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# Very Cost Effective Partitions in Graphs

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Very Cost Effective Partitions in Graphs

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A thesis

presented to

the faculty of the Department of Mathematics

East Tennessee State University

In partial fulfillment

of the requirements for the degree

Master of Science in Mathematical Sciences

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by

Inna Vasylieva

May 2013

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Keywords: very cost effective partition, Cartesian product, cactus graph

## ABSTRACT

### Very Cost Effective Partitions in Graphs

by

Inna Vasylieva

For a graph  $G = (V, E)$  and a set of vertices  $S \subseteq V$ , a vertex  $v \in S$  is said to be very cost effective if it is adjacent to more vertices in  $V \setminus S$  than in  $S$ . A bipartition  $\pi = \{S, V \setminus S\}$  is called very cost effective if both  $S$  and  $V \setminus S$  are very cost effective sets. Not all graphs have a very cost effective bipartition, for example, the complete graphs of odd order do not. We consider several families of graphs  $G$ , including Cartesian products and cacti graphs, to determine whether  $G$  has a very cost effective bipartition.

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## DEDICATION

I lovingly dedicate this thesis to my mother, Larisa Vasilieva, for her kindness and endless support throughout my whole life. I could never thank you enough for your love and selflessness. You have always been my inspiration and I am eternally grateful to have you as my parent.

## ACKNOWLEDGMENTS

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It gives me great pleasure in acknowledging the support and help of professor from my previous university, Yuriy Zakhariyenko. His outstanding ways of teaching mathematics has inspired me to pursue a career in that field. I would also like to thank my family and friends for their encouragement and support during my life and, in particular, my graduate school career. A special thank you to my best friend Sophia. I cannot imagine having more kind, sincere, thoughtful and supportive friends than mine.

## CONTENTS

ABSTRACT . . . . .	2
DEDICATION . . . . .	4
ACKNOWLEDGMENTS . . . . .	5
LIST OF FIGURES . . . . .	7
1 INTRODUCTION . . . . .	8
1.1 Basic Terminology of Graph Theory . . . . .	8
1.2 Very Cost Effective Partitions in Graphs . . . . .	11
1.3 Motivation and Applications . . . . .	12
2 LITERATURE SURVEY . . . . .	14
2.1 Unfriendly Partitions . . . . .	14
2.2 Cost Effective Domination . . . . .	16
2.3 Variations of the Aharoni, Milner, Prikry Theorem . . . . .	21
3 PRELIMINARY RESULTS . . . . .	24
4 MAIN RESULTS . . . . .	30
4.1 General Results . . . . .	30
4.2 Cartesian Product and Cacti Graphs . . . . .	31
5 CONCLUDING REMARKS . . . . .	39
BIBLIOGRAPHY . . . . .	40
VITA . . . . .	43

## LIST OF FIGURES

1	House graph . . . . .	8
2	$C_5, W_5, K_5$ . . . . .	10
3	Inequalities of Corollary 2.5. . . . .	18
4	Graph $G$ has an odd cycle endblock . . . . .	31
5	$C_3 \square C_3$ . . . . .	32
6	$K_2 \square C_3$ . . . . .	34
7	Example of a branch . . . . .	35
8	Graph $G$ has property $P$ . . . . .	38



# 1 INTRODUCTION

## 1.1 Basic Terminology of Graph Theory

As defined in [4], a graph  $G = (V, E)$  is a nonempty, finite set of elements called *vertices* together with a (possibly empty) set of unordered pairs of distinct vertices of  $G$  called *edges*. The vertex set of  $G$  is denoted by  $V(G)$  and the edge set of  $G$  is denoted by  $E(G)$ . When there is no risk of ambiguity, these are denoted  $V$  and  $E$ , respectively. In Figure 1, we have an example of a graph.

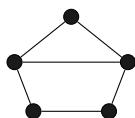


Figure 1: House graph

In this thesis, we will be studying *simple graphs*, that is, graphs for which there exists at most one edge between any two vertices and for which the endpoints of any edge are distinct. If  $e = \{u, v\}$  is an edge of a graph  $G$ , then  $u$  and  $v$  are *adjacent* vertices, while  $u$  and  $e$  are *incident*, as are  $v$  and  $e$ . Two adjacent vertices are called *neighbors* of each other. The *degree* of a vertex  $v$  in a graph  $G$  is the number of edges incident to  $v$ . A vertex of degree 0 in  $G$  is called an *isolated vertex*, while a vertex of degree of 1 is referred to as a *leaf* or *pendant*. A vertex  $v$  is said to be *even* or *odd*, according to whether its degree in  $G$  is even or odd.

Given any graph  $G$ , the *order* of  $G$ , denoted  $n(G) = |V(G)|$ , is the number of vertices in  $G$ . The *size* of  $G$ , denoted  $m(G) = |E(G)|$ , is the number of edges in  $G$ . If there is no risk of ambiguity, these are written as  $n$  and  $m$ , respectively. For

example, for the graph  $G$  in Figure 1, the order  $n(G) = 5$  and the size  $m(G) = 6$ . A graph of order 1 is called a *trivial graph*, and a graph of order at least 2 is called a *nontrivial graph*. A graph of size 0 is called an *empty graph*. A nonempty graph has one or more edges. A  $u$ - $v$  *walk*  $W$  of  $G$  is a finite, alternating sequence  $W : u = u_0, e_1, u_1, e_2, \dots, u_{k-1}, e_k = v$  of vertices and edges, beginning with vertex  $u$  and ending with vertex  $v$ , such that  $e_i = u_{i-1}u_i$  for  $i = 1, 2, \dots, k$ . The number  $k$  (the number of occurrences of edges) is called the *length* of  $W$ . A  $u$ - $v$  walk is *closed* or *open* depending on whether  $u = v$  or  $u \neq v$ . A  $u$ - $v$  *trail* is a  $u$ - $v$  walk in which no edge is repeated. A  $u$ - $v$  *path* is a  $u$ - $v$  walk in which no vertex is repeated. A nontrivial closed trail of a graph  $G$  is referred to as a *circuit* of  $G$ . A circuit  $v_1, v_2, \dots, v_n, v_1$  ( $n \geq 3$ ) whose  $n$  vertices  $v_i$  are distinct is called a *cycle*. Paths on  $n$  vertices are denoted  $P_n$  and cycles on  $n$  vertices are denoted  $C_n$ . A *wheel graph*  $W_n$  on  $n$  vertices is a graph consisting of a cycle  $C_n$  and a single vertex which is adjacent to all vertices in the cycle (see Figure 2).

A graph of  $n$  vertices is *complete* if every two of its vertices are adjacent. This is denoted  $K_n$ . A graph  $G$  is  $k$ -*partite*,  $k \geq 1$ , if it is possible to partition  $V(G)$  into  $k$  subsets  $V_1, V_2, \dots, V_k$  (called partite sets) such that every element of  $E(G)$  joins a vertex of  $V_i$  to a vertex of  $V_j, i \neq j$ . For  $k = 2$ , such graphs are called *bipartite*. A *complete  $k$ -partite graph*  $G$  is a  $k$ -partite graph with partite sets  $V_1, V_2, \dots, V_k$  having the added property that if  $u \in V_i$  and  $v \in V_j, i \neq j$ , then  $uv \in E(G)$ . A complete bipartite graph with partite sets  $V_1$  and  $V_2$ , where  $|V_1| = r$  and  $|V_2| = s$  is denoted by  $K_{r,s}$ . The graph  $K_{1,s}$  is called a *star*.

A vertex  $v$  of a connected graph  $G$  is called a *cut vertex* of  $G$  if its removal

produces a disconnected graph. A nontrivial connected graph with no cut vertices is called a *nonseparable graph*. A labeled graph  $H$  is a *subgraph* of a labeled graph  $G$  if  $V(H) \subseteq V(G)$  and  $E(H) \subseteq E(G)$ . A *block* of a graph  $G$  is a maximal nonseparable subgraph of  $G$ . A *cactus graph* is a connected graph where each block is either an edge or a cycle. A cactus graph having one cycle is called a *unicyclic graph* and a connected cactus graph with no cycle is called a *tree*.

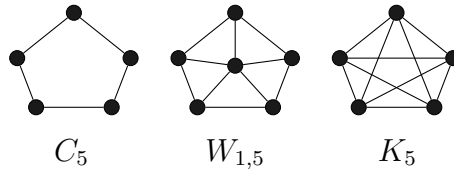


Figure 2:  $C_5, W_5, K_5$

The *complement* of  $G$ , denoted  $\overline{G}$ , is a graph with  $V(\overline{G}) = V(G)$  and  $E(\overline{G}) = \{ab \mid ab \notin E(G)\}$ . In other words, the complement  $\overline{G}$  of a graph  $G$  is the graph with vertex set  $V(G)$  such that two vertices are adjacent in  $\overline{G}$  if and only if these vertices are not adjacent in  $G$ . This means that both  $G$  and its complement  $\overline{G}$  have the same vertices, but  $\overline{G}$  has precisely the edges that  $G$  lacks.

For any vertex  $v \in V(G)$ , the *open neighborhood* of  $v$  is  $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$ , and the *closed neighborhood*  $N_G[v] = N_G(v) \cup \{v\}$ . For a set  $S \subseteq V(G)$ , its *open neighborhood*  $N_G(S) = \cup_{v \in S} N_G(v)$ , and its *closed neighborhood*  $N_G[S] = N_G(S) \cup S$ . These are sometimes denoted  $N(v)$ ,  $N[v]$ ,  $N(S)$  or  $N[S]$ , respectively, if there is no risk of ambiguity.

A set  $S \subseteq V(G)$  is a *dominating set* (abbreviated DS) if  $N[S] = V(G)$ . In other words, every vertex in  $V(G) \setminus S$  is adjacent to a vertex in  $S$ . Every graph has a DS

since  $V(G)$  is such a set. Among all DS of  $G$ , a DS with minimum cardinality is said to be a  $\gamma(G)$ -set. Its cardinality is known as the *domination number* of  $G$  and it is denoted by  $\gamma(G)$ . An *independent dominating set*, abbreviated IDS, of a graph  $G$  is a set  $S \subseteq V(G)$  that is a DS of  $G$  and is an independent set. The *independent domination number* of  $G$ , denoted by  $i(G)$ , is the minimum cardinality of an IDS of  $G$ . An IDS of  $G$  of cardinality  $i(G)$  is called an  $i(G)$ -set.

## 1.2 Very Cost Effective Partitions in Graphs

Very cost effective partitions in graphs were first introduced in [11] and were motivated by the studies of unfriendly partitions (i.e. [1, 8, 19, 20]). According to [11], a vertex  $v$  in a set  $S$  is said to be *cost effective* if it is adjacent to at least as many vertices in  $V \setminus S$  as in  $S$ . A vertex  $v$  is *very cost effective* if it is adjacent to more vertices in  $V \setminus S$  than in  $S$ . A set  $S$  is a (very) cost effective set if every vertex  $v \in S$  is (very) cost effective.

A bipartition  $\pi = \{S, V \setminus S\}$  is called *cost effective* if (i) for every vertex  $i \in S$ ,  $|N(i) \cap (V \setminus S)| \geq |N(i) \cap S|$ , and (ii) for every vertex  $j \in V \setminus S$ ,  $|N(j) \cap S| \geq |N(j) \cap (V \setminus S)|$ . Thus, given a cost effective partition  $\pi = \{S, V \setminus S\}$ , every vertex in  $S$  is cost effective with respect to the set  $S$ , and every vertex in  $V \setminus S$  is cost effective with respect to the set  $V \setminus S$ .

**Theorem 1.1** [11] *Every connected graph  $G$  of order  $n \geq 2$  has a cost effective partition.*

**Proof.** Let  $\pi = \{S, V \setminus S\}$  be any bipartition of  $V(G)$  having the property that the number of edges between  $S$  and  $V \setminus S$  is a maximum. To show that  $\pi$  is a cost

effective partition, we assume to the contrary, that it is not. Then, without loss of generality, we may assume  $S$  is not cost effective. Hence, there is a vertex, say  $v \in S$ , having more neighbors in  $S$  than in  $V \setminus S$ . In this case, moving  $v$  to  $V \setminus S$  will increase the number of edges between  $S$  and  $V \setminus S$ , contradicting the assumption that  $\pi$  has a maximum number of edges between the two sets.  $\square$

A bipartition  $\pi = \{S, V \setminus S\}$  of the vertices  $V$  of a graph  $G = (V, E)$  is *very cost effective* if (i) for every vertex  $u \in S$ ,  $|N[u] \cap (V \setminus S)| \geq |N[u] \cap S|$ , and (ii) for every vertex  $v \in V \setminus S$ ,  $|N[v] \cap S| \geq |N[v] \cap (V \setminus S)|$ . Equivalently, every vertex in  $S$  is very cost effective with respect to  $S$ , and every vertex in  $V \setminus S$  is very cost effective with respect to  $V \setminus S$ . A graph  $G$  is called *very cost effective* if it has a very cost effective bipartition.

Thus, we have seen that every nontrivial graph  $G$  has a cost effective bipartition, but not every graph has a very cost effective bipartition. In this work, we study several families of graphs and determine whether they have a very cost effective bipartition.

### 1.3 Motivation and Applications

In terms of an application, we assume that maintaining edges in a network has an associated cost, and thus they should be used effectively. We assume that an edge between a vertex in a set  $S$  and a vertex in  $V \setminus S$  is being used effectively, while an edge between two vertices in  $S$  is not necessarily being used cost effectively. Thus, a vertex is considered to be cost effective if at least as many edges incident to it are being used cost effectively as not.

Also, (very) cost effective partitions are motivated by business applications. For

example, a company that offers service to both customers and employees would want to be certain to make more money than it is spending. Let the edges inside  $S$  represent services that employees are using (internal cost), and let edges between  $S$  and  $V \setminus S$  represent income from customers paying the company for services. If the company allows employees to use the services it offers for free or at a discounted price, then the company needs to have more edges between  $S$  and  $V \setminus S$  to be able to make a profit. Thus, for each vertex,  $v \in S$  it would be necessary for  $v$  to have at least as many neighbors in  $V \setminus S$  as in  $S$  in order for the company to make money.

## 2 LITERATURE SURVEY

### 2.1 Unfriendly Partitions

Very cost effective partitions are derived from the study of unfriendly partitions of graphs, as follows. Let  $C$  be a two-coloring of the vertices of a graph  $G$ ,  $C : V \rightarrow \{Red, Blue\}$ . For every vertex  $i \in V$ , define  $B(i) = \{j \in N(i), C(j) = Blue\}$  and  $R(i) = \{j \in N(i), C(j) = Red\}$ . Similarly, define  $B(V) = \{j \in V, C(j) = Blue\}$  and  $R(V) = \{j \in V, C(j) = Red\}$ . A two-coloring produces a bipartition of  $V$ ,  $\pi = \{B(V), R(V)\}$ . Given such a bipartition  $\pi$ , we say that an edge  $uv \in E$  is *bicolored* if  $C(u) \neq C(v)$ . A bipartition  $\pi$  is called an *unfriendly partition* if every vertex  $i \in B(V)$  has at least as many neighbors in  $R(V)$  as it does in  $B(V)$ , and every vertex  $j \in R(V)$  has at least as many neighbors in  $B(V)$  as it does in  $R(V)$ . That is, if  $C(i) = Blue$ , then  $|B(i)| \leq |R(i)|$ , and if  $C(i) = Red$ , then  $|R(i)| \leq |B(i)|$ . These types of partitions were defined and studied by Borodin and Koshtochka [3], Aharoni, Milner, and Prikry [1], and Shelah and Milner [20], who called these *unfriendly partitions*. They observed the following, simple proof of which we provide here since it formed the basis for Theorem 1.1.

**Theorem 2.1** [1] *Every finite connected graph  $G$  of order  $n \geq 2$  has an unfriendly partition.*

**Proof.** Let  $\pi = \{B(V), R(V)\}$  be any bipartition of  $V(G)$  having the property that the number of bicolored edges is a maximum. It is easy to see that such a partition is unfriendly. If  $\pi$  is not an unfriendly partition then there must exist a vertex, say  $v \in R(V)$ , having more red neighbors than blue neighbors. In this case, moving  $v$  to

$B(V)$  will increase the number of bicolored edges, contrary to the assumption that  $\pi$  has a maximum number of bicolored edges.  $\square$

Unfriendly partitions have shown up indirectly in several other lines of research. In [5, 6] the concept of  $\alpha$ -domination in graphs is defined and studied. A set  $S \subseteq V$  of vertices in a graph  $G = (V, E)$  is called an  $\alpha$ -*dominating set* if for every vertex  $v \in V \setminus S$ ,  $|N(v) \cap S|/|N[v]| \geq \alpha$ , where  $0 \leq \alpha < 1$ . In the case where  $\alpha \geq 1/2$  every vertex in  $V \setminus S$  meets the *unfriendly condition* in that it has at least as many neighbors in  $S$  as it has in  $V \setminus S$ . However, no unfriendly condition is imposed on the vertices in  $S$ .

Similarly, in [7, 12, 13, 15, 17] global offensive alliances in graphs are defined and studied. A set  $S \subseteq V$  of vertices is called a *global offensive alliance* if for every vertex  $v \in V \setminus S$ ,  $|N(v) \cap S| \geq |N[v] \cap (V \setminus S)|$ . As with  $\alpha$ -domination, if  $S$  is a global offensive alliance, then every vertex  $v \in V \setminus S$  satisfies the unfriendly condition, in that it has at least as many neighbors in  $S$  as it has in  $V \setminus S$ , if you count the vertex  $v$  as one of its own neighbors. But no unfriendly condition is imposed on the vertices in  $S$ .

A partition that is in some sense dual to an unfriendly partition is a bipartition  $\pi = \{B(V), R(V)\}$  called a *satisfactory partition* such that every vertex  $i \in B(V)$  has at least as many neighbors in  $B(V)$  as it does in  $R(V)$ , and every vertex  $i \in R(V)$  has at least as many neighbors in  $R(V)$  as it has in  $B(V)$ . That is, if  $C(i) = \text{Blue}$ , then  $|B(i)| \geq |R(i)|$ , and if  $C(i) = \text{Red}$ , then  $|R(i)| \geq |B(i)|$ . Satisfactory partitions have been studied in [8, 9, 10, 19]. However, unlike unfriendly partitions, not every graph has a satisfactory partition. In fact, it is an NP-complete problem to decide if



an arbitrary graph has a satisfactory partition [2].

## 2.2 Cost Effective Domination

A related topic to very cost effective partitions is (very) cost effective domination. Cost effective domination was introduced in [11] and studied further in [14, 16].

**Definition 2.2** *A set  $S$  is a cost effective dominating set if  $S$  is both a cost effective set and a dominating set.*

**Definition 2.3** *The cost effective domination number  $\gamma_{ce}(G)$  of a graph  $G$  equals the minimum cardinality of a cost effective dominating set in  $G$ . The upper cost effective domination number  $\Gamma_{ce}(G)$  equals the maximum cardinality of a cost effective dominating set in  $G$ . A cost effective set of  $G$  with cardinality  $\gamma_{ce}(G)$  is called a  $\gamma_{ce}$ -set of  $G$ . The very cost effective domination number and the upper very cost effective domination number are defined similarly.*

It should be pointed out that while the property of being a dominating set is *superhereditary*, that is, every superset of a dominating set is also a dominating set, the property of being a cost effective dominating set is not superhereditary. This explains why the definition of the upper cost effective domination number does not include the word “minimal” as it does in the definition of the upper domination number. Without the word *minimal* in the definition of  $\Gamma(G)$ , the value of  $\Gamma(G)$  would equal  $n = |V|$  for all graphs.

**Proposition 2.4** [11] *Every independent dominating set  $S$  in a graph  $G$  is a cost effective dominating set.*

**Proof.** Let  $S \subseteq V$  be an independent dominating set of  $G$ . Then for each  $v \in S$ ,  $|N(v) \cap S| = 0 \leq |N(v) \cap (V \setminus S)|$ . Thus, every vertex  $v \in S$  is cost effective.  $\square$

**Corollary 2.5** [11] *For any graph  $G$ ,  $\gamma(G) \leq \gamma_{ce}(G) \leq i(G)$ .*

The *corona* of graphs  $G$  and  $H$ , denoted  $G \circ H$ , is the graph formed from one copy of  $G$  and  $|V(G)|$  copies of  $H$ , where the  $i^{th}$  vertex in  $V(G)$  is adjacent to every vertex in the  $i^{th}$  copy of  $H$ . We note that all four combinations of the inequalities in Corollary 2.5 are possible (see Figure 3 where the darkened vertices represent a  $\gamma_{ce}$ -set). In particular, for  $\gamma(G) < \gamma_{ce}(G) < i(G)$ , let  $G$  be the corona  $G = K_t \circ \overline{K}_{t-3}$ , for  $t \geq 5$ . Then  $\gamma(G) = t$ ,  $\gamma_{ce}(G) = 2t - 4$ , and  $i(G) = t^2 - 4t + 4$ . Figure 1(a) illustrates  $K_5 \circ \overline{K}_2$ . For graphs  $G$  having  $\gamma(G) < \gamma_{ce}(G) = i(G)$ , let  $G$  be the tree  $K_{1,3} \circ \overline{K}_2$ . That is, the tree formed by adding two leaf vertices adjacent to each vertex of the claw  $K_{1,3}$ . This graph has  $\gamma(G) = 4 < \gamma_{ce}(G) = i(G) = 5$  (see Figure 3(b)). The cycles  $C_n$  have  $\gamma(C_n) = \gamma_{ce}(C_n) = i(C_n) = \lceil n/3 \rceil$  (see Figure 3(c) for the example,  $C_{10}$ ). The *doublestar* is the tree having exactly two non-leaf vertices. Finally, for a graph  $G$  with  $\gamma(G) = \gamma_{ce}(G) < i(G)$ , let  $G$  be the doublestar where each non-leaf vertex is adjacent to  $k \geq 2$  leaves, that is,  $G = K_2 \circ \overline{K}_k$ . Here  $\gamma(G) = \gamma_{ce}(G) = 2 < k + 1 = i(G)$  (see Figure 3(d)).

The trees in Figure 3(b) show that not all trees have cost effective  $\gamma$ -sets. Similarly, we can consider characterizing the graphs obtaining equalities in any of the bounds. Also, [14] provides us with upper bound on the cost effective domination number of trees and characterizes the trees obtaining this bound.

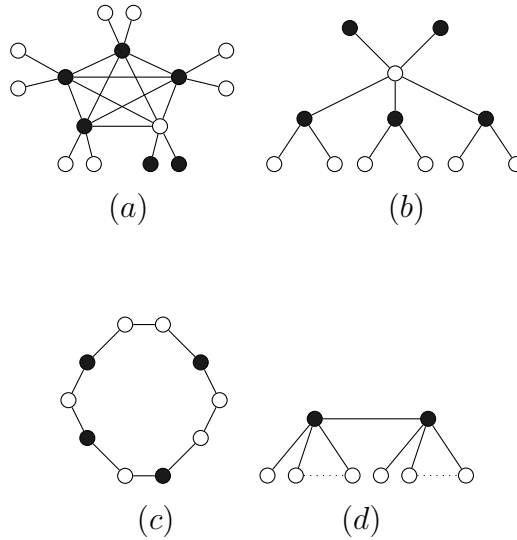


Figure 3: Inequalities of Corollary 2.5.

**Theorem 2.6** [14] *If  $T$  is a tree with  $\gamma(T) \geq 3$ , then  $\gamma(T) \leq \gamma_{ce}(T) \leq 2\gamma(T) - 3$ , and these bounds are sharp.*

In order to characterize the trees obtaining the bound, in [14] the authors define the family  $F$  of trees  $T_t$ , which are obtained from the star  $K_{1,t}$  with center  $x$  and leaves  $x_1, x_2, \dots, x_t$  as follows. Add exactly  $t - 1$  new vertices adjacent to  $x$ , and for  $1 \leq i \leq t$ , add at least  $t - 1$  new vertices adjacent to  $x_i$ .

In general, we use notation and terminology of [11]. We will employ the following terminology and results from [16].

**Definition 2.7** [16] *The minimum (very) cost effectiveness of a graph  $G$ , denoted  $(vce(G))$ ,  $ce(G)$  equals the minimum order of a maximal (very) cost effective set in  $G$ , and the maximum (very) cost effectiveness of  $G$ ,  $(VCE(G))$ ,  $CE(G)$  equals the maximum cardinality of a (very) cost effective set in  $G$ .*

**Proposition 2.8** [16] *For any connected graph  $G$  of order  $n > 1$ ,  $1 \leq vce(G) \leq VCE(G) \leq n - 1$ , and these bounds for  $vce(G)$  and  $VCE(G)$  are sharp.*

**Proof.** Trivially,  $vce(K_1) = vce(K_3) = 1$ . In order to show that the upper bound for  $VCE(G)$  is sharp note that for the star  $K_{1,n}$  of order  $n + 1$ ,  $VCE(K_{1,n}) = n$ .  $\square$

It is natural to ask: for which classes of graphs is  $vce(G) = 1$  and for which classes of graphs is  $VCE(G) = n - 1$ ?

**Theorem 2.9** [16] *Let  $G$  be a connected graph of order  $n > 1$ . Then (i)  $vce(G) = 1$  if and only if  $G = K_1 + (rK_2 \cup sK_1)$ , for  $r + s \geq 1$ , and (ii)  $VCE(G) = n - 1$  if and only if  $G = K_{1,n_1}$ .*

**Proposition 2.10** [16] *Every independent set  $S$  in a graph  $G$  without isolated vertices is a very cost effective set.*

The proof is similar to a proof of Proposition 1.4.

**Observation 2.11** *All bipartite graphs are very cost effective.*

**Corollary 2.12** [16] *For any connected graph  $G$  of order  $n > 1$ ,  $VCE(G) \leq CE(G)$ .*

An interesting relation between very cost effective sets and the dominating sets is established in the next proposition.

**Proposition 2.13** [16] *Every maximal very cost effective set in a connected graph  $G$  is a dominating set.*

**Proof.** Let  $S \subseteq V$  be a maximal very cost effective set, and assume that  $S$  is not a dominating set. Let  $v \in V \setminus N[S]$  be a vertex that is not dominated by any vertex in  $S$ . Then it follows that  $S \cup \{v\}$  is a very cost effective set since  $|N(v) \cap S| = 0 < |N(v) \cap (V \setminus S)|$ , contradicting the assumption that  $S$  is a maximal very cost effective set.  $\square$

We have already discussed the properties of cost effective domination. At this point, we turn our attention to very cost effective domination since it is more closely related to the field of our research. Thus, [16] suggests the following definitions.

**Definition 2.14** *A set  $S$  is a very cost effective dominating set if  $S$  is both a very cost effective set and a dominating set.*

**Definition 2.15** *The very cost effective domination number of a graph  $G$ , denoted  $\gamma_{vce}(G)$ , equals the minimum cardinality of a very cost effective dominating set in  $G$ . The upper very cost effective domination number  $\Gamma_{vce}(G)$  equals the maximum cardinality of a minimal dominating set in  $G$  that is very cost effective.*

It should be pointed out that while the property of being a dominating set is superhereditary, that is, every superset of a dominating set is also a dominating set, the property of being a very cost effective dominating set is not superhereditary. It is easy to see that every isolate-free graph has a very cost effective dominating set since every maximal independent set is both dominating and very cost effective.

**Proposition 2.16** [16] *For any graph  $G$  without isolated vertices,  $\gamma(G) \leq \gamma_{ce}(G) \leq \gamma_{vce}(G) \leq \{vce(G), i(G)\}$ , and all of these inequalities can be strict.*

Also, in terms of  $\Gamma$  we have the following inequalities.

**Corollary 2.17** [16] *For any connected graph  $G$  without isolated vertices,  $\Gamma_{vce}(G) \leq \Gamma_{ce}(G) \leq \Gamma(G)$ , and these bounds are sharp.*

### 2.3 Variations of the Aharoni, Milner, Prikry Theorem

In Section 2, we gave a simple proof of Theorem 2.1 that every graph of order  $n \geq 2$  has an unfriendly partition. This theorem can be represented equivalently by the following algorithm, which was given in [1].

**Algorithm** Cost Effective 2-Coloring

**Input:** A graph  $G$  of order  $n \geq 2$ .

**Output:** A partition  $V(G)$  into two cost effective sets.

1. Arbitrarily color the vertices red and blue.
2. **While** there exists a vertex  $v \in V$  having more neighbors of its own color than the other color **do** change  $v$ 's color to the other color **od**.

This algorithm must terminate, since every change of color increases the number of bi-colored edges. When this algorithm is finished, the set of vertices colored red and the set of vertices colored blue will both be cost effective sets. It is easy to see that one of these two sets must have cardinality at least  $n/2$ .

**Corollary 2.18** [16] *For any graph of order  $n \geq 2$ ,  $CE(G) \leq n/2$ .*

The following results hold for odd-regular graphs. They represent a family of very cost effective graphs.

**Corollary 2.19** [16] *The vertices of every graph of order  $n \geq 2$ , in which every vertex has odd degree, can be partitioned into two maximal very cost effective sets.*

**Corollary 2.20** [14] *If  $G$  has a  $\gamma_{ce}(G)$ -set that consists of only odd vertices, then  $\gamma_{ce}(G) = \gamma_{vce}(G)$ .*

Note that in particular,  $\gamma_{ce}(G) = \gamma_{vce}(G)$  for cubic graphs.

We have previously observed that every nontrivial graph can be partitioned into two cost effective sets, but not every graph can be partitioned into two very cost effective sets, for example the odd order complete graphs  $K_{2n+1}$ . However most graphs can be partitioned into three very cost effective sets, using the following simple algorithm given in [16].

**Algorithm** Very Cost Effective 3-Coloring (VCE3)

**Input:** A graph  $G$  of order  $n \geq 3$  having no isolated vertices.

**Output:** A partition  $V(G)$  into three non-empty, very cost effective sets.

1. Color one vertex red, one vertex white and one vertex blue. Then, arbitrarily color each remaining vertex red, white or blue.
2. **While** there exists a vertex  $v \in V$  that has more neighbors of its own color than it has of one of the other two colors **do** change  $v$ 's color to the color appearing least frequently in  $N(v)$ . **od**

Algorithm VCE3 must terminate since every change of color increases the number of bicolored edges, or equivalently, decreases the number of mono-colored edges.

**Corollary 2.21** [16] *The vertices of any graph having order  $n \geq 3$  and no isolated vertices can be partitioned into three very cost effective sets.*

**Corollary 2.22** [16] *For any graph having order  $n \geq 3$  and no isolated vertices,  $VCE(G) \geq n/3$ , and this bound is sharp. Notice that for cycles of order  $n = 3k$ ,  $VCE(G) = n/3 = k$ .*



### 3 PRELIMINARY RESULTS

The research for this thesis was conducted in two phases. Results from the first phase have already been published in [11]. In this section, we present results that serve as an introduction to the main results of this thesis found in Section 4.

As we have observed, all nontrivial, connected graphs are cost effective, in that they have a cost effective bipartition, but not all graphs are very cost effective. The class of graphs that are not very cost effective includes the complete graphs of odd order. We first establish four families of very cost effective graphs.

**Proposition 3.1** *The following classes of graphs are very cost effective:*

(i) *connected bipartite graphs  $G = (X, Y, E)$  of order  $n \geq 2$ ,*

(ii) *complete graphs  $K_{2k}$  of even order,*

(iii) *the corona  $G \circ K_1$ , where  $G$  is a nontrivial connected graph, and*

(iv) *the wheel  $W_{1,k}$  for  $k \neq 4$ .*

**Proof.**

- (i) Since  $G = (X, Y, E)$  has no isolated vertices, it is easy to see that the natural bipartition  $\pi = \{X, Y\}$  into two disjoint independent sets  $X$  and  $Y$  is very cost effective.

- (ii) Simply partition the vertices of  $K_{2k}$  into two sets  $X$  and  $Y$  of order  $k$ .
- (iii) Since  $G$  is assumed to be connected and of order  $n \geq 2$ , by Theorem 1.1,  $G$  has a cost effective bipartition  $\pi = \{R, B\}$ . But this may not be a very cost effective bipartition. However, if each leaf is colored differently than the vertex to which it is adjacent, the resulting bipartition is very cost effective.
- (iv) For the wheel,  $W_{1,k}$ , we consider two cases based on the parity of  $k$ .

**(Case 1)** Let  $k = 2t+1$ . Color the vertices of the cycle  $C_{2t+1}$ ,  $1, 2, 1, 2, \dots, 1, 2, 3$ , and color the central vertex  $x$  with color 4. Let  $R$  be the set of vertices colored 1 together with the single vertex, say  $y$ , colored 3, and let  $B$  be the set of vertices colored 2 together with the central vertex  $x$  colored 4. We show that the bipartition  $\pi = \{R, B\}$  is very cost effective.

Notice that the set of vertices colored 1 is independent. Thus, if  $v$  is a vertex colored 1, then  $1 \leq |N[v] \cap R| \leq 2$ , while  $2 \leq |N(v) \cap B| \leq 3$ . Also for  $y$ , the only vertex colored 3,  $2 = |N[y] \cap R| = |N(y) \cap B|$ . Similarly, the set of vertices colored 2 is independent. Thus, if  $w$  is a vertex colored 2, then  $|N[w] \cap B| = |N(w) \cap R| = 2$ . Also for  $x$ , the only vertex colored 4,  $2 \leq |N[x] \cap B| = |N(x) \cap R| = t + 1$ . Thus, the bipartition  $\pi = \{R, B\}$  is very cost effective.

**(Case 2)** Let  $k \geq 6$  be even, and let the vertices of the cycle  $C_k$  be labeled in order  $v_1, v_2, \dots, v_k$ . First color the center vertex  $x$  red. For the vertices  $v_i$ , for

$1 \leq i \leq 6$ , color  $v_1$  and  $v_4$  red and  $v_2, v_3, v_5, v_6$  blue. Then for  $v_i$ ,  $7 \leq i \leq k$ , alternately color the vertices red and blue (beginning with red). This coloring forms a bipartition  $\pi = \{R, B\}$ , where  $R$  is the set of red vertices and  $B$  is the set of blue vertices. Consider the set  $R$ . For the vertex  $x$ ,  $|N[x] \cap R| = (k - 6)/2 + 3 < |N(x) \cap B| = (k - 6)/2 + 4$ . For every other vertex, say  $v \in R$ ,  $|N[v] \cap R| = 2 = |N(v) \cap B|$ . For  $v \in B$ ,  $1 \leq |N[v] \cap B| \leq 2$  and  $2 \leq |N(v) \cap R| \leq 3$ . Thus, the bipartition  $\pi = \{R, B\}$  is a very cost effective partition.  $\square$

We note that the wheel  $W_{1,4}$  is not very cost effective. We next consider other families of graphs that are not very cost effective.

**Proposition 3.2** *No cycle  $C_{2k+1}$  of odd order is very cost effective.*

**Proof.** Any bipartition  $\pi = \{R, B\}$  of the vertices of an odd cycle into two non-empty sets must place two adjacent vertices into either set  $R$  or set  $B$ . This follows since odd cycles do not have partitions into two independent sets. Suppose that two adjacent vertices, say  $u$  and  $v$ , are in the same set, say  $R$ . Let vertex  $w$  be the second vertex adjacent to  $v$  on the cycle. If  $w \in R$ , then vertex  $v$  does not meet the very cost effective condition, having no neighbors in  $B$ . While if  $w \in B$ , then  $|N[v] \cap R| = 2 > |N(v) \cap B| = 1$ . Thus,  $C_{2k+1}$  does not have a very cost effective bipartition.  $\square$

**Proposition 3.3** *Let  $H = C_5 - C_5$  be a graph obtained from two disjoint cycles  $C_5$  by adding one edge between any vertex in one  $C_5$  and any vertex in the other  $C_5$ . Then  $H$  is not very cost effective.*

**Proof.** Let the vertices of the first five-cycle of  $H$  be labelled in order  $u_1, u_2, u_3, u_4, u_5$ , and those of the second five cycle of  $H$  be labelled  $v_1, v_2, v_3, v_4, v_5$ . Assume that vertices  $u_1$  and  $v_1$  are joined by an edge. We must show that this graph does not have a very cost effective bipartition.

Assume that  $\pi = \{R, B\}$  is a very cost effective bipartition of the graph  $H$ . Assume without loss of generality that  $u_1 \in R$ . Vertex  $u_1$  is adjacent to vertices  $u_2$  and  $u_5$ . If  $u_2 \in R$ , then it cannot be very cost effective, having two vertices in its closed neighborhood colored red, but at most one vertex in its neighborhood colored blue. Thus, vertex  $u_2$  must be colored blue. The same argument applies to vertex  $u_5$ , and thus vertex  $u_5$  must be colored blue. Now by the same reasoning, vertex  $u_3$  cannot be colored blue, else vertex  $u_2$  is not very cost effective, having two vertices in its closed neighborhood colored blue, and at most one colored red. Therefore, vertex  $u_3$  must be colored red. But the same argument can be used to show that vertex  $u_4$  must also be colored red. Thus, neither  $u_3$  nor  $u_4$  is very cost effective.  $\square$

**Corollary 3.4** *No graph  $G$  containing a  $C_5$  attached to the rest of the graph by a connecting edge is very cost effective.*

By much the same reasoning, it can be shown that no graph containing an attached odd cycle has a very cost effective bipartition. This result is generalized even further in Section 4 (see Theorem 4.5). A *cactus* is any connected graph having the property that no edge is contained in two or more cycles. Thus, all trees are cacti, as are all cycles. The class of cacti provide more examples of graphs that are not very cost effective. Since all bipartite cacti are very cost effective, a cactus that is not very

cost effective must contain an odd cycle. First note that a *unicyclic graph* is any connected graph  $G$  of order  $n$  and size  $n$ . Equivalently, it is a graph obtained from a tree by adding an edge between any two non-adjacent vertices of  $T$ . Of course, every unicyclic graph is a cactus. It follows that if a unicyclic graph is not very cost effective, then its only cycle has odd order. However, as we shall see in Section 4, a unicyclic graph with an odd cycle can be very cost effective.

Recall that the *Cartesian product* of two graphs,  $G$  and  $H$ , is the graph denoted  $G \square H = (V(G) \times V(H), E(G \square H))$ , where two vertices  $(u, v)$  and  $(w, x)$  are adjacent in  $G \square H$  if and only if either  $u = w$  and  $v$  is adjacent to  $x$  in  $H$ , or  $u$  is adjacent to  $w$  in  $G$  and  $v = x$ . In  $G \square H$ , there exists a copy of  $H$  for each vertex in  $V(G)$  and a copy of  $G$  for each vertex in  $H$ . To aid in our discussion, we let  $H_v$  represent the copy of  $H$  in  $G \square H$  corresponding to the vertex  $v$  in  $G$ . In other words,  $H_v$  is the subgraph induced by the vertices of  $V(G \square H)$  whose first coordinate is  $v$ .

**Theorem 3.5** *Let  $G = (X, Y, E)$  be a bipartite graph with no isolated vertices and  $H$  be a nontrivial, connected graph. Then the Cartesian product  $G \square H$  is very cost effective.*

**Proof.** Let  $G$  be an isolate-free, bipartite graph with partite sets  $X$  and  $Y$ , and let  $H$  be a nontrivial, connected graph. Since  $H$  is connected, Theorem 1.1 implies that  $H$  has a cost effective bipartition  $\pi = (V_1, V_2)$ . We build a very cost effective partition of  $G \square H$  as follows. Let  $(u, v) \in V(G \square H)$ . First, assume that  $u$  is the partite set  $X$  in  $G$ . If  $v \in V_1(H)$ , then color  $(u, v)$  red; and if  $v \in V_2(H)$ , then color  $(u, v)$  blue. Next, let  $u \in Y(G)$ . If  $v \in V_1(H)$ , then color  $(u, v)$  blue; and if  $v \in V_2(H)$ , then color  $(u, v)$  red. In other words, for a copy of  $H$  corresponding to a vertex in the partite

set  $X$  of  $G$ , we use the cost effective partition of  $H$ ,  $\pi = (V_1, V_2)$ ; and for a copy of  $H$  corresponding to a vertex in the other partite set  $Y$ , we reverse the colors of those in  $\pi$ . We note that since  $\pi = (V_1, V_2)$  is a cost effective partition for  $H$ , each vertex  $(x, y)$  is cost effective in  $H_x$ . Also, every neighbor of  $(x, y)$  that is not in  $H_x$  is colored a different color than  $(x, y)$ . Since  $G$  is isolate-free, each vertex  $(x, y)$  has at least one neighbor outside of  $H_x$ . Hence, each vertex  $(x, y)$  is very cost effective with respect to its color class in  $G \square H$ . It follows that the bipartition  $(R, B)$ , where  $R$  is the set of red vertices and  $B$  is the set of blue vertices, is a very cost effective bipartition of  $G \square H$ .  $\square$

This result from our preliminary work raised several questions that we were able to answer in the second phase of our research. Answers to the following questions can be found in Section 4.

**Question 1** *If  $G$  is very cost effective, is  $G \square H$  very cost effective for all graphs  $H$ ?*

**Question 2** *If  $G$  and  $H$  are both very cost effective, is  $G \square H$  very cost effective?*

**Question 3** *Is every Cartesian product  $G \square H$  very cost effective?*

## 4 MAIN RESULTS

In this section we present the results from phase 2 of our research. In particular, we present results on very cost effective partitions and explore families of graphs, such as Cartesian products and cacti graphs. To aid in our discussion, given a partition  $\pi = \{R, B\}$  of a graph  $G$ , we say the vertices of  $R$  are colored red and the vertices of  $B$  are colored blue under  $\pi$ . If  $\pi$  is a very cost effective partition of  $G$ , then we say that  $G$  is very cost effective under  $\pi$ .

### 4.1 General Results

The following result is shown in [16].

**Theorem 4.1** [16] *Every connected graph  $G$  of order  $n \geq 3$  can be partitioned into three very cost effective sets.*

**Theorem 4.2** *No very cost effective partition of  $G$  is a very cost effective partition of its complement  $\overline{G}$ .*

**Proof.** Let  $\pi = \{V_1, V_2\}$  be a very cost effective partition of  $G$ . Assume to the contrary that  $\pi$  is a very cost effective partition of  $\overline{G}$ . Without loss of generality, let  $u$  be a vertex in  $V_1$ . Let  $|N_G(u) \cap V_1| = k_1$  and  $|N_G(u) \cap V_2| = k_2$ . Then,  $k_1 < k_2$  because  $u$  is very cost effective in  $G$ . Also,  $|N_{\overline{G}}(u) \cap V_1| = |V_1| - k_1 - 1 < |V_2| - k_2 = |N_{\overline{G}}(u) \cap V_2|$ , because according to our assumption,  $u$  is very cost effective in  $\overline{G}$  under  $\pi$ . Since  $k_1 < k_2$ , the last inequality can be written as  $|V_1| - k_2 < |V_2| - k_2$ . Hence,  $|V_1| < |V_2|$ . On the other hand, an analogous argument for an arbitrary vertex in  $V_2$  shows that  $|V_2| < |V_1|$ , a contradiction.  $\square$

As promised in Section 3, our next result generalizes Corollary 3.4.

**Lemma 4.3** *If  $G$  is a graph that has an odd cycle as an endblock, then  $G$  is not very cost effective.*

**Proof.** Let  $G$  contain an odd cycle endblock  $C$ , and label the vertices on  $C$ ,  $u_1, u_2, \dots, u_{2k+1}$  such that  $u_1$  is the cut vertex on  $C$ . Suppose to the contrary that  $\pi = \{R, B\}$  is a very cost effective bipartition of the graph  $G$ . Assume, without loss of generality, that  $u_1 \in R$ . Since  $C$  is an endblock and  $u_1$  is the cut vertex on  $C$ ,  $\deg(u_j) = 2$  for  $2 \leq j \leq 2k+1$ . Since  $\pi$  is a very cost effective partition, it follows that  $u_2 \notin R$ , that is,  $u_2 \in B$ . Further,  $u_{2i+1} \in R$  and  $u_{2i} \in B$  for  $1 \leq i \leq k$ . But then  $u_{2k+1} \in R$  and  $|N(u_{2k+1}) \cap R| = |N(u_{2k+1}) \cap B| = 1$ , so  $u_{2k+1}$  is not very cost effective under  $\pi$ , a contradiction.  $\square$

Figure 4 is an example of a graph  $G$  that has an odd cycle endblock.

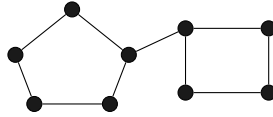


Figure 4: Graph  $G$  has an odd cycle endblock

#### 4.2 Cartesian Product and Cacti Graphs

Here we answer the questions raised in Section 3 concerning Cartesian products. Not every Cartesian product is very cost effective. For example, the  $C_3 \square C_3$  shown in Figure 5 is not very cost effective.



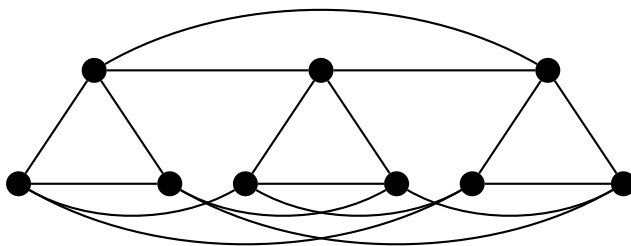


Figure 5:  $C_3 \square C_3$

On the other hand, in our next result, we prove that the Cartesian product of a very cost effective graph and an arbitrary connected graph is very cost effective.

**Theorem 4.4** *Let  $G$  be a very cost effective graph. Then  $G \square H$  is a very cost effective graph for any connected graph  $H$ .*

**Proof.** Let  $G$  be a graph with a very cost effective partition  $\pi_1 = (V_1(G), V_2(G))$ . If  $H$  is the trivial graph, then  $G \square H = G$ , and, clearly, the result holds. Let  $H$  be a non-trivial, connected graph. By Theorem 1.1,  $H$  has a cost effective partition  $\pi_2 = (V_1(H), V_2(H))$ . We define two colorings,  $C_1$  and  $C_2$ , of  $H$ . Let  $C_1$  color the vertices of  $V_1(H)$  red and the vertices of  $V_2(H)$  blue; and let the coloring  $C_2$  swap the colors of  $C_1$ . We consider  $u \in V(G)$  to determine the coloring of  $H_u$  in  $G \square H$ . If  $u \in V_1(G)$ , then use the coloring  $C_1$  for  $H_u$ ; while if  $u \in V_2(G)$ , then use the coloring  $C_2$  for  $H_u$ . Let  $R$  be the set of red vertices in  $G \square H$ , and let  $B$  be the set of blue ones. To see that  $\pi = (R, B)$  is a very cost effective partition of  $G \square H$ , consider an arbitrary vertex  $(u, v) \in V(G \square H)$ . Without loss of generality, assume that  $(u, v)$  is colored red and  $H_u$  is colored by  $C_1$ . Since  $C_1$  is a red-blue cost effective partition of  $H_u$ ,  $(u, v)$  has at least as many blue neighbors in  $H_u$  as red ones. It suffices to show that  $(u, v)$  has more blue neighbors than red ones in  $G_v$ . Since  $(u, v)$  is red under  $C_1$  of  $H_u$ , it follows that  $(x, v)$  is red for all  $x \in V_1(G_v)$ . Moreover,  $(y, v)$  is blue for all

$y \in V_2(G_v)$ . Since  $\pi_2 = (V_1(G_v), V_2(G_v))$  is a very cost effective partition of  $G_v$ , we have that  $(u, v)$  has more blue neighbors than red ones in  $G_v$ . Hence,  $\pi = (R, B)$  is a very cost effective partition of  $G \square H$ .  $\square$

Figure 6 illustrates the Cartesian product  $K_2 \square C_3$ , in which  $K_2$  is a very cost effective graph and  $C_3$  is not.

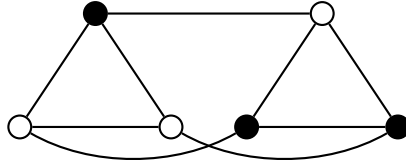


Figure 6:  $K_2 \square C_3$

Recall that a cactus graph is a connected graph where each block is either an edge or a cycle. The cacti is a family of graphs that has many applications. According to [18], cacti graphs are used as a data structure for comparing sets of related genomes. Cacti graphs can represent duplications and general genomic rearrangements. Additionally, they naturally decompose the common substructures in a set of related genomes into a hierarchy of chains that can be visualized as two-dimensional multiple alignments and nets that can be visualized in circular genome plots.

Given a block  $B$  and a vertex  $v$  in  $B$  of a cactus graph  $G$ , we define a  $v$ -branch  $B_v$  to be a connected subgraph such that  $V(B) \cap V(B_v) = \{v\}$ , and  $B_v$  is maximal with this property. We say that  $v$  is the root of the  $v$ -branch  $B_v$  of  $B$ , and that  $B$  supports  $B_v$ . To aid in our discussion, we say that a branch is *bad* if it is not very cost effective. In Figure 7, we give an example of a  $v$ -branch. Let  $G$  be the graph in Figure 7 and  $B$  be a block of  $G$  induced by the set  $\{v_6, v_7, v_8\}$ . The  $v_6$ -branch of  $B$  is the subgraph induced by the set  $\{v_6, v_9, v_{10}, \dots, v_{20}\}$ , and the  $v_7$ -branch of  $B$  is the subgraph induced by the set  $\{v_7, v_1, v_2, \dots, v_5\}$ . Also note that if  $B'$  is the block induced by the set  $\{v_6, v_9, v_{10}, v_{11}, v_{12}\}$ . Then the  $v_6$ -branch of  $B'$  is the graph  $G \setminus \{v_9, v_{10}, v_{11}, v_{12}\}$ .

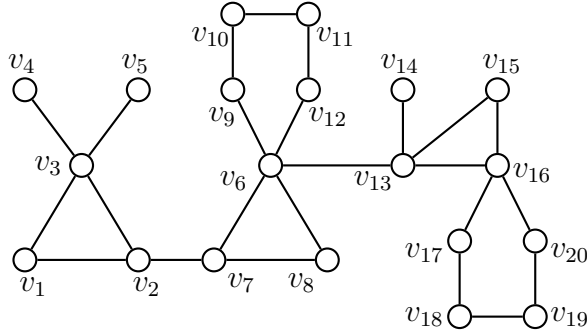


Figure 7: Example of a branch

**Theorem 4.5** *A cactus graph  $G$  is very cost effective if and only if every odd cycle block  $C$  of  $G$  supports two branches  $C_u$  and  $C_v$ , where  $u$  and  $v$  are adjacent vertices on  $C$  and each of  $C_u$  and  $C_v$  is a very cost effective cactus.*

**Proof.** We say that a cactus  $G$  has Property  $P$  if every odd cycle block  $C$  of  $G$  supports two branches  $C_u$  and  $C_v$ , where  $u$  and  $v$  are adjacent vertices on  $C$  and each of  $C_v$  and  $C_u$  is a very cost effective cactus.

We show that if  $G$  is a very cost effective cactus graph, then  $G$  has Property  $P$  by proving the contrapositive. Assume that  $G$  is a cactus graph that does not have Property  $P$ . Since Property  $P$  vacuously holds for graphs with no odd cycles,  $G$  has an odd cycle, say  $C$ , that does not comply with Property  $P$ .

Suppose, for the purpose of a contradiction, that  $G$  is very cost effective, and let  $\pi = \{R, B\}$  be a very cost effective partition of  $G$ . Since  $C$  is an odd cycle, under  $\pi$  at least two adjacent vertices, say  $u$  and  $v$ , of  $C$  are colored the same color, say red. Assume first that adjacent vertices  $u$  and  $v$  are roots of branches  $C_u$  and  $C_v$ , respectively. Since  $G$  does not have Property  $P$ , we may assume without loss of generality that  $C_u$  is not very cost effective. Since  $\pi$  is a very cost effective partition

of  $G$ , all the vertices of  $C_u$ , except possibly  $u$ , are very cost effective under  $\pi$  in  $C_u$ . Thus,  $u$  is not very cost effective under  $\pi$  in  $C_u$ , that is,  $u$  has at least as many red neighbors as blue ones in  $C_u$ . Since  $u$  is very cost effective under  $\pi$  in  $G$ , that is,  $u$  has more blue neighbors than red ones in  $G$ , we deduce that the two neighbors of  $u$  on  $C$  are both blue. In particular,  $v \in B$ , a contradiction.

Hence, we may assume that at least one of  $u$  and  $v$ , say  $u$ , is not the root of a branch. But since  $u$  and  $v$  are red under  $\pi$  and  $\deg_G(u) = 2$ ,  $u$  is not very cost effective in  $G$  under  $\pi$ , a contradiction. Thus, the result holds.

For the converse, assume that  $G$  has Property  $P$ . To show that  $G$  is very cost effective, we proceed by induction on the number  $c$  of odd cycles in  $G$ . If  $G$  has no odd cycles, then  $G$  is bipartite and by Observation 2.11,  $G$  is very cost effective. Thus, we may assume that  $G$  has an odd cycle. If  $c = 1$ , then let  $C$  be the odd cycle of  $G$ . Label the vertices of  $C$  as  $u_1, u_2, \dots, u_k, u_1$ , such that  $u_1$  and  $u_k$  are a pair of adjacent vertices with very cost effective branches,  $C_{u_1}$  and  $C_{u_k}$ , respectively. We show that  $G$  is very cost effective by giving a very cost effective partition  $\pi = \{R, B\}$  of  $G$ . Let  $u_i \in R$  if  $i$  is odd and  $u_i \in B$  if  $i$  is even. Then, both  $u_1$  and  $u_k$  are red and each have exactly one blue neighbor on  $C$ . Let  $p_i = \{V_1, V_2\}$  be a very cost effective partition of  $H' = C_{u_1}$ . If  $u_1 \in V_i$ , then, in  $H'$ , color the vertices of  $V_i$  red and the vertices of  $V_{3-i}$  blue. Hence,  $u_1$  is very cost effective in  $H'$ , that is,  $u_1$  has more blue neighbors than red ones in  $H'$ , implying that  $|N_G(u_1) \cap R| = |N_{H'}(u_1) \cap R| + 1 < |N_{H'}(u_1) \cap B| + 1 = |N_G(u_1) \cap B|$ . Hence,  $u_1$  is very cost effective in  $G$ . A similar argument shows that  $u_k$  is also very cost effective under  $\pi$  in  $G$ . For every  $u_i$ ,  $2 \leq i \leq k - 1$ , if  $\deg_G(u_i) = 2$ , then  $u_i$  is very cost effective in  $G$ . Assume that  $\deg(u_i) \geq 3$ , for some  $i$ , where  $2 \leq i \leq k - 1$ ,

and let  $C_{u_i}$  be the branch rooted at  $u_i$  from  $C$ . Since  $C_{u_i}$  has no odd cycles,  $C_{u_i}$  is bipartite, and so, by Observation 2.11,  $C_{u_i}$  is very cost effective. Let  $\pi' = \{V_1, V_2\}$  be a very cost effective partition of  $C_{u_i}$ . Relabeling the sets  $V_1$  and  $V_2$  if necessary, we may assume that  $u_i \in V_1$  of  $C_{u_i}$ . If  $i$  is odd, then color the vertices of  $V_1$  red and the vertices of  $V_2$  blue. If  $i$  is even, then we color the vertices of  $V_1$  blue and the vertices of  $V_2$  red. Then in  $G$ , the vertices of  $C_{u_i}$  are very cost effective under  $\pi = (R, B)$ . Hence if  $c = 1$ , then  $G$  is very cost effective, establishing our base case.

Let  $c > 1$ , and assume that any cactus graph with Property  $P$  having fewer than  $c$  odd cycles is very cost effective. Let  $G$  be a cactus graph having  $c \geq 2$  odd cycles and Property  $P$ . Begin with an odd cycle  $C$  of  $G$ . If every branch supported by  $C$  is very cost effective, then using an argument similar to the case for  $c = 1$ , we can show that  $G$  is very cost effective. Hence, we may assume that  $G$  has an odd cycle that supports a bad branch. Among all odd cycles with bad branches, select  $C$  to be one that minimizes the number of vertices in a bad branch supported by  $C$ . Label the vertices of  $C$  as  $u_1, u_2, \dots, u_k, u_1$ , such that  $u_1$  and  $u_k$  are a pair of adjacent vertices with very cost effective branches,  $C_{u_1}$  and  $C_{u_k}$ , respectively. Let  $C_{u_i}$  be a bad branch of  $C$  having the minimum number of vertices. Then,  $C_{u_i}$  has at least one odd cycle. Moreover, since  $C$  is not a cycle in  $C_{u_i}$ ,  $C_{u_i}$  has fewer than  $c$  cycles. If  $C_{u_i}$  has Property  $P$ , then applying our inductive hypothesis to  $C_{u_i}$ , we have that  $C_{u_i}$  is very cost effective. This is a contradiction. Thus, we may assume that  $C_{u_i}$  does not have Property  $P$ . Since  $G$  has Property  $P$ , we conclude that  $u_i$  is a vertex on an odd cycle, say  $C'$ , of  $C_{u_i}$ , and  $C'_{u_i}$  is a very cost effective branch supported by  $C'$ . Since  $G$  has Property  $P$ , there is a neighbor of  $u_i$ , say  $x$ , on  $C'$ , such that  $C'_x$  is very cost

effective. Coloring  $x$  the same color as  $u_i$  and proceeding as in our base case, we can show that all the vertices of  $C'$  are very cost effective. If every branch supported by  $C'$  is very cost effective, then using an argument similar to the case for  $c = 1$ , we can show that  $G$  is very cost effective. Thus, we may assume that for some vertex  $y$  on  $C'$ ,  $C'_y$  is a bad branch. But, since  $C'_y$  is a proper subgraph of  $C_{u_i}$ ,  $C'_y$  has fewer vertices than  $C_{u_i}$ , contradicting our choice of  $C$ . Hence, we conclude that  $G$  is very cost effective.  $\square$

Figure 8 illustrates how a graph having property  $P$  can be partitioned into two very cost effective sets.

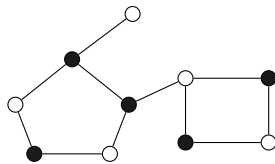


Figure 8: Graph  $G$  has property  $P$

## 5 CONCLUDING REMARKS

We have characterized the very cost effective cacti graphs and showed that if a graph  $G$  or  $H$  has a very cost effective bipartition, then so does their Cartesian product  $G \square H$ . We conclude with some open problems suggested by this work:

1. Characterize the very cost effective graphs.
2. We have seen that the odd-regular graphs are very cost effective. Characterize the even-regular graphs that are very cost effective.
3. Study very cost effective partitions in other families of graphs.
4. Characterize the very cost effective Cartesian products.



## BIBLIOGRAPHY

- [1] R. Aharoni, E. C. Milner, and K. Prikry. Unfriendly partitions of a graph. *J. Combin. Theory, Ser. B* 50(1):1-10, 1990.
- [2] C. Bazgan, Z. Tuza and D. Vanderpooten. The satisfactory partition problem. *Discrete Appl. Math.* 154:1236-1245, 2006.
- [3] A. V. Borodin and A. V. Kostochka. On an upper bound of a graph's chromatic number, depending on the graph's degree and density. *J. Combin. Theory Ser. B* 23:247-250, 1977.
- [4] G. Chartrand and L. Lesniak. *Graphs & Digraphs*, Fifth Edition, CRC Press Company, 2008.
- [5] F. Dahme, D. Rautenbach, and L. Volkmann. Some remarks on  $\alpha$ -domination. *Discuss. Math. Graph Theory*, 24:423-430, 2004.
- [6] J. E. Dunbar, D. G. Hoffman, R. C. Laskar, and L. R. Markus.  $\alpha$ -domination. *Discrete Math.*, 211:11-26, 2000.
- [7] O. Favaron, G. H. Fricke, W. Goddard, S. M. Hedetniemi, S. T. Hedetniemi, P. Kristiansen, R. C. Laskar, and R. D. Skaggs. Offensive alliances in graphs. *Discuss. Math. Graph Theory*, 24:263-275, 2004.
- [8] M. U. Gerber and D. Kobler. Algorithmic approach to the satisfactory graph partitioning problem. *European J. Oper. Res.* 125(2):283-291, 2000.

- [9] M. U. Gerber and D. Kobler. Classes of graphs that can be partitioned to satisfy all their vertices. *Australas. J. Combin.* 29:201-214, 2004.
- [10] M. U. Gerber and D. Kobler. Partitioning a graph to satisfy all vertices. *Tech. Rept. Swiss Federal Inst. Tech.*, Lausanne, 1998.
- [11] T. W. Haynes, S. M. Hedetniemi, S. T. Hedetniemi, T. L. McCoy, and I. Vasylieva. Cost effective domination in graphs. *Congr. Numer.* 211:197-209, 2012.
- [12] T. W. Haynes, S. T. Hedetniemi, and M. A. Henning. Global defensive alliances. *Proc. 17<sup>th</sup> Int. Symp. Comput. Inform. Sci., I, ISCIS XVII (Orlando, FL, 2002)*, 303-307, CRC Press, 2002.
- [13] T. W. Haynes, S. T. Hedetniemi, and M. A. Henning. Global defensive alliances in graphs. *Electron. J. Combin.*, 10:Research Paper 47, 13 pp, 2003.
- [14] T. W. Haynes, S. T. Hedetniemi, and T. L. McCoy. Cost Effective Domination. To appear in *Discrete Appl. Math.*
- [15] S. M. Hedetniemi, S. T. Hedetniemi, and P. Kristiansen. Alliances in graphs. *J. Combin. Math. Combin. Comput.*, 48:157-177, 2004.
- [16] S. M. Hedetniemi, S. T. Hedetniemi, and A. A. McRae. Cost effective sets in graphs. Submitted
- [17] P. Kristiansen, S. M. Hedetniemi, and S. T. Hedetniemi. Introduction to alliances in graphs. *Proc. 17<sup>th</sup> Int. Symp. Comput. Inform. Sci., ISCIS XVII (Orlando, FL, 2002)*, 308-312, CRC Press, 2002.

- [18] B. Paten, M Diekhans, D Earl, J. St. John, J. Ma, B. Suh, and D. Haussler. Cactus graphs for genome comparisons. *Journal of Computational Biology*, Volume 18, Number 3 : 469481, 2011
- [19] K. H. Shafique and R. D. Dutton. On satisfactory partitioning of graphs. *Congr. Numer.* 154:183-194, 2002.
- [20] S. Shelah and E. C. Milner. Graphs with no unfriendly partitions. In A. Baker, B. Bollobás, and A. Hajnal (eds.), *A Tribute to Paul Erdős*, Cambridge University Press, Cambridge, 1990, 373-384.

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