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ABSTRACT
This article proposes a procedure for fitting a pure exploratory bifactor solution in which the general factor is orthogonal to the group factors, but the loadings on the group factors can satisfy any orthogonal or oblique rotation criterion. The proposal combines orthogonal Procrustes rotations with analytical rotations and consists of a sequence of four steps. The basic input is a semispecified target matrix that can be (a) defined by the user, (b) obtained by using Schmid-Leiman orthogonalization, or (c) automatically built from a conventional unrestricted solution based on a prescribed number of factors. The relevance of the proposal and its advantages over existing procedures is discussed and assessed via simulation. Its feasibility in practice is illustrated with two empirical examples in the personality domain.

KEYWORDS
Bifactor solutions; exploratory factor analysis; orthogonal Procrustes rotations; orthogonal and oblique analytical rotations; semispecified target matrices

Factor analysis (FA) applications to item and test scores are generally based on one of these two models: (a) the unidimensional (Spearman) model or (b) the correlated-factors model. In the first case, the scores are assumed to be indicators of a single dimension, with no local dependencies or correlated uniquenesses among them. In the second case, the scores are assumed to measure two or more related dimensions. Furthermore, the pattern of the relations between the indicators and the factors is generally expected to approach a simple structure (Thurstone, 1935).

The bifactor model combines the two specifications above and allows the hypothesis of a general dimension to be maintained, while the additional common variance among the scores is modeled using group factors that are expected to approach a simple structure. The idea of this modeling dates back to at least 1937 (Holzinger & Swineford, 1937). However, for more than 50 years, the correlated factor model was the model of choice and the bifactor model practically fell into disuse. However, there was a resurgence in the 1990s (e.g., Gibbons & Hedeker, 1992, Mulaik & Quartetti, 1997) and interest has been growing spectacularly ever since (e.g., Morin, Arens, & Marsh, 2016, Reise, 2012).

An "ideal" bifactor pattern with $m=6$ indicators and $r=2$ group factors is shown below, with asterisks denoting the loading parameters that are free, and zeros denoting those expected to be zero.

$$\mathbf{P} = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & 0 \\ * & 0 & * \\ * & 0 & * \\ * & 0 & * \end{pmatrix}$$

The first column of **P** defines the general factor, and its loading values are all usually defined as free parameters; the next columns define the group factors. The indicators (which are the rows in the loading matrix) usually have only two free loading parameters: one related to the general factor, and the other related to a single group factor. In this regard, the "ideal" structure for the group factors is a simple structure. The general factor is assumed to be uncorrelated with the group factors. Furthermore, in "traditional" bifactor modeling the group factors are also assumed to be uncorrelated between them (Holzinger & Swineford, 1937). However, this assumption can be relaxed, and bifactor solutions with oblique group factors can also be specified (Jennrich & Bentler, 2012). In an oblique solution, the correlations among the group factors would model the additional common variance among them that cannot be accounted for by the general factor. The relevance of this type of solution is discussed below.

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If enough information is available for specifying an "ideal" solution of the type so far described, then the bifactor model can be fitted as a confirmatory FA (CFA) model in a rather direct way (see e.g., Reise, 2012). However, in many scenarios, the CFA approach is unfeasible or problematic. First, in many measures that were developed as essentially unidimensional, dependencies in content among items that can be modeled as additional dimensional structures do exist (e.g., Furnham, 1990). However, these structures cannot be generally anticipated "a priori." Second, and also in this type of measure, dependencies due to shared noncontent-related specificities (doublets, triplets or testlets) are also quite common, but they generally have to be discovered. Finally, there is the most general problem of the nonnegligible small cross-loadings that are forced to be zero, a restriction which is a potential source of misfit and biased parameter estimates in CFA applications (Ferrando & Lorenzo-Seva, 2000, Reise, 2012). All the situations discussed so far (among others) call for an exploratory FA (EFA) application of the bifactor model. Interest in applications of this type has been growing in recent years (e.g., Morin et al., 2016), possibly due to growing dissatisfaction with the results of strict CFA approaches as well as a general trend towards more flexible forms of modeling (e.g., Ferrando & Lorenzo-Seva, 2017a, 2017b, Morin et al., 2016).

The developments that have been made in the EFA bifactor model to date are reviewed below. Unlike CFA, however, the EFA approaches proposed so far have acknowledged shortcomings (e.g., Mansolf & Reise, 2016) and it is safe to say that there is still room for improvement in this domain. This scenario is the starting point of the present article, in which we propose a multistep general procedure that allows a pure exploratory bifactor solution to be fitted. Our proposal is general in that it allows (a) an initial target matrix to be specified using both existing and new procedures; (b) the group solution to be orthogonally or obliquely rotated using any standard rotation procedure, and (c) solutions that are not feasible with currently available methods to be obtained.

A review of existing bifactor EFA proposals

In principle, two different strategies have been considered for fitting an EFA bifactor model. The first is to use second-order FA solutions that are transformed into bifactor solutions. The second is to modify conventional rotation methods (designed to arrive at a simple structure) so that they can arrive at a bifactor structure instead.

The proposals derived from the first strategy have mostly been based on the Schmid-Leiman (SL) orthogonalization (Schmid & Leiman, 1957), which, in its most basic form, can be summarized in three steps. First, an oblique solution in r primary factors is obtained from the sample correlation matrix. Second, the Spearman model is fitted to the interfactor correlation matrix, so a second-order factor is extracted. Finally, on the basis of the second-order results, the rotated solution in r factors is expanded into an SL orthogonal solution with $r+1$ factors.

The basic problem with the SL approach is that this expansion from r to $r+1$ factors imposes proportionality constraints on the SL pattern that are not intrinsic to bifactor models. In particular, the first column of the SL pattern (corresponding to the general factor) is a linear combination of the remaining columns. And, if the data has been generated by a bifactor model in which this constraint does not apply, then the estimates obtained by applying the SL solution will be biased with respect to the corresponding parameters in the "true" solution (see Jennrich & Bentler, 2011).

Reise, Moore, and Haviland (2010), and Reise, Moore, and Maydeu-Olivares (2011) considered that the biases described above are of no great concern when the SL pattern is only used to identify a pattern of salient and nonsalient loadings in a bifactor solution. These authors went on to suggest using the SL solution as a target for performing a semispecified Procrustes rotation (Browne, 1972), which can be regarded as less restricted than the SL solution. Simulations carried out by the proponents suggest that the target proposal works well in many conditions. However, there still remains the problem that the SL solution which is used as a target can be a biased estimate of the "true" population in many cases (Abad, Garcia-Garzon, Garrido, & Barrada, 2017; Reise et al., 2011).

In order to minimize this problem of a (possibly) biased initial target, Abad et al. (2017) proposed to empirically update the initial SL target by using the iterated target rotation procedure developed by Moore, Reise, Depaoli, & Haviland (2015). The extensive simulation study by Abad at al. (2017) suggests that their proposal is, to date, the best strategy for orthogonal exploratory bifactor analysis (i.e., exploratory bifactor models in which all group factors are uncorrelated). In particular, it appears to perform substantially better than the fixed-target proposal when there are both (a) complex structures with many cross-loadings and (b) pure indicators of the general factor.

At the time, this article was submitted, a paper by Waller (2017) that we were not aware of was accepted for publication. Waller's proposal is based on a direct SL approach and bears a close resemblance to the present proposal. For this reason, we have decided to present our proposal first, and then compare it with Waller's in a specific section.

We turn now to the second strategy above. Bifactor rotation procedures based on an initial EFA solution in $r+1$ factors have been proposed by Jennrich and Benter (JB) for the orthogonal (2011) and the oblique (2012) cases. Jennrich and Bentler (2011, 2012) considered only two rotation criteria: bi-quartimin and bi-geomin, and chose the gradient projection algorithm (GPA) to minimize the proposed criteria.

Unlike the SL approaches discussed above, the JB solutions are free from proportionality constraints. However, as well as the limited number of rotation criteria available, they have other drawbacks which are clearly discussed in Mansolf and Reise (2016). First, when a solution in r factors or an SL solution with proportionality constraints holds, the JB rotation with $r+1$ factors breaks down. Second, although the JB rotation criteria does not depend on the first column of the pattern (i.e., the general factor), this factor is also rotated in the process, and this implicit rotation might shift variance to the general factor and lead to local minima problems. At the empirical level, the simulation study by Abad et al. (2017) suggests that the JB procedures do not work better than SL-based procedures, and, in particular, that the bi-quartimin rotation is not a method to be recommended.

A new proposal on pure exploratory bifactor analysis

In a pure exploratory bifactor analysis (PEBI), a correlation matrix $\mathbf{R}$ between m indicators is analyzed, and the solution in $r+1$ factors is rotated to approach as much as possible the "ideal" bifactor pattern described above according to some specified criterion function. In more detail, $\mathbf{R}$ is decomposed as

$$\mathbf{R} = \mathbf{P}\mathbf{\Phi}\mathbf{P}' + \mathbf{\Psi} \tag{1}$$

where $\mathbf{P}$ is a loading matrix of order $m \times (r+1)$, $\mathbf{\Phi}$ is the interfactor correlation matrix of order $(r+1) \times (r+1)$, and $\mathbf{\Psi}$ is a diagonal matrix of order $m \times m$. In the loading matrix $\mathbf{P}$, r columns describe the relationship between the m items and the group factors in such a way that a given simplicity criterion is maximized. The extra column in $\mathbf{P}$ contains the loadings of the m items on the general factor. While the group factors can be correlated (if the simplicity criterion that is maximized allows them to be), the correlation between the general factor and the group factors is restricted to zero. Our proposal for assessing this pure bifactor model is based on the following four-step procedure.

Step 1. Define a partially specified target matrix

The aim in the first step is to build a target matrix $\mathbf{H}$ of the general form $\mathbf{P}$ above. So, the target matrix $\mathbf{H}$ can be partitioned as

$$\mathbf{H} = \left[\boldsymbol{h}_g | \boldsymbol{H}_s\right] \tag{2}$$

where $\mathbf{h}_g$ is a vector of m free parameters (related to the general factor), and $\mathbf{H}_s$ is a target of order $m \times r$ (related to the group factors). Because only some of the values in $\mathbf{H}_s$ are specified (typically those that are expected to be zero in the population model), $\mathbf{H}$ can be defined as a *partially specified target matrix*, and the main issue is to identify the free parameters in $\mathbf{H}_s$ (usually there is just one free parameter per item). Our procedure can use various approaches to do this:

1. The researcher can propose a target submatrix $\mathbf{H}_s$ based on previous research results.
1. The SL-based target matrix proposed by Reise et al. (2011) or the final iterated target in the proposal by Abad et al. (2017) discussed above can be used. The columns of group factors in the target matrix obtained with any of these procedures corresponds to the target submatrix $\mathbf{H}_s$ in our proposal.

In addition to these approaches, we propose a third approach in which an initial factor solution in r factors is obtained from $\mathbf{R}$, and then, a partially specified target based on the r retained factors is automatically built. Procedures for obtaining this type of target matrix have already been proposed in the literature on simple structure rotation criteria. Two examples are Simplimax (Kiers, 1994) and Promin (Lorenzo-Seva, 1999). The target matrix obtained by either of these two rotation methods corresponds to the target submatrix $\mathbf{H}_s$ in the present proposal.

Step 2. Identify the loadings on the general factor

In the second step, the correlation matrix $\mathbf{R}$ is factor analyzed again, but now $r+1$ factors are specified. If $\mathbf{A}$ is the initial loading matrix of order $m \times (r+1)$, and $\mathbf{H}$ is the target matrix obtained in step 1, an orthogonal semi-specified Procrustes rotation (Browne, 1972) is performed in order to determine the transformation matrix $\mathbf{G}$ that minimizes the distance between the product $\mathbf{AG}$ and the values specified in $\mathbf{H}$,

$$f(\mathbf{G}) = \mathrm{Q}(\mathbf{AG}, \mathbf{H}). \tag{3}$$

Note that $\mathbf{A}$ (of order $m \times (r+1)$) is rotated against a target $\mathbf{H}$ of the same order, which has a first column of free loadings intended to model the general factor. Therefore, if we define the rotated loading matrix as $\mathbf{B} = \mathbf{AG}$, then $\mathbf{B}$ can again be partitioned as,

$$\mathbf{B} = [\mathbf{b}_g | \mathbf{B}_s] \tag{4}$$

where $\mathbf{b}_g$ is a vector that contains the loadings of the m items on the general factor, and $\mathbf{B}_s$ is a matrix of order $m \times r$ that contains the loadings of the m items on the r group factors.

Step 3. Rotate the loadings on the group factors to maximize factor simplicity

Once $\mathbf{B}_s$ is available from step 2, it can be further rotated to maximize any orthogonal or oblique criterion. For example, applied researchers generally prefer to rotate loading matrices with the same rotation criterion used in previous studies (e.g., the popular Varimax). In addition, researchers would like to inspect whether the group factors are correlated with each other or not.

In order to rotate the group factors and maximize, for example the Varimax criterion, the transformation matrix $\mathbf{S}$ must be obtained as

$$f(\mathbf{S}) = \mathrm{vmax}(\mathbf{B_s S}). \tag{5}$$

In the same way, $\mathbf{S}$ can be an oblique transformation matrix that maximizes an oblique rotation criterion. For example, $\mathbf{S}$ could be obtained using Promin rotation

$$f(\mathbf{S}) = \mathrm{promin}(\mathbf{B_s S}). \tag{6}$$

Step 4. Obtain the final exploratory bifactor solution

The final transformation matrix is obtained as the product

$$\mathbf{T} = \mathbf{G}\begin{bmatrix} 1 & 0' \\ 0 & \mathbf{S} \end{bmatrix}. \tag{7}$$

where $\mathbf{0}$ is a column vector of r zero values. The final rotated loading matrix is obtained as

$$\mathbf{P} = \mathbf{AT}, \tag{8}$$

and the interfactor correlation matrix is obtained as

$$\Phi = \mathbf{T}^{-1}\mathbf{T}^{-1\prime}. \tag{9}$$

It is noted that, in this fourth step, only the group factors are rotated while the general factor loadings are obtained in the same way as in the second step. It follows from this proposal that potential biases and misspecifications at step 2 (due e.g., to an inappropriate target) are also expected to propagate to step 3. How important this problem is in practice is assessed in the simulation studies below.

The single group factor ($r = 1$) case

The situation in which a set of items is essentially unidimensional, but in which a (generally small) subgroup of items share specific variance is relatively common in practice. Among other cases, this consistent clustering might arise because of "method" effects (e.g., similar item wording) or content specificity, as illustrated in one of the examples below. Addressing this situation involves fitting the bifactor model with $r = 1$. So, the target matrix $\mathbf{H}$ in (2) can now be partitioned as

$$\mathbf{H} = [\mathbf{h}_g | \mathbf{h}_s] \tag{10}$$

where $\mathbf{h}_g$ is a vector of m free parameters (related to the general factor), and $\mathbf{h}_s$ is a vector related to the single group factor.

The PEBI approach can be easily applied to the single group case if a target vector $\mathbf{h}_s$ can be specified a priori in step 1 above. Once the full $\mathbf{H}$ matrix in (10) has been obtained, then only Step 2 above has to be computed, and the rotated loading matrix $\mathbf{B}$ in (4) is the final loading matrix.

If the information available is not sufficient to specify $\mathbf{h}_s$, then we propose the full exploratory procedure that follows. Assume that the loading matrix $\mathbf{A}$ above (of order $m \times 2$ in this case) is in the usual canonical form (e.g., Harman, 1962). If it is, the second column of $\mathbf{A}$ is bipolar, with one half of the loadings positive and the other half negative, thus separating the items into two clusters. Next, the cluster that explains of the least variance (i.e., the set that has the lowest sum of squared loadings) is selected as the set of items that form the group factor. So, the corresponding values in $\mathbf{h}_s$ are set as free values, while the remaining values are defined as zero.

A comparison with Waller's direct SL approach (BiFAD)

From the point of view of our proposal, Waller's (2017) approach can be considered to have the same general structure as PEBI but with alternative solutions in steps 1 and 2. It should be noted that

Waller's paper has a wider scope, but we shall only focus on the exploratory part of his proposal which we shall call BiFAD, the name of the function used (Waller, 2017).

The general structure common to PEBI and BiFAD consists of setting a target for form (2) above, and performing a rotation of an $r+1$ factor pattern against this target. Now, with regards to target-setting step one, Waller proposes to obtain the submatrix $\mathbf{H}_s$ by (a) fitting an oblique solution in r factors, (b) setting an arbitrary threshold value, and (c) dichotomizing each loading at this threshold to produce a signed target of zeroes and ones. As for the $r+1$ pattern to be rotated (step 2), Waller uses the initial factor solution in r factors with a vector of zeroes appended. Finally, with regards to the remaining PEBI proposals, BiFAD is only concerned with orthogonal rotations and PEBI steps 3 and 4 above are not considered. The single-factor case is not dealt with either.

The present proposal views the target specification in BiFAD as a fourth possible alternative within PEBI's first step. More specifically, in comparison with the Promin specification, Waller's procedure is simpler but has an unavoidable component of arbitrariness in the choice of the threshold. In contrast, Promin automatically sets the threshold values as a function of the distribution of the loadings in each column (see Lorenzo-Seva, 1999).

With regards to the **A** initial matrix to be rotated, Waller's proposal is clearly simpler, as only the initial matrix in r factors is required and there is no need to then fit the $r+1$ solution in step 2. Now, if the hierarchical bifactor model is correct, our $r+1$ column in **A** (which is in canonical form) should be a column of zeros, so the second extraction will be totally unnecessary. For this reason, Waller's approach can be considered to be more confirmatory than PEBI: Our approach assumes that the bifactor solution in r factors is only an approximation and that a certain amount of common variance might not be accounted for by this model. It should be noted, however, that Waller (2017, Section 4.3) also considered the full $r+1$ solution in conditions in which the hierarchical bifactor model was not expected to be correct.

The more exploratory orientation of PEBI with respect to BIFAD can also be seen in the choice of the target rotation. BiFAD uses a Procrustes rotation which assigns the same weights (i.e., 1) to the nonzero loadings. This means that the method expects all the items to contribute equally to the general and corresponding specific factor. In contrast, in PEBI the rotation is semi-specified, and the nonzero loadings are freely estimated, which means that no hypothesis is advanced about the amount of common variance of each item related to the general and corresponding specific factor.

The additional third and fourth steps proposed in PEBI are expected to improve factor simplicity and they are needed if an oblique solution is to be obtained (the importance of oblique solutions is discussed below). The extent to which they lead to important improvements in practice, however, is a matter that must be empirically assessed, and the simulation study below provides some initial evidence on this issue.

Simulation studies

An extensive set of simulation studies was undertaken to assess the functioning of PEBI under four general scenarios: (a) the single-group-factor case, (b) the uncorrelated (orthogonal) group-factor case, (c) the orthogonal case when there is no general factor (i.e., the multiple orthogonal FA model), and (d) the correlated (oblique) group-factor case. In scenarios (b), (c), and (d) the performance of PEBI was compared to that of previously proposed approaches.

The single group factor (r = 1) case

Independent variables

The study was based on a $3 \times 2 \times 2 \times 2$ design with a total of 24 conditions and 100 replicas per condition. The independent variables were: (1) sample size $N = 200$, 500, 2,000; (2) number of observed variables $m = 6$, 12; (3) loading value sizes: low (largest communality .65) and high (largest communality .85); and (4) size of the general factor: the general factor defined with loadings equal to the group factor loadings (GF = SF), the general factor defined with loadings larger than the group factor loadings (GF > SF).

Population loading matrices were built as follows. Loading values in the general factor were randomly chosen in the range [.50, .65] for condition GF = SF, and in the range [.80, .95] for condition GF > SF. The group factor was defined by two observed variables (when $m = 6$), or by four observed variables (when $m = 12$). Salient loading values in the group factor were randomly chosen in the range [.50, .65]. The nonsalient loadings in the group factor were all set to zero. Once the initial loading had been generated, the whole loading matrix was row scaled so that the maximum communality of any item was .65 or .85, depending on the "loading value size" condition at hand (low or large). Interfactor correlation matrix $\mathbf{\Phi}$ was set as a 2×2 identity matrix.

Data generation and model-data fitting

A total of 100 sample data matrices were simulated for each condition according to the common factor model. First, the reproduced population correlation matrix (with communalities in the diagonal) was computed as

$$\mathbf{R}^* = \mathbf{P}\Phi\mathbf{P}' \tag{11}$$

(see Equation (1)). The population correlation matrix $\mathbf{R}$ was then obtained by inserting unities in the diagonal of $\mathbf{R}^*$. Then, we computed the Cholesky decomposition of $\mathbf{R} = \mathbf{L}'\mathbf{L}$, where $\mathbf{L}$ is an upper triangular matrix. The sample data matrix of continuous variables $\mathbf{X}$ was finally obtained as $\mathbf{X} = \mathbf{Z}\mathbf{L}$, where $\mathbf{Z}$ is a matrix of random standard normal scores with rows equal to the corresponding sample size, and number of columns equal to the corresponding number of variables.

In all cases, the sample data matrices were fitted by using procedures implemented in Matlab. Variables were always treated as continuous and fitted using the unweighted least squares (ULS) criterion. In order to assess the bi-factor model in the simulated data, PEBI was computed for the case of $r = 1$ (i.e., single group factor case).

Dependent variables

Congruence and discrepancy indices were used to assess the degree to which the true generated structures were recovered. The congruence index was the Burt-Tucker coefficient of congruence, a measure of profile similarity (see Lorenzo-Seva & ten Berge, 2006) that is defined as

$$\phi(x, y) = \frac{\sum x_i y_i}{\sqrt{\sum x_i^2 \sum y_i^2}}. \tag{12}$$

Expression 12 was used to assess the congruence between the columns of the population loading matrix and the columns of the fitted loading matrices. The overall congruence between two loading matrices is usually reported by calculating the average of the column congruence. Lorenzo-Seva and ten Berge (2006) pointed out that a value in the range [.85–.94] corresponds to a fair similarity, while a value higher than .95 implies that the factor solutions compared can be considered equal. The discrepancy index was the root mean squared residual (RMSR) between the population model and the data fitted model, a measure of profile distance that is defined as

$$\mathrm{RMSR}(\boldsymbol{X}, \boldsymbol{Y}) = \sqrt{(1/mr)\sum_i^m \sum_j^r (x_{ij} - y_{ij})^2}. \tag{13}$$

Results

Table 1 shows the congruence and RMSRs related to the general factor and the group factor. The recovery of the general factor was very good in all conditions: congruencies above .95 (the cut-off reference suggested by Lorenzo-Seva & ten Berge, 2006) and RMSRs of about .053 or less. On the other hand, the recovery of the group factor was generally worse. The two most difficult conditions were: small sample size ($N = 200$), and few observed variables ($m = 6$).

To assess effect sizes, analyses of variance were carried out with the IBM SPSS Statistics v. 20 program. Cohen (1988, pp. 413–414) suggested that threshold values for eta squared (η^2) effect sizes of .02 represent small effects, .13 medium effects, and .26 or more large effects. Sample size to some extent affected the congruence of the general factor recovery ($\eta^2 = .108$). Sample size ($\eta^2 = .451$ for congruence index) and number of variables ($\eta^2 = .261$ for discrepancy index) substantially affected the congruence and the discrepancy of the group factor recovery. Finally, the effect sizes of the interactions among independent variables were generally small (lower than .035).

The uncorrelated group factors case

The main aim of this study was to use various orthogonal and oblique rotation criteria to assess the functioning of PEBI when the bifactor solution in the population was actually orthogonal. We also aimed to compare it to existing procedures that allow for both orthogonal and oblique rotations of the group factors (i.e., the GPA-based approaches). Finally, although Waller's (2017) BiFAD approach is only intended for orthogonal solutions, we also included it in this study

Table 1. Averages and standard deviations (given in parenthesis) of congruence and discrepancy indices for condition $r = 1$.

	General factor		Group factor	
Condition	Congruence	Discrepancy	Congruence	Discrepancy
OVERALL	**.998** (.005)	.034 (.075)	**.965** (.085)	.071 (.059)
$N = 200$	**.996** (.007)	.053 (.024)	.941 (.134)	.095 (.063)
$N = 500$	**.999** (.002)	.033 (.014)	**.972** (.043)	.069 (.050)
$N = 2000$	**.999** (.001)	.016 (.006)	**.981** (.035)	.049 (.055)
$m = 6$	**.997** (.006)	.038 (.027)	.942 (.112)	**.101** (.068)
$m = 12$	**.999** (.001)	.030 (.016)	**.988** (.029)	.041 (.023)
Low loadings	**.997** (.006)	.040 (.026)	**.951** (.112)	.081 (.066)
High loadings	**.999** (.001)	.028 (.016)	**.978** (.039)	.061 (.050)
GF = SF	**.997** (.006)	.039 (.025)	**.973** (.059)	.070 (.056)
GF > SF	**.999** (.002)	.030 (.019)	**.956** (.104)	.072 (.062)

Note: GF = SF: General factor has been defined with loadings equal to the loadings of group factor; GF > SF: General factor has been defined with larger loadings than the group factor loadings. Congruence values larger than .95 and discrepancies larger than .10 are printed in bold face.

given its similarity to the PEBI approach discussed above.

The PEBI analyses considered in the simulation study were based on nine rotation criteria aimed at maximizing the simplicity of the group factors: Varimax, Quartimax, Equamax, Orthogonal Promin, Orthogonal Quartimin, Orthogonal Geomin, Oblique Promin, Oblique Quartimin, and Oblique Geomin. After running the analyses, however, we found that the correlations among the outcomes ranged between .918 and .951. The GPA-based analyses were based on four rotation criteria: Orthogonal Quartimin, Orthogonal Geomin, Oblique Quartimin, and Oblique Geomin. In this case, the outcomes correlated with one another in the range between .360 and .676. Overall, to simplify the reported results, we decided to report only the PEBI- and GPA-based outcomes obtained by using Orthogonal Quartimin, the criterion that led to the best performance of the GPA-based analyses in the simulation study. Finally, to compute BiFAD, a previous oblique rotation must be chosen. We opted for oblique quartimin because it seemed to be the most accurate in the illustrative example provided by Waller (2017). With these settings, all the reported outcomes are based on the same rotation criterion (quartimin), so any differences can be attributed to the different bifactor approaches.

In general terms, the design in this section attempted to mimic the conditions expected to be found in empirical applications and was partly based on the study by Abad et al. (2017). The main differences were (a) the number of group factors in our study ranged from 2 to 5; and (b) we manipulated the size of the general factor. Abad et al. (2017) considered only high-dimensionality solutions, starting from 4 group factors, and in his study, the size of the general factor was not manipulated. In addition, Abad et al. (2017) included items that were pure indicators of the general factor. We did not include this variable because, in a previous simulation study, we observed that pure indicators did not add much information to the outcomes. As the study by Abad et al. (2017) was not carried out in the same conditions as our own simulation study, our outcomes cannot be directly compared with theirs.

Independent variables

A $3 \times 4 \times 2 \times 2 \times 3 \times 2$ design with a total of 288 conditions and 100 replicas per condition was used. The independent variables were: (1) sample size $N = 200$, 500, 2,000; (2) number of group factors $r = 2$, 3, 4, 5; (3) number of variables per group factor $m = 6$, 12; (4) loading value sizes: low (largest communality .65) and high (largest communality .85); (5) size of the general factor: the general factor defined with loadings lower than the group factor loadings (GF < SF), the general factor defined with loadings equal to the group factor loadings (GF = SF), the general factor defined with loadings larger than the group factor loadings (GF > SF); and (6) cross-loadings: No (no cross-loadings in group factors) and Yes (one item from each group factor has a cross-loading in another group factor).

Population loading matrices were built as follows. Loading values in the general factor were randomly chosen in the range [.20, .35] for condition GF < SF, in the range [.50, .65] for condition GF = SF, and in the range [.80, .95] for condition GF > SF. To define the group factors, a second value randomly chosen in the range [.50, .65] was assigned as the loading related to the corresponding group factor for each observed variable. The nonsalient loadings were all set to zero. Finally, in the conditions in which cross loadings were present, a third loading value on r items (one item per group factor) was randomly chosen in the range [.50, .65] from a uniform distribution. Once the initial loading matrix was available, the whole loading matrix was row scaled so that the maximum communality of any item was .65 or .85, depending on the "loading value size" condition at hand (low or large).

Data generation and model-data fitting

Data was generated as in the first simulation study with the difference that $\mathbf{\Phi}$ was now a unit matrix of order $(r+1)\times(r+1)$.

As in the previous study, variables were always treated as continuous and fitted using the ULS criterion. As mentioned above, the performance of PEBI was compared to that of GPA-based, and BiFAD approaches.

Dependent variables

As in the previous study, congruence and discrepancy indices were used to assess the degree to which the true generated structures were recovered, and the size of the effect sizes were inspected using eta squared (η^2).

Results

Table 2 shows the congruence and RMSRs related to the overall bifactor solution, the general factor, and the group factors. Overall, the GPA-based approach had more difficulties in recovering the population model, while PIBE and BiFAD performed quite well with congruences above .95 and RMSRs lower than .10. BiFAD was the approach that achieved the best results.

A better understanding of the performance of the two best approaches can be obtained by inspecting Table 3. The most "difficult" situation for PEBI was when there were 2 group factors. Other difficult situations were: a small sample, a small number of observed variables, the presence of cross-loadings, and when the general factor has lower loadings than the group factors. The general factor was best recovered when it was better defined than the group factors, and the group factors were best recovered when they had larger loadings than the general factor.

While BiFAD was very successful here, its profile performance was similar to that of PEBI except for the number of group factors: the larger the number of factors, the worse the performance of BiFAD in terms of recovering the group factors. In contrast, PEBI seemed to improve as the number of group factors increased.

Table 2. Averages and standard deviations (given in parenthesis) of congruence and discrepancy indices related to orthogonal Quartimin-based rotations.

Fit index	Pure exploratory bifactor	Gradient projection algorithm	Direct Schmid–Leiman
Congruence			
Overall	**.956** (.057)	.913 (.100)	**.983** (.010)
General factor	**.952** (.126)	**.961** (.109)	**.994** (.009)
Group factors	**.957** (.065)	.897 (.137)	**.980** (.012)
Discrepancy			
Overall	.082 (.055)	**.105** (.068)	.067 (.028)
General factor	.090 (.065)	.097 (.064)	.073 (.041)
Group factors	.077 (.055)	**.104** (.076)	.064 (.024)

Note: Congruence values larger than .95 and discrepancies larger than .10 are printed in bold face.

In terms of discrepancy-based results (Table 4). PEBI performed less well when there were 2 group factors, a small sample, and the general factor was lower than the group factor. It should be pointed out that, in terms of discrepancy, BiFAD performed like PEBI: the largest was the number of group factors, the lowest was the distance between the population model and the sample estimates.

The largest effect size for PEBI was the number of group factors ($\eta^2 = .239$ and $\eta^2 = .516$ for correspondence and discrepancy indices, respectively). As for BiFAD, effect sizes are difficult to assess because the ceiling effect means that variances are very low since the values are so close to their upper limit. The effect size is largest for the *size of the general factor* ($\eta^2 = .240$ and $\eta^2 = .560$ for correspondence and discrepancy indices, respectively): the outcomes were optimal when the general factor and the group factors were equal. Finally, interaction-related effect sizes were generally small (lower than .066).

The case of uncorrelated group factors when a general factor is not present in the population

The study above assessed the performance of PEBI and competing procedures when the bifactor was the correct population model. In contrast, this third study aims to determine how they perform when there is no general factor. In other words, when the "correct" model in the population is the multiple orthogonal model. For a well-functioning approach, the expected outcome in this case would be as follows: (a) the general factor would be residual, and (b) the group factors should approach the "true" orthogonal solution. At the opposite extreme, an outcome consisting of a

Table 3. Averages and standard deviations (given in parenthesis) of congruence indices for orthogonal population models.

Condition	Pure exploratory bifactor			Direct Schmid–Leiman		
	Overall	General factor	Group factors	Overall	General factor	Group factors
$N = 200$	.940 (.060)	.938 (.138)	.941 (.073)	**.977** (.013)	**.992** (.010)	**.972** (.015)
$N = 500$	**.957** (.057)	**.950** (.133)	**.959** (.064)	**.984** (.008)	**.994** (.009)	**.981** (.009)
$N = 2000$	**.972** (.048)	**.967** (.103)	**.973** (.054)	**.988** (.006)	**.995** (.008)	**.985** (.008)
$r = 2$	.909 (.803)	.919 (.171)	.904 (.106)	**.987** (.007)	**.992** (.011)	**.984** (.009)
$r = 3$	**.964** (.047)	.934 (.163)	**.973** (.033)	**.984** (.009)	**.994** (.008)	**.980** (.011)
$r = 4$	**.976** (.022)	**.974** (.058)	**.977** (.022)	**.981** (.010)	**.993** (.009)	**.978** (.012)
$r = 5$	**.976** (.020)	**.980** (.040)	**.976** (.022)	**.979** (.013)	**.995** (.007)	**.976** (.015)
$m/r = 6$	**.949** (.061)	.949 (.132)	.948 (.075)	**.982** (.011)	**.993** (.010)	**.979** (.013)
$m/r = 12$	**.963** (.051)	**.956** (.120)	**.967** (.053)	**.984** (.010)	**.994** (.007)	**.981** (.011)
Low Loadings	**.951** (.058)	**.950** (.123)	**.950** (.069)	**.981** (.012)	**.993** (.009)	**.977** (.014)
High Loadings	**.962** (.056)	**.953** (.129)	**.964** (.061)	**.985** (.008)	**.994** (.009)	**.982** (.010)
Cross-loadings = No	**.971** (.047)	**.973** (.072)	**.971** (.062)	**.987** (.008)	**.997** (.003)	**.984** (.010)
Cross-loadings = Yes	.941 (.062)	.931 (.160)	.944 (.067)	**.978** (.010)	**.990** (.011)	**.975** (.013)
GF < SF	.949 (067)	.873 (.194)	**.976** (.043)	**.980** (.007)	**.985** (.011)	**.978** (.008)
GF = SF	**.963** (.050)	**.987** (.022)	**.954** (.069)	**.990** (.009)	**.997** (.003)	**.988** (.010)
GF > SF	**.957** (.051)	**.996** (.004)	.942 (.075)	**.979** (.011)	**.999** (.001)	**.973** (.013)

Note: GF < SF: General factor has been defined with lower loadings than the group factors loadings; GF = SF: General factor has been defined with loadings equal to the loadings of group factors; GF > SF: General factor has been defined with larger loadings than the group factors loadings. Congruence values larger than .95 are printed in bold face.

Table 4. Averages and standard deviations (given in parenthesis) of discrepancy indices for orthogonal population models.

Condition	Pure exploratory bifactor			Direct Schmid–Leiman		
	Overall	General factor	Group factors	Overall	General factor	Group factors
$N = 200$	**.100** (.049)	**.110** (.061)	.094 (.050)	.076 (.025)	.079 (.039)	.074 (.021)
$N = 500$	.082 (.053)	.090 (.063)	.076 (.053)	.066 (.027)	.072 (.041)	.062 (.023)
$N = 2000$	.064 (.057)	.070 (.065)	.059 (.056)	.060 (.030)	.068 (.042)	.056 (.025)
$r = 2$	**.146** (.060)	**.148** (.074)	**.144** (.065)	.086 (.035)	.094 (.051)	.081 (.029)
$r = 3$	.072 (.036)	.086 (.063)	.063 (.028)	.069 (.024)	.074 (.038)	.066 (.021)
$r = 4$	.056 (.023)	.066 (.039)	.052 (.021)	.060 (.020)	.066 (.034)	.058 (.018)
$r = 5$	.050 (.020)	.059 (.033)	.048 (.018)	.054 (.015)	.058 (.029)	.052 (.016)
$m/r = 6$	.090 (.061)	.098 (.071)	.084 (.062)	.069 (.028)	.076 (.041)	.066 (.024)
$m/r = 12$	.074 (.047)	.082 (.057)	.069 (.046)	.065 (.028)	.070 (.040)	.062 (.024)
Low loadings	.082 (.052)	.089 (.059)	.078 (.051)	.066 (.025)	.070 (.037)	.063 (.022)
High loadings	.081 (.059)	.090 (.071)	.075 (.058)	.069 (.030)	.075 (.044)	.065 (.026)
Cross-loadings = No	.070 (.048)	.078 (.056)	.065 (.049)	.062 (.031)	.066 (.044)	.060 (.027)
Cross-loadings = Yes	.093 (.059)	**.102** (.072)	.088 (.058)	.072 (.023)	.080 (.037)	.068 (.020)
GF < SF	.092 (.061)	**.135** (.078)	.071 (.052)	.092 (.022)	**.120** (.028)	.080 (.018)
GF = SF	.081 (.056)	.081 (.049)	.082 (.060)	.041 (.016)	.039 (.017)	.041 (.016)
GF > SF	.071 (.044)	.054 (.030)	.077 (.051)	.069 (.017)	.059 (.019)	.072 (.018)

Note: GF < SF: General factor has been defined with lower loadings than the group factors loadings; GF = SF: General factor has been defined with loadings equal to the loadings of group factors; GF > SF: General factor has been defined with larger loadings than the group factors loadings. Discrepancies larger than .10 are printed in bold face.

Table 5. Averages and standard deviations (given in parenthesis) of congruence and common variance related to orthogonal Quartimin-based rotations when a general factor is fitted at the sample data, but is not modeled at the population model.

Fit index	Pure exploratory bifactor	Gradient projection algorithm	Direct Schmid–Leiman
Congruence of group factors	**.989** (.020)	.888 (.112)	**.959** (.007)
Common variance			
Total	20.361 (8.602)	20.361 (8.602)	19.991 (8.598)
General factor	0.879 (0.706)	2.513 (1.723)	3.789 (1.470)
Average of group factors	4.326 (1.784)	3.960 (1.895)	3.591 (1.518)

Note: Congruence values larger than .95 are printed in bold face.

strong general factor together with group factors that depart from the "true" pattern, should be considered as unsuccessful recovery.

The present study used the same design as that above with two differences: (a) only group factors were present in the population, and (b) only 4 and 5 group factors were considered. Limitation (b) was applied because in the previous simulation all the methods performed well at these levels. Overall, the study used a $3 \times 2 \times 2 \times 2 \times 2$ design with a total of 48 conditions and 100 replicas per condition. The data were generated as in the previous example, and congruence was computed to assess the degree to which the population group factors were recovered. We also computed the amount of variance explained by the fitted general factor, and the average variance explained by the fitted group factors.

Results

Overall congruences between sample fitted group factors and the population group factors are displayed in Table 5. The GPA-based approach performed worst, BiFAD was just over the threshold of .95, and PEBI gave the largest congruence value. In terms of amount of variance, both PEBI and GPA introduced more variance into the fitted bifactor loading matrix, which is an expected result because both methods extract $r + 1$ factors. The amount of extra variance, however, was very low when compared to that produced by BiFAD (which extracts just r factors). In terms of relevance of the general factor, the best solutions were obtained by PEBI because this factor was clearly residual. At the opposite extreme, BiFAD arrived at solutions in which the general factor explained even more variance than the group factors. Finally, the GPA-based approach arrived at solutions in which the fitted general factor explained a substantial amount of variance, but this variance was less than that explained by the group factors. The best performance of PEBI here suggests that it is the most suitable approach for a truly exploratory study in which the presence of a general factor in the population is not warranted.

The case of correlated group factors

In this final study, we assessed the outcomes of PEBI- and GPA-based methods when the bifactor model with correlated group factors held in the population. The design and conditions considered were the same as those in the second study with two exceptions. First, two of the factors were correlated in the range [.40; .70] and the actual value in this range was randomly drawn from a uniform distribution. Second, the RMSR between the population and the fitted interfactor correlation matrices was included as a dependent variable.

PEBI analyses here were based on three maximizing-simplicity criteria: Oblique Promin, Oblique Quartimin, and Oblique Geomin. However, once the outcomes were available, we observed that they correlated with one another between .914 and .966. In addition, the GPA-based analyses were based on two rotation criteria: Oblique Quartimin, and Oblique Geomin. Once again, we observed a correlation of .713 in this case. Overall, to simplify the reported results, we decided to report both the PEBI- and GPA-based outcomes based on Oblique Quartimin, which was the criterion that produced the best performance of the GPA-based procedures in the simulation. Because all the reported outcomes are based on the same rotation criterion, the differences obtained can be attributed to the different bifactor approaches.

Results

The results are summarized in Table 6. In terms of the overall congruence, PEBI was the best approach.

Table 6. Averages and standard deviations (given in parenthesis) of congruence and discrepancy indices related to oblique Quartimin-based rotations.

Fit index	Pure exploratory bifactor	Gradient projection algorithm
Congruence		
Overall	.920 (.078)	.864 (.117)
General factor	**.956** (.071)	**.968** (.062)
Group factors	.902 (.113)	*.823* (.171)
Discrepancy		
Overall	.126 (.065)	.141 (.076)
General factor	.143 (.080)	.139 (.078)
Group factors	.119 (.066)	.139 (.085)
Inter-factor correlations	.178 (.094)	.233 (.166)

Note: Congruence values larger than .95 are printed in bold face. Congruence values lower than .85 are printed in italics.

However, the GPA-based approach best replicated the general factor, while PEBI best replicated the group factors. In fact, the GPA-based procedure produced a congruence value for the group factors that was under the minimum threshold of .85 proposed by Lorenzo-Seva and ten Berge (2006). The pattern is also the same for discrepancy. Finally, PEBI was the approach that best replicated the interfactor correlation matrix. Overall, the outcomes in Table 8 suggest that PEBI outperformed the GPA-based approach except for the recovery of the general factor.

For a better understanding of the performance of the two approaches, Table 7 shows the congruence indices among the different levels of the independent variables. As can be seen, the most complex situation handled by PEBI was when there were two group factors. In this situation the general factor was recovered very well (congruence of .984), but the group factors were recovered very deficiently (congruence of .790, which is under the critical threshold of .85). Except for this condition, PEBI outcomes suggest a balance between the recovery of the general factor and the recovery of the group factors, which is slightly biased in favor of the recovery of the general factor. In contrast, the GPA-based procedure clearly focuses on the recovery of the general factor whereas the recovery of the group factors is frequently under the critical threshold of .85.

Table 8 shows the discrepancy results across the different conditions. Overall, PEBI seems to perform also systematically better in terms of discrepancy except in the conditions in which there are no cross-loadings. In fact, under this condition, the GPA-based approach also gave the best results in terms of the general and the content factors. Furthermore, the

Table 7. Averages and standard deviations (given in parenthesis) of congruence indices for oblique population models.

Condition	Pure exploratory bifactor			Gradient projection algorithm		
	Overall	General factor	Group factors	Overall	General factor	Group factors
$N = 200$	.892 (.085)	**.950** (.084)	.867 (.125)	*.823* (.119)	**.967** (.060)	*.769* (.173)
$N = 500$	.921 (.075)	**.957** (.070)	.903 (.111)	.867 (.113)	**.969** (.061)	*.826* (.167)
$N = 2000$	.946 (.062)	**.961** (.056)	.936 (.091)	.903 (.106)	**.969** (.065)	.875 (.156)
$r = 2$	.854 (.099)	**.984** (.015)	*.790* (.147)	*.763* (.123)	**.982** (.056)	*.654* (.187)
$r = 3$	.922 (.066)	**.952** (.063)	.911 (.085)	.868 (.108)	**.968** (.062)	*.835* (.145)
$r = 4$	**.950** (.043)	.944 (.083)	**.950** (.051)	.906 (.086)	**.965** (.059)	.891 (.107)
$r = 5$	**.955** (.043)	.945 (.089)	**.957** (.046)	.920 (.078)	**.959** (.068)	.913 (.092)
$m/r = 6$	.907 (.085)	**.953** (.081)	.886 (.123)	*.847* (.122)	**.965** (.073)	*.802* (.178)
$m/r = 12$	.932 (.068)	**.959** (.059)	.918 (.101)	.882 (.110)	**.972** (.049)	*.845* (.161)
Low loadings	.907 (.083)	**.954** (.077)	.885 (.121)	*.849* (.119)	**.967** (.068)	*.802* (.174)
High loadings	.933 (.070)	**.958** (.065)	.919 (.103)	.880 (.114)	**.970** (.056)	*.844* (.166)
Cross-loadings = No	.946 (.065)	**.964** (.050)	.935 (.098)	.928 (.089)	**.967** (.073)	.908 (.133)
Cross-loadings = Yes	.893 (.080)	.948 (.086)	.869 (.118)	*.801* (.108)	**.970** (.049)	*.738* (.163)
GF < SF	.915 (.077)	.892 (.094)	.914 (.111)	.872 (.116)	.923 (.089)	*.847* (.169)
GF = SF	.927 (.074)	**.981** (.012)	.905 (.107)	.868 (.116)	**.986** (.021)	*.823* (.168)
GF > SF	.917 (.082)	**.995** (.004)	.887 (.120)	.854 (.119)	**.996** (.007)	*.800* (.174)

Note: GF < SF: General factor has been defined with lower loadings than the group factors loadings; GF = SF: General factor has been defined with loadings equal to the loadings of group factors; GF > SF: General factor has been defined with larger loadings than the group factors loadings. Congruence values equal or larger than .95 are printed in bold face. Congruence values lower than .85 are printed in italics.

Table 8. Averages and standard deviations (given in parenthesis) of discrepancy indices for oblique population models.

Condition	Pure exploratory bifactor			Gradient projection algorithm		
	Overall	General factor	Group factors	Overall	General factor	Group factors
$N = 200$	**.139** (.061)	.150 (.077)	**.134** (.063)	.162 (.074)	**.149** (.080)	.164 (.082)
$N = 500$	**.126** (.065)	.142 (.080)	**.119** (.066)	.141 (.077)	**.138** (.078)	.139 (.085)
$N = 2000$	**.115** (.067)	.136 (.083)	**.106** (.066)	.120 (.073)	**.131** (.073)	.114 (.082)
$r = 2$	**.209** (.055)	**.196** (.087)	**.211** (.050)	.236 (.068)	.200 (.090)	.247 (.074)
$r = 3$	**.130** (.037)	.152 (.066)	**.119** (.032)	.139 (.053)	**.137** (.068)	.135 (.059)
$r = 4$	**.094** (.033)	.124 (.068)	**.083** (.025)	.104 (.038)	**.116** (.055)	.097 (.043)
$r = 5$	**.073** (.029)	**.099** (.063)	**.064** (.023)	.085 (.032)	.104 (.053)	.078 (.034)
$m/r = 6$	**.133** (.069)	.148 (.084)	**.126** (.069)	.150 (.078)	**.147** (.081)	.148 (.086)
$m/r = 12$	**.120** (.061)	.137 (.076)	**.112** (.061)	.132 (.074)	**.131** (.073)	.130 (.083)
Low loadings	**.124** (.061)	.136 (.074)	**.118** (.062)	.141 (.073)	**.135** (.075)	.141 (.081)
High loadings	**.129** (.069)	.149 (.085)	**.120** (.069)	.141 (.080)	**.144** (.080)	.138 (.089)
Cross-loadings = No	.127 (.069)	.147 (.086)	.118 (.070)	**.122** (.081)	**.132** (.080)	**.114** (.092)
Cross-loadings = Yes	**.126** (.061)	**.139** (.073)	**.121** (.061)	.160 (.066)	.147 (.074)	.164 (.070)
GF < SF	**.163** (.071)	.230 (.062)	**.135** (.073)	.165 (.090)	**.213** (.077)	.143 (.099)
GF = SF	**.120** (.056)	**.124** (.043)	**.119** (.063)	.140 (.071)	.127 (.045)	.144 (.083)
GF > SF	**.096** (.049)	**.075** (.030)	**.104** (.057)	.118 (.059)	.079 (.030)	.129 (.071)

Note: GF < SF: General factor has been defined with lower loadings than the group factors loadings; GF = SF: General factor has been defined with loadings equal to the loadings of group factors; GF > SF: General factor has been defined with larger loadings than the group factors loadings. The lowest discrepancy value for each condition and type of column (overall, general factor, and group factors) is printed in bold face.

GPA-based approach was frequently the method that performed best in terms of the general factor.

For PEBI, the largest effect size was the number of group factors ($\eta^2 = .262$ and $\eta^2 = .624$ for correspondence and discrepancy indices, respectively), as it happened for GPA-based approach ($\eta^2 = .274$ and $\eta^2 = .578$ for correspondence and discrepancy indices, respectively). In addition, the GPA-based approach showed a considerable effect size for the cross-loadings main effect ($\eta^2 = .289$ and $\eta^2 = .063$ for correspondence and discrepancy indices, respectively). The effect sizes of the interactions among independent variables were generally small (none of them larger than .02).

Illustrative examples with real data

Example 1: The statistical anxiety scale

A 16-item version of the Statistical Anxiety Scale (SAS; Vigil-Colet, Lorenzo-Seva, and Condon, 2008), a measure of anxiety towards statistics, was administered to a sample of 384 undergraduate students. The reduced version used here is designed to assess (a) two related dimensions of anxiety: Examination Anxiety (EX; 8 items), and interpretation anxiety (IN; 8 items), as well as (b) a general dimension of statistical-related anxiety (Vigil-Colet et al., 2008). All 16 items are positively worded and use a five-point Likert response format, ranging from “no anxiety” (1) to “considerable anxiety” (5).

Examination of the item scores showed that the distributions were generally skewed. So, the item scores were treated as ordered-categorical variables, and the FA based on the polychoric interitem correlations was the model chosen to fit the data. This model is an alternative parameterization of the multidimensional graded Item Response model (see Ferrando & Lorenzo-Seva, 2013).

The interitem polychoric correlation matrix had good sample adequacy, Kaiser-Meyer-Olkin (KMO) Test for Sampling Adequacy = .908 (Kaiser & Rice, 1974), and Schwarz’s Bayesian information criterion suggested that a two-factor model was the most appropriate. Next, a bidimensional EFA solution was fitted by using Robust FA based on the Diagonally Weighted Least Squares (DWLS) criterion as implemented in the program FACTOR (Ferrando & Lorenzo-Seva, 2017a), and reached acceptable goodness-of-fit levels: RMSEA = .072 (95% confidence interval .057 and .075), CFI = .982 (95% confidence interval .971 and .988), and Weighted Root Mean Square Residual (WRMR) = 0.056 (95% confidence interval .049 and .057). The columns on the left of Table 9 show the Promin-rotated solution.

The rotated pattern in Table 9 agrees with the theoretically expected structure and has acceptable fit. So, the conclusion reached by conventional EFA is that the oblique two-factor model is quite appropriate for this data. However, the estimated interfactor correlation was .562, which suggests that a bifactor solution (a general statistical anxiety factor and two group factors) would also be appropriate, and more so given the purposes for which the SAS was designed. Therefore, a pure bifactor exploratory solution as proposed in this paper was then fitted. The target matrix $\mathbf{H}_s$ in (2) was obtained by using the target matrix obtained during the previous EFA rotation process based on Promin. The bifactor solution was again based on DWLS, and the rotation criterion to obtain the $\mathbf{S}$ matrix was Promin (see Equation (6)). So, the

Table 9. Outcomes of exploratory factor analysis (EFA) and exploratory bifactor analysis (EBIFA) related to SAS. Loading values larger than .20 are printed in bold face.

Item	EFA		EBIFA			ECV	95% confidence interval	
	IN	EX	GF	IN	EX	Point estimate	inf	sup
1 IN	**.98**	−.09	**.51**	**.78**	−.11	.30	.23	.40
2 IN	**.90**	−.16	**.38**	**.72**	−.12	.22	.13	.37
3 IN	**.85**	−.11	**.38**	**.70**	−.06	.23	.11	.43
4 IN	**.76**	−.14	**.36**	**.57**	−.16	.27	.13	.41
5 IN	**.61**	.01	**.46**	**.41**	−.16	.52	.28	.78
6 IN	**.42**	**.30**	**.66**	**.21**	−.12	.88	.71	.98
7 IN	**.37**	**.34**	**.60**	**.24**	.01	.86	.69	.98
8 IN	**.28**	**.31**	**.56**	.13	−.05	.94	.79	1.00
9 EX	−.05	**.91**	**.61**	.18	**.68**	.43	.31	.58
10 EX	−.04	**.90**	**.85**	−.05	**.30**	.89	.53	.96
11 EX	−.12	**.89**	**.79**	−.11	**.31**	.85	.65	.94
12 EX	−.02	**.82**	**.60**	.14	**.55**	.52	.37	.66
13 EX	.02	**.79**	**.56**	**.21**	**.60**	.44	.26	.69
14 EX	.04	**.76**	**.75**	.03	**.27**	.88	.67	.97
15 EX	.10	**.74**	**.64**	**.20**	**.45**	.63	.47	.78
16 EX	.09	**.71**	**.76**	.04	**.20**	.93	.73	1.00

Note: ECV: Explained common variance.

group factors were allowed to correlate. Goodness-of-fit results were now better than in the previous model: RMSEA = .064 (95% confidence interval .052 and .068), CFI = .988 (95% confidence interval .977 and .990), and WRMR = 0.045 (95% confidence interval .040 and .046). This result is only to be expected given that the bifactor model is more parameterized.

The right-hand columns in Table 9 show the rotated bifactor pattern, which is quite a plausible solution given the SAS design. Furthermore, as intended, the solution for the group factors approaches a simple structure. In order to assess which items contribute most to the general factor, the item explained common variance (I-ECV) was computed (see Ferrando & Lorenzo-Seva, 2017b). Three items on the IN subscale (items 6, 7 and 8), and four items on the EX subscale (items 10, 11, 14, and 16) had I-ECV values higher than .85.

Finally, the interfactor correlation between the group factors in the solution above was .03, which did not significantly differ from zero. This result suggests that the two group factors become independent after the general anxiety factor is modeled.

Example 2: Rotter's locus of control scale

The second example illustrates the purely exploratory procedure we have proposed for the single-group-factor case. For many years, the popular locus of control scale (LOC) scale (Rotter, 1966) was the reference instrument for measuring the bipolar personality dimension of Locus of Control (internal vs. external pole). So, the LOC was initially intended to be a unidimensional measure. Few measures, however, have been so questioned and factor analyzed as the LOC. After more than 50 years and countless FA studies, dimensionality proposals range from essential unidimensionality (Ferrando, Demestre, Anguiano-Carrasco, & Chico, 2011, Lefcourt, 1991) to solutions between 2 and 9 factors (Parkes, 1985). This scenario is only to be expected because the LOC is a broad bandwidth general-purpose scale, and its items purposely refer to a series of well-differentiated domains that can easily be identified as separate dimensions by using FA methods (Rotter, 1990).

A parsimonious solution of the type above that is found with some regularity is a bidimensional oblique solution with a general factor that reflects Rotter's construct as initially defined, and a "political" factor that reflects the respondent's ability to control political or large social institutions (Lefcourt, 1991, Mirels, 1970, Parkes, 1985). This solution has also been found in our studies with the scale, and, given the interpretation of the factors above, it appears to be more appropriately modeled as a bifactor solution (a general factor and a single group factor) than by a bidimensional oblique solution.

The Spanish version of the LOC scale (Ferrando et al. 2011) was administered to a sample of 1299 undergraduate students. This version is a translation of the original scale in which neither the item content nor the presentation are modified. So, it consists of 23 dichotomously scored items. As in the previous example, item scores were treated as ordered-categorical and the FA based on the tetrachoric interitem correlations, an alternative parameterization of the multidimensional two-parameter normal-ogive model, was fitted to the data. Sampling adequacy was acceptable, KMO Test for Sampling Adequacy = .80.

First, the unidimensional model was fitted by using the same procedures as in the previous example. The fit was only marginally acceptable: RMSEA = .070 (95% confidence interval .066 and .070), CFI = .893 (95% confidence interval .884 and .919), and WRMR = .056 (95% confidence interval .056 and .089). The solution had positive manifold, with most of the loadings in the range .30 - .40, and the explained common variance (see e.g., Ferrando & Lorenzo-Seva, 2017b) was 0.71. To sum up, the results support the hypothesis that there is a general factor running through the 23 LOC items. However, the unidimensional model is still unable to satisfactorily explain all the interitem covariation.

The single-group bifactor solution was then fitted using the exploratory procedure proposed in this article. The weakest negative pole of the second canonical

Table 10. Outcomes of exploratory bifactor analysis (EBIFA) related to LOC loading values larger than .20 are printed in bold face.

Item	General factor	Group factor
1	**.41**	−.06
2	.18	**.50**
3	**.38**	.09
4	**.45**	−.03
5	**.31**	−.03
6	**.27**	.07
7	**.29**	.09
8	**.46**	−.05
9	**.52**	.08
10	**.31**	**.74**
11	**.50**	.16
12	**.59**	−.20
13	**.62**	−.11
14	**.34**	**.64**
15	**.55**	.00
16	**.22**	.03
17	.12	−.01
18	**.30**	**.71**
19	**.51**	.04
20	**.61**	.00
21	**.28**	**.23**
22	**.52**	.04
23	**.25**	**.29**

factor identified six items that were specified as free parameters in the group-factor column. Inspection of the item content clearly revealed that these items are those systematically identified as defining the Political factor in previous studies. The fit was now quite acceptable: RMSEA = .046 (95% confidence interval .046 and .050), CFI = .957 (95% confidence interval .956 and .961), and WRMR = .041 (95% confidence interval .040 and .041). The final bifactor solution is in Table 10.

The solution in Table 10 reveals some well-known weaknesses of the LOT. Most of the item loadings on the general factor (i.e., discriminating power) are moderate to weak, and item 17 is a "noise" item with no significant loadings on any of the factors. Furthermore, item 2 makes practically no contribution to the general factor. Apart from these limitations, however, the solution is quite clear. The general factor is well defined by most of the items and, more specifically, 11 items have Explained Common Variance values above .85. As for the group factor, it is defined by the six "Political" items, as expected.

Discussion

In this article, we have proposed a purely exploratory bifactor approach that incorporates known procedures but which attempts to overcome certain limitations noted in the approaches proposed so far. Methodologically, our proposal combines semispecified Procrustes rotations, pure analytical rotations and target rotations in which the target is built from the initial solution. While these procedures are known, combining and structuring them, as we propose here, seems to be a new contribution. Its main potential advantages are simplicity, flexibility and versatility. With regards to simplicity, for example, once the general-factor vector has been obtained, it is not involved in any of the successive rotations. As for versatility and flexibility, the initial target can be obtained from three different approaches, and the group factors can be rotated to satisfy any orthogonal or oblique criterion to maximize factor simplicity. Our proposal can also be used in the single group-factor case, a scenario that does not appear to have been considered in previous bifactor proposals.

The PEBI approach is closely related to Waller's (2017) BiFAD proposal, an approach that we were unaware of when the first version of this article was submitted. The comparison above and the results of the simulation study suggest it is better to consider the two approaches as complementary rather than as competing alternatives. This point is discussed below in more detail.

Because, so far, most of the proposed bifactor approaches have assumed orthogonal group factors, the relevance and advantages of fitting an oblique solution deserve further discussion. To start with, the main motivation in Jennrich & Bentler's (2012) paper was that an oblique rotation was expected to produce pattern matrices that better approximate a simple bifactor structure. Furthermore, in this pragmatic line of reasoning, we note that an oblique solution can always be first specified as default and, if the correlation among group factors turns out to be negligible (as in the first illustrative example), then the more restricted orthogonal scenario can be modeled next. The simulation results suggests that this strategy, which was recommended by Browne (2001), works well with PEBI and appears also to work well in applications. For example, Bellier-Teichmann, Golay, Bonsack, and Pomini (2016) computed oblique bifactor analysis in a model with three group factors and reported that the correlations among group factors ranged between .020 and .089, whereas Olino, McMakin, and Forbes (2016) in another oblique bifactor analysis reported that the correlation among group factors ranged between .20 and .44. The negative side to this proposal, however, is that an oblique solution is potentially less stable than an orthogonal solution. So, replication studies are highly recommended if an oblique solution is to be adopted.

From a substantive point of view, an oblique solution might be hard to justify if the group factors are viewed as mere disturbances, but could be meaningful in some cases in which they are viewed as substantive dimensions. In fact, Mulaik and Quartetti (1997) considered that in ability and personality domains the orthogonality of the group factors might well be an artifact. As an example in the first domain, in the analysis of cognitive abilities, and when rotations were performed by using semianalytical approaches, it was usual to leave the general "g" factor unrotated, keep it orthogonal to the group factors, and rotate obliquely these group factors viewed as additional components of intelligence (e.g., Bernstein, 2012). As a second example in the personality domain, consider the analysis of theoretically related dimensions, and a general factor of response bias such as extreme responding or acquiescence which generalizes across the items that measure the different dimensions (e.g., Ferrando, Lorenzo-Seva, & Chico, 2009). It seems reasonable here to assume that the group factors are still related by something other than the common influence of the general factor.

The results of the simulation study suggest that our proposal functions quite well, although it is not better than the alternatives that were considered in all cases. Thus, in the oblique case, the GPA-based procedure tends to recover the general factor better than the group factors, and, when an orthogonal bifactor model holds in the population, Waller's (2017) BiFAD tends to perform better. Overall, however, in purely exploratory scenarios in which there is little information regarding the correctness of the model in $r+1$ factors, the strength of the general factor, or the relations among the group factors (i.e., orthogonal or oblique) we believe that PEBI is the best option available at present. In more confirmatory scenarios and in the orthogonal case, BiFAD would probably be the method of choice. Indeed, the results are only generalizable for the scenarios considered, and both the simulation and the empirical studies have their share of limitations. Thus, in the simulation study, only continuous variables fitted by ULS were considered, so we do not know if we would have obtained different results if we had used other estimation procedures or types of variable. However, we also note that in the illustrative examples, the variables were treated as ordered-categorical and were fitted with a different estimation procedure (DWLS) and PEBI also appeared to work well in these conditions.

In the context of exploratory bifactor models, further research should be done in order to assess the biasing effects of cross-loadings and correlated errors. Even though the methodology for fitting exploratory bifactor models has been available in the literature for some time, these topics have not received a deep treatment. Given that these misspecifications adversely affect statistical fit in the context of exploratory factor analysis, their effect in the context of bifactor models should be properly studied.

Finally, we should mention the convenience of fitting a bifactor model for a particular data set. It must be noted that for any multidimensional data set in which the dimensions are correlated with each other, a bifactor model can always be computed using the pure exploratory approach that we have proposed here. However, the question is whether such a model is well suited to the particular data set at hand. The alternative to the bifactor model would be Thurstone's correlated-factors model. To decide which model is most suitable for a particular data set, a number of considerations must be taken into account. First, the bifactor model must be coherent with the substantive model that underlies the data set: if the theoretical background is completely against the proposal of a main general factor, then the bifactor model should not be an option. Second, if both models are fitted, the goodness-of-fit indices will systematically indicate that the bi-factor model gives the best fit ($r+1$ factors vs. r factors). However, the researcher must inspect the factor solution to assess if the fitted model can be considered a true bifactor model. Examples of defective bifactor models would be: (a) when the general factor mainly shows lower loading values than the salient group factor loadings (i.e., the general factor explains a low amount of variance); (b) when the general factor is defined by items related to a limited number of group factors (i.e., none of the items related to a particular group factor shows a salient loading on the general factor); or (c) when the general factor mainly shows much larger values than the salient group factor loadings (i.e., the group factors can be viewed as residual factors). If the visual inspection of the loading values reveals some of these situations, then the bifactor model should not be an option. It must be said that applied researchers are already using these kinds of strategy when assessing the suitability of a bifactor model. For example, Cho et al. (2015) reported that they discarded the bifactor model due to the lack of large values in the loadings related to the group factors.

The authors' experience suggests that proposals such as the present one are only used in practical applications if they are implemented in user-friendly and easily available software. In this respect, the procedure proposed here has been implemented in the 10.6 version of the program FACTOR (Ferrando & Lorenzo-Seva, 2017a). Furthermore, given the results discussed above, we also expect to implement Waller's (2017) procedures in the near future. Thus, to start with, Waller's proposal for defining the target matrix can be included as an additional option in PEBI's first step.

Article information

Conflict of interest disclosures

Each author signed a form for disclosure of potential conflicts of interest. No authors reported any financial or other conflicts of interest in relation to the work described.

Ethical principles

The authors affirm having followed professional ethical guidelines in preparing this work. These guidelines include obtaining informed consent from human participants, maintaining ethical treatment and respect for the rights of human or animal participants, and ensuring the privacy of participants and their data, such as ensuring that individual participants cannot be identified in reported results or from publicly available original or archival data.

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