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A new discrete monotonicity formula with application to a two-phase free boundary problem in dimension two

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ABSTRACT

We continue the analysis of the two-phase free boundary problems initiated by ourselves, studying where we studied the linear growth of minimizers in a Bernoulli-type free boundary problem at the non-flat points and the related regularity of free boundary. There, among other things, we also defined the functional

$$\varphi_p(r,u,x_0) = \frac{1}{r^4}\int_{B_r(x_0)} \frac{|\nabla u^+(x)|^p}{|x-x_0|^{N-2}}dx \int_{B_r(x_0)} \frac{|\nabla u^-(x)|^p}{|x-x_0|^{N-2}}dx,$$

where x_0 is a free boundary point, i.e. $x_0 \in \partial\{u>0\}$ and u is a minimizer of the functional

$$J(u) := \int_\Omega |\nabla u|^p + \lambda_+^p\ \chi_{\{u>0\}} + \lambda_-^p\ \chi_{\{u\le 0\}},$$

for some bounded smooth domain $\Omega \subset \mathbb{R}^N$ and positive constants $\lambda_\pm$ with $\Lambda := \lambda_+^p - \lambda_-^p > 0$.
Here we show $\varphi_p(r,u,x_0)$ in discrete monotone at non-flat points x_0, when $N=2$ and p is sufficiently close to 2, and then establish the linear growth of u. A new feature of our approach is the anisotropic scaling argument discussed in Section 4.

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1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded planar domain such that any function in the Sobolev space $W^{1,p}(\Omega)$ has well-defined trace and $p \in (2, 2+\varepsilon)$, for a small, fixed $\varepsilon > 0$. Assume that u is a local minimizer of

$$J(u) := \int_\Omega |\nabla u|^p + \lambda_+^p\ \chi_{\{u>0\}} + \lambda_-^p\ \chi_{\{u\le 0\}}, \quad u-g \in W_0^{1,p}(\Omega), \tag{1.1}$$

where λ_+ and λ_- are positive constants such that $\lambda_+^p - \lambda_-^p > 0$, and $g \in W^{1,p}(\Omega)$ is a prescribed boundary datum. In what follows, χ_U denotes the characteristic function of the set $U \subset \mathbb{R}^2$.

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The variational problem for the functional (1.1) is called the Bernoulli-type free boundary problem and it models a number of interesting phenomena, notably planar cavitational flow of one or two perfect fluids (see [1, Chapter 9.11]), the equilibrium configuration for heat or electrostatic energy optimization in higher dimensions (e.g. heat flow with power Fourier law) and the dynamics of non-Newtonian fluids when the velocity obeys the power law $\mathbf{v} = \nabla\psi|\nabla\psi|^{\frac{1}{s}-1}$, where ψ is a stream function. Notice that $s=1$ corresponds to the Newtonian fluids, and s is a physical parameter, see [2].

For $p=2$, both the one-phase and two-phase problems have been extensively studied for variational [3] as well as viscosity solutions [4]. There is a significant difference between the one-phase and two-phase problems stipulated by a sign change of u across the free boundary. The main and only known method for proving the optimal regularity for the two-phase problem is based on the monotonicity formula of Alt et al. [3] given by

$$\varphi(r,x_0) = \frac{1}{r^4}\int_{B_r(x_0)} \frac{|\nabla u^+(x)|^2}{|x-x_0|^{N-2}}dx \int_{B_r(x_0)} \frac{|\nabla u^-(x)|^2}{|x-x_0|^{N-2}}dx, \tag{1.2}$$

where $r>0$ and $x_0 \in \partial\{u>0\}$. It is well-known that if u is a minimizer of (1.1) and $u^+ := \max\{0,u\}$ and $u^- := -\min\{0,u\}$, then $\varphi(r,x_0)$ is a non-decreasing function of r. The monotonicity of φ, combined with the $C_0>0$ and $Q_{\frac{3}{2}R}$, for $x_0 \in \partial\{u>0\}$, gives uniform local upper linear bound for u, see [3].

The key ingredient in the proof of the monotonicity formula in [3] is the following geometric property of the eigenvalues of the Laplace-Beltrami operator on the unit sphere ∂B_1: let γ_1,γ_2 be the characteristic numbers corresponding to two complementary domains Γ_1,Γ_2 on ∂B_1, that is

$$\gamma_i(\gamma_i+N-2) := \inf_{v\in W_0^{1,2}(\Gamma_i)} \frac{\int_{\Gamma_i}|\nabla_\theta v|^2}{\int_{\Gamma_i} v^2}, \quad i=1,2,$$

then

$$\gamma_1+\gamma_2 \geq 2 \tag{1.3}$$

and the equality holds if and only if Γ_1,Γ_2 are two complementary hemispheres, see [4, Chapter 12].

If $N=2$ then γ_i is the square root of the eigenvalue of the Laplace-Beltrami operator corresponding to the portion Γ_i of the unit circle.

In Section 7, we present some results related to the characteristic numbers and the eigenvalues of the p-Laplace-Beltrami operator for $p\neq 2$.

There are fewer results established for the two-phase problem when $p\neq 2$. A partial result on the optimal regularity of u is given in [5] under a smallness assumption on the Lebesgue density of the set $\{u\leq 0\}$, and recently it has been extended to a more general class of functionals in [6].

Our paper contributes in the direction of optimal regularity and monotonicity formula techniques for a class of two-phase nonlinear problems. More precisely, we show that in two spatial dimensions, $N=2$ (and for p sufficiently close to 2) the functional

$$\varphi_p(r,u,x_0) := \frac{1}{r^4}\int_{B_r(x_0)}|\nabla u^+|^p \int_{B_r(x_0)}|\nabla u^-|^p \tag{1.4}$$

is discrete monotone (see Theorem 2.1 for the precise statement). Here $r>0$ is small and $x_0 \in \Gamma := \partial\{u>0\}$, being Γ the free boundary. Consequently, we prove that φ_p is bounded if the free boundary is not flat at x_0.

The discrete monotone quantity φ_p has wider applications. For instance, for the viscosity solutions of the two-phase problem with the p-Laplacian, for p close to 2 and $N=2$, we can show that at the non-flat points x_0 of the free boundary the function φ_p is bounded, which in turn implies that the solution u has linear growth at x_0. Another closely related result, which is of independent interest, is the analog of the relation of the characteristic numbers for the p-Laplace-Beltrami operators for $N=2$ and $p>2$, namely we have the inequality

$$\sqrt{\lambda_1\big(\lambda_1(p-1)+2-p\big)}+\sqrt{\lambda_2\big(\lambda_2(p-1)+2-p\big)}\geq 2,$$

see the forthcoming Theorem 2.4.

In fact, we establish a dichotomy for φ_p: either the free boundary in the ball $B_r(x_0)$ is contained between two parallel planes which are hr apart, or φ_p is discrete monotone. Moreover, if h is sufficiently small then the free boundary is $C^{1,\alpha}$ smooth near x_0, see the forthcoming Theorem 6.5.

For this, we introduce a suitable notion of flatness for the free boundary points characterizing the *flat* points. It follows from the results of [7, 8] that at such points the free boundary must be regular provided that u is also a viscosity solution in the sense of Definition 6.1, see also the discussion in Section 6. The fact that the minimizers of J are also viscosity solutions has been established in [9].

On the other hand, at *non-flat* points, we prove that φ_p is discrete monotone and we deduce from this the linear growth of u near these points.

In the subsequent section, we present our main results. A detailed plan about the organization of the paper will then be presented at the end of Section 2.

2. Main results

In this section, we formulate our main results. We will denote by $\Gamma := \partial\{u>0\}$ the free boundary. Fix $x_0\in\Gamma$ and $h>0$, and consider the slab

$$S(h;x_0,\nu) := \big\{x\in\mathbb{R}^n : -h<(x-x_0)\cdot\nu<h\big\} \tag{2.1}$$

where ν is a unit vector. Let $h_{\min}(x_0,r,\nu)$ be the minimal height of the slab in the unit direction ν containing the free boundary in $B_r(x_0)$, i.e.

$$h_{\min}(x_0,r,\nu) := \inf\big\{h : \partial\{u>0\}\cap B_r(x_0)\subset S(h;x_0,\nu)\cap B_r(x_0)\big\}. \tag{2.2}$$

If we set

$$h(x_0,r) := \inf_{\nu\in\mathbb{S}^n} h_{\min}(x_0,r,\nu) \tag{2.3}$$

then $h(x_0,r)$ is non-decreasing in r.

Theorem to follow deals with the points where the free boundary is not sufficiently flat.

Theorem 2.1. *Let $N=2$ and u be a local minimizer of the functional J defined in (1.1). Then, there exist tame constants $p_0>2, r_0>0$ and $h_0>0$ such that if*

$$2<p<p_0 \quad and \quad r<r_0 \tag{2.4}$$

$$thentheinequality\ h(x_0,r)\geq h_0 r, \qquad for\ x_0\in\Gamma\cap B_{3r}, \tag{2.5}$$

implies that

$$\varphi_p(r,u,x_0)\leq\varphi_p(3r,u,x_0), \tag{2.6}$$

where h(x, r) is defined by (2.3) and φ_p by (1.4). Moreover, p_0 does not depend on u.

Theorem 2.1 says that if at the level r the free boundary is not sufficiently flat then the φ_p energy at the level r is controlled by the same energy at the tripled level $3r$.

It is worthwhile to point out that in the proof of Theorem 2.1, we use a compactness argument based on an anisotropic scaling in order to assure the non-degeneracy of an appropriately scaled function, thus avoiding the use of the knowledge of the linear growth from [9]. In the proof of Theorem 2.1, the only places where we use that u is a minimizer is Step 2, Cases (2b) and ($2b_2$), where we utilize the continuity of u. Therefore, we have the following:

Corollary 2.2. *Let $N=2$ and let u be a continuous viscosity solution in the sense of Definition 6.1. Then, the conclusion of Theorem 2.1 remains true.*

As another consequence, we have:

Theorem 2.3. *Let u be a local minimizer of the functional J defined in (1.1), and let $x_0\in\Gamma$ be a non-flat point of the free boundary, i.e. for any $r<r_0$ and for $p\in(2,p_0)$, we have that $h(x_0,r)\geq h_0 r$, where r_0 and p_0 are as in Theorem 2.1.*

Then, u has linear growth near x_0.

Observe that we always have that u^+ and u^- have comparable rates of growth from the free boundary, thanks to Corollary 3.4, i.e.

$$\frac{1}{r}\fint_{B_r(x_0)}u^+\sim\fint_{B_r(x_0)}u^-.$$

In order to conclude that each of these terms is bounded we apply Theorem 2.1 to infer that the product $\frac{1}{r^2}\fint_{B_r(x_0)}u^+\fint_{B_r(x_0)}u^-$ is also bounded. This is where φ_p enters into the game and provides the necessary bound, see Section 5.

Finally, we state our main estimate for the characteristic numbers of the p-Laplace-Beltrami operator which is of independent interest:

Theorem 2.4. *Let λ_1 be the solution of*

$$\begin{cases} -\dfrac{d}{d\theta}\left\{\left(\lambda^2\varphi^2+\varphi_\theta^2\right)^{\frac{p-2}{2}}\varphi_\theta\right\}=\lambda\left(\lambda(p-1)+2-p\right)\left(\lambda^2\varphi^2+\varphi_\theta^2\right)^{\frac{p-2}{2}}\varphi \text{ in } S_1,\\ \varphi(\theta)=0 \text{ on } \partial S_1,\end{cases}$$

for $S_1:=(0,\omega)$, and λ_2 for the complementary arc $S_2:=(\omega,2\pi)$. Then

$$\sqrt{\lambda_1\big(\lambda_1(p-1)+2-p\big)}+\sqrt{\lambda_2\big(\lambda_2(p-1)+2-p\big)}\ge 2. \tag{2.7}$$

Furthermore equality holds if and only if $\lambda_1=\lambda_2=1$, i.e. for half circles $S=(0,\pi)$.

Outline

In Section 3, we collect some basic material that we will use throughout the paper. We also show a coherence result (see Proposition 3.1, P.4) by using a different strategy with respect to the case $p=2$ (see [3]), that we think has an independent interest.

Sections 4 and 5 are devoted to the proofs of Theorems 2.1 and 2.3.

tIn Section 6, we discuss the fact that any minimizer of the functional in (1.1) is also a viscosity solution, according to Definition 6.1. This, together with the notion of slab flatness, will allow us to apply the regularity theory developed in [7, 8] for viscosity solutions.

Finally, in Section 7 we recall some results concerning the relation between the characteristic numbers corresponding to two complementary cones for $p\neq 2$ and prove Theorem 2.4.

Notations

$C, C_0, C_N, \ldots$ generic constants,
$\bar{U}$ closure of a set U,
∂U boundary of a set U,
$B_r(x), B_r$ ball centered at x with radius $r>0$, $B_r := B_r(0)$,
Γ the free boundary$\partial\{u>0\}$,
$\Gamma_\pm B_1\cap\partial\{u^\pm>0\}$,
$\fint$ mean value integral,
ω_N volume of unit ball,
$\lambda(u)\lambda_+^p\ \chi_{\{u>0\}}+\lambda_-^p\ \chi_{\{u\le 0\}}$,
$\Lambda=\lambda_+^p-\lambda_-^p$ Bernoulli constant.

3. Technicalities

In this section, we gather some basic facts that we shall use in the forthcoming sections. One of the important results to be proved is the coherence estimate (3.1). For $p=2$ this estimate was showed in [3] (see Theorem 4.1 there), and the proof uses the Poisson representation formula, that we do not have for $p\neq 2$. However, a combination of the methods from [3, 10] and [11] will give the result.

3.1. Some basic properties of the local minimizers of J

In the proposition to follow all claims are valid in any dimension.

Proposition 3.1. *Let $u\in W^{1,p}(\Omega)$ be a local minimizer of J with $\lambda_+^p-\lambda_-^p>0$. Then*
P.1 $\Delta_p u^\pm\ge 0$ *in the sense of distributions and* $\Delta_p u=0$ *in* $\{u>0\}\cup\{u<0\}$,
P.2*there is* $c_0>0$ *such that if*

$$\limsup_{r\to 0}\frac{|B_r(x_0)\cap\{u<0\}|}{|B_r(x_0)|}\le c_0,\quad x_0\in\Gamma,$$

then u has linear growth near x_0 depending only on $\frac{1}{c_0}$ times some tame constant,

P.3$\nabla u \in L^q$ *locally, for any finite* $q > 1$, *and* u *is locally log-Lipschitz continuous,*

P.4*for any* $D \Subset \Omega$ *there exist* $\bar{r} > 0$ *and* $C > 0$ *depending on* $p, \sup|u|, \text{dist}(D, \partial\Omega)$ *such that for any* $x_0 \in \Gamma \cap B_1$

$$\left| \fint_{\partial B_r(x_0)} u \right| \leq Cr, \qquad \text{for any } r \leq \bar{r}. \tag{3.1}$$

Remark 3.2. *Note that P.4 in Proposition 3.1 says that either both* $\frac{1}{r}\fint_{\partial B_r(x_0)} u^+$ *and* $\frac{1}{r}\fint_{\partial B_r(x_0)} u^-$ *go to* $+\infty$ *as* $r \to 0$ *or they both remain bounded.*

Remark 3.3. *We stress on the fact that the results in Proposition 3.1 hold in any dimension.*

Proof. P.1 follows from a standard comparison of u and $u + \varepsilon\varphi$ for a suitable smooth compactly supported function φ, and the proof of P.2 can be found in [5].

Now we focus on the proofs of P.3 and P.4. For this, we observe that it is enough to show that

$$\text{locally} \quad \nabla u \in BMO. \tag{3.2}$$

Indeed, if this is true then $\nabla u \in L^q$ locally, for any $1 < q < +\infty$. Moreover, the log-Lipschitz estimate follows from [12], Theorem 3. This proves P.3.

Also, u is continuous and

$$\lim_{r \to 0} \fint_{\partial B_r(x_0)} u = 0 \qquad \text{for any} x_0 \in \Gamma.$$

Now, we notice that, for $\varepsilon > 0$,

$$\begin{aligned}\left| \frac{1}{\varepsilon^{N-1}} \int_{B_\varepsilon(x)} \nabla u(x) \cdot \frac{x - x_0}{|x - x_0|}\ dx \right| &= \left| \frac{1}{\varepsilon^{N-1}} \int_{B_\varepsilon(x)} \left(\nabla u(x) - \fint_{B_\varepsilon(x)} \nabla u \right) \cdot \frac{x - x_0}{|x - x_0|}\ dx \right| \\ &\leq \varepsilon \left(\frac{1}{\varepsilon^N} \int_{B_\varepsilon(x)} |\nabla u(x) - \fint_{B_\varepsilon(x)} \nabla u|\ dx \right) \to 0,\end{aligned}$$

as $\varepsilon \to 0$, thanks to the BMO estimate in (3.2). Thus

$$\begin{aligned}\frac{1}{r^{N-1}} \int_{\partial B_r(x)} u &= \int_0^r \left(\frac{d}{dt} \int_{\partial B_1} u(x_0 + t\omega)\ d\mathcal{H}^1 \right)\ dt \\ &= \int_0^r \frac{1}{t^{N-1}} \int_{\partial B_t} \nabla u(x_0 + \nu) \cdot \nu\ d\mathcal{H}^1\ dt = \\ &= \int_0^r \frac{1}{t^{N-1}} \frac{d}{dt} \left(\int_{B_t(x_0)} \nabla u(x) \cdot \frac{x - x_0}{|x - x_0|}\ dx \right)\ dt = \\ &= \frac{1}{r^{N-1}} \int_{B_r(x_0)} \nabla u(x) \frac{x - x_0}{|x - x_0|}\ dx + (N-1) \int_0^r \frac{1}{t^N} \int_{B_t(x_0)} \nabla u(x) \frac{x - x_0}{|x - x_0|}\ dx\ dt \\ &= \frac{1}{r^{N-1}} \int_{B_r(x_0)} \left[\nabla u(x) - \fint_{B_r(x_0)} \nabla u \right] \frac{x - x_0}{|x - x_0|}\ dx \\ &\quad + (N-1) \int_0^r \frac{1}{t^N} \int_{B_t(x_0)} \left[\nabla u(x) - \fint_{B_t(x_0)} \nabla u \right] \frac{x - x_0}{|x - x_0|}\ dx\ dt.\end{aligned}$$

Therefore, the BMO estimate in (3.2) yields

$$\left|\frac{1}{r^{N-1}}\int_{\partial B_r(x)} u\right| \le 3r||\nabla u||_{BMO},$$

which gives the desired result in P.4.

Hence, it remains to show (3.2), that is locally $\nabla u \in BMO$. In order to prove it, fix $R \ge r > 0$ and let v be the solution of $\Delta_p v = 0$ in $B_{2R}(x_0)$ and $v = u$ on $\partial B_{2R}(x_0)$. If follows from [13, p. 100] that

$$\int_{B_{2R}(x_0)} |\nabla(u-v)|^p \le CR^N,$$

for some tame constant $C > 0$. Notice that, by Hölder inequality,

$$\int_{B_{2R}(x_0)} |\nabla(u-v)|^2 \le CR^N, \tag{3.3}$$

up to renaming C.

Now, we denote by

$$(\nabla u)_{x_0,\rho} := \fint_{B_\rho(x_0)} \nabla u,$$

and we observe that, using Hölder inequality,

$$\begin{aligned}\int_{B_r(x_0)} |(\nabla v)_{x_0,r} - (\nabla u)_{x_0,r}|^2 &= \int_{B_r(x_0)} |\left(\fint_{B_r(x_0)} \nabla v - \nabla u\right)|^2 \\ &= |B_r(x_0)||\left(\fint_{B_r(x_0)} \nabla v - \nabla u\right)|^2 \\ &\le \int_{B_r(x_0)} |\nabla v - \nabla u|^2.\end{aligned} \tag{3.4}$$

Furthermore, we have the following Campanato growth type estimate (see [10, Theorem 5.1])

$$\int_{B_r(x_0)} |\nabla v - (\nabla v)_{x_0,R}|^2 \lesssim \left(\frac{r}{R}\right)^{N+\alpha} \int_{B_R(x_0)} |\nabla v - (\nabla v)_{x_0,R}|^2, \tag{3.5}$$

where the symbol $\lesssim$ means that the inequality is true up to a positive tame constant.

Therefore, using (3.3), (3.4) and (3.5), we have

$$\begin{aligned}\int_{B_r(x_0)} |\nabla u - (\nabla u)_{x_0,r}|^2 &\lesssim \int_{B_r(x_0)} |\nabla u - \nabla v|^2 + \int_{B_r(x_0)} |\nabla v - (\nabla v)_{x_0,r}|^2 \\ &\quad + \int_{B_r(x_0)} |(\nabla v)_{x_0,r} - (\nabla u)_{x_0,r}|^2 \\ &\lesssim \int_{B_r(x_0)} |\nabla u - \nabla v|^2 + \int_{B_r(x_0)} |\nabla v - (\nabla v)_{x_0,r}|^2 \\ &\lesssim \int_{B_r(x_0)} |\nabla u - \nabla v|^2 + \left(\frac{r}{R}\right)^{N+\alpha} \int_{B_R(x_0)} |\nabla v - (\nabla v)_{x_0,R}|^2\end{aligned}$$

$$
\begin{aligned}
&\lesssim \int_{B_r(x_0)} |\nabla u - \nabla v|^2 \\
&\quad + \left(\frac{r}{R}\right)^{N+\alpha} \left[\int_{B_R(x_0)} |\nabla v - \nabla u|^2 + \int_{B_R(x_0)} |\nabla u - (\nabla u)_{x_0,R}|^2\right] \\
&\quad + \left(\frac{r}{R}\right)^{N+\alpha} \int_{B_R(x_0)} |(\nabla u)_{x_0,R} - (\nabla v)_{x_0,R}|^2 \\
&\lesssim \int_{B_r(x_0)} |\nabla u - \nabla v|^2 \\
&\quad + \left(\frac{r}{R}\right)^{N+\alpha} \left[\int_{B_R(x_0)} |\nabla v - \nabla u|^2 + \int_{B_R(x_0)} |\nabla u - (\nabla u)_{x_0,R}|^2\right] \\
&\lesssim \int_{B_R(x_0)} |\nabla u - \nabla v|^2 + \left(\frac{r}{R}\right)^{N+\alpha} \int_{B_R(x_0)} |\nabla u - (\nabla u)_{x_0,R}|^2 \\
&\lesssim (R)^N + \left(\frac{r}{R}\right)^{N+\alpha} \int_{B_R(x_0)} |\nabla u - (\nabla u)_{x_0,R}|^2.
\end{aligned}
$$

Now, we define

$$\psi(r) := \sup_{t \le r} \int_{B_t(x_0)} |\nabla u - (\nabla u)_{x_0,t}|^2.$$

It follows from [10] that

$$\psi(r) \le A \left(\frac{r}{R}\right)^{N+\alpha} \psi(R) + BR^N$$

for some positive constants A, B and α. Applying Lemma 2.1 from [11, Chapter 3] we conclude that there exist $c > 0$ and $R_0 > 0$ such that

$$\psi(r) \le cr^N \left(\frac{\psi(R)}{R^N} + B\right)$$

for all $r \le R \le R_0$, and hence

$$\int_{B_r(x_0)} |\nabla u - (\nabla u)_{x_0,r}|^2 \le Cr^N$$

for some tame constant $C > 0$. This shows that ∇u is locally BMO and concludes the proof of (3.2). The proof of Proposition 3.1 is then complete. □

As a consequence, we have:

Corollary 3.4. *Let $u \in W^{1,p}(\Omega)$ be a local minimizer of J. Then for any subdomain $D \Subset \Omega$ there is a constant $C > 0$ depending on $p, \sup|u|$ and $\operatorname{dist}(D, \partial\Omega)$ such that*

$$\left| \fint_{B_r(x_0)} u \right| \le Cr \quad \text{for any } x_0 \in \partial\{u>0\} \quad \text{and} \quad r>0 \quad \text{such that} \quad B_r(x_0) \subset D. \tag{3.6}$$

3.2. A remark on the form of the functional

We can write the functional in (1.1) as

$$J(u) = \int_\Omega |\nabla u|^p + \Lambda \chi_{\{u>0\}} + \lambda_-^p |\Omega|,$$

with $\Lambda := \lambda_+^p - \lambda_-^p > 0$. Notice that the last term does not affect the minimization problem, and so if u is a minimizer for J, then it is also a minimizer for

$$\widetilde{J}(u) := \int_\Omega |\nabla u|^p + \Lambda \chi_{\{u>0\}}. \tag{3.7}$$

Observe that the free boundary $\partial\{u>0\} \cap \partial\{u \le 0\}$ for the minimizer u of J coincides with $\partial\{u>0\}$ if $\Lambda > 0$, see e.g. Section 3.4 in [9].

3.3. Alt-Caffarelli-Friedman monotonicity formula

Here we recall a result obtained in [14], see in particular Lemmata 2.2 and 2.3 there.

Theorem 3.5. *Let $p=2$ and $u^\pm$ be two continuous subharmonic functions with disjoint supports in B_1 such that $u^\pm(0)=0$.*

Then we have that

$$\varphi_2'(r,u,x_0) \ge \frac{2}{r} \varphi_2(r,u,x_0)\big(\gamma(\Gamma_+) + \gamma(\Gamma_-) - 2\big),$$

where φ_2 has been introduced in (1.4) and

$$\gamma(\Gamma_\pm)(\gamma(\Gamma_\pm) + N - 2) = \inf_{v \in W_0^{1,2}(\Gamma_\pm)} \frac{\int_{\Gamma_\pm} |\nabla_\theta v|^2}{\int_{\Gamma_\pm} v^2}.$$

Furthermore, let $\gamma(r) := \gamma(\Gamma_+) + \gamma(\Gamma_-) - 2$. Then $\gamma(r) \ge 0$ for all small r. Moreover the strict inequality holds unless $\Gamma_\pm^*$ are both half-spheres. In particular if any of the $\Gamma_\pm^*$ digresses from being a half-spherical cap by an area-size of ε, say, then

$$\gamma(r) > C\varepsilon^2,$$

for some $C>0$. Here E^* stands for the spherical symmetrization of E.

We will use here only the two-dimensional version of Theorem 3.5.

3.4. Some estimates for capacity

In this section, we gather some well-known facts about the capacity on the plane and the one-dimensional Hausdorff measure. So, we fix $N=2$ and, for $\rho>0$, we define

$$\mathcal{H}^1_\rho(E) := \inf \sum_i r_i, \tag{3.8}$$

where the infimum is taken over all the coverings of $E \subset \mathbb{R}^2$ by countably many balls of radii $r_i \le \rho$. Clearly $\mathcal{H}^1_\rho(E)$ is a decreasing function of ρ, hence if $\mathcal{H}^1(E) := \lim_{\rho \to 0} \mathcal{H}^1_\rho(E)$ exists then it is called the one-dimensional Hausdorff measure of E. It is also useful to define the set function $\mathcal{H}^1_\infty$ called the Hausdorff content.

Throughout this paper the $C_{1,\ell}$ capacity, defined in [15, p. 20], is denoted by cap_ℓ. Let $\mathrm{cap}_\ell(E, Q)$ be the ℓ capacity of $E \subset Q$ where $Q \subset \mathbb{R}^2$ is a square and $1 < \ell < 2$.

We have the following lower estimate for the capacity in terms of the Hausdorff content, see e.g. Corollary 5.1.14, inequality (5.1.3) in [15]:

$$\mathrm{cap}_\ell(E, Q) \ge \mathrm{cap}_\ell\left(E, \mathbb{R}^2\right) \ge A\left(\mathcal{H}^1_\infty(E)\right)^{2-\ell} \tag{3.9}$$

for a tame constant $A > 0$.

It is convenient to formulate a version of (3.9) replacing the Hausdorff content with the measure $\mathcal{H}^1$. For this, let $E := \partial\{v > 0\}$, for some continuous function $v \in C(Q)$ such that $\partial\{v > 0\}$ is connected, the center of the square Q belongs to $\partial\{v > 0\}$ and $\partial\{v > 0\} \cap \partial Q \ne \emptyset$. If ℓ_0 is a line passing though the center of Q and a point on $\partial\{v > 0\} \cap \partial Q$ then the $\mathcal{H}^1$ measure of the projection of $\partial\{v > 0\}$ on ℓ_0 is at least $\frac{\mathrm{diam} Q}{2\sqrt{2}}$. Let $\sigma > 0$ be such that $\mathcal{H}^1(E) > \sigma$. Observe that there is $\rho_0 > 0$ such that $\sum_i r_i \ge \sigma$ if $r_i \le \rho_0$, for all coverings of E by countably many balls of radii $r_i \le \rho_0$. Moreover, for all the other coverings we have that $\sum_i r_i \ge \sum_{i=1}^{\left[\frac{\sigma}{2\rho_0}\right]} \rho_0 \ge \frac{\sigma}{4}$. Thus, choosing $\sigma := \frac{1}{2}\mathcal{H}^1(E)$, we get from (3.9)

$$\mathrm{cap}_\ell(E, Q) \ge \mathrm{cap}_\ell\left(E, \mathbb{R}^2\right) \ge A\left(\frac{\mathcal{H}^1(E)}{8}\right)^{2-\ell}. \tag{3.10}$$

We will also need another lower estimate for the capacity, see e.g. [16, p. 5]:

$$\mathrm{cap}_\ell\left(E, \mathbb{R}^2\right) \ge A|E|^{1-\frac{\ell}{2}}, \tag{3.11}$$

where $|E|$ is the Lebesgue measure on the plane.

Finally, we state the Poincaré inequality for $v \in W^{1,\ell}$: there is a tame constant $c > 0$ such that

$$\int_D |v|^\ell \le \frac{c}{\mathrm{cap}_\ell(\{v = 0\}, D)} \int_D |\nabla v|^\ell, \tag{3.12}$$

where D is a ball or a square, see [16, p. 15–16].

3.5. Gehring's lemma

Here we recall the Gehring's result on the higher integrability, see [11, Proposition 1.1, p. 122].

Proposition 3.6. *Let Q be a square and $r > q \ge 1$. Suppose that $f \in L^r(Q), g \in L^q(Q)$ are nonnegative, and that*

$$\fint_{Q_R(x_0)} g^q \le b\left(\fint_{Q_{2R}(x_0)} g\right)^q + \fint_{Q_{2R}(x_0)} f^q + \theta \fint_{Q_{2R}(x_0)} g^q \tag{3.13}$$

for each $x_0 \in Q$ *and each* $R < \min\{\frac{1}{2}\operatorname{dist}(x_0, \partial Q), R_0\}$, *where* R_0, b *and* θ *are constants with* $b > 1$, $R_0 > 0$ *and* $0 \le \theta < 1$.

Then there exist $\epsilon > 0$ *and* $c > 0$ *such that* $g \in L^p_{\mathrm{loc}}(Q)$ *for* $p \in [q, q+\epsilon)$ *and*

$$\left(\fint_{Q_R} g^p\right)^{\frac{1}{p}} \le c\left\{\left(\fint_{Q_{2R}} g^q\right)^{\frac{1}{q}} + \left(\fint_{Q_{2R}} f^p dx\right)^{\frac{1}{p}}\right\}, \tag{3.14}$$

for any $R < R_0$ *such that* $Q_{2R} \subset Q$, *where* c *and* ϵ *are positive constants depending on* b, θ, q *and* r.

4. Proof of Theorem 2.1

4.1. Step 0: Heuristic discussion

We will prove Theorem 2.1 using a contradiction argument. That is, we assume that there exist $p_j \to 2$, with $p_j > 2$, minimizers U_j, $x_j \in \Gamma_j$ and $r_j \to 0$, as $j \to +\infty$, such that $h(x_j, r_j) > h_0\ r_j$ and

$$\varphi_{p_j}(r_j, U_j, x_j) > \varphi_{p_j}(3r_j, U_j, x_j). \tag{4.1}$$

We set

$$S_j^\pm := \left(\fint_{B_{3r_j}(x_j)} |\nabla U_j^\pm(y)|^{p_j}\ dy\right)^{1p_j}, \tag{4.2}$$

and introduce the scaled functions

$$u_j^\pm(x) := \frac{U_j^\pm(x_j + r_j x)}{r_j\ S_j^\pm}. \tag{4.3}$$

By construction we have

$$||\nabla u_j^\pm||_{L^{p_j}(B_3)} = 3^{2p_j} \le 3. \tag{4.4}$$

Hence, from (4.1) we deduce that

$$\varphi_{p_j}(1, u_j, 0) > \varphi_{p_j}(3, u_j, 0) = 1, \tag{4.5}$$

or equivalently

$$\int_{B_1} |\nabla u_j^+|^{p_j} \int_{B_1} |\nabla u_j^-|^{p_j} > 1. \tag{4.6}$$

Thanks to the uniform bound (4.4) we can extract a subsequence $\{u_{j_m}^\pm\}$ that weakly converges to some $u_0^\pm \in W^{1,2}(B_1)$. Consequently, from the semicontinuity of the Dirichlet's integral we have that

$$\lim \varphi_{p_j}(1, u_j, 0) \geq \lim \varphi_{p_j}(3, u_j, 0) \geq \varphi_2(3, u_0, 0). \tag{4.7}$$

In order to handle the limit on the left hand side we need strong convergence of $\nabla u_j^{\pm}$ in, say, $L^2(B_1)$ (e.g. it will suffice to have *uniform higher integrability* of $\{\nabla u_j\}$, for instance $|\nabla u_j| \in L^q(B_1)$ for some fixed $q > 2$, which we will prove using Gehring's lemma). Suppose for a moment that this is true, then passing to the limit in (4.7) we infer the inequality

$$\varphi_2(1, u_0, 0) \geq \varphi_2(3, u_0, 0). \tag{4.8}$$

Note that thanks to the uniform convergence $u_j \to u_0$, as $j \to +\infty$, (due to the uniform estimate $|\nabla u_j| \in L^q_{loc}$, with $q > 2$, and the Sobolev embedding), we obtain that (2.5) translates to

$$h(0, 1) \geq h_0. \tag{4.9}$$

Furthermore, from P.1 in Proposition 3.1 we have that $\Delta_{p_j} u_j^{\pm} \geq 0$, and this translates to $\Delta u_0^{\pm} \geq 0$ in view of the $W^{1,q}$ estimate for $q > 2$.

If both functions $u_0^{\pm}$ do not vanish (i.e. both u_j^+ and u_j^- are *non-degenerate*) then $u_0^{\pm}$ are admissible functions in Theorem 3.5, and we infer from (4.8) that $u_0 = u_0^+ - u_0^-$ is a two-plane solution in $B_3 \setminus B_1$. Consequently, employing some standard unique continuation results for harmonic functions we shall conclude that u_0 is a two-plane solution in B_3 which, however, will be in contradiction with (4.9) and the proof will follow.

Now we begin with the actual proof of Theorem 2.1. It is convenient to split the proof into a number of steps, which in combination shall yield the proof of Theorem 2.1. In Step 1 below, we prove that the scaled functions, defined in (4.3), remain uniformly non-degenerate in $L^2(B_2)$. Step 2, which is the most technical one, takes care of the higher integrability of the gradient of the scaled functions ∇u_j, allowing us to pass to the limit in (4.7). To do so we employ Gehring's Lemma (recall Proposition 3.6) and the Caccioppoli's inequality. One more technical issue that arises here is to establish a Poincaré type estimate for the scaled functions $u_0^{\pm}$. In Step 3 and Step 4, we perform a gap filling argument based on some ideas from the unique continuation theory, allowing us to extend the linearity of u_0 from $B_3 \setminus B_1$ into B_1.

4.2. Step 1: Non-degeneracy

In order to take the limit of the scaled functions $u_j^{\pm}$ as $j \to +\infty$ (recall (4.3)), we need to ensure that both u_j^+ and u_j^- do not vanish identically. Lemma to follow provides a lower bound in term of L^p integrals.

Lemma 4.1. *Let $u_j^{\pm}$ be as in (4.3). Then, there exists $C_0 > 0$ independent of j such that*

$$\int_{B_2} |u_j^+|^{p_j} \int_{B_2} |u_j^-|^{p_j} \geq C_0.$$

Proof. From the scaling properties of the operator Δ_p it follows that u_j^+ is p_j-subharmonic in B_3. Therefore, we have that, for any $\psi \in C_0^1(B_3)$, with $\psi \geq 0$,

$$\int_{B_3} |\nabla u_j^+|^{p_j-2} \nabla u_j^+ \cdot \nabla \psi \le 0. \tag{4.10}$$

Now, we consider a cutoff function $\eta \in C^\infty(B_3)$ such that $\eta \ge 0, \eta \equiv 0$ in $B_3 \setminus B_2$ and $\eta \equiv 1$ in B_1, and we take $\psi := u_j^+ \ \eta^{p_j}$ in (4.10). We obtain

$$\int_{B_3} |\nabla u_j^+|^{p_j} \eta^{p_j} + p_j \int_{B_3} |\nabla u_j^+|^{p_j-2} u_j^+ \ \eta^{p_j-1} \nabla u_j^+ \cdot \nabla \eta \le 0,$$

which implies, using Hölder's inequality,

$$\begin{aligned}\int_{B_3} |\nabla u_j^+|^{p_j} \eta^{p_j} &\le p_j \int_{B_3} \left(|\nabla u_j^+|^{p_j-1} \eta^{p_j-1} \right) \left(u_j^+ |\nabla \eta| \right) \\ &\le p_j \left(\int_{B_3} |\nabla u_j^+|^{p_j} \eta^{p_j} \right)^{\frac{p_j-1}{p_j}} \left(\int_{B_3} |u_j^+|^{p_j} \ |\nabla \eta|^{p_j} \right)^{\frac{1}{p_j}}.\end{aligned}$$

This gives that

$$\int_{B_3} |\nabla u_j^+|^{p_j} \eta^{p_j} \le p_j^{p_j} \int_{B_3} |u_j^+|^{p_j} \ |\nabla \eta|^{p_j}.$$

Therefore, recalling the properties of η, we obtain that

$$\int_{B_1} |\nabla u_j^+|^{p_j} \le \int_{B_3} |\nabla u_j^+|^{p_j} \eta^{p_j} \le p_j^{p_j} \int_{B_3} |u_j^+|^{p_j} \ |\nabla \eta|^{p_j} \le C \ p_j^{p_j} \int_{B_2} |u_j^+|^{p_j}, \tag{4.11}$$

for some $C > 0$ independent of j.

Notice that a similar result holds if we substitute u_j^+ with u_j^- in the previous computations. Namely,

$$\int_{B_1} |\nabla u_j^-|^{p_j} \le C \ p_j^{p_j} \int_{B_2} |u_j^-|^{p_j}.$$

Combining this and (4.11) and using (4.6), we get

$$\begin{aligned}\int_{B_2} |u_j^+|^{p_j} \int_{B_2} |u_j^-|^{p_j} &\ge \frac{1}{C^2 \ p_j^{2p_j}} \int_{B_1} |\nabla u_j^+|^{p_j} \int_{B_1} |\nabla u_j^-|^{p_j} \\ &> \frac{1}{C^2 \ p_j^{p_j}} \\ &\ge C_0,\end{aligned}$$

for a suitable $C_0 > 0$ independent of j (recall that $2 < p_j < p_0$). This concludes the proof of Lemma 4.1. □

4.3. Step 2: Higher integrability

The next result is based on Gerhing's Lemma (see [11 p. 122] and Proposition 3.6 here) and allows us to obtain higher integrability of $\nabla u_j^\pm$ and thus to justify the passage to the limit and infer (4.8).

Lemma 4.2. *Let $u_j^\pm$ be as in (4.3). Then there exist $q > 2$ and $C > 0$ independent of j such that*

$$||\nabla u_j^{\pm}||_{L^q(B_1)} \le C.$$

Proof. We first claim that there exists a universal constant $\bar{C} > 0$ such that, for any square $Q_{2R} \subset B_3$ (with $R > 1$) there holds

$$\left(\fint_{Q_R} |\nabla u_j^{\pm}|^2 \right)^{12} \le \bar{C} \left(\fint_{Q_{2R}} |\nabla u_j^{\pm}|^{\ell} \right)^{1\ell}, \tag{4.12}$$

for any fixed ℓ satisfying $p_0 2 < \ell < 2$ (recall that p_0 is the constant in (2.4)). However, one may also take $\ell := \frac{3}{2}$, since here it is only important to have $\ell \in (1, 2)$, i.e. the lower order norm controls the higher one.

We show (4.12) only for u_j^+, since the proof for u_j^- is analogous. We denote by

$$\ell_j := \frac{2p_j}{2 + p_j}, \tag{4.13}$$

that is p_j is the Sobolev exponent corresponding to ℓ_j. Notice that $1 < \ell_j < p_0 2$, therefore, if $\ell > p_0 2$ then $\ell > \ell_j$. Also, $\ell_j \to 1$ as $j \to +\infty$.

So we fix ℓ independent of j such that $p_0 2 < \ell < 2$ and consider three possibilities:

Case (1): $Q_{2R} \cap \partial\{u_j^+ = 0\} \neq \emptyset$ and $\operatorname{cap}_\ell(\{u_j^+ = 0\}, Q_{2R}) \ge \delta R^{2-\ell}$, for any j, for some $\delta > 0$ independent of j,

Case (2): $Q_{2R} \cap \partial\{u_j^+ = 0\} \neq \emptyset$ but the capacity $\operatorname{cap}_\ell(\{u_j^+ = 0\}, Q_{2R})$ is small,

Case (3): $Q_{2R} \cap \partial\{u_j^+ = 0\} = \emptyset$.

Case (1): We use the fact that u_j^+ is p-subharmonic in B_{3R} (recall P.1 in Proposition 3.1) to deduce that, for any $\psi \in C_0^1(B_{3R})$, with $\psi \ge 0$, we have

$$\int_{B_{3R}} |\nabla u_j^+|^{p_j - 2} \nabla u_j^+ \cdot \nabla \psi \le 0. \tag{4.14}$$

Now we take a cutoff function $\eta \in C^\infty(B_{3R})$ such that $\eta \ge 0$, $\eta \equiv 1$ in Q_R, $\eta \equiv 0$ outside Q_{2R} and $|\nabla \eta| \le \frac{C}{R}$ for some $C > 0$. Then, we choose $\psi := u_j^+ \ \eta^{p_j}$ in (4.14) and we obtain that

$$\int_{B_{3R}} |\nabla u_j^+|^{p_j} \eta^{p_j} + p_j \int_{B_{3R}} |\nabla u_j^+|^{p_j - 2} u_j^+ \ \eta^{p_j - 1} \nabla u_j^+ \cdot \nabla \eta \le 0.$$

After applying Hölder's inequality, this yields

$$\int_{B_{3R}} |\nabla u_j^+|^{p_j} \eta^{p_j} \le p_j^{p_j} \int_{B_{3R}} |u_j^+|^{p_j} |\nabla \eta|^{p_j}.$$

Therefore, recalling the properties of η, we have

$$\int_{Q_R} |\nabla u_j^+|^{p_j} \le \frac{C^{p_j} \ p_j^{p_j}}{R^{p_j}} \int_{Q_{2R}} |u_j^+|^{p_j},$$

which implies that

$$\frac{R^{p_j}}{C^{p_j} \ p_j^{p_j}} \fint_{Q_R} |\nabla u_j^+|^{p_j} \le 2^2 \fint_{Q_{2R}} |u_j^+|^{p_j}. \tag{4.15}$$

Rescaling u_j^+ and setting

$$v_j^+(x) := u_j^+(Rx), \tag{4.16}$$

we observe that p_j is the Sobolev exponent corresponding to ℓ_j, see (4.13), hence the Sobolev embedding gives that

$$\left(\int_{Q_2} |v_j^+|^{p_j}\right)^{1p_j} \leq C\left(\int_{Q_2} |v_j^+|^{\ell_j} + |\nabla v_j^+|^{\ell_j}\right)^{1\ell_j} \leq C\left(\int_{Q_2} |v_j^+|^{\ell} + |\nabla v_j^+|^{\ell}\right)^{1\ell}, \tag{4.17}$$

for some $C > 0$ (recall that $\ell > \ell_j$). Furthermore, using the scaling properties of the $\ell-$capacity and applying the Poincaré inequality (3.12), we get

$$\left(\int_{Q_2} |v_j^+|^{\ell}\right)^{1\ell} \leq \left(\frac{c}{\mathrm{cap}_\ell\left(\left\{v_j^+ = 0\right\}, Q_2\right)}\right)^{1\ell}\left(\int_{Q_2} |\nabla v_j^+|^{\ell}\right)^{1\ell} \leq c_0\left(\int_{Q_2} |\nabla v_j^+|^{\ell}\right)^{1\ell}, \tag{4.18}$$

where c_0 is a positive constant independent of j.

Now, putting together (4.17) and (4.18), we obtain that

$$\left(\int_{Q_2} |v_j^+|^{p_j}\right)^{1p_j} \leq C(1 + c_0)\left(\int_{Q_2} |\nabla v_j^+|^{\ell}\right)^{1\ell}. \tag{4.19}$$

Now we observe that, by (4.16) and by making the change of variable $y = Rx$, we have that

$$\begin{aligned}\int_{Q_2} |v_j^+(x)|^{p_j}\ dx &= \int_{Q_2} |u_j^+(Rx)|^{p_j}\ dx \\ &= R^{-2}\int_{Q_{2R}} |u_j^+(y)|^{p_j}\ dy = 2^2 \int_{Q_{2R}} |u_j^+(y)|^{p_j}\ dy.\end{aligned} \tag{4.20}$$

Moreover, from (4.16) we deduce that

$$\nabla v_j^+(x) = R\ \nabla u_j^+(Rx),$$

which implies

$$\begin{aligned}\int_{Q_2} |\nabla v_j^+(x)|^{\ell}\ dx &= R^{\ell} \int_{Q_2} |\nabla u_j^+(Rx)|^{\ell}\ dx \\ &= R^{\ell-2} \int_{Q_{2R}} |\nabla u_j^+(y)|^{\ell}\ dy = 2^2\ R^{\ell} \int_{Q_{2R}} |\nabla u_j^+(y)|^{\ell}.\end{aligned} \tag{4.21}$$

Plugging (4.20) and (4.21) into (4.19), we get

$$2^{2p_j}\ \left(\int_{Q_{2R}} |u_j^+|^{p_j}\right)^{1p_j} \leq C(1 + c_0)\ 2^{2\ell}\ R\ \left(\int_{Q_{2R}} |\nabla u_j^+|^{\ell}\right)^{1\ell}.$$

From this and (4.15), we obtain

$$\begin{aligned}\frac{R}{C\ p_j}\ \left(\int_{Q_R} |\nabla u_j^+|^{p_j}\right)^{1p_j} &\leq 2^{2p_j}\ \left(\int_{Q_{2R}} |u_j^+|^{p_j}\right)^{1p_j} \\ &\leq C(1 + c_0)\ 2^{2\ell}\ R\ \left(\int_{Q_{2R}} |\nabla u_j^+|^{\ell}\right)^{1\ell},\end{aligned}$$

or equivalently

$$\left(\fint_{Q_R} |\nabla u_j^+|^{p_j}\right)^{1p_j} \leq C \left(\fint_{Q_{2R}} |\nabla u_j^+|^{\ell}\right)^{1\ell},$$

up to renaming constants.

Now, since $p_j > 2$, for any fixed ℓ such that $p_0 2 < \ell < 2$ we have

$$\left(\fint_{Q_R} |\nabla u_j^+|^2\right)^{12} \leq C \left(\fint_{Q_R} |\nabla u_j^+|^{p_j}\right)^{1p_j} \leq C \left(\fint_{Q_{2R}} |\nabla u_j^+|^{\ell}\right)^{1\ell},$$

which establishes (4.12) in the Case (1).

Case (2): Suppose that $\mathrm{cap}_\ell(\{u_j^+ = 0\}, Q_{2R}) < \delta R^{2-\ell}$. We take the square $Q_{\frac{3}{2}R}$ and we consider two subcases:

Case (2a): $Q_{\frac{3}{2}R} \cap \{u_j^+ = 0\} = \emptyset$,

Case (2b): $Q_{\frac{3}{2}R} \cap \{u_j^+ = 0\} \neq \emptyset$.

In Case (2a), thanks to P.1 in Proposition 3.1 we have that u_j^+ is p_j-harmonic in $Q_{\frac{3}{2}R}$, and so

$$\int_{Q_{\frac{3}{2}R}} |\nabla u_j^+|^{p_j - 2} \nabla u_j^+ \cdot \nabla \psi = 0, \tag{4.22}$$

for any $\psi \in C_0^1(Q_{\frac{3}{2}R})$. Now we take a cutoff function $\eta \in C^\infty(B_3)$ such that $\eta \geq 0$, $\eta \equiv 1$ in Q_R, $\eta \equiv 0$ outside $Q_{\frac{3}{2}R}$ and $|\nabla \eta| \leq CR$ for some positive C. We also set

$$\bar{u}_j^+ := \left(\frac{3}{2}R\right)^{-2} \int_{Q_{\frac{3}{2}R}} u_j^+(x)\ dx.$$

Therefore, taking $\psi := (u_j^+ - \bar{u}_j^+)\eta$ in (4.22), we obtain that

$$\int_{Q_{\frac{3}{2}R}} |\nabla u_j^+|^{p_j} + p_j \int_{Q_{\frac{3}{2}R}} |\nabla u_j^+|^{p_j - 2} \left(u_j^+ - \bar{u}_j^+\right) \eta^{p_j - 1} \nabla u_j^+ \cdot \nabla \eta = 0.$$

So, by Hölder's inequality,

$$\int_{Q_{\frac{3}{2}R}} |\nabla u_j^+|^{p_j} \eta^{p_j} \leq p_j^{p_j} \int_{Q_{\frac{3}{2}R}} |u_j^+ - \bar{u}_j^+|^{p_j} |\nabla \eta|^{p_j},$$

which implies that

$$\int_{Q_R} |\nabla u_j^+|^{p_j} \leq \frac{C^{p_j}\ p_j^{p_j}}{R^{p_j}} \int_{Q_{\frac{3}{2}R}} |u_j^+ - \bar{u}_j^+|^{p_j},$$

thanks to the properties of η. Thus

$$\frac{R^{p_j}}{C^{p_j}\ p_j^{p_j}} \int_{Q_R} |\nabla u_j^+|^{p_j} \leq \left(\frac{3}{2}\right)^2 \int_{Q_{\frac{3}{2}R}} |u_j^+ - \bar{u}_j^+|^{p_j}. \tag{4.23}$$

Now we rescale u_j^+ in the following way: we set $v_j^+(x) := u_j^+(Rx)$ and $\bar{v}_j^+ := \frac{4}{9} \int_{Q_{\frac{3}{2}}} v_j^+(x)\ dx$. Notice that

Figure 1. The two subcases of Case 2b.

$$\bar{v}_j^+ = \left(\frac{2}{3}\right)^2 \int_{Q_{\frac{3}{2}}} u_j^+(Rx)\ dx = \left(\frac{2}{3R}\right)^2 \int_{Q_{\frac{3}{2}R}} u_j^+(y)\ dy = \bar{u}_j^+. \tag{4.24}$$

From the Sobolev embedding and Poincaré's inequality we get

$$\begin{aligned} \left(\fint_{Q_{\frac{3}{2}}} |v_j^+ - \bar{v}_j^+|^{p_j}\right)^{1p_j} &\le C\left(\fint_{Q_{\frac{3}{2}}} |v_j^+ - \bar{v}_j^+|^{\ell_j} + |\nabla v_j^+|^{\ell_j}\right)^{1\ell_j} \\ &\le C\left(\fint_{Q_{\frac{3}{2}}} |v_j^+ - \bar{v}_j^+|^{\ell} + |\nabla v_j^+|^{\ell}\right)^{1\ell} \\ &\le C\left(\fint_{Q_{\frac{3}{2}}} |\nabla v_j^+|^{\ell}\right)^{1\ell}, \end{aligned} \tag{4.25}$$

where ℓ_j is given by (4.13), $p_0 2 < \ell < 2$, and the constant $C > 0$ may vary from line to line but it is independent of j (recall (2.5)).

Using the change of variable $y = Rx$ and (4.24), we have that

$$\begin{aligned} \int_{Q_{\frac{3}{2}}} |v_j^+(x) - \bar{v}_j^+|^{p_j}\ dx &= \int_{Q_{\frac{3}{2}}} |u_j^+(Rx) - \bar{v}_j^+|^{p_j}\ dx \\ = R^{-2}\int_{Q_{\frac{3}{2}R}} |u_j^+(y) - \bar{u}_j^+|^{p_j}\ dy &= \left(\frac{3}{2}\right)^2 \omega_2 \fint_{Q_{\frac{3}{2}R}} |u_j^+(y) - \bar{u}_j^+|^{p_j}\ dy. \end{aligned}$$

Similarly, one can check that

$$\int_{Q_{\frac{3}{2}}} |\nabla v_j^+|^{\ell} = \left(\frac{3}{2}\right)^2 R^{\ell} \omega_2 \fint_{Q_{\frac{3}{2}R}} |\nabla u_j^+|^{\ell}.$$

Inserting the last two formulas into (4.25) we obtain that

$$\left(\frac{3}{2}\right)^{2/p_j} \left(\fint_{Q_{\frac{3}{2}R}} |u_j^+ - \bar{u}_j^+|^{p_j}\right)^{1/p_j} \le C\left(\frac{3}{2}\right)^{2/\ell} R\left(\fint_{Q_{\frac{3}{2}R}} |\nabla u_j^+|^{\ell}\right)^{1/\ell},$$

which, together with (4.23), implies that

$$\left(\fint_{Q_R} |\nabla u_j^+|^{p_j}\right)^{1p_j} \le C\, p_j \left(\frac{3}{2}\right)^{2\ell} \left(\fint_{Q_{\frac{3}{2}R}} |\nabla u_j^+|^{\ell}\right)^{1\ell} \le C \left(\fint_{Q_{\frac{3}{2}R}} |\nabla u_j^+|^{\ell}\right)^{1\ell},$$

up to renaming C. Notice that C is independent on j, thanks to (2.4) and the fact that $\ell < 2$. Thus

$$\left(\fint_{Q_R} |\nabla u_j^+|^{p_j}\right)^{1p_j} \le C \left(\fint_{Q_{2R}} |\nabla u_j^+|^{\ell}\right)^{1\ell}.$$

This, together with the fact that $p_j > 2$, implies (4.12) for any $p_0 2 < \ell < 2$. This finishes Case 2a).

Now we suppose that Case (2b) holds true. Since the ℓ-capacity of $\{u_j^+ = 0\}$ in Q_{2R} is small relative to $R^{2-\ell}$, it cannot happen that $Q_{\frac{3}{2}R} \subseteq \{u_j^+ = 0\}$, otherwise we would have a uniform bound from below for the capacity (see e.g. [16] or the estimate (3.11)). Therefore, there exists a point $q \in \partial\{u_j^+ > 0\} \cap Q_{\frac{3}{2}R}$. Let $\Gamma_j^+ := \partial\{u_j^+ > 0\}$ and $K \subseteq \{u_j^+ = 0\}$ be the component of $\{u_j^+ = 0\}$ such that $q \in \partial K$.

Suppose first that K is the unique component of $\{u_j^+ = 0\}$ such that $K \cap Q_{\frac{3}{2}R} \neq \emptyset$. Since u is a minimizer, then it is log-Lipschitz continuous, see Proposition P.3 in 3.1, therefore u_j is continuous. Hence,

Case (2b$_1$):either $\partial K \cap \partial Q_{2R} \neq \emptyset$, see Figure 1A,

Case (2b$_2$):or $\partial K \Subset Q_{2R}$, see Figure 1B.

In Case (2b$_1$), that is when $\partial K \cap \partial Q_{2R} \neq \emptyset$, we have that

$$\mathcal{H}^1\left(\partial\left\{u_j^+ = 0\right\} \cap \left(Q_{2R} \setminus Q_{\frac{3}{2}R}\right)\right) \ge \frac{R}{8}, \tag{4.26}$$

since u_j is a continuous functions. Indeed, let ξ_1 and ξ_2 be the intersection points of ∂K with $\partial Q_{\frac{3}{2}R}$ and ∂Q_{2R}, respectively, and let K_0 be the orthogonal projection of $\tau := \partial K \cap (Q_{2R} \setminus Q_{\frac{3}{2}R})$ on the line joining ξ_1 and ξ_2. We consider a covering of τ, namely $\tau \subset \cup_{i\in I} B_i(x_i)$, such that $diam B_i(x_i) < \varepsilon$ for every $i \in I$. Hence, denoting by $\bar{x}_i, i \in I$, the projection of x_i on the line that joins ξ_1 and ξ_2, we find a covering for K_0, that is $K_0 \subset \cup_{i\in I} B_i(\bar{x}_i)$, with $diam B_i(\bar{x}_i) < \varepsilon$. Consequently,

$$\mathcal{H}^1_\varepsilon(\tau) \ge \mathcal{H}^1_\varepsilon(K_0) = \inf \sum_{i\in I} \mathrm{diam} B_i(\bar{x}_i) \ge \frac{R}{8}, \tag{4.27}$$

where the infimum is taken over all the coverings of K_0 such that $diam B_i(\bar{x}_i) < \varepsilon$. Hence, sending ε to zero we obtain (4.26).

We notice that (4.26) gives a lower bound of the capacity, thanks to (3.10), and so we conclude as in Case 1).

In **Case (2b$_2$)**, that is when $\partial K \Subset Q_{2R}$, we recall Section 3.2 in order to conclude that the free boundary is given just by $\partial\{u > 0\}$.

That said, we observe that if $u_j \le 0$ in K and $u_j \le 0$ outside, then actually $u_j \equiv 0$ inside K, since $u_j = 0$ on ∂K and it is p_j-subharmonic inside. Thus

$$u_j \ge 0 \quad \text{ in } Q_{\frac{3}{2}R}. \tag{4.28}$$

Thus, we can consider the pure one-phase minimization problem (3.7) in $Q_{\frac{3}{2}R}$ (recall that $\Lambda > 0$).

Now, if $Q_{\frac{3}{4}R}$ is contained in the set K, then we have a uniform lower bound for the capacity, and so we conclude as in Case 1).

Hence we suppose that $Q_{\frac{3}{4}R}$ is not contained in K, and we take a small square centered at q, say $Q_{\frac{R}{8}}(q)$, such that $Q_{\frac{R}{8}}(q) \subset Q_{\frac{3}{4}R}$ (see Figure 1B).

Now, recalling (4.28), we have that we can deal with a one-phase problem in the square $Q_{\frac{R}{8}}(q)$. Since u is continuous then we have, as in the proof of (4.27), that

$$\mathcal{H}^1\left(\left\{u_j^+ = 0\right\} \cap Q_{\frac{3}{8}R}(q)\right) \geq \frac{3R}{4},$$

see Figure 1B. Again this implies a lower bound for the capacity, thanks to (3.10), and so we conclude as in Case (1).

Suppose now that there is another component $K_2 \subset \{u_j^+ = 0\}$ such that $K_2 \cap Q_{\frac{3}{2}R} \neq \emptyset$ (that is u_j may change sign). Then, as before, either $\partial K_2 \cap \partial Q_{2R} \neq \emptyset$ or $\partial K \Subset Q_{2R}$. In the first case, we obtain a lower bound for the capacity reasoning as in Case (2b$_1$). In the second case, we use again the maximum principle to reduce the argument to a one-phase minimization problem and, from the density estimate for the zero set, we get a lower bound for the capacity.

Case (3): Finally we deal with the last case, which is the easiest one. In fact, the proof follows as in Case (2a) if we replace there $Q_{\frac{3R}{2}}$ with Q_{2R}.

Thus, since $p_j > 2$, for any $p_0 2 < \ell < 2$ we obtain the claim in (4.12) also for squares that do not touch $\partial\{u_j^+ = 0\}$.

Combining all the cases treated above, we can see that for any square $Q_{2R} \subset B_3$ and some fixed ℓ with $p_0 2 < \ell < 2$ there exists a tame constant $C > 0$ such that there holds

$$\left(\fint_{Q_R} |\nabla u_j^+|^2\right)^{12} \leq C\left(\fint_{Q_{2R}} |\nabla u_j^+|^\ell\right)^{1\ell}.$$

Therefore we can apply Gehring's Lemma (see Proposition 3.6, and for instance [11] for the proof) and we get that there exists $q > 2$ such that

$$||\nabla u_j^+||_{L^q(Q_R)} \leq C,$$

for a suitable $C > 0$. By a covering argument, this implies the desired result. □

From the uniform estimates in $W^{1,q}_{loc}(B_3)$, with $q > 2$, and the Sobolev's embedding Theorem we immediately get the following:

Corollary 4.3. The functions $u_j^\pm$ are uniformly continuous in B_2.

4.4. Step 3: Linearity in $B_3 \setminus B_1$

Thanks to Lemma 4.2 and a standard compactness argument, we conclude that

$$\nabla u_j^\pm \text{ converges strongly in } L^{q'}(B_1), \text{ for any } q' < q, \text{ with } q > 2, \text{ to some } \nabla u_0^\pm. \tag{4.29}$$

Moreover, Lemma 4.1 implies that both u_0^+ and u_j^- are non-degenerate. Therefore, since $p_j \to 2$ as $j \to +\infty$, from (4.5) we deduce that

$$\begin{aligned}\liminf_{j\to\infty} \varphi_{p_j}(1,u_j,0) &\geq \liminf_{j\to\infty} \varphi_{p_j}(3,u_j,0)\\ &= \liminf_{j\to\infty} 3^{-4}\int_{B_3} |\nabla u_j^+|^{p_j} \int_{B_3} |\nabla u_j^-|^{p_j}\\ &\geq 3^{-4}\int_{B_3} |\nabla u_0^+|^2 \int_{B_3} |\nabla u_0^-|^2 \equiv \varphi_2(3,u_0,0),\end{aligned} \tag{4.30}$$

where the last line follows from the semicontinuity of the Dirichlet's integral.

On the other hand, (4.29) implies strong convergence of the gradient in $L^2(B_1)$, since $q > 2$ in Lemma 4.2. Therefore

$$\liminf_{j\to\infty} \varphi_{p_j}(1,u_j,0) = \liminf_{j\to+\infty} \int_{B_1} |\nabla u_j^+|^{p_j} \int_{B_1} |\nabla u_j^-|^{p_j} = \int_{B_1} |\nabla u_0^+|^2 \int_{B_1} |\nabla u_0^-|^2.$$

Hence,

$$\varphi_2(3,u_0,0) \leq \int_{B_1} |\nabla u_0^+|^2 \int_{B_1} |\nabla u_0^-|^2 = \varphi_2(1,u_0,0). \tag{4.31}$$

Now, we observe that u_0^+ and u_0^- are non-negative subharmonic functions with disjoint supports fulfilling the conditions of Theorem 3.5, and so the monotonicity of φ_2 implies that

$$\varphi_2(1,u_0,0) \leq \varphi_2(3,u_0,0).$$

This and (4.31) give that φ_2 is constant in $B_3 \setminus B_1$. Thus, Theorem 3.5 yields that u_0^+ and u_0^- must be linear in $B_3 \setminus B_1$, say, $u_0^+ = \alpha x_1^+$ and $u_0^- = \beta x_1^-$, for some α and $\beta > 0$.

4.5. Step 4: Filling in the gap

In this subsection, we want to show that u_0^+ and u_0^- are linear in B_3, and this will give a contradiction with (4.9). For this, we will prove that either u_0^+ in $\{u_0 > 0\}$ or u_0^- in $\{u_0 < 0\}$ is harmonic, in order to employ some unique continuation result.

Let us show that

$$u_0^+ \text{ is harmonic in } \{u_0 > 0\} \tag{4.32}$$

(the proof for u_0^- is analogous). We take a point $x_0 \in \Omega$ such that $u_0(x_0) > 0$, then, thanks to the uniform convergence of u_j to u_0 (see Corollary 4.3), we have that $u_j(x_0) > 0$ for j large enough. Therefore, Corollary 4.3 implies that there exists a small $\delta = \delta(x_0) > 0$ such that $u_j > 0$ in $B_\delta(x_0)$, and so we can use P.1 in Proposition 3.1 to obtain that

$$\Delta_{p_j} u_j = 0 \qquad \text{in } B_\delta(x_0).$$

Therefore, for any $\psi \in C_0^\infty(B_\delta(x_0))$, we have that

$$\int_{B_\delta(x_0)} |\nabla u_j|^{p_j} \leq \int_{B_\delta(x_0)} |\nabla u_j + \nabla\psi|^{p_j}.$$

Taking the limit as $j \to +\infty$ we have that

$$\int_{B_\delta(x_0)} |\nabla u_0|^2 \le \int_{B_\delta(x_0)} |\nabla u_0 + \nabla\psi|^2, \qquad \text{for any } \psi \in C_0^\infty\big(B_\delta(x_0)\big) \tag{4.33}$$

(recall that φ is fixed and that we have strong convergence of ∇u_j to ∇u_0 in $L^2_{loc}(B_3)$). By a density argument, from (4.33) we get

$$\int_{B_\delta(x_0)} |\nabla u_0|^2 \le \int_{B_\delta(x_0)} |\nabla v|^2, \qquad \text{for any } v \in W^{1,2}\big(B_\delta(x_0)\big) \text{ s.t. v}-u_0 \in W^{1,2}\big(B_\delta(x_0)\big).$$

Thus, we conclude that

$$\Delta u_0 = 0 \quad \text{ in } B_\delta(x_0).$$

Since u_0 is a continuous function, this implies (4.32).

From Step 3 and (4.32), and applying the Unique Continuation Theorem (see [17]), we obtain that

$$u_0^+ \text{ and } u_0^- \text{ are linear in } B_3. \tag{4.34}$$

On the other hand, the uniform convergence of u_j to u_0, as $j \to +\infty$, implies that (4.9) holds true, and so the level sets of u_0 are not flat in B_1. Indeed, by the uniform convergence, for any $\varepsilon > 0$ there is j_0 such that $|Cx_1 - u_j^+(x)| < \varepsilon$ whenever $j > j_0$, where we assume that $u_0^+(x) = Cx_1$ for some constant $C > 0$. Since $\partial\{u_j > 0\}$ is h_0 thick in B_1 it follows that there is $y_j \in \partial\{u_j > 0\} \cap B_1$ such that $y_j = e_1 h_0 2 + t_j e_2$, for some $t_j \in \mathbb{R}$, where e_1 is the unit direction of the x_1-axis and $e_2 \perp e_1$. Then we have that $|C\frac{h_0}{2} - 0| = |u_0^+(y_j) - u_j^+(y_j)| < \varepsilon$ which is in contradiction with (4.34), and thus concludes the proof of Theorem 2.1.

5. Proof of Theorem 2.3

In this section we prove Theorem 2.3. For this, we recall Corollary 3.4 and we square (3.6): we have

$$\left(\frac{1}{r}\int_{B_r(x_0)} u^+\right)^2 + \left(\frac{1}{r}\int_{B_r(x_0)} u^-\right)^2 \le C^2 + \frac{2}{r^2}\int_{B_r(x_0)} u^+ \int_{B_r(x_0)} u^-, \tag{5.1}$$

where $C > 0$ is the constant appearing in Corollary 3.4.

Now we set $u_r^\pm(x) := u^\pm(rx)$. So from the Hölder inequality, the Poincaré inequality (3.12) and (3.10), we have that, for any $1 < \ell < 2$,

$$\begin{aligned}
\int_{B_r(x_0)} u^\pm(x)\ dx &\le \left(\int_{B_r(x_0)} (u^\pm)^\ell(x)\ dx\right)^{\frac{1}{\ell}} = \left(\int_{B_1(x_0)} (u_r^\pm)^\ell(y)\ dy\right)^{\frac{1}{\ell}} \\
&\le C_1\left(\int_{B_1(x_0)} |\nabla u_r^\pm(y)|^\ell\ dy\right)^{\frac{1}{\ell}} = C_1\left(r^\ell \int_{B_r(x_0)} |\nabla u^\pm|^\ell\right)^{\frac{1}{\ell}} \\
&= C_1\ r\left(\int_{B_r(x_0)} |\nabla u^\pm|^\ell\right)^{\frac{1}{\ell}},
\end{aligned}$$

for some $C_1 > 0$. However, from Hölder's inequality we have for $p > 2 > \ell > 1$

$$\left(\fint_{B_r(x_0)} |\nabla u^\pm|^\ell\right)^{\frac{1}{\ell}} \le \left(\fint_{B_r(x_0)} |\nabla u^\pm|^p\right)^{\frac{1}{p}}.$$

Therefore

$$\left(\frac{1}{r}\fint_{B_r(x_0)} u^+\right)^2 + \left(\frac{1}{r}\fint_{B_r(x_0)} u^-\right)^2 \le C^2 + C_2\big(\varphi_p(r,u,x_0)\big)^{\frac{1}{p}},$$

for some $C_2 > 0$.

Let now $r_k := 3^{-k_0-k}$, for any $k \in \mathbb{N}$, where k_0 is the smallest positive integer such that $3^{-k_0} < r_0$. If $3^{-m-1} \le r \le 3^{-m}$, for some $m \in \mathbb{N}$, then

$$\varphi_p(r,u,x_0) \le C_3\varphi_p(3^{-m},u,x_0) \le C_3\varphi_p\big(3^{-k_0},u,x_0\big)$$

for some $C_3 > 0$, implying that

$$\left(\frac{1}{r}\fint_{B_r(x_0)} u^\pm\right)^2 \le C^2 + C_4\Big(\varphi_p\big(3^{-k_0},u,x_0\big)\Big)^{\frac{1}{p}},$$

for suitable $C_4 > 0$. Hence, P.4 in Proposition 3.1 and the weak maximum principle (see Corollary 3.10 in [18]) imply the estimate $\sup_{B_r(x_0)} |u| \le Cr$. This completes the proof of Theorem 2.3.

6. Viscosity solutions

In order to apply the regularity theory for free boundary problems developed for the viscosity solutions in [7, 8] we shall observe that any weak $W^{1,p}$ minimizer is also viscosity solution (see Definition 2.4 in [4] for the case $p = 2$). For this, we denote by $\Omega^+(u) := \{u > 0\}$ and $\Omega^-(u) := \{u < 0\}$. Moreover,

$$G\big(u_\nu^+, u_\nu^-\big) := \big(u_\nu^+\big)^p - \big(u_\nu^-\big)^p - \frac{\Lambda}{p-1}$$

is the flux balance across the free boundary, where u_ν^+ and u_ν^- are the normal derivatives in the inward direction to $\Omega^+(u)$ and $\Omega^-(u)$, respectively (recall that $\Lambda = \lambda_+^p - \lambda_-^p$).

We recall the definition of viscosity solutions for the case $p \ne 2$ (see Definition 4.1 in [9]).

Definition 6.1. Let Ω a bounded domain in $\mathbb{R}^2$ and let u be a continuous function in Ω. We say that u is a viscosity solution in Ω if

i. $\Delta_p u = 0$ in $\Omega^+(u)$ and $\Omega^-(u)$,
ii. along the free boundary $\Gamma = \partial\{u > 0\} \cup \partial\{u < 0\}$, u satisfies the free boundary condition, in the sense that:
   a. if at $x_0 \in \Gamma$ there exists a ball $B \subset \Omega^+(u)$ such that $B \cap \Gamma = \{x_0\}$ and

$$u^+(x) \ge \alpha\langle x - x_0, \nu\rangle^+ + o(|x - x_0|), \quad \text{for } x \in B, \tag{6.1}$$

$$u^-(x) \le \beta\langle x - x_0, \nu\rangle^- + o(|x - x_0|), \quad \text{for } x \in B^c, \tag{6.2}$$

for some $\alpha > 0$ and $\beta \geq 0$, with equality along every non-tangential domain, then the free boundary condition is satisfied

$$G(\alpha, \beta) \leq 0,$$

b. if at $x_0 \in \Gamma$ there exists a ball $B \subset \Omega^-(u)$ such that $B \cap \Gamma = \{x_0\}$ and

$$u^-(x) \geq \beta\langle x - x_0, \nu\rangle^- + o(|x - x_0|), \ \text{for } x \in B,$$
$$u^+(x) \leq \alpha\langle x - x_0, \nu\rangle^+ + o(|x - x_0|), \ \text{for } x \in \partial B,$$

for some $\alpha \geq 0$ and $\beta > 0$, with equality along every non-tangential domain, then

$$G(\alpha, \beta) \geq 0.$$

With this notion of viscosity solutions, in [9] we prove the following:

Theorem 6.2. *Let $u \in W^{1,p}(\Omega)$ be a minimizer of (1.1). Then, u is also a viscosity solution in the sense of Definition 6.1.*

See Theorem 4.2 in [9] for the proof of Theorem 6.2.

We also recall the notion of $\epsilon-$monotonicity of a viscosity solution to our free boundary problem.

Definition 6.3. We say that u is ε-monotone if there are a unit vector e and an angle θ_0 with $\theta_0 > \frac{\pi}{4}$ (say) and $\varepsilon > 0$ (small) such that, for every $\varepsilon' \geq \varepsilon$,

$$\sup_{B_{\varepsilon' \sin\theta_0}(x)} u\left(y - \varepsilon' e\right) \leq u(x). \tag{6.3}$$

We define $\Gamma(\theta_0, e)$ the cone with axis e and opening θ_0.

Definition 6.4. We say that u is ε-monotone in the cone $\Gamma(\theta_0, \epsilon)$ if it is ε-monotone in any direction $\tau \in \Gamma(\theta_0, \varepsilon)$.

One can interpret the $\epsilon-$monotonicity of u as closeness of the free boundary to a Lipschitz graph with Lipschitz constant sufficiently close to 1 if we depart from the free boundary in directions e at distance ϵ and higher. The exact value of the Lipschitz constant is given by $(\tan \frac{\theta_0}{2})^{-1}$. Then the ellipticity propagates to the free boundary via Harnack's inequality giving that Γ is Lipschitz. Furthermore, Lipschitz free boundaries are, in fact, $C^{1,\alpha}$ regular.

For $p = 2$ this theory was founded by L. Caffarelli, see [19–21]. Recently J. Lewis and K. Nyström proved that this theory is valid for all $p > 1$, see [7, 8].

For viscosity solutions we replace the ε-monotonicity with slab flatness measuring the thickness of $\partial\{u > 0\} \cap B_r(x)$ in terms of the quantity $h(x, r)$ introduced in (2.3). In other words, $h(x, r)$ measures how close the free boundary is to a pair of parallel planes in a ball $B_r(x)$ with $x \in \Gamma$. Clearly, planes are Lipschitz graphs in the direction of the normal, therefore the slab flatness of Γ is a particular case of $\epsilon-$monotonicity of u.

Hence, under h_0-flatness of the free boundary we can reformulate the regularity theory "flatness implies $C^{1,\alpha}$" as follows, see [7, 8]:

Theorem 6.5. *Let $x_0 \in \partial\{u>0\}$ and $r>0$ such that $B_r(x_0) \subset \Omega$. Then there exists $h>0$ such that if $\Gamma \cap B_r(x_0) \subset \{x \in \mathbb{R}^N : -hr < (x-x_0)\cdot\nu < hr\}$ then $\Gamma \cap B_{r2}(x_0)$ is locally $C^{1,\alpha}$ in the direction of ν, for some $\alpha \in (0,1)$.*

7. Geometry of eigenvalues

Here we present some results that are related to the characteristic numbers and the eigenvalues of the p-Laplace-Beltrami operator for $p \neq 2$.

7.1. Homogeneous p-harmonic functions in complementary cones

Let us consider

$$\varphi_p(R, u_1, u_2, 0) := \frac{1}{R^4}\int_{B_R} |\nabla u_1|^p \int_{B_R} |\nabla u_2|^p,$$

for given $u_i = r^{\lambda_i} g_i(\theta)$, with $i = 1, 2$ such that u_1, u_2 are p-harmonic in two complementary cones. Here r, θ are the polar coordinates. We will show an estimate on the eigenvalues λ_1 and λ_2 of the p-Laplace-Beltrami operator, namely we prove that

$$\sqrt{\lambda_1\big(\lambda_1(p-1)+2-p\big)} + \sqrt{\lambda_2\big(\lambda_2(p-1)+2-p\big)} \geq 2, \tag{7.1}$$

with equality if and only if both functions are linear

In turn, this implies that $\varphi_p(R, u_1, u_2, 0)$ is non-decreasing in R. Furthermore, $\varphi_p(R, u_1, u_2, 0)$ is constant if and only if $\lambda_1 = \lambda_2 = 1$.

7.2. Properties of eigenvalues

In this section, we prove a relation between the eigenvalues of the p-Laplace-Beltrami operator that correspond to two complementary cones. We begin with an existence result of P. Tolksdorf [22, p. 780, Theorem 2.1.1, Corollary 2.1].

Theorem 7.1. *Let $S := (0, \omega)$, with $\omega \in [0, 2\pi]$. Then there exists a solution $(\lambda, \varphi(\theta))$, with $\theta \in S$ of*

$$\begin{cases} -\dfrac{d}{d\theta}\left\{\left(\lambda^2\varphi^2+\varphi_\theta^2\right)^{\frac{p-2}{2}}\varphi_\theta\right\} = \lambda\big(\lambda(p-1)+2-p\big)\left(\lambda^2\varphi^2+\varphi_\theta^2\right)^{\frac{p-2}{2}}\varphi \text{in} S, \\ \varphi(\theta) = 0 \text{ on } \partial S, \end{cases} \tag{7.2}$$

such that

$$\lambda > \max\left\{0, \frac{p-2}{p-1}\right\}, \varphi > 0 \text{ in } S, \\ \text{and } \varphi^2 + \varphi_\theta^2 > 0 \text{ in } S.$$

Furthermore any two solutions are constant multiples of each other.

M. Dobrowolski computed explicitly the value of λ in (7.2), see [23 p. 187], Theorem 1:

Theorem 7.2. *Let φ be given by Theorem 7.1. Then*

$$\lambda = \begin{cases} s+\sqrt{s^2+\frac{1}{\rho}}, & \omega \le \pi, \\ s-\sqrt{s^2+\frac{1}{\rho}}, & \pi \le \omega < 2\pi, \\ \frac{p-1}{p}, & \omega = 2\pi, \end{cases} \tag{7.3}$$

where

$$\rho := \left(\frac{\omega}{\pi}-1\right)^2 - 1, \tag{7.4}$$

$$\text{and } s := \frac{(\rho-1)p-2\rho}{2\rho(p-1)} = \frac{p-2}{2(p-1)} + \frac{p}{2(p-1)}\left[-\frac{1}{\rho}\right]. \tag{7.5}$$

Now we are ready to prove Theorem 2.4.

Proof of Theorem 2.4. Without loss of generality we may assume that $\omega \le \pi$. Next let us notice that the eigenvalue λ is determined by the size of the arc only. Hence for S_2 we have by (7.4)

$$\rho_2 = \left(\frac{2\pi-\omega}{\pi}-1\right)^2 - 1 = \left(1-\frac{\omega}{\pi}\right)^2 - 1.$$

Thus, $\rho_1 = \rho_2 = \rho$ and from (7.5) we infer that $s_1 = s_2 = s$. In order to prove (2.7) it is enough to check that

$$\begin{aligned} & I + 2\sqrt{II} \ge 4, \\ \text{where} \quad & I := \lambda_1\big(\lambda_1(p-1)+2-p\big) + \lambda_2\big(\lambda_2(p-1)+2-p\big) \\ \text{and} \quad & II := \lambda_1\lambda_2\big(\lambda_1(p-1)+2-p\big)\big(\lambda_2(p-1)+2-p\big). \end{aligned} \tag{7.6}$$

In order to prove this, we notice that, by (7.3),

$$\begin{cases} \lambda_1+\lambda_2 &= 2s \\ \lambda_1\lambda_2 &= -\frac{1}{\rho} \\ \lambda_1^2+\lambda_2^2 &= 4s^2+\frac{2}{\rho} \end{cases}$$

which gives

$$I = \big(\lambda_1^2+\lambda_2^2\big)(p-1) + (2-p)(\lambda_1+\lambda_2) = (p-1)\left(4s^2+\frac{2}{\rho}\right) + 2s(2-p). \tag{7.7}$$

For convenience we introduce a new quantity

$$t := -\frac{1}{\rho} = \frac{1}{1-\left(\frac{\omega}{\pi}-1\right)^2} \ge 1 \tag{7.8}$$

and notice that, by (7.4), we have that $t \ge 1$ and, by (7.5), one has

$$s = \frac{1}{2(p-1)}(p-2+pt). \tag{7.9}$$

Hence (7.7) can be manipulated further in the following way:

$$\begin{aligned}
I &= 2\left[(p-1)\left(2s^2 + \frac{1}{\rho}\right) + s(2-p)\right] \\
&= 2\left[s\{2s(p-1) + (2-p)\} + (p-1)\frac{1}{\rho}\right] \\
&= 2\left[s\{(p-2+pt) + (2-p)\} + (p-1)\frac{1}{\rho}\right] \\
&= 2[spt - t(p-1)] \\
&= 2t[sp-(p-1)].
\end{aligned}$$

Similarly, using (7.3) and (7.9) we get

$$\begin{aligned}
II &= \lambda_1\lambda_2\big(\lambda_1(p-1) + 2-p\big)\big(\lambda_2(p-1) + 2-p\big) \\
&= -\frac{1}{\rho}\left[\lambda_1\lambda_2(p-1)^2 + (p-1)(2-p)(\lambda_1 + \lambda_2) + (2-p)^2\right] \\
&= -\frac{1}{\rho}\left[-\frac{1}{\rho}(p-1)^2 + 2s(p-1)(2-p) + (2-p)^2\right] \\
&= t\left[t(p-1)^2 + 2s(p-1)(2-p) + (p-2)^2\right] \\
&= t\left[t(p-1)^2 + (p-2+pt)(2-p) + (2-p)^2\right] \\
&= t\left[t(p-1)^2 + pt(2-p)\right] \\
&= t^2.
\end{aligned}$$

Thus, putting together the last two formulas,

$$\begin{aligned}
I + 2\sqrt{II} &= 2t[sp - (p-1)] + 2t \\
&= 2t[sp - (p-1) + 1] \\
&= 2t\left[\frac{p}{2(p-1)}(p-2+pt) - (p-1) + 1\right]] \\
&= \frac{t}{p-1}[p(p-2+pt) - 2(p-1)(p-2)] \\
&= \frac{t}{p-1}\left[p^2 - 2p + p^2 t - 2p^2 + 6p - 4\right) \\
&= \frac{t}{p-1}\left[p^2(t-1) + 4(p-1)\right] \\
&= 4t + \frac{p^2}{p-1}t(t-1) \geq 4
\end{aligned} \tag{7.10}$$

since $t \geq 1$ (see (7.8)), which implies (7.6) and finishes the proof of Theorem 2.4. □

7.3. Computing the logarithmic derivative

In what follows, fixed u_1 and u_2, we put $\varphi(R) := \varphi_p(R, u_1, u_2, 0)$. In order to prove that φ is non-decreasing in R, it is enough to prove that $\varphi'(R) \geq 0$ for $R=1$, since φ is scale-invariant. For this, let $J_i := \int_{B_R} |\nabla u_i|^p$. Then we have

$$\begin{aligned}\left(\log \varphi(R)\right)' &= \frac{\varphi'(R)}{\varphi(R)} = \frac{J_1'(R)}{J_1(R)} + \frac{J_2'(R)}{J_2(R)} - \frac{4}{R} \\ &= \frac{\int_{\partial B_R} |\nabla u_1|^p}{\int_{B_R} |\nabla u_1|^p} + \frac{\int_{\partial B_R} |\nabla u_2|^p}{\int_{B_R} |\nabla u_2|^p} - \frac{4}{R} \\ &= \frac{1}{R}\left(\frac{R\int_{\partial B_R} |\nabla u_1|^p}{\int_{B_R} |\nabla u_1|^p} + \frac{R\int_{\partial B_R} |\nabla u_2|^p}{\int_{B_R} |\nabla u_2|^p} - 4\right).\end{aligned} \tag{7.11}$$

Next, we notice that

$$\begin{aligned}\int_{B_1} |\nabla u_i|^p &= \int_{\partial B_1} |\nabla u_i|^{p-2} u_i \frac{\partial u_i}{\partial \nu} \\ &\leq \left[\int_{\partial B_1} |\nabla u_i|^{p-2} u_i^2 \int_{\partial B_1} |\nabla u_i|^{p-2} u_{i,\mathrm{rad}}^2\right]^{\frac{1}{2}}, \quad i = 1, 2,\end{aligned} \tag{7.12}$$

where $u_{i,\nu} = u_{i,\mathrm{rad}}$ is the radial derivative (in direction of the outer unit normal ν of unit circle).

Next decomposing $|\nabla u_i|^2$ into the sum of the squares of the radial and tangential derivative, $u_{i,\theta}$, we obtain

$$\begin{aligned}\int_{\partial B_1} |\nabla u_i|^p &= \int_{\partial B_1} |\nabla u_i|^{p-2} \left(u_{i,\mathrm{rad}}^2 + u_{i,\theta}^2\right) \geq \\ &\geq 2\left[\int_{\partial B_1} |\nabla u_i|^{p-2} u_{i,\mathrm{rad}}^2 \int_{\partial B_1} |\nabla u_i|^{p-2} u_{i,\theta}^2\right]^{\frac{1}{2}}.\end{aligned} \tag{7.13}$$

Hence to prove that φ is monotone, it is enough to check that

$$\underbrace{\left[\frac{\int_{S_1} |\nabla u_1|^{p-2} u_{1,\theta}^2}{\int_{S_1} |\nabla u_1|^{p-2} u_1^2}\right]^{\frac{1}{2}}}_{H(u_1)} + \underbrace{\left[\frac{\int_{S_2} |\nabla u_2|^{p-2} u_{2,\theta}^2}{\int_{S_2} |\nabla u_2|^{p-2} u_2^2}\right]^{\frac{1}{2}}}_{H(u_2)} - 2 \geq 0.$$

Here, $S_i := \mathrm{supp} u_i \cap \partial B_1$. For the solutions (λ_i, φ_i) of the eigenvalue problem on S_i stated in Theorem 7.1 and (2.7) we infer that $H(u_1) + H(u_2) \geq 2$, thanks to (2.7).

Recalling the notation in (7.6) and observing that in (7.10) the equality $I + 2\sqrt{II} = 4$ holds if and only if $(t-1)(4 + \frac{tp^2}{p-1}) = 0$, we conclude that $t=1$. On the other hand, the equality holds in (2.7) if and only if $t=1$, i.e. by (7.8) when $\rho = -1$, and hence, in view of (7.4), when $\omega = \pi$, which corresponds to the half circle. This implies that φ is non-decreasing, and in turn shows (7.1).

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