the 3-circuit with a pendent arc from D_v such that the number of arcs remaining is a minimum or such that the number of arcs repeated is a minimum?*" These concepts are called *packings* and *coverings*, respectively, of the complete digraph on v vertices and we talk of "*the packing problem*" as well as "*the covering problem*" for complete digraphs. The remaining arcs are often referred to as *the leave* of the packing while the repeated arcs are called *the padding* of the covering. We shall consider the packing problem for D_v with isomorphic copies of m_1 and of m_2. More precisely, a *maximal packing* of a directed graph G with isomorphic copies of a graph g is a set $\{g_1, g_2, ..., g_n\}$ where $g_i \cong g$ and $V(g_i) \subset V(G)$ for all i and $A(g_i) \cap A(g_j) = \phi$ for $i \neq j$ and $\bigcup_i^n g_i \subset G$ and

$$|A(l)| = |A(G)/\bigcup_i^n g_i|$$

is minimal, where $V(G)$ is the vertex set and $A(G)$ is the arc set of the graph G. The leave of the packing is represented by l.

A number of graphs have been studied in connection with the problem of finding maximal packing (with minimal leaves). Maximal C_3 packings for K_v were explored by Schönheim and Spencer [17, 18]. Schöheim and Bialostocki, 1975, studied packings of Complete Graph with 4-cycles and established the following theorem [2].

Theorem 1.1 [2] *A C_4 packing of K_v with minimal leave l exist if and only if*

1) if $v \equiv$ *0* $(mod\ 2)$ *then* $|E(l)| = v/2$

2) if $v \equiv$ *1* $(mod\ 8)$ *then* $|E(l)| = 0$

3) if $v \equiv$ *3* $(mod\ 8)$ *then* $|E(l)| = 3$

4) if $v \equiv$ *5* $(mod\ 8)$ *then* $|E(l)| = 6$ *and*

5) if $v \equiv$ *7* $(mod\ 8)$ *then* $|E(l)| = 5$.

K_4-packings of K_v have been studied in [3] and C_6 packings of K_v in [10, 11]. Some packings of noncomplete graphs have been studied, for example some cycle packings of $K_v - K_u$ are studied in [4, 13]

A *minimal covering* of a simple graph G with isomorphic copies of a graph g is a set $\{g_1, g_2, \ldots, g_n\}$ where $g_i \cong g$ and $V(g_i) \subset V(G)$ for all i, $G \subset \cup_{i=1}^{n} g_i$, and

$$|A(P)| = |\cup_{i=1}^{n} A(g_i) \setminus A(G)|$$

is minimal (the graph $\cup_{i=1}^{n} g_i$ may not be simple and $\cup_{i=1}^{n} E(g_i)$ may be a multiset). The graph P is called the *padding* of the covering.

A number of graphs have been studied in connection with the problem of finding minimal coverings (with minimal paddings). Minimal C_3 coverings of K_v were explored by Fort and Hedlund [5]. Schöheim and Bialostocki, 1975, studied coverings of complete graph with 4-cycles and establish the theorem below [7].

Theorem 1.2 [7] *A* C_4 *covering of* K_v *with minimal padding* P *exist if and only if*

1) if $v \equiv$ *0* $(mod\ 4)$ *then* $|E(P)| = v/2$

2) if $v \equiv$ *2* $(mod\ 4)$ *then* $|E(P)| = v/2 + 2$

3) if $v \equiv$ *1* $(mod\ 8)$ *then* $|E(P)| = 0$

4) if $v \equiv$ *3* $(mod\ 8)$ *then* $|E(P)| = 5$

5) if $v \equiv 5 \ (mod\ 8)$ then $|E(P)| = 2$ and

6) if $v \equiv 7 \ (mod\ 8)$ then $|E(P)| = 5$.

C_6-coverings of K_v have also been studied in [12]. A minimal cyclic C_4 coverings of the complete graph have been studied by Gardner, Gwellem and Lwenczuk [7]. Coverings have not been as extensively studied as packings.

As an example, a minimal covering of K_5 with isomorphic copies of C_3 has a padding of $P = 2 \times K_2$ as illustrated in Figure 5.

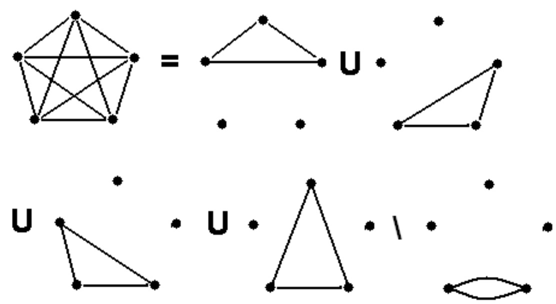

Figure 5: C_3 Covering of K_5.

In the studies of the complete digraph, D_v, one question we try to answer is *" For what v does there exist a decomposition for D_v into the 3-circuit with a pendent arc?"* And also if a decomposition of D_v does not exist, then another question to address is *"Can we efficiently remove isomorphic copies of the 3-circuit with pendent arc from D_v such that the number of arcs remaining is a minimum or can we efficiently build up D_v from isomorphic copies of the 3-circuit with pendent arc such that the number of arcs repeated is a minimum?"*

2 DECOMPOSITIONS

2.1 Introduction

A *decomposition* of a *complete digraph* with *isomorphic* copies of digraph g is a set $\{g_1, g_2, ..., g_n\}$ where $g_i \cong g$ and $V(g_i) \subset V(G)$ for all i and $A(g_i) \cap A(g_j) = \phi$ for $i \neq j$ and the union over all g_i's gives the graph D_v. The g_i's are called the *blocks* of the decomposition. A graph (respectively digraph) decomposition into isomorphic copies of a graph (digraph) on 3 vertices is equivalent to a triple system. A K_3 decomposition of a complete graph on v vertices, K_v, is called a *Steiner Triple system*, $STS(v)$, which is known to exist if and only if $v \equiv 1$ or 3 (mod 6) [16].

Putting orientations on K_3 generates a 3-circuit and a transitive triple denoted by C_3 and T respectively as illustrated in Figure 2. A Mendelsohn triple system is equivalent to a 3-circuit (C_3) decomposition of D_v and exists if and only if $v \equiv 0$ or 1 (mod 3), $v \neq 6$ [14]. A directed triple system is equivalent to a transitive triple T (see Figure 2) decomposition of D_v and exists if and only if $v \equiv 0$ or 1 (mod 3) [9].

Also of relevance to my results are decompositions of K_v into copies of K_3 with a pendant edge (the graph L of Figure 4). Such decompositions exist if and only if $v \equiv 0$ or 1 (mod 8) [1]. There are 8 orientations of a K_3 with a pendant edge. In this chapter, we will give results of decompositions of the complete digraph, D_v, into the two orientations given in Figure 4.

2.2 Examples, Theorems and Proofs

In this subsection we will give specific examples to illustrate the *difference method*. The difference method involves direct construction. We will illustrate how a complete digraph, D_v, admits a cyclic decomposition if $v \equiv 1 \pmod 4$ by the difference method. We will also use the difference method to illustrate that if $v \equiv 0 \pmod 4$ then D_v can undergo a rotational decomposition which is with one fixed point. At the end, we will generalize the results with theorems and proofs.

Example 1: Suppose we have a complete digraph D_v on 72 vertices. This implies that $v \equiv 0 \pmod 4$ and so the decomposition in this case is rotational with one fixed point denoted by ∞. We note that a *base block*, m_1, $(a\ b\ c)$- d has associated differences of a-b, b-c, c-a and a-d. The 3-circuit with a pendent arc difference method for a graph of v vertices is given by verifying if the sum of the first three difference is equal to 0 (mod $v-1$).The fourth difference can appear in any order since it is a pendent arc and has no restriction. To illustrate the *difference method* concept, we use the m_1-*decomposition* of a complete digraph D_v of 72 vertices as shown by Figure 6.

On the other hand, suppose we have a complete digraph D_v on 73 vertices. This tells us that $v \equiv 1 \pmod 4$ and so the decomposition in this case is cyclic with no fix point. We will need to carry out the same procedure as above for the decomposition except for the fact that the sum of the first three differences will be equal to 0 (mod v). We can, for example, generalized the case for $v \equiv 1 \pmod 4$ and it is from this generalization that the results and the proofs stated below originates. This m_1-decomposition is illustrated in Figure 7.

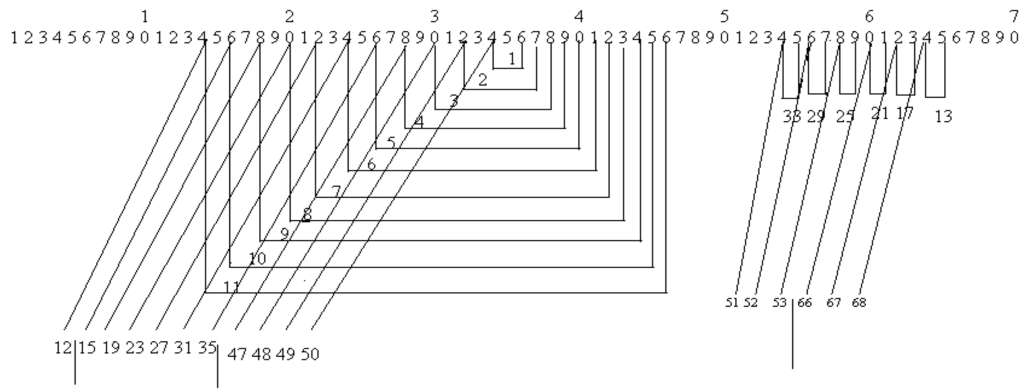

Figure 6: m_1-Decomposition of the Directed Graph with v Vertices.

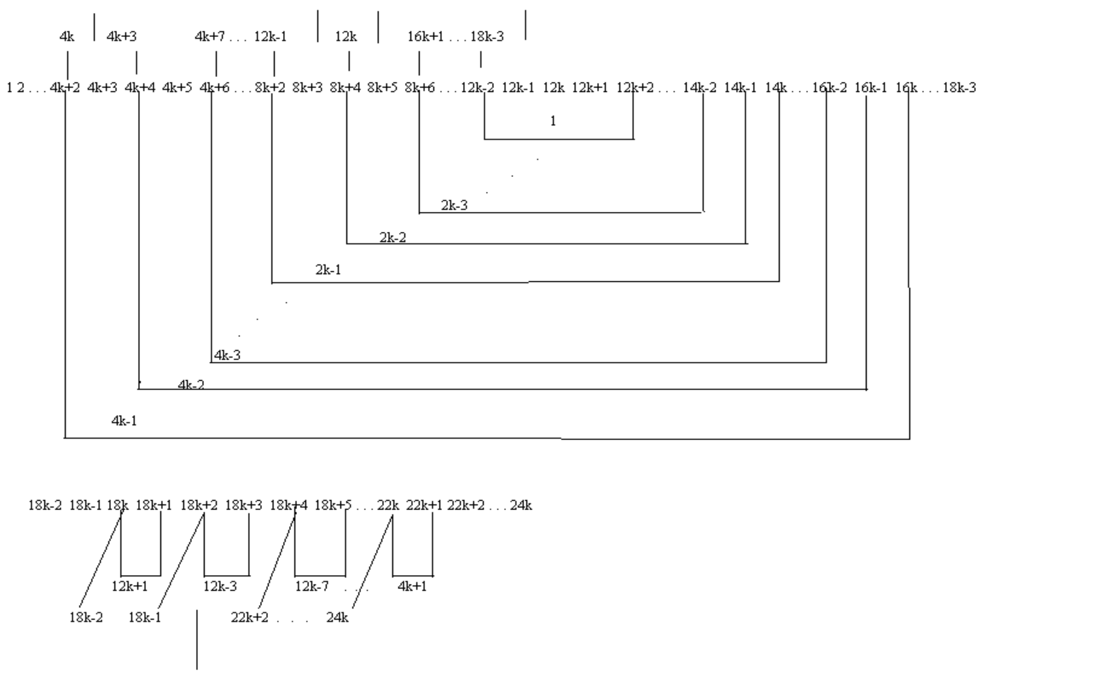

Figure 7: m_1-Decomposition of the Directed Graph with v Vertices.

In this figure, the lines show the pattern in which the differences are grouped so that the sum should be equal to 0 (mod v).

Theorem 2.1 *A m_1-decomposition of D_v exists if and only if $v \equiv 0$ or 1 (mod 4).*

Proof.

In this proof we consider several cases as outlined below.

Case 1. Suppose $v \equiv 1$ (mod 24), say $v = 24k + 1$.

Consider the blocks: $\{(j, 4k+4+2i+j, 8k+2+i+j) - (20k-2-4i+j)_{m1} \mid i = 0, 1, \ldots, 2k-1, j = 0, 1, \ldots, 24k\} \bigcup \{(j, 8k+6+2i+j, 10k+3+i+j) - (8k-1-i+j)_{m1} \mid i = 0, 1, \ldots, 2k-4, j = 0, 1, \ldots, 24k\} \bigcup \{(j, 18k+4+2i+j, 12k+8+4i+j) - (2k-1-i+j)_{m1} \mid i = 0, 1, \ldots, 2k-2, j = 0, 1, \ldots, 24k\} \bigcup \{(j, 4k+2+j, 8k+1+j) - (20k+1+J)_{m1}, (j, 8k+4+j, 10k+2+j) - (12k+1+j)_{m1}, (j, 18k+j, 6k+j) - (6k+3+j)_{m1}, (j, 18k+2+j, 6k-2+j) - (6k+2+j)_{m1} \mid j = 0, 1, \ldots, 24k\}$.

Case 2. Suppose $v \equiv 5$ (mod 24), say $v = 24k + 5$.

Consider the blocks: $\{(j, 4k+2+2i+j, 8k+2+i+j) - (20k+2-4i+j)_{m1} \mid i = 0, 1, \ldots, 2k, j = 0, 1, \ldots, 24k+4\} \bigcup \{(j, 8k+6+2i+j, 10k+4+i+j) - (8k+1-i+j)_{m1} \mid i = 0, 1, \ldots, 2k-3, j = 0, 1, \ldots, 24k+4\} \bigcup \{(j, 18k+8+2i+j, 12k+12+4i+j) - (2k-1-i+j)_{m1} \mid i = 0, 1, \ldots, 2k-2, j = 0, 1, \ldots, 24k+4\}$ $\{(j, 8k+4+j, 10k+3+j) - (12k+3+j)_{m1}, (j, 18k+4+j, 6k+j) - (6k+3+j)_{m1}, (j, 18k+6+j, 6k-2+j) - (6k+2+j)_{m1} \mid j = 0, 1, \ldots, 24k+4\}$.

Case 3. Suppose $v \equiv 9$ (mod 24), say $v = 24k + 9$.

Consider the blocks: $\{(j, 4k+2+2i+j, 8k+3+i+j) - (20k+4-4i+j)_{m1} \mid i =$

$0,1,\ldots,2k, j=0,1,\ldots,24k+8\} \bigcup\{(j, 8k+6+2i+j, 10k+5+i+j)-(8k+2-i+j)_{m1} \mid i=0,1,\ldots,2k-2, j=0,1,\ldots,24k+8\} \bigcup\{(j, 18k+9+2i+j, 12k+10+4i+j)-(2k-i+j)_{m1} \mid i=0,1,\ldots,2k-1, j=0,1,\ldots,24k+8\} \bigcup\{(j, 8k+4+j, 10k+4+j)-(12k+5+J)_{m1}, (j, 18k+7+j, 6k+1+j)-(6k+3+j)_{m1} \mid j=0,1,\ldots,24k+8\}$.

Case 4. Suppose $v \equiv 13 \pmod{24}$, say $v = 24k+13$.

Consider the blocks: $\{(j, 4k+6+2i+j, 8k+6+i+j)-(20k+8-4i+j)_{m1} \mid i=0,1,\ldots,2k, j=0,1,\ldots,24k+12\} \bigcup\{(j, 8k+10+2i+j, 10k+8+i+j)-(8k+4-i+j)_{m1} \mid i=0,1,\ldots,2k-3, j=0,1,\ldots,24k+12\} \bigcup\{(j, 18k+13+2i+j, 12k+14+4i+j)-(2k-i+j)_{m1} \mid i=0,1,\ldots,2k-1, j=0,1,\ldots,24k+12\} \bigcup\{(j, 4k+4+j, 8k+5+j)-(20k+11+j)_{m1}, (j, 8k+8+j, 10k+7+j)-(12k+7+j)_{m1} \mid j=0,1,\ldots,24k+12\} \bigcup\{(j, 18k+9+j, 12k+6+j)-(6k+6+j)_{m1}, (j, 18k+11+j, 12k+10+j)-(6k+5+J)_{m1} \mid j=0,1,\ldots,24k+12\}$.

Case 5. Suppose $v \equiv 17 \pmod{24}$, say $v = 24k+17$.

Consider the blocks: $\{(j, 4k+4+2i+j, 8k+6+i+j)-(20k+12-4i+j)_{m1} \mid i=0,1,\ldots,2k+1, j=0,1,\ldots,24k+16\} \bigcup\{(j, 8k+10+2i+j, 10k+9+i+J)-(8k+5-i+J)_{m1} \mid i=0,1,\ldots,2k-2, j=0,1,\ldots,24k+16\} \bigcup\{(j, 18k+17+2i+j, 12k+18+4i+J)-(2k-i+J)_{m1} \mid i=0,1,\ldots,2k-1, j=0,1,\ldots,24k+16\} \bigcup\{(j, 8k+8+j, 10k+8+j)-(12k+9+j)_{m1}, (j, 18k+3+j, 6k+3+j)-(6k+6+j)_{m1}, (j, 18k+15+j, 6k+1+j)-(6k+5+J)_{m1} \mid j=0,1,\ldots,24k+16\}$.

Case 6. Suppose $v \equiv 21 \pmod{24}$, say $v = 24k+21$.

Consider the blocks: $\{(j, 4k+4+2i+j, 8k+7+i+j)-(20k+14-4i+j)_{m1} \mid i=0,1,\ldots,2k+1, j=0,1,\ldots,24k+20\} \bigcup\{(j, 8k+10+2i+j, 10k+10+i+j)-(8k+6-i+j)_{m1} \mid i=0,1,\ldots,2k-1, j=0,1,\ldots,24k+20\} \bigcup\{(j, 18k+18+2i+$

$j, 12k+16+4i+j) - (2k+1-i+j)_{m1} \mid i = 0, 1, \ldots, 2k, j = 0, 1, \ldots, 24k+20\}$ $\bigcup\{(j, 8k+8+j, 10k+9+j) - (12k+11+j)_{m1}, (j, 18k+16+j, 6k+4+j) - (6k+6+j)_{m1} \mid j = 0, 1, \ldots, 24k+20\}$. In each of Cases 1–6, the given set of blocks forms a decomposition of D_v where $V(D_v) = \{0, 1, \ldots, v-1\}$ and vertex labels in the blocks are reduced modulo v.

Case 7. Suppose $v \equiv 0 \pmod{24}$, say $v = 24k$.

Consider the blocks: $\{(j, 4k+4+2i+j, 8k+2+i+j) - (20k-4-4i+j)_{m1} \mid i = 0, 1, \ldots, 2k-1, j = 0, 1, \ldots, 24k-2\} \bigcup\{(j, 8k+4+2i+j, 10k+2+i+j) - (8k-i+j)_{m1} \mid i = 0, 1, \ldots, 2k-3, , j = 0, 1, \ldots, 24k-2\} \bigcup\{(j, 18k+6+2i+j, 12k+14+4i+j) - (2k-1-i+j)_{m1} \mid i = 0, 1, \ldots, 2k-4, j = 0, 1, \ldots, 24k-2\} \bigcup\{(j, \infty, 2+j) - (1+j)_{m1}, (j, 4k+2+j, 8k+1+j) - (20k-1+j)_{m1}, (j, 18k+j, 6k-2+j) - (6k+2+j)_{m1}, (j, 18k+2+j, 6k-4+j) - (6k+1+j)_{m1}, (j, 18k+4+j, 6k-6+j) - (6k+j)_{m1} \mid j = 0, 1, \ldots, 24k-2\}$.

Case 8. Suppose $v \equiv 4 \pmod{24}$, say $v = 24k+4$.

Consider the blocks: $\{(j, 4k+2+2i+j, 8k+2+i+j) - (20k-4i+j)_{m1} \mid i = 0, 1, \ldots, 2k-1, j = 0, 1, \ldots, 24k+2\} \bigcup\{(j, 8k+4+2i+j, 10k+3+i+J) - (8k+1-i+j)_{m1} \mid i = 0, 1, \ldots, 2k-2, j = 0, 1, \ldots, 24k+2\} \bigcup\{(j, 18k+10+2i+j, 12k+18+4i+j) - (2k-1-i+j)_{m1} \mid i = 0, 1, \ldots, 2k-4, j = 0, 1, \ldots, 24k+2\} \bigcup\{(j, \infty, 2+j) - (1+j)_{m1}, (j, 8k+2+j, 10k+2+j) - (12k+2+j)_{m1}, (j, 18k+4+j, 6k-2+j) - (6k+2+j)_{m1}, (j, 18k+6+j, 6k-4+j) - (6k+1+j)_{m1}, (j, 18k+8+j, 6k-6+j) - (6k+j)_{m1} \mid j = 0, 1, \ldots, 24k+2\}$.

Case 9. Suppose $v \equiv 8 \pmod{24}$, say $v = 24k+8$.

Consider the blocks: $\{(j, 4k+2+2i+j, 8k+3+i+j) - (20k+2-4i+j)_{m1} \mid$

$i = 0, 1, \ldots, 2k-1, j = 0, 1, \ldots, 24k+6\} \bigcup \{(j, 8k+4+2i+j, 10k+4+i+j) - (8k+2-i+j)_{m1} \mid i = 0, 1, \ldots, 2k-1, j = 0, 1, \ldots, 24k+6\} \bigcup \{(j, 18k+11+2i+j, 12k+16+4i+j) - (2k-i+j)_{m1} \mid i = 0, 1, \ldots, 2k-3, j = 0, 1, \ldots, 24k+6\}$ $\bigcup \{(j, \infty, 2+j) - (1+j)_{m1}, (j, 8k+2+j, 10k+3+j) - (12k+4+j)_{m1}, (j, 18k+7+j, 6k-1+j) - (6k+2+j)_{m1}, (j, 18k+9+j, 6k-3+j) - (6k+1+J)_{m1} \mid j = 0, 1, \ldots, 24k+6\}$.

Case 10. Suppose $v \equiv 12 \pmod{24}$, say $v = 24k+12$.

Consider the blocks: $\{(j, 4k+6+2i+j, 8k+6+i+j) - (20k+6-4i+j)_{m1} \mid i = 0, 1, \ldots, 2k, j = 0, 1, \ldots, 24k+10\} \bigcup \{(j, 8k+8+2i+j, 10k+7+i+j) - (8k+4-i+j)_{m1} \mid i = 0, 1, \ldots, 2k-2, j = 0, 1, \ldots, 24k+10\} \bigcup \{(j, 18k+15+2i+j, 6k-5-2i+j) - (2k-i+j)_{m1} \mid i = 0, 1, \ldots, 2k-3, j = 0, 1, \ldots, 24k+10\} \bigcup \{(j, \infty, 2+j) - (1+j)_{m1}, (j, 4k+4+j, 8k+5+j) - (20k+9+j)_{m1}, (j, 18k+9+j, 6k+1+j) - (6k+5+j)_{m1}, (j, 18k+11+j, 6k-1+j) - (6k+9+j)_{m1}, (j, 18k+13+j, 6k-3+j) - (6k+3+j)_{m1} \mid j = 0, 1, \ldots, 24k+10\}$.

Case 11. Suppose $v \equiv 16 \pmod{24}$, say $v = 24k+16$.

Consider the blocks: $\{(j, \infty, 2+j) - (1+j)_{m1}, (j, 8k+6+j, 10k+7+j) - (12k+8+j)_{m1}, (j, 18k+13+j, 6k+1+j) - (6k+5+j)_{m1}, (j, 18k+15+j, 6k-1+j) - (6k+4+j)_{m1}, (j, 18k+17+j, 6k-3+j) - (6k+3+j)_{m1} \mid j = 0, 1, \ldots, 24k+14\}$ $\bigcup \{(j, 4k+4+2i+j, 8k+6+i+j) - (20k+10-4i+j)_{m1} \mid i = 0, 1, \ldots, 2k, j = 0, 1, \ldots, 24k+14\} \bigcup \{(j, 8k+8+2i+j, 10k+8+i+j) - (8k+5-i+j)_{m1} \mid i = 0, 1, \ldots, 2k-1, j = 0, 1, \ldots, 24k+14\} \bigcup \{(j, 18k+19+2i+j, 6k-5-2i+j) - (2k-i+j)_{m1} \mid i = 0, 1, \ldots, 2k-3, j = 0, 1, \ldots, 24k+14\}$.

Case 12. Suppose $v \equiv 20 \pmod{24}$, say $v = 24k+20$.

Consider the blocks: $\{(j, 4k+4+2i+j, 8k+7+i+j) - (20k+12-4i+j)_{m1} \mid i =$

$0, 1, \ldots, 2k, j = 0, 1, \ldots, 24k+18\} \bigcup \{(j, 8k+8+2i+j, 10k+9+i+j)-(8k+6-i+j)_{m1} \mid i = 0, 1, \ldots, 2k, j = 0, 1, \ldots, 24k+18\} \bigcup \{(j, 18k+20+2i+j, 6k-2-2i+j)-(2k+1-i+j)_{m1} \mid i = 0, 1, \ldots, 2k-2, j = 0, 1, \ldots, 24k+18\}$ $\{j, \infty, 2+j)-(1+j)_{m1}, (j, 8k+6+j, 10k+8+j)-(12k+10+j)_{m1}, (j, 18k+16+j, 6k+2+j)-(6k+5+j)_{m1}$ $(j, 18k+18+j, 6k+j)-(6k+4+j)_{m1} \mid j = 0, 1, \ldots, 24k+18\}$.

In each of Cases 7–12, the given set of blocks forms a decomposition of D_v where $V(D_v) = \{\infty, 0, 1, \ldots, v-2\}$ and the numerical vertex labels in the blocks are reduced modulo $v-1$.

Corollary 2.2 *A m_2-decomposition of D_v exists if and only if $v \equiv 0$ or 1 (mod 4).*

Proof. The necessary condition follows as in Theorem 2.1. Since the converse of m_1 is m_2 and the D_v is self converse, the result follows trivially from Theorem 2.1.

3 PACKINGS AND COVERINGS

In this chapter, we will target the following question: "*When a decomposition does not exist, how close to it can we get?*" There are two approaches to this question: packings and coverings. A *g-packing* of a directed graph D with isomorphic copies of a graph g is a set $\{g_1, g_2, ..., g_n\}$ where $g_i \cong g$ and $V(g_i) \subset V(D)$ for all i and $A(g_i) \cap A(g_j) = \phi$ for $i \neq j$ and $\bigcup_i^n g_i \subset D$ and

$$|A(l)| = |A(D)/\bigcup_i^n g_i|$$

is minimal , where $V(D)$ is the vertex set and $A(D)$ is the arc set of the graph D. The leave of the packing is represented by l. The packings are said to be maximal (optimal) when the leave is minimal. Maximum packings of complete graphs with hexagons was studied by J. Kennedy [10, 11]. Gardner, Gwellem and Lewenczuk studied maximal cyclic C_4 packings of complete graphs [7]. Maximal packings of complete digraphs D_v with 3-circuit and transitive triples have been studied by R. Gardner [6]. In his studies, he came out with the following results.

Theorem 3.1 [6] *A maximal packing of D_v, where $v \neq 6$, with copies of the 3-circuit, C_3, and a leave l satistfies:*

1) $|A(l)| = 0$ if $v \equiv 0$ or 1 (mod 3), $v \neq 6$ or

2) $|A(l)| = 2$ and l=C_2 if $v \equiv 2$ (mod 3) .

Theorem 3.2 [6] *A maximal packing of D_v, where $v \neq 6$, with copies of the transitive triple, T, and a leave l satisfies:*

1) $|A(l)| = 0$ if $v \equiv 0$ or 1 (mod 3), or

2) $|A(l)| = 2$ and l=C_2 if $v \equiv 2$ (mod 3) .

A *g-covering* of a directed graph D with isomorphic copies of a graph g is a set $\{g_1, g_2, \ldots, g_n\}$ where $g_i \cong g$ and $V(g_i) \subset V(D)$ for all i, $G \subset \cup_{i=1}^{n} g_i$, and

$$|A(P)| = |\cup_{i=1}^{n} A(g_i) \setminus A(D)|$$

The graph P is called the *padding* of the covering. The covering is said to be minimal if the padding is minimal. J. Kennedy explored minimal coverings of complete graphs with hexagons [12]. Minimal cyclic C_4 coverings of complete graphs was studied by Gardner, Gwellem and Lewenczuk [7]. R. Gardner studied the minimal coverings problem of complete digraph D_v with 3-circuit and transitive triples and came out with the following theorems [6].

Theorem 3.3 [6] *A minimal covering of D_v, where $v \neq 6$, with copies of the 3-circuit, C_3, and padding P satisfies:*

1) $|A(P)| = 0$ if $v \equiv 0$ or 1 (mod 3), $v \neq 6$ or

2) $|A(l)| = 3$ and P=C_3 if $v = 6$, or

3) $|A(P)| = 4$ if $v \equiv 2$ (mod 3) and P may be two disjoint copies of C_2, a 4-circuit or two osculating 2-circuits OC_2.

Theorem 3.4 [6] *A minimal covering of D_v, where $v \neq 6$, with copies of the transitive triple ,T, and padding P satisfies:*

1) $|A(P)| = 0$ if $v \equiv 0$ or 1 (mod 3), or

2) $|A(P)| = 4$ if $v \equiv 2$ (mod 3) and P may be two disjoint copies of C_2, any orientation of a 4-cycle or two isolating 2-circuits OC_2.

The main purpose of this chapter is to carry out maximal packings and minimal coverings of complete digraph, D_v, with m_1 and m_2. This work is accomplished by the following main theorems, corollaries and proofs.

Theorem 3.5 *A maximal m_1-packing of D_v with leave L satisfies*

1) $|A(L)| = 0$ if $v \equiv 0$ or 1 $(mod\ 4)$, and

2) $|A(L)| = 2$ if $v \equiv 2$ or 3 $(mod\ 4)$.

Proof. If $v \equiv 0$ or 1 (mod 4), then there is a decomposition by Theorem 2.1 and the result follows. For $v \equiv 2$ or 3 (mod 4), we consider several cases.

We need to put in the packing for D_7 since we have seven fixed points in each of the cases of $v \equiv 2$ (mod 4). Here is the packing of D_7 with m_1's: We first of all keep two of the vertices as fixed points and call them ∞_1 and ∞_2. The 10 m_1's are: $(0\ 3\ \infty_1)$ - (4), $(1\ 4\ \infty_1)$ - (0), $(2\ 0\ \infty_1)$ - (1), $(3\ 1\ \infty_1)$ - (2), $(4\ 2\ \infty_1)$ - (3), $(0\ 2\ \infty_2)$ - (1), $(1\ 3\ \infty_2)$ - (2), $(2\ 4\ \infty_2)$ - (3), $(3\ 0\ \infty_2)$ - (4), $(4\ 1\ \infty_2)$ - (0)

Leave $= \{[\infty_1\ \infty_2], [\infty_2\ \infty_1]\}$.

Case 1. Suppose $v \equiv 2$ (mod 12), say $v = 12k + 2$. Consider the blocks: $\{(j, 9k-7-2i+j, \infty_{i+1}) - (3k+1+2i+j)_{m1} \mid i = 0,1,2,3,4, j = 0,1,\ldots,12k-6\}$ $\bigcup\{j, 9k+3+j, \infty_6) - (3+j)_{m1}, (j, 12k-7+j, \infty_7) - (1+j)_{m1} \mid j = 0,1,\ldots,12k-6\}$ $\bigcup\{(j, 1+i+j, 6k-3-i+j) - (6k-2+2i+j)_{m1} \mid i = 0,1,\ldots,6, j = 0,1,\ldots,12k-6\}$ $\bigcup\{(j, k+3-i+j, 5k-8+i+j) - (10k-8-4i+j)_{m1} \mid i = 0,1,\ldots,k-5, j = 0,1,\ldots,12k-6\}$ $\bigcup\{(j, 6k-19-4i+j, 3k-10+2i+j) - (k-1-i+j)_{m1} \mid i = 0,1,\ldots,k-5, j = 0,1,\ldots,12k-6\}$ $\bigcup\{(j, 2k-1-i+j, 4k-1+i+j) - (4k-2-i+j)_{m1} \mid i = 0,1,\ldots,k-5, j = 0,1,\ldots,12k-6\}$.

Case 2. Suppose $v \equiv 6$ (mod 12), say $v = 12k + 6$.

Consider the blocks: $\{(j, 9k-5+2i+j, \infty_{1+i}) - (3k+3-2i+j)_{m1} \mid i = 0, 1, \ldots, 4, j = 0, 1, \ldots, 12k-2\} \bigcup \{(j, 9k+5+j, \infty_6) - (3+j)_{m1}, (j, 12k-3+j, \infty_7) - (1+j)_{m1} \mid j = 0, 1, \ldots, 12k-2\} \bigcup \{(j, 6k-15+j, 3k-8+j) - (6k+j)_{m1}, (j, k+3+j, 5k-3+j) - (10k-1+j)_{m1} \mid j = 0, 1, \ldots, 12k-2\} \bigcup \{(j, 1+i+j, 6k-1-i+j) - (6k+2+2i+j)_{m1} \mid i = 0, 1, \ldots, 5, j = 0, 1, \ldots, 12k-2\} \bigcup \{(j, 2k-1-i+j, 4k+1+i+j) - (4k-i+j)_{m1} \mid i = 0, 1, \ldots, k-5, j = 0, 1, \ldots, 12k-2\} \bigcup \{(j, k+2-i+j, 5k-2+i+j) - (10k-4-4i+j)_{m1} \mid i = 0, 1, \ldots, k-5, j = 0, 1, \ldots, 12k-2\} \bigcup \{(j, 6k-19-4i+j, 3k-10-2i+j) - (k-1-i+j)_{m1} \mid i = 0, 1, \ldots, k-5, j = 0, 1, \ldots, 12k-2\}$.

Case 3. Suppose $v \equiv 10 \pmod{12}$, say $v = 12k + 10$.

Consider the blocks: $\{(j, 9k-1+2i+j, \infty_{i+1}) - (3k+3-2i+j)_{m1} \mid i = 0, 1, 2, 3, 4, j = 0, 1, \ldots, 12k+2\} \bigcup \{(j, 9k+9+j, \infty_6) - (3+j)_{m1}, (j, 12k+1+j, \infty_7) - (1+j)_{m1} \mid j = 0, 1, \ldots, 12k+2\} \bigcup \{(j, 6k-15+j, 3k-8+j) - (6k+1+j)_{m1} \mid j = 0, 1, \ldots, 12k+2\} \bigcup \{(j, 1+i+j, 6k+1-i+j) - (6k+4+2i+j)_{m1} \mid i = 0, 1, \ldots, 6, j = 0, 1, \ldots, 12k+2\} \bigcup \{(j, k+3-i+j, 5k+5+i+j) - (10k-4i+j)_{m1} \mid i = 0, 1, \ldots, k-5, j = 0, 1, \ldots, 12k+2\} \bigcup \{(j, 2k-i+j, 4k+2+i+j) - (4k+1-i+j)_{m1} \mid i = 0, 1, \ldots, k-5, j = 0, 1, \ldots, 12k+2\} \bigcup \{(j, 6k-19-4i+j, 3k-10-2i+j) - (k-1-i+j)_{m1} \mid i = 0, 1, \ldots, k-5, j = 0, 1, \ldots, 12k+2\}$.

In each of the cases below we need to put in the packing for D_6 since we have six fixed points in the cases of $v \equiv 3 \pmod 4$. Here is the packing of D_6 with m_1's. The 7 m_1's are: (0 1 5) - (2), (0 5 1) - (3), (4 0 2) - (1), (4 1 3) - (0), (3 5 4) - (0), (5 3 2) - (4), (2 3 1) - (5)

Leave = {[4 2], [2 1]}.

Case 4. Suppose $v \equiv 3 \pmod{12}$, say $v = 12k + 3$.

Consider the blocks: $\{(j, 9k-6+2i+j, \infty_{1+i})-(3k+2-2i+j)_{m1} \mid i = 0,1,2,3,4, j = 0,1,\ldots,12k-4\} \bigcup \{(j, 12k-5+j, \infty_6)-(1+j)_{m1}, (j, 1+j, 6k-3+j)-(6k-1+j)_{m1}, j = 0,1,\ldots,12k-4\} \bigcup \{(j, 2+i+j, 6k-4-i+j)-(6k-2+2i+j)_{m1} \mid i = 0,1,\ldots,6, j = 0,1,\ldots,12k-4\} \bigcup \{(j, k+4-i+j, 5k-6+i+j)-(10k-6-4i)_{m1} \mid i = 0,1,\ldots,k-5, j = 0,1,\ldots,12k-4\} \bigcup \{(j, 2k-1-i+J, 4k-1+i+j)-(4k-2-i+j)_{m1} \mid i = 0,1,\ldots,k-6, j = 0,1,\ldots,12k-4\} \bigcup \{(j, 6k-15-4i+j, 3k-8-2i+j)-(k-1-i+j)_{m1} \mid i = 0,1,\ldots,k-1, j = 0,1,\ldots,12k-4\}$.

Case 5. Suppose $v \equiv 7 \pmod{12}$, say $v = 12k+7$.

Consider the blocks: $\{(j, 9k-4+2i+j, \infty_{i+1})-(3k+4-2i+j)_{m1} \mid i = 0,1,2,3,4, j = 0,1,\ldots,12k\} \bigcup \{(j, 12k-1+j, \infty_6)-(1+j)_{m1}, (j, k+3+j, 5k-5+j)-(10k+1+j)_{m1}, (0, 6k-11, 3k-6)-(6k+1)_{m1} \mid j = 0,1,\ldots,12k\} \bigcup \{(j, 1+i+j, 6k-1-i+j)-(6k+2i+j)_{m1} \mid i = 0,1,2,3,4,5, j = 0,1,\ldots,12k\} \bigcup \{(j, k+3-i+j, 5k-3+i+j)-(10k-2-4i+j) \mid i = 0,1,\ldots,k-4, j = 0,1,\ldots,12k\} \bigcup \{(j, 2k-1-i+j, 4k+1+i+j)-(4k-i+j)_{m1} \mid i = 0,1,\ldots,k-6, j = 0,1,\ldots,12k\} \bigcup \{(j, 6k-15-4i+j, 3k-8-2i+j)-(k-1-i+j)_{m1} \mid i = 0,1,\ldots,k-4, j = 0,1,\ldots,12k\}$.

Case 6. Suppose $v \equiv 11 \pmod{12}$, say $v = 12k+11$.

Consider the blocks: $\{(j, 9k+2i+j, \infty_{i+1})-(3k+4-2i+j)_{m1} \mid i = 0,1,2,3,4, j = 0,1,\ldots,12k+4\} \bigcup \{(j, 12k+4+j, \infty_6)-(1+j)_{m1}, (j, 6k-11+j, 3k-6+j)-(6k+3+j)_{m1} \mid j = 0,1,\ldots,12k+4\} \bigcup \{(j, 1+i+j, 6k+1-i+j)-(6k+2+2i+j)_{m1} \mid i = 0,1,\ldots,6, j = 0,1,\ldots,12k+4\} \bigcup \{(j, k+4-i+j, 5k-2+i+j)-(10k+2-4i+j)_{m1} \mid i = 0,1,\ldots,k-4, j = 0,1,\ldots,12k+4\} \bigcup \{(j, 2k-i+j, 4k+2+i+j)-(4k+1-i+j)_{m1} \mid i = 0,1,\ldots,k-5, j = 0,1,\ldots,12k+4\} \bigcup \{(j, 6k-11-4i+j, 3k-4-2i+j)-(k-1-i+j)_{m1} \mid$

$i = 0, 1, \ldots, k-5, j = 0, 1, \ldots, 12k+4\}$.

□

Corollary 3.6 *A maximal m_2-packing of D_v with leave L satisfies*

(i) $|A(L)| = 0$ *if* $v \equiv 0$ *or* 1 *(mod* 4*), and*

(ii) $|A(L)| = 2$ *if* $v \equiv 2$ *or* 3 *(mod* 4*).*

Proof. Since the converse of m_1 is m_2 and the D_v is self converse, the result follows trivially from Theorem 3.5. □

Corollary 3.7 *A minimal m_1-covering of D_v with padding P satisfies*

(i) $|A(P)| = 0$ *if* $v \equiv 0$ *or* 1 *(mod* 4*), and*

(ii) $|A(P)| = 2$ *if* $v \equiv 2$ *or* 3 *(mod* 4*).*

Proof. What needs to be added is just the covering of D_7 for $v \equiv 2$ (mod 4) and D_6 for $v \equiv 3$ (mod 4) and the result follows trivially from Theorem 3.5.

Here is the covering of D_7 with m_1's: The 11 m_1's are: (0 6 1) - (4), (0 1 6) - (2), (5 1 3) - (0), (5 0 2) - (1), (4 3 0) - (6), (3 4 6) - (0), (3 6 2) - (5), (5 2 4) - (6), (4 2 1) - (5), (6 3 2) - (5), (1 5 2) - (4)

Padding= {[6 3], [5 3]}.

Here is the covering of D_6 with m_1's: The 8 m_1's are: (0 1 5) - (2), (0 5 1) - (3), (4 0 2) - (1), (4 1 3) - (0), (3 5 4) - (0), (5 3 2) - (4), (2 3 1) - (5), (2 1 3) - (4)

Padding = {[1 3], [3 2]}.

□

Corollary 3.8 *A minimal m_2-covering of D_v with padding P satisfies*

(i) $|A(P)| = 0$ *if* $v \equiv 0$ *or* 1 $(mod\ 4)$, *and*

(ii) $|A(P)| = 2$ *if* $v \equiv 2$ *or* 3 $(mod\ 4)$.

Proof. What needs to be added is just the covering of D_7 for $v \equiv 2 \pmod 4$ and D_6 for $v \equiv 3 \pmod 4$ and the result follows trivially from Theorem 3.5 and Corollary 3.7 since the converse of m_1 is m_2 and the D_v is self converse. □

4 CONCLUSION

In this thesis, we studied decompositions, packings and coverings of complete digraphs with a 3-circuit and a pendent arc. We outlined the necessary and sufficient conditions for a decomposition and this exists if and only if $v \equiv 0$ or 1 (mod 4). For $v \equiv 2$ or 3 (mod 4) we showed that maximal packings and minimal coverings exist with a leave of size two and a padding of size two respectively.

BIBLIOGRAPHY

[1] J. C. Bermond and J. Schönheim, $G-$Decompositions of K_n where G has Four Vertices or Less, *Discrete Math.* **19** (1977) 113-120.

[2] J. Schöheim and A. Bialostocki, Packing and Covering of Complete Graph with 4-cycles, *Canadian Mathematics Bulletin*, **18**(5) (1975), 703-708.

[3] A. Brower, Optimal Packins of K_4 into K_n, *Journal of Combinatorial Theory*, Series A, **26**(A) (1979), 278-297.

[4] D. Bryant, C. Rodger, and E. Spicer, Embeddings of m-cycle systems and Incomplete m-cycle systems: $m \leq 14$, *Discrete Mathematics*, **171** (1997), 55-75.

[5] M. K. Fort, Jr. and G. A. Hedlund, Minimal Coverings of Pairs by Triples, *Pacific Journal of Math.* **8** (1958), 709–719.

[6] R. Gardner, Optimal Packings and Coverings of the Complete Directed Graph with 3-Circuits and with Transitive Triples, *Congressus Numerantium*, **127** (1997) 161-170.

[7] R. Gardner, C. Gwellem, and J. Lewenczuk, Maximal cyclic C_4 packings and minimal Cyclic C_4 Coverings of Complete Graph, *Congressus Numerantium* **180**(2006) 193-199.

[8] D. G. Hoffman and K. S. Kirkpatrick, Another Doyen-Wilson Theorem, *Ars Combinatoria* **54** (2000), 87–96.

[9] S. Hung and N. Mendelsohn, Directed Triple Systems, *Journal of Combin. Theory Series A* **1**4 (1973), 310–318.

[10] J. Kennedy, Maximum Packings of K_n with Hexagons, *Australasian Journal of Combinatorics*, **7** (1993), 101-110.

[11] J. Kennedy, Maximum Packings of K_n with Hexagons: Corrigendum, *Australasian Journal of Combinatorics*, **1**0 (1994), 293.

[12] J. Kennedy, Minimal Coverings of K_n with Hexagons, *Australasian Journal of Combinatorics*, **1**6 (1997), 295-303.

[13] D. Bryant, and A. Khodkar, Maximum Packings of $K_v - K_u$ with Triples, *Ars Combinatoria*, **5**5 (2000) 259-270.

[14] N. Mendelsohn, A Natural Generalization of Steiner Triple Systems, *Computers in Number Theory*, eds. A. O. Atkin and B. Birch, Academic Press, London, 1971.

[15] R. Gardner , C. Nguyen, and S. Lavoie, 4-cycle coverings of the Complete Graph with a Hole, submitted.

[16] R. Peltesohn, Eine Losung der beiden HeffterDifferenzenprobleme, *Compositio Math*, **6** (1939), 251-257.

[17] J. Schönheim, On Maximal Systems of k-tuples, *Studia Sci. Math. Hungarica* (1966), 363–368.

[18] J. Spencer, Maximal Consistent Families of Triples, *Journal of Combinatorial Theory* **5** (1968), 1-8.

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