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## Power Analysis for the Likelihood-Ratio Test in Latent Markov Models: Shortcutting the Bootstrap p-Value-Based Method

Dereje W. Gudicha, Verena D. Schmittmann, Fetene B. Tekle, and Jeroen K. Vermunt

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### ABSTRACT

The latent Markov (LM) model is a popular method for identifying distinct unobserved states and transitions between these states over time in longitudinally observed responses. The bootstrap likelihood-ratio (BLR) test yields the most rigorous test for determining the number of latent states, yet little is known about power analysis for this test. Power could be computed as the proportion of the bootstrap p values (PBP) for which the null hypothesis is rejected. This requires performing the full bootstrap procedure for a large number of samples generated from the model under the alternative hypothesis, which is computationally infeasible in most situations. This article presents a computationally feasible shortcut method for power computation for the BLR test. The shortcut method involves the following simple steps: (1) obtaining the parameters of the model under the null hypothesis, (2) constructing the empirical distributions of the likelihood ratio under the null and alternative hypotheses via Monte Carlo simulations, and (3) using these empirical distributions to compute the power. We evaluate the performance of the shortcut method by comparing it to the PBP method and, moreover, show how the shortcut method can be used for sample-size determination.

### KEYWORDS

Latent Markov; number of states; Likelihood Ratio; bootstrap; Monte Carlo simulation; longitudinal data; Power Analysis

In recent years, the latent Markov (LM) model has proven useful to identify distinct underlying states and the transitions over time between these states in longitudinally observed responses. In LM models, as in latent class models, or more generally in finite mixture models, the observed responses are governed by a set of discrete underlying categories, which are named states, classes, or mixture components. Moreover, the LM model allows transitions between these states from one timepoint to another; that is, the state membership of respondents can change during the period of observation. The LM model finds its application, for example, in educational sciences to study how the interests of students in certain subjects changes over time (Vermunt, Langeheine, & Bockenholt, 1999) and in medical sciences to study the change in health behavior of patients suffering from certain diseases (Bartolucci, Farcomeni & Pennoni, 2010). Various examples of applications in social, behavioral, and health sciences are presented in the textbooks by Bartolucci et al. (2013) and Collins and Lanza (2010).

In most research situations, including those just mentioned, the number of states is unknown and must be inferred from the data itself. The bootstrap likelihood-ratio (BLR) test, proposed by McLachlan (1987) and extended by Feng and McCulloch (1996) and Nylund,

Asparouhov and Muthén (2007), is often used to test hypotheses about the number of mixture components. These previous studies focused on p value computation rather than on power computation for the BLR test, which is the topic of the current study.

Power computation is straightforward if, under certain regularity conditions, the theoretical distributions of the test statistic under the null and the alternative hypothesis are known. This is not the case for the BLR test in LM models. The power of a statistical test can be computed as the proportion of the p values smaller than the chosen alpha. When using the BLR statistic to test for the number of states in LM models, such a power calculation becomes computationally expensive because it requires performing the bootstrap p value computation for multiple sets of data. As explained in detail in the following, it requires generating  $M$  data sets from the model under the alternative hypothesis, and for each data set, estimating the models under the null and alternative hypotheses to obtain the LR value. Whether the null hypothesis will be rejected for a particular generated data set is determined by computing the bootstrap p value, which in turn requires (a) generating  $B$  data sets from the model estimates under the null hypothesis and (b) estimating the models under the null and alternative hypotheses using these  $B$  data sets.

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Hereafter, we refer to this computationally demanding procedure, which involves calculating the power as the proportion of the bootstrap p value for which the model under the null hypothesis is rejected, as the PBP method.

Because using the PBP method is infeasible in most situations, we propose an alternative method that we refer to as the shortcut method. Computing the power using the shortcut method involves constructing the empirical distributions of the LR under both the null and alternative hypotheses. We show how the asymptotic values of the parameters of the model under the null hypothesis can be obtained from a certain large data set, and these parameters will in turn be used in the process to obtain the distribution of the LR statistic under the null hypothesis. As explained in detail in the following, the distribution of the LR under the null hypothesis is used to obtain the critical value, given a predetermined level of significance. Given this critical value, we compute the power by simulating the distribution of the LR under the alternative hypothesis. Using numerical experiments, we examine the data requirements (e.g., the sample size, the number of timepoints, and the number of response variables) that yield reasonable levels of power for given population characteristics.

The remaining part of the article is organized as follows. We first describe the LM model and the BLR test for determining the number of states. We then provide power computation methods for the BLR test and discuss how these methods can be applied to determine the required sample size. We also present numerical experiments that illustrate the proposed methods of power and sample size computation. The article ends with a discussion and conclusions.

## The LM models

Let  $\mathbf{Y}_t = (Y_{t1}, Y_{t2}, Y_{t3}, \dots, Y_{tP})$  for  $t = 1, 2, 3, \dots, T$  be the  $P$ -dimensional response variable of interest at timepoint  $t$ . Denoting the latent variable at timepoint  $t$  by  $X_t$ , in an LM model the relationships among the latent and observed response variables at the different timepoints can be represented using the simple path diagram shown in Figure 1.

An LM model is a probabilistic model defining the relationships between the time-specific latent variables  $X_t$  (e.g., between  $X_1$ ,  $X_2$ , and  $X_3$ ) and the relationships between the latent variables  $X_t$  and the time-specific vectors of observed responses  $\mathbf{Y}_t$  (e.g.,  $X_1$  with  $\mathbf{Y}_1$ ). In the basic LM model, the latent variables are assumed to follow a first-order Markov process (i.e., the state membership at  $t + 1$  depends only on the state occupied at timepoint  $t$ ),

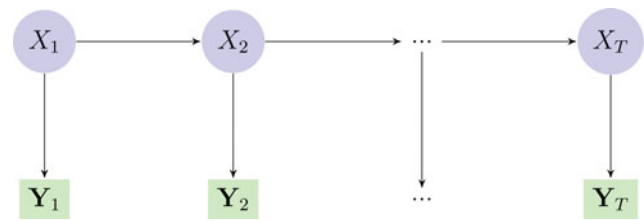


Figure 1. Simple path diagram.

and the response variables are assumed to be locally independent given the latent states. From these assumptions, we define the  $S$ -state LM model as a mixture density of the form

$$p(\mathbf{y}_i, \Phi) = \sum_{x_1=1}^S \sum_{x_2=1}^S \sum_{x_3=1}^S \dots \sum_{x_T=1}^S p(x_1) \times \prod_{t=2}^T P(x_t | x_{t-1}) \prod_{j=1}^P p(y_{tji} | x_t),$$

where  $\mathbf{y}_i$  denotes the vector of responses for subject  $i$  over all the timepoints,  $y_{tji}$  the response of subject  $i$  to the  $j$ th variable measured at timepoint  $t$ ,  $x_t$  a particular latent state at timepoint  $t$ , and  $\Phi$  the vector of model parameters (Vermunt et al., 1999; Bartolucci et al., 2013).

The LM model has three sets of parameters:

1. The initial state probabilities (or proportions)  $p(X_1 = s) = \pi_s$  satisfying  $\sum_{s=1}^S \pi_s = 1$ . That is, the probability of being in state  $s$  at the first timepoint;
2. The transition probabilities  $p(X_t = s | X_{t-1} = r) = \pi_{s|r}^t$  satisfying  $\sum_{s=1}^S \pi_{s|r}^t = 1$ . These transition probabilities indicate the probabilities of remaining in a state or switching to another state, conditional on the state membership at the previous timepoint. All transition probabilities are conveniently collected in a transition matrix in which the entry in row  $r$  and column  $s$  represents the probability of a transition from state  $r$  at timepoint  $(t - 1)$  to state  $s$  at timepoint  $t$ ;
3. The state-specific parameters of the density function  $p(y_{tji} | x_t)$ , which govern the association between the latent states and the observed response variables. The choice of the specific density form for  $p(y_{tji} | x_t)$ , which depends on the scale type of the response variable, determines the state-specific parameters for this density function. With continuous responses, one may, for example, define the state-specific density to be a normal distribution, for which the parameters are the mean  $\mu_{j|s}^t$  and the variance  $\sigma_{j|s}^{2t}$  (Schmittmann, Dolan, van der Maas & Neale, 2005). With dichotomous and nominal responses, the multinomial distribution is assumed, for which the parameters

become the conditional response probabilities  $p(y_{tji}|x_t = s) = \theta_{j|s}^t$  (Collins & Wugalter, 1992; Vermunt, Tran, & Magidson, 2008). The state-specific parameters and the transition probabilities may vary across time, hence the subscript  $t$ , but are assumed to be time-homogeneous during the remainder of this article.

Given a sample of size  $n$ , the parameters are typically estimated by maximizing the log-likelihood function:

$$l(\Phi) = \sum_{i=1}^n \log p(\mathbf{y}_i, \Phi). \tag{1}$$

The search for the values of  $\Phi$  that maximize the log-likelihood function in Equation (1) can be carried out with the expectation-maximization (EM) algorithm (Dempster, Laird, & Rubin, 1977; McLachlan & Krishnan, 2007), which alternates between computing the expected complete data log-likelihood function (E step) and updating the unknown parameters of interest by maximizing this function (M step). For LM models, a special version of the EM algorithm with a computationally more efficient implementation of the E step may be used. This algorithm is referred to as the Baum-Welch or forward-backward algorithm (Bartolucci et al., 2010; Baum, Petrie, Soules, & Weiss 1970; Vermunt et al., 2008).

As already discussed, identifying the number of latent states is a common goal in LM modeling and typically the first step in the analysis. Testing hypotheses about the number of states involves estimating LM models with increasing numbers of states and checking whether the model fit is significantly improved by adding one or more states. More formally, the hypotheses about the number of states may be specified as  $H_0: S = r$  versus  $H_1: S = s$ , where  $r < s$ . Usually, the  $r$  and  $s$ -state models differ by one state. For example, the test for  $H_1: 3$ -state LM model against  $H_0: 2$ -state LM model. However, in principle, the comparison can also be between the 3-state and the 1-state LM model. In this article, we restrict ourselves to the situation in which  $r = s - 1$ .

The LR statistic for this type of test is defined as

$$LR = 2(l(\hat{\Phi}_s) - l(\hat{\Phi}_r)), \tag{2}$$

where  $l(\cdot)$  is the log-likelihood function and  $\hat{\Phi}_s$  and  $\hat{\Phi}_r$  are the maximum likelihood estimates under the alternative and null hypothesis, respectively. In the standard case, under certain regularity conditions, it is generally assumed that the LR statistic in Equation (2) follows a central chi-square distribution under the null hypothesis and a noncentral chi-square distribution under the alternative hypothesis (Steiger, Shapiro, & Browne, 1985). In such a case, one may use the (theoretical) chi-square distribution with the appropriate number of degrees of freedom to

compute the p value of the LR test given a predetermined level of significance  $\alpha$  or the power of the LR test given the population characteristics of  $H_1$  model. These asymptotic distributions however do not apply when using the LR statistic for testing the number of latent states (Aitkin, Anderson, & Hinde, 1981). One reason is that the  $H_0$  model with  $S - 1$  states is obtained from the  $H_1$  model by restricting the initial probability for state  $S$  and the transition probabilities toward state  $S$  to 0. This violates the regularity condition that restriction should not be on the boundary of the parameter space. In addition, when state  $S$  is assumed to have a zero probability of occurrence, the parameters for this state are unidentified, which yields a violation of the regularity condition that all parameters in the  $H_0$  should be identifiable.

One may however apply the method of parametric bootstrapping to construct the empirical distribution of the LR and subsequently use the constructed empirical distribution for p value computation. Due to advances in computing facilities, this can be applied readily. Using parametric bootstrapping, the empirical distribution of the LR statistic under the null hypothesis is constructed by generating  $B$  independent (bootstrap) samples according to a parametric (probability) model  $P(\mathbf{y}, \hat{\Phi}_r)$ , where  $\hat{\Phi}_r$  itself is an estimate based on a sample of size  $n$  (Feng & McCulloch, 1996; McLachlan, 1987; Nylund et al., 2007). Denoting the bootstrap samples by  $\mathbf{y}^b$  (for  $b = 1, 2, 3, \dots, B$ ), Equation (2) becomes

$$BLR_b = 2 \left( l(\hat{\Phi}_s^b) - l(\hat{\Phi}_r^b) \right), \tag{3}$$

where  $BLR_b$  denotes the BLR, computed for (bootstrap) sample  $\mathbf{y}^b$ .

So sampling  $B$  data sets from the  $r$ -state LM model defined by  $P(\mathbf{y}, \hat{\Phi}_r)$  and computing the BLR statistic as shown in Equation (3) for each of these data sets yields the BLR distribution under the null hypothesis. This distribution is then employed in the bootstrap p value computation. In short, the bootstrap p value computation proceeds as follows:

- Step 1. Treating the ML parameter estimates as if they were the “true” parameter values for the  $r$ -state LM model, generate  $B$  independent (bootstrap) samples from the  $r$ -state LM model.
- Step 2. Compute the  $BLR_b$  values as shown in Equation (3), which requires us to fit the  $r$ - and  $s$ -state models using the bootstrap samples generated in Step 1.
- Step 3. Compute the bootstrap p value as  $p = \frac{1}{B} \sum_{b=1}^B I(BLR_b > LR)$ , where  $I(\cdot)$  is the indicator function, which takes on the value 1 if the argument  $BLR_b > LR$  holds and 0 otherwise. The decision concerning whether

the  $r$ -state LM model should be retained or rejected in favor of the  $s$ -state model is then determined by comparing this  $p$  value with the predetermined significance level  $\alpha$ .

### Power analysis for the BLR test

Statistical power analyses are often performed to (a) determine the post hoc power of a study (i.e., given a certain sample size, number of timepoints, and number of response variables) and (b) a priori determine the sample size (or other design factors such as the number of timepoints or the number of response variables) required to achieve a certain power level. In both cases, we assume that the population parameters are known (in a priori analyses a range of expected parameter values may be used) and other factors such as the number of indicator variables and the number of classes are fixed. In what follows, we first show how the bootstrapping procedure discussed previously can be used for power computation and subsequently present the computationally more efficient shortcut method for power and sample-size computation in LM models.

### Power computation

In this subsection, we present two alternative methods for computing the power of the BLR test. The first option, the PBP method, involves computing the power as the proportion of the bootstrap  $p$  values (PBP) for which  $H_0$  is rejected. More specifically, the PBP method for power computation involves the following steps:

- Step 1. Generate  $M$  independent samples, each of size  $n$ , from the parametric model  $P(y, \Phi_s)$ , where  $\Phi_s$  is the given parameter values under  $H_1$ .
- Step 2. For each sample  $m$  ( $m = 1, 2, 3, \dots, M$ ) in Step 1, compute the likelihood ratio  $LR_m$  as shown in Equation (2).
- Step 3. Obtain the bootstrap  $p$  value of each sample  $m$  as  $p_m = \frac{1}{B} \sum_{b=1}^B I(BLR_{bm} > LR_m)$ , where  $LR_m$  is the LR of sample  $m$  from the  $H_1$  population;  $BLR_{bm}$  is the corresponding BLR for bootstrap sample  $b$ ; and  $I(\cdot)$  is the indicator function as defined in the preceding.
- Step 4. The actual power associated with a sample of size  $n$  is computed as the proportion of the  $H_1$  data sets in which  $H_0$  is rejected. That is,

$$PBP = \frac{1}{M} \sum_{m=1}^M I(p_m < \alpha), \quad (4)$$

where the indicator function  $I(\cdot)$  and  $\alpha$  are as defined in the preceding.

As mentioned previously, such a method of power computation is computationally expensive and requires a considerable amount of computer memory. For example, setting  $M = 500$  and  $B = 99$  requires us to generate and analyze  $M(B + 1) = 50,000$  data sets. In addition, to achieve a good approximation to the sampling distribution, which, if not well approximated, could affect the  $p$  value (and subsequently the power), both  $M$  and  $B$  should be large enough.

For LM models, for which model fitting requires iterative procedures, power computation by using the PBP method is computationally too intensive in practice. We propose a computationally more efficient method, which we call the shortcut method. It works very much as the standard power computation (see for example, Brown, Lovato, & Russell, 1999), with the difference that we construct the distributions under  $H_0$  and  $H_1$  by Monte Carlo simulation. In Figure 2, these two distributions are indicated with curve  $H_0$  and  $H_1$ , respectively. As explained in the following, the distribution under  $H_0$  is used to obtain the critical value (CV), and the distribution under  $H_1$  is used to compute the power given the CV.

First, the  $H_0$  "population" parameters needed to compute the CV should be obtained. This can be achieved by creating an exemplary data set, which is a data file with all possible response patterns and the relative frequencies of the response patterns under  $H_1$  as weights (O'Brien, 1986; Self, Mauritsen, & Ohara, 1992). Because in LM models with more than a few indicators and/or timepoints the number of possible response pattern is very large, this method cannot always be applied. Therefore, as an alternative, using the parameter values of the  $H_1$  model, we

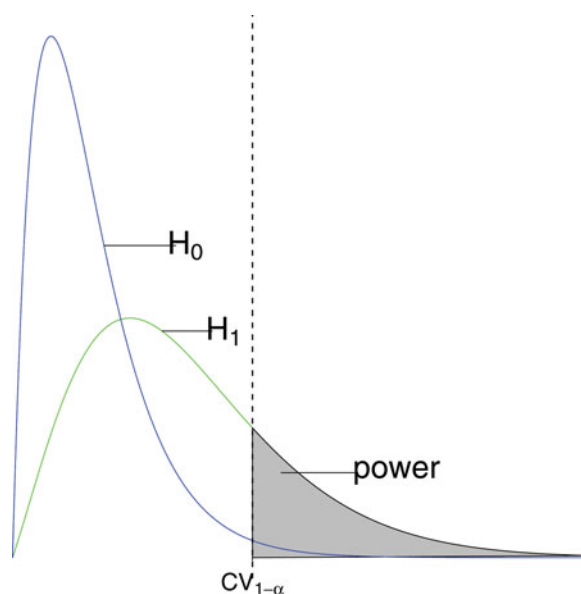


Figure 2. Distributions of LR under the null and alternative hypotheses.

generate a large data set (e.g., 10,000 observations), which is assumed to represent the hypothetical  $H_1$  population. Estimating the  $H_0$  model (i.e., the  $r$ -state LM model) using this large data set yields the pseudo parameter values for the  $r$ -state model. These  $H_0$  parameters are then employed to construct the distribution of the LR under the null hypothesis. That is, given the estimated parameters of the  $H_0$  model, generate  $K$  data sets (each of size  $n$ ), and for each of these data sets, compute the LR as shown in Equation (2). Next, order the LR values in such a way that  $LR_{[1]} \leq LR_{[2]} \leq LR_{[3]} \leq \dots \leq LR_{[K]}$ . Given the nominal level  $\alpha$ , compute the CV as

$$CV_{(1-\alpha)} = \{LR_k : p(LR > LR_{[k]}|H_0) = \alpha\}. \quad (5)$$

Similarly, the distribution of the LR under the alternative hypothesis is constructed using  $M$  samples of the  $H_1$  model. That is, given the parameters of the  $H_1$  model, we generate  $M$  independent samples from the  $s$ -state LM model and for each of these samples, compute the LR as shown in Equation (2). For sufficiently large  $M$ , the distribution of the LR under the alternative hypothesis approximates the  $H_1$  curve in Figure 2. The power is then computed as the probability that the LR value belongs to the shaded region of Figure 2. That is,

$$\text{power} = p(LR > CV_{(1-\alpha)}|H_1) = \frac{\sum_{m=1}^M I(LR_m > CV_{(1-\alpha)})}{M}, \quad (6)$$

where  $I(\cdot)$  is the indicator function, indicating whether the LR value (based on the  $b$  sample of the  $H_1$  population) exceeds the  $CV_{1-\alpha}$  value.

So both the PBP and the shortcut methods require  $M$  samples given  $H_1$  and the calculation of the LR for each of these samples (i.e., steps 1 and 2 of the PBP power calculation). The saving in computation time of the shortcut method lies in the omission of the full bootstrap for each of the  $M$  samples from the  $H_1$  model. Rather, the LRs given  $H_1$  are now evaluated against the approximated distribution of LRs given  $H_0$ . Therefore, compared to the PBP-based power computation, the number of data sets to be generated and analyzed is much smaller when using the shortcut method. For example, for  $M = 500$  and  $K = 500$ , we analyze  $M + K = 1,000$  data sets. To further explain the computational time gain, let the time required to calculate the PBP-based power by analyzing  $M(B + 1)$  data sets be  $\omega$ . The time required to compute the power by the shortcut method—which requires analyzing  $M + K$  data sets—can be shown to be  $(\frac{1}{B+1} + \frac{K}{M+B})\omega$ . For large  $M$ , and under the setting with  $B = K = M$ , this computational time may simplify to  $(\frac{2}{M})\omega$ . In other words, the shortcut method is  $M/2$  times faster than the PBP method.

The shortcut method of power computation presented in the preceding can easily be implemented using statistical software for LM analysis as outlined in the following.

1. Obtain the  $H_0$  population parameters: Given the parameters of the  $H_1$  model, generate a large dataset (e.g., 10,000 observations) from the  $H_1$  population. For this purpose, any software that allows generating a sample from an LM model with fixed parameter values can be used. For the numerical studies shown in the following, we used the syntax module of the Latent GOLD 5.0 program (Vermunt & Magidson, 2013). Using this large data set, then estimate the parameters of the  $H_0$  model.
2. Compute the CV: Given the estimated parameters of the  $H_0$  model, generate  $K$  data sets (each of size  $n$ ) and for each of these data sets, compute the LR as shown in Equation (2). Note that this requires estimating both the  $r$ - and the  $s$ -state models. For a sufficiently large  $K$ , the LR distribution approximates the population distribution of the LR under the null hypothesis (i.e., the  $H_0$  curve in Figure 2). We use this distribution to compute the CV of the LR test as shown in Equation (5).
3. Compute the power: Given the parameters of the  $H_1$  model, obtain the empirical distribution of the LR. That is, generate  $M$  data sets from  $H_1$  model, and using these data sets, compute the LR as shown in (2). Given the CV and the empirical distribution of the LR under  $H_1$ , compute the power as shown in Equation (6).

### Sample-size computation

In this section, we show how the procedure described in the preceding for power computation using the shortcut method can be applied for sample-size determination. For sample-size determination, step 1 of the power computation procedure (discussed under power computation) remains the same. The last two steps are however repeated for different trial sample sizes. More specifically, suppose the investigator wishes to achieve a certain prespecified power level (say, power = .8 or larger) while avoiding the sample size becoming unnecessarily large. Then, the LR power computation is performed as outlined in steps 2 and 3, starting with a certain sample size  $n_1$ . In the following we provide power curves that can be used as a guidance to locate this starting sample size. If the power obtained based on these  $n_1$  observations is lower than .8, repeat step 2 and 3 by choosing  $n_2$  larger than  $n_1$ . If the chosen  $n_1$  results in larger power instead (and we want to optimize the sample size), choose  $n_2$  smaller than  $n_1$ .

and repeat steps 2 and 3. In this way, the power computation procedure is repeated for different trial samples of varying sizes, and from these trial samples, the one that best approximates the desired power level is used as the sample size for the study concerned. In our numerical study, we repeated this power computation procedure for different sample sizes, which resulted in a series of power values. By plotting these power values against the corresponding sample size, we obtain a power curve from which one can easily determine the minimum sample size that satisfies the power requirements, for example, that the power should be larger than .8.

When designing a longitudinal study, it is also of interest to determine the number of timepoints required to achieve a certain power level. For a fixed sample size, a fixed number of response variables, and a priori specified  $H_1$  parameter values, the procedures discussed in the preceding for sample-size determination can be applied to the number of timepoints determination as well. More specifically, in steps 2 and 3 of the power computation procedures, the number of timepoints  $T$  should be varied instead of the sample size  $n$ .

## Numerical study

A numerical study was conducted to (a) illustrate the proposed power and sample-size computation methods and (b) investigate whether the shortcut method and the PBP method give similar results. This numerical study has an additional benefit for applied researchers using the LM model: Given the population characteristics, the resulting BLR power tables and the power curves shown in the following may help to make an informed decision about the data requirements in testing the number of states for the LM model. More specifically, the results of this numerical study may be used as a guidance by applied researchers to locate the initial trial sample size when computing the required sample size to achieve a desired power level, as discussed in the preceding.

### Numerical study setup

The power of the BLR test for the number of states in LM models depends on several design factors and population characteristics. See, for example, Gudicha, Schmittmann and Vermunt (2016), who studied factors affecting the power in LM models. The design factors include the sample size, the number of timepoints, and the number of response variables. The number of latent states and the various model parameter values (i.e., parameter values for the initial state proportions, for the state transition probabilities, and for the state-specific densities) define the population characteristics (Collins & Wugalter, 1992).

In this numerical study, we varied both the design factors and the population characteristics. The design factors varied were the sample size ( $n = 300, 500, \text{ or } 700$ ), the number of timepoints ( $T = 3 \text{ or } 5$ ), and the number of response variables ( $P = 6 \text{ or } 10$ ). The population characteristics under the alternative hypothesis (i.e., the  $s$ -state LM model for  $S = 3 \text{ or } 4$ ) were specified to meet varying levels of (a) initial state proportions (balanced, moderately imbalanced, highly imbalanced), (b) stability of state membership (stable, moderately stable, unstable), and (c) state-response associations (weak, moderate, strong) as follows.

In line with Dias (2006), the initial state proportions were specified using  $\pi_s = \frac{\delta^{s-1}}{\sum_{h=1}^S \delta^{h-1}}$ . We set the values of  $\delta$  to 1, 2, and 3, which correspond to balanced, moderately imbalanced, highly imbalanced initial state proportions, respectively. For the transition matrix, we used the specification suggested by Bacci, Pandolfi, and Pennoni (2014), which under the assumption of time homogeneity gives  $\pi_{s|r} = \frac{\rho^{s-r}}{\sum_{h=1}^S \rho^{h-r}}$ . Setting the values of  $\rho$  to  $\rho = .1, .15, \text{ and } .3$  yields what we referred to above as stable, moderately stable, and unstable state membership. In this numerical study, we restricted ourselves to the situation that the response variables of interest are binary and that the state specific conditional response probabilities are time-homogeneous. For  $S = 3$ , we set the state-specific conditional response probabilities to high for state 1 ( $\theta_{j|1} = .75, .8, \text{ and } .85$  for all the response variables), low for state 3 ( $\theta_{j|3} = 1 - .75, 1 - .8, \text{ and } 1 - .85$  for all the response variables), and medium for state 2 ( $\theta_{j|2} = .58, .65, \text{ and } .7$  for all the response variables). These three settings of the conditional response probabilities result in what we referred to in Table 1 as weak, medium, and strong state-response variables association, respectively. For  $S = 4$ , we used the same setting of conditional response probabilities as for  $S = 3$ , but now defined the conditional response probabilities of the remaining state as high ( $= \theta_{j|1}$ ) for half of the response variables and low ( $= \theta_{j|s}$ ) for the other half.

The design factors and population characteristics were fully crossed, resulting in 3 (sample size)  $\times$  2 (number of timepoints)  $\times$  2 (number of response variables)  $\times$

**Table 1.** Distribution of conditional response probabilities across states.

State-responses association levels	S = 3			S = 4			
	s = 1	s = 2	s = 3	s = 1	s = 2	s = 3	s = 4
Weak	.75	.58	.25	.75	.58	.75 or .25	.25
Moderate	.80	.65	.20	.80	.65	.80 or .20	.20
Strong	.85	.70	.15	.85	.70	.85 or .15	.15

Note. Weak = low conditional probabilities for all the response variables; moderate = medium conditional probabilities for all the response variables; strong = high conditional probabilities for all the response variables.

**Table 2.** Power of the BLR test for  $H_0: S = 2$  versus  $H_1: S = 3$ : The case of equal initial state size ( $\delta = 1$ ) and six indicator variables ( $P = 6$ ).

		State-responses associations									
		Weak			Moderate			Strong			
		Index of state transition			Index of state transition			Index of state transition			
Sample size	Method	$\rho = 0.1$	$\rho = 0.15$	$\rho = 0.3$	$\rho = 0.1$	$\rho = 0.15$	$\rho = 0.3$	$\rho = 0.1$	$\rho = 0.15$	$\rho = 0.3$	
$T = 3$	300	shortcut	.188	.145	.104	.301	.260	.176	.568	.494	.339
		PBP	.180	.148	.116	.286	.250	.131	.550	.496	.320
	500	shortcut	.398	.301	.178	.581	.534	.294	.869	.809	.631
		PBP	.394	.280	.150	.558	.493	.296	.858	.804	.610
	700	shortcut	.642	.439	.238	.842	.704	.405	.978	.957	.796
		PBP	.592	.442	.224	.802	.691	.387	.968	.960	.800
$T = 5$	300	shortcut	.698	.559	.228	.849	.727	.394	.972	.927	.687
		PBP	.683	.516	.212	.846	.730	.380	.966	.947	.700
	500	shortcut	.955	.868	.416	.990	.959	.726	1	.999	.942
		PBP	.958	.870	.422	.986	.960	.650	1	.993	.952
	700	shortcut	.995	.970	.654	1	.999	.887	1	1	.992
		PBP	.998	.969	.640	1	.997	.892	1	1	.988

Note.  $T$  = number of timepoints;  $P$  = number of response variables,  $\delta$  = initial state proportion index;  $\rho$  = state transition probability index; PBP = proportion bootstrap p value rejected. Weak = low conditional probabilities for all the response variables; moderate = medium conditional probabilities for all the response variables; and strong = high conditional probabilities for all the response variables.

2 (number of states)  $\times$  3 (initial state proportions)  $\times$  3 (transition probability matrices)  $\times$  3 (state-response variables association levels) = 572 simulation conditions. For each simulation condition, a large data set (of 10,000 observations) was generated according to the  $H_1$  model, and the  $H_0$  parameters were estimated using this data set. Next, for each simulation condition,  $K = 1,000$  samples were generated according to the  $H_0$  parameters, and the CV was computed, assuming  $\alpha = .05$ . Given a specified sample size, number of timepoints, and the parameter values under the alternative hypothesis, the power was then computed according to  $M = 1000$  samples generated according to the  $H_1$  model as discussed in the preceding. To minimize the problem of local maxima, we use multiple random start sets for parameter

estimation in combination with specifying the true population parameter value as the starting value.

**Results**

The results obtained from the numerical study for power computation by the shortcut and PBP methods are shown in Tables 2–5. As can be seen from these tables, the power values of the two methods are in general comparable. Although the power values obtained by the shortcut method seem to be slightly larger, overall differences do not lead to different conclusions regarding the hypotheses about the number of states. The most important added value of the shortcut method is, however, that it is  $\frac{M}{2}$  times faster than the PBP method, where  $M$  refers to the

**Table 3.** Power of the BLR test for  $H_0: S = 2$  versus  $H_1: S = 3$ : The case of imbalanced initial state size ( $\delta = 2$  or 3) with six indicator variables ( $P = 6$ ) and three timepoints ( $T = 3$ ).

		State-responses associations									
		Weak			Moderate			Strong			
		Index of state transition			Index of state transition			Index of state transition			
Sample size	Method	$\rho = 0.1$	$\rho = 0.15$	$\rho = 0.3$	$\rho = 0.1$	$\rho = 0.15$	$\rho = 0.3$	$\rho = 0.1$	$\rho = 0.15$	$\rho = 0.3$	
$\delta = 2$	300	shortcut	.147	.130	.080	.247	.197	.135	.346	.308	.249
		PBP	.142	.130	.104	.204	.170	.128	.332	.272	.214
	500	shortcut	.290	.210	.127	.357	.296	.244	.637	.559	.457
		PBP	.274	.208	.132	.336	.280	.252	.604	.544	.410
	700	shortcut	.445	.367	.193	.594	.517	.337	.801	.763	.574
		PBP	.416	.336	.186	.571	.484	.295	.790	.775	.566
$\delta = 3$	300	shortcut	.114	.075	.073	.138	.099	.090	.171	.147	.125
		PBP	.108	.092	.084	.106	.100	.095	.164	.134	.110
	500	shortcut	.146	.112	.104	.196	.173	.131	.307	.281	.220
		PBP	.135	.112	.107	.176	.168	.146	.298	.228	.166
	700	shortcut	.231	.186	.124	.306	.245	.195	.515	.456	.378
		PBP	.193	.176	.124	.289	.246	.196	.482	.374	.244

Note.  $T$  = number of timepoints;  $P$  = number of response variables;  $\delta$  = initial state proportion index;  $\rho$  = state transition probability index; and PBP = proportion bootstrap p value rejected. Weak = low conditional probabilities for all the response variables; moderate = medium conditional probabilities for all the response variables; strong = high conditional probabilities for all the response variables.



**Table 4.** Power of the BLR test for  $H_0: S = 2$  versus  $H_1: S = 3$ : The case of equal initial state size ( $\delta = 1$ ) and three timepoints ( $T = 3$ ).

		State-responses associations									
		Weak			Moderate			Strong			
		Index of state transition			Index of state transition			Index of state transition			
Sample size	Method	$\rho = 0.1$	$\rho = 0.15$	$\rho = 0.3$	$\rho = 0.1$	$\rho = 0.15$	$\rho = 0.3$	$\rho = 0.1$	$\rho = 0.15$	$\rho = 0.3$	
$P = 6$	300	shortcut	.188	.145	.104	.301	.260	.176	.568	.494	.339
		PBP	.180	.148	.116	.286	.250	.151	.550	.496	.320
	500	shortcut	.398	.301	.178	.581	.534	.294	.869	.809	.631
		PBP	.394	.280	.150	.558	.493	.296	.858	.804	.610
700	shortcut	.642	.439	.238	.842	.704	.405	.978	.957	.796	
	PBP	.592	.442	.224	.802	.691	.387	.968	.960	.800	
$P = 10$	300	shortcut	.791	.646	.402	.872	.786	.551	.987	.952	.885
		PBP	.766	.618	.394	.848	.746	.540	.970	.948	.848
	500	shortcut	.973	.941	.702	.993	.976	.866	1	1	.993
		PBP	.975	.934	.674	.992	.981	.874	1	1	.992
	700	shortcut	.989	.941	.831	.993	.976	.893	1	1	1
		PBP	.996	.958	.859	1	1	.970	1	1	.998

Note.  $T$  = number of timepoints;  $P$  = number of response variables;  $\delta$  = initial state proportion index;  $\rho$  = state transition probability index; PBP = proportion bootstrap p value rejected. Weak = low conditional probabilities for all the response variables; moderate = medium conditional probabilities for all the response variables; strong = high conditional probabilities for all the response variables.

number of Monte Carlo and bootstrap samples for the shortcut and the PBP methods, respectively.

If we now turn to the power values for various combinations of data and population characteristics, we see in Table 2 that the power of the BLR test increases with sample size and the number of timepoints. Comparison of the effect of sample size and the number of timepoints shows that holding the other factors constant, increasing the number of timepoints has a larger effect on the power than increasing the sample size. Keeping the other design factors constant, the power of the BLR test in general increases with stronger measurement conditions (i.e., weak to moderate to strong state-response variable associations) and with more stable state memberships (smaller transition probabilities).

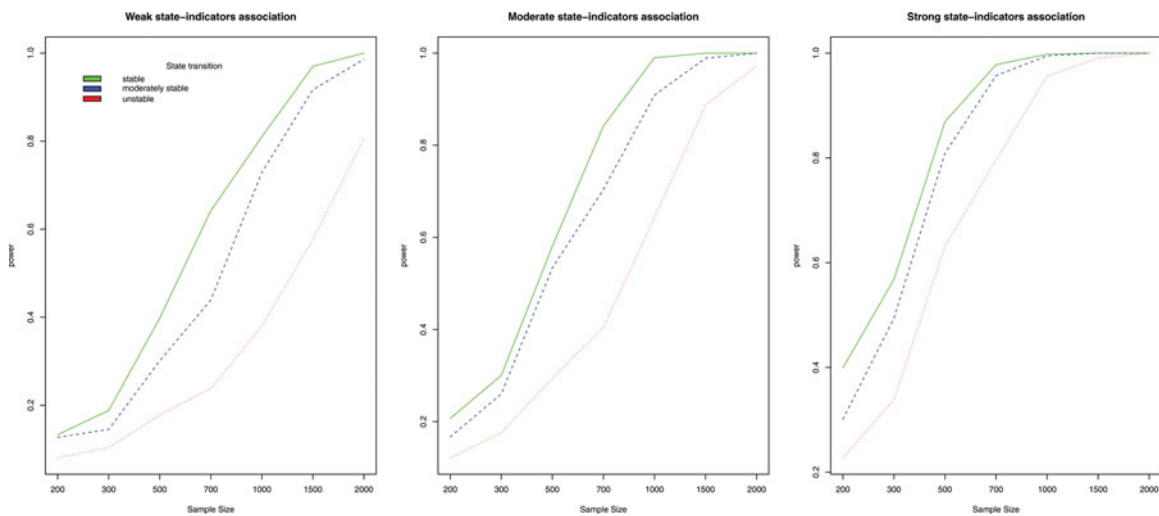
While in Table 2 we reported the results for equal initial state proportions, in Table 3, we report the results for unequal initial state proportions. As can be seen, the BLR power drops when the initial state size is imbalanced. The more imbalanced the initial state sizes the smaller the power. Table 4 shows the effect of the number of indicator variables on the power of the BLR test: Power generally increases when the number of indicator variables increases. Comparing the results in Table 2 with those in Table 5, holding the other factors constant, the power of the BLR test to reject  $H_0: S = 2$  in favor of  $H_1: S = 3$  is in general larger than for  $H_0: S = 3$  against  $H_1: S = 4$ .

In summary, the results reported in Tables 2–5 show that in the weak measurement condition, the power of the BLR test is in general very low, indicating that very

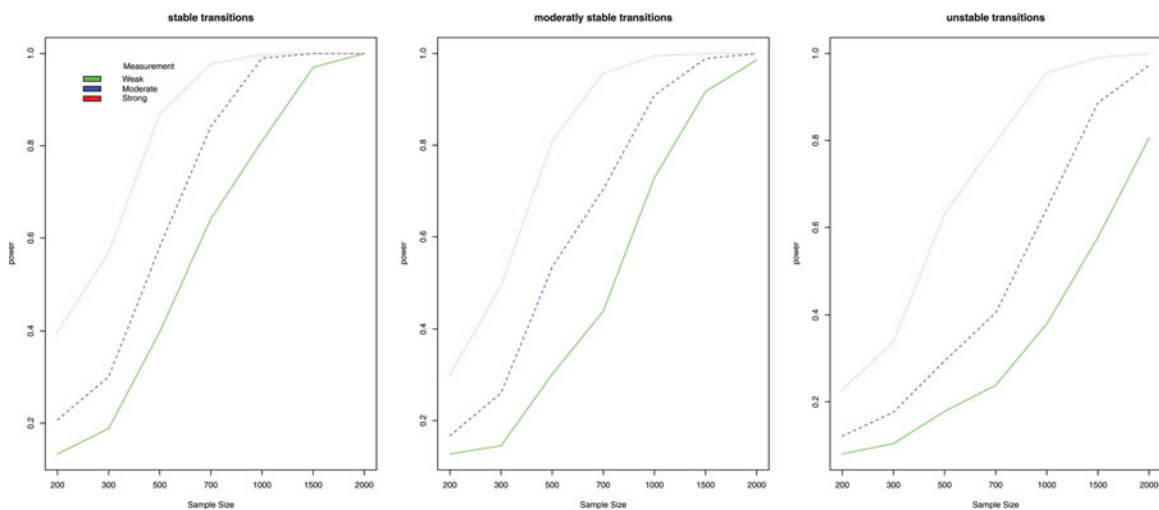
**Table 5.** The power of the BLR test for testing  $H_0: S = 3$  versus  $H_1: S = 4$ : The case of equal initial state size and six indicator variables.

		State-responses associations									
		Weak			Moderate			Strong			
		Index of state transition			Index of state transition			Index of state transition			
Number of timepoints	Sample size	$\rho = 0.1$	$\rho = 0.15$	$\rho = 0.3$	$\rho = 0.1$	$\rho = 0.15$	$\rho = 0.3$	$\rho = 0.1$	$\rho = 0.15$	$\rho = 0.3$	
$T = 3$	300	shortcut	.121	.099	.074	.170	.120	.093	.377	.3007	.195
		PBP	.114	.096	.062	.162	.142	.110	.338	.284	.178
	500	shortcut	.199	.158	.122	.272	.230	.171	.643	.539	.341
		PBP	.213	.162	.108	.294	.224	.154	.610	.512	.302
	700	shortcut	.273	.218	.151	.464	.387	.233	.811	.717	.516
		PBP	.258	.222	.126	.474	.354	.238	.794	.718	.498
$T = 5$	300	shortcut	.387	.237	.147	.534	.483	.212	.872	.737	.401
		PBP	.351	.244	.136	.516	.448	.200	.868	.754	.388
	500	shortcut	.738	.551	.214	.882	.802	.361	.994	.962	.706
		PBP	.694	.504	.208	.864	.758	.342	.994	.970	.698
	700	shortcut	.919	.736	.356	.985	.918	.572	1	1	.886
		PBP	.924	.732	.315	.980	.912	.542	1	1	.894

Note.  $T$  = number of timepoints;  $\delta$  = initial state proportion index;  $\rho$  = state transition probability index; PBP = proportion bootstrap p value rejected. Weak = low conditional probabilities for all the response variables; moderate = medium conditional probabilities for all the response variables; strong = high conditional probabilities for all the response variables.



**Figure 3.** Power by sample size for a 3-state LM population model with varying levels of measurement parameters, equal initial state proportions, six response variables, and three timepoints.



**Figure 4.** Power by sample size for a 3-state LM population model with varying levels of transition parameters, equal initial state proportions, six response variables, and three timepoints.

large sample sizes may be required to achieve an acceptable power level in these conditions. Although the quality of state-response association plays a dominant role, the power computed for the weak measurement condition improved substantially by increasing the number of response variables or timepoints. In addition, situations in which the state membership is unstable (e.g.,  $\rho = .3$  or larger) need special care, since the power is low in such situations.

Figures 3 and 4 present a power curve (as a function of sample size) for different settings of the parameter values of the 3-state LM population model with equal initial state proportions, six response variables, and three timepoints. Figure 3 shows that when the state-response associations are weak, to achieve a power of .8 or larger, we may require a sample of 1,000 or more when state membership is stable, and a sample of 2,000 or more when state

membership is unstable. We can also see from the same figure that when the state-response associations are rather strong, the required sample sizes may drop to less than 500 and 700, respectively, for stable and unstable state membership conditions. As can be seen from Figure 4, to achieve a power level of .8 when the state memberships are moderately stable, sample sizes of at least 1,200, 850, and 500, may be required in the weak, medium, and strong measurement conditions, respectively.

## Discussion and conclusion

This study addressed methods of power analysis for the BLR when testing hypotheses on the number of states in LM models. Two alternative methods of power computation were discussed: the proportion of significant bootstrap  $p$  values (PBP) and the shortcut method. Using the

PBP method, power is computed by first generating a number of independent data sets under the alternative hypothesis and then, for each of these data sets, computing the  $p$  value by applying a parametric bootstrap procedure (McLachlan, 1987). The PBP method is computationally very demanding as it requires performing the full bootstrap for each of  $M$  samples from the  $H_1$  model. We proposed solving this computational problem using the shortcut method. The shortcut method works very much as a standard power computation, with the difference that instead of relying on the theoretical distributions (a central chi-square under the null hypothesis and a noncentral chi-square under the alternative hypothesis), the distributions under  $H_0$  and  $H_1$  are constructed by Monte Carlo simulation.

A numerical study was conducted to (a) illustrate the proposed power analysis methods and (b) compare the power obtained by the shortcut and the PBP methods. As expected, the power of the BLR test in the LM models increased with sample size. Likewise, power increased with more timepoints and more response variables. In addition to these design factors, the power of the BLR test was shown to depend on the following population characteristics: the initial state proportions, the state transition probabilities, and the state-response associations. Holding the other design factors constant, power was larger with more balanced initial state proportions, more stable state memberships, and stronger state-response associations. Contrary to this, when initial state proportions are highly imbalanced, state membership is unstable, and the state-response association is weak, the power of the BLR test is low.

The overall power is strongly dependent on the power at the individual timepoints. More specifically, the stronger the time-specific measurement models, the larger the power. But the reverse is also true; that is, the overall power also affects the class separation and thus the power at a specific timepoint. The latter effect is stronger when timepoints are more strongly related (when transitions are less likely). In the most extreme case in which all transition probabilities are equal to 0, the time-specific and overall power values are exactly the same.

For the simulation conditions that we have considered in this study, the sample size required to achieve a power level of .8 or larger ranged from a few hundred to thousands of cases. In addition, the required sample size depended on other design factors and population characteristics, which are highly interdependent. In general, the more timepoints, the more response variables, the more balanced the initial state proportions, the more stable the state memberships, and the stronger the state-response associations, the smaller the sample

size needed to achieve a certain power level. Because of mutual dependencies among the LM model parameters, and since the required sample size is also influenced by the number of timepoints, response variables, and state-indicator variable associations, a sample size of 300 or 500 will often not suffice in LM analysis. Therefore, we strongly suggest applied researchers perform a power analysis for their specific research situation instead of relying on certain rules of thumb about the sample size. The same applies to questions about the minimum number of timepoints and/or response variables.

Both the shortcut and PBP methods discussed in this article make use of parameter estimates obtained by maximizing the log-likelihood function. In LM models, as in other mixture models, the log-likelihood function can have multiple maxima, meaning that the estimates found do not always correspond to the global maximum of the log-likelihood function. This may have an effect on the computed power (or sample size). In this article, we dealt with this problem of local maxima by using multiple sets of random starting values for the parameters, in addition to a set of start values corresponding to the known population parameter values.

The  $p$  values,  $p = 6$  and  $p = 10$ , were chosen to illustrate how the power of the BLR test can be affected by the number of response variables in the typical latent Markov analysis applications (with not very small number of indicators and not very large number of timepoints). In other application types of latent Markov models, the number of response variables can be smaller than this (sometimes just one), which will typically be compensated by a much larger number of timepoints. Our power computation method can be applied without any modification in those situations as well to determine the required sample size and/or number of timepoints. A limitation of our numerical study is that it does not give much information on the power of such studies.

The procedure we proposed can also be used when there is missing data, either by design or by some known missing at random (MAR) mechanism. In our numerical study, we did not pay attention to the possible effect of missing data on the power since that would be a study on its own. However, without any modification, our method can be used to compute (and thus compare) the power or the required sample size under different MAR missing data scenarios.

Limitations to the current numerical experiments need to be acknowledged. First, in this study, we assumed time homogeneity for both state transition and conditional response probabilities. Future research should assess the power of the BLR test if this assumption is relaxed. Second, the conditional response probabilities of the binary response variables were set to equal values,

and for simplicity, we considered a specific structure of the transition matrix:  $\pi_{s|r} = \frac{\rho^{|s-r|}}{\sum_{h=1}^s \rho^{h-r}}$ . However, in practice the conditional response probabilities may differ across response variables; the response variables may be nominal with more than two categories, continuous or of mixed type; and the structure of the transition matrix can be completely unconstrained, or, for example, symmetric or triangular (Bartolucci, 2006). Third, this article focused on power and sample-size computation. A further study with more focus on determining the required number of measurement occasions is suggested. Power analysis for the number of timepoints depends not only on the state transition probabilities, but also on the time scale and on whether the dynamics of the system are stationary. Fourth, in our study, we illustrated the proposed power computation methods considering tests for 3-state against 2-state LM models and 4-state against 3-state LM models. In practice, one may encounter tests for larger numbers of states.

It can be concluded that more intensive simulations that address these different scenarios concerning the  $H_1$  population model may be needed to establish more knowledge and guidelines about the power and sample size requirements of the BLR test for the number of states in LM models. What is clear is one should not rely on certain rules of thumb about the required sample size, number of timepoints, or number of indicator variables, but instead perform a power analysis tailored to the specific situation of interest. The proposed shortcut method makes this computationally feasible.

## Article information

**Conflict of Interest Disclosures:** Each author signed a form for disclosure of potential conflicts of interest. No authors reported any financial or other conflicts of interest in relation to the work described.

**Ethical Principles:** The authors affirm having followed professional ethical guidelines in preparing this work. These guidelines include obtaining informed consent from human participants, maintaining ethical treatment and respect for the rights of human or animal participants, and ensuring the privacy of participants and their data, such as ensuring that individual participants cannot be identified in reported results or from publicly available original or archival data.

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