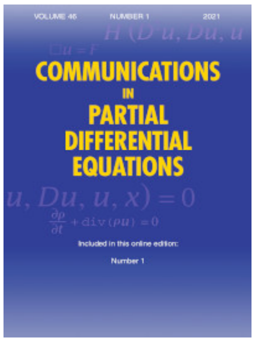

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Attractors of the Einstein-Klein-Gordon system

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ABSTRACT
It is shown that negative Einstein metrics are attractors of the Einstein-Klein-Gordon system. As an essential part of the proof we upgrade a technique that uses the continuity equation complementary to L^2-estimates to control massive matter fields. In contrast to earlier applications of this idea we require a correction to the energy density to obtain sufficiently strong pointwise bounds.

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1. Introduction

Nonlinear stability results are milestones in the study of the Einstein vacuum equations. Nonlinear stability for the vacuum Einstein flow is known for two particular spacetimes, those of Minkowski spacetime [1] and the Milne model [2]. If the cosmological constant is non-vanishing a large class of de Sitter type universes and black holes are known to be stable [3–5] and also toward the singularity of certain cosmological solutions, stability has been established recently [6]. Restricting to stability results toward a complete direction of spacetime and the case of vanishing cosmological constant, i.e. to the Milne model and Minkowski space, several results appeared recently that generalize these works to the non-vacuum setting. The matter models that have been considered in these generalizations include Maxwell fields [7–11], collisionless matter [12–16] and scalar fields [17–20], in particular Klein-Gordon fields, which are the subject of this paper.

1.1. The Einstein-Klein-Gordon system

The Einstein-Klein-Gordon system (EKGS) describes a non-vacuum spacetime with a massive scalar field(s) as the matter model. The EKGS, for the unknown metric tensor $g_{\mu\nu}$ and scalar field ϕ of mass $m > 0$, reads

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$$\begin{aligned} R[g]_{\mu\nu} - \frac{1}{2} R[g] g_{\mu\nu} &= 2T_{\mu\nu}, \\ \nabla^\mu T_{\mu\nu} &= 0, \end{aligned} \tag{1.1}$$

where we use natural units ($c = 1$, $4\pi G = 1$) and where the stress-energy tensor is given by

$$T_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \left(g^{\rho\sigma} \nabla_\rho \phi \nabla_\sigma \phi + m^2 \phi^2 \right). \tag{1.2}$$

The system emerges as a projection of the Einstein equations on massive modes of the Fourier expansion of five-dimensional Kaluza-Klein metrics [19]. Independently, the system poses an interesting mathematical model as it consists of a system of quasilinear massless and massive wave equations. The presence of massive waves in the system makes its treatment substantially more difficult than that of, for instance, the Einstein-Scalar field system where only a massless wave is coupled to the Einstein equations. The present paper considers the stability problem of the Milne model in the expanding direction as a solution to the Einstein-Klein-Gordon system.

The EKGS has been studied intensely in recent years with an emphasis on the nonlinear stability problem [17, 19, 20] (cf. also [21] where the decay of the KG field on a Kerr-AdS background is investigated).

1.2. Cosmological spacetimes and stability

The Milne model $((0,\infty) \times M, \bar{g})$ with metric

$$\bar{g} = -dt_c^2 + \frac{t_c^2}{9} \gamma_{ij} dx^i dx^j, \tag{1.3}$$

where M is a closed 3-manifold admitting an Einstein metric γ with negative curvature, i.e. $R_{ij}[\gamma] = -\frac{2}{9}\gamma_{ij}$, is a solution to the vacuum Einstein equations in $3+1$ dimensions. It models a universe emanating from a big bang singularity at $t_c = 0$ expanding for all time with a linear scale factor. It is a member of the FLRW family of cosmologies and, in comparison with related isotropically expanding models for the Einstein equations with a positive cosmological constant (such as de Sitter space), has the slowest expansion rate. This feature makes it difficult to establish stability results for the Milne model as decay rates of fields in cosmological spacetimes correspond inversely proportional to the rate of expansion.

The nonlinear stability of the Milne model in the expanding direction is known due to a series of works by Andersson and Moncrief who resolved this problem in general dimensions [2]. Their approach uses the CMCSH gauge, developed in [22], which casts the Einstein equations into an elliptic-hyperbolic system. This gauge enables a crucial decomposition of the spatial Ricci tensor into an elliptic operator and perturbation terms. A corrected L^2-energy based on this operator in combination with control of its kernel then allows for a sufficiently strong energy estimate that yields decay of perturbations at a rate that implies future completeness of the spacetime.

Recently, the stability of the Milne model has been generalized to the presence of a variety of matter models such as collisionless matter [12] by Andersson and the first author, fields emanating from generalized Kaluza-Klein spacetimes, in particular electromagnetic fields [8] by Branding, the first author and Kröncke and also Klein-Gordon fields [19] by Wang. While the former two works use the CMCSH gauge to control the evolution of perturbations, in [19] a CMC-vanishing-shift gauge and Bel-Robinson-type energies (cf. [23]) are used.

A crucial difficulty that arises for massive matter models coupled to the Einstein equations for data close to the Milne model results from the slow decay of the lapse gradient. This is due to the matter quantity appearing in the elliptic equation for the lapse function ($\tau\eta$ in (2.7c)). Roughly speaking this implies that the decay of the gradient of the lapse, after rescaling, takes the form $\nabla N \approx \varepsilon e^{-T}$ where ε denotes the size of the initial perturbation. Then in the evolution of the L^2 energy of the Klein-Gordon field, the critical term at lowest order (see (6.8) for higher orders) reads

$$m^2 \nabla N \nabla \phi, \tag{1.4}$$

when written in rescaled variables. Given the coupling to the lapse gradient, this leads to a small growth of $e^{\varepsilon T}$ in the L^2 energy of the Klein-Gordon field. When the matter field couples back into the lapse equation (via $\tau\eta$) it reduces the decay of the gradient of the lapse to $\varepsilon e^{(-1+\varepsilon)T}$ and consequently one cannot close the bootstrap argument. This issue was first observed for the Einstein-Vlasov system in [12], and also arises for the Klein-Gordon field as discussed in [19] Einstein equations in CMCSH gauge take and the present paper.

1.3. Upgrading decay estimates for massive fields by the continuity equation

The main motivation for the present paper is a rough similarity between the Milne stability problem for the EKGS and the corresponding one for the massive Einstein-Vlasov system considered in [12]. Therein, the crucial step to overcome the problem of slow decay of the lapse was the utilization of the continuity equation, which turns into a first order evolution equation for the energy density. This evolution equation has a beneficial structure that allows one to obtain better estimates for the energy density than for a generic component of the energy-momentum tensor. As it is precisely the energy density that causes the lapse gradient to lose decay, this auxiliary estimate is the essential tool to obtain sharp bounds on the lapse and close the bootstrap argument. In [12] it was conjectured that the continuity equation has similarly powerful applications in corresponding *massive* matter models.

In the present paper we show that this conjecture is true for the EKGS. In particular, we use the continuity equation to obtain improved bounds for a suitable *corrected energy density* on lowest order and based on this initialization construct a hierarchy of estimates increasing in regularity. In particular, we consider the rescaled energy density ρ (defined in (2.9)) and correct it with a small indefinite term to obtain the corrected energy density

$$\widehat{\rho} = \rho - \frac{1}{2}\tau^2 \phi \left(\frac{3}{2} N^{-1} \phi - \phi' \right), \tag{1.5}$$

where τ, ϕ and ϕ' are mean-curvature, Klein-Gordon field and its time derivative defined in Section 2. The corrected energy density fulfills an evolution equation, given in (5.4), with only time-integrable terms on the right-hand side yielding uniform pointwise bounds on the energy density and, in turn, for the Klein-Gordon field. This approach is sufficiently strong to close a bootstrap argument for the full system.

We remark that our main theorem is also shown by work of Wang [19], who uses the CMC-vanishing-shift gauge and Bel-Robinson-type energies (cf. [23]) to control the geometric perturbations. The issue of the slow decay of the lapse gradient is resolved therein by a hierarchy which is initiated at lowest order of regularity using an estimate for the Klein-Gordon field that has been proven in the work on Minkowski stability by LeFloch and Ma [17] but surprisingly applies in the cosmological setting as well.

By contrast we work in CMC and spatially harmonic gauge. Consequently we have access to the energy-method of [2, 12] based on the modified Einstein operator (4.6) to control the perturbation of the geometry, which is significantly more concise than the one based on Bel-Robinson energies used in [19, 23]. Defining the shift vector field through the spatial harmonicity condition is necessary for this technique. We have access to this approach since we do not require the shift vector to vanish as our auxiliary estimate for the energy density, based on the continuity equation, is sufficiently robust to handle a non-vanishing shift vector field. In consequence, we obtain a significantly shorter proof of the nonlinear stability problem which avoids many of the technical details used in [19].

1.4. Main theorem

We formulate the main theorem using terminology introduced in Section 2. To summarize this briefly, we let $(\tilde{g}, \tilde{k})$ denote the unknown Riemannian metric and second fundamental form induced on a hypersurface of constant mean curvature $\tau = tr_{\tilde{g}}\tilde{k} < 0$. On the background the mean curvature is related to the physical time variable in (1.3) via $t_c = -3\tau^{-1}$ and so $\tau \nearrow 0$ corresponds to the direction of cosmological expansion. Following the convention of [2, 22] we also introduce rescaled variables (g, k, ϕ), see (2.6), so that on the background we have $(g, k, \phi) = \left(\gamma, \frac{1}{3}\gamma, 0\right)$.

For functions and symmetric tensor fields on M we denote the standard Sobolev norm with respect to the fixed metric γ of order $k \geq 0$ by $||\cdot||_{H^k}$. The corresponding function spaces are denoted by $H^k = H^k(M)$. We let $\mathscr{B}_\varepsilon^{j,k,l,m}\left(\gamma, \frac{1}{3}\gamma, 0, 0\right)$ denote the ball of radius ε in the space $H^j \times H^k \times H^l \times H^m$ centered at $\left(\gamma, \frac{1}{3}\gamma, 0, 0\right)$.

Theorem 1. *Let (M, γ) be a negative, closed 3-dimensional Einstein manifold with Einstein constant $\mu = -2/9$. Let $\varepsilon > 0$ and $(g_0, k_0, \phi_0, \dot{\phi}_0)$ be rescaled initial data for the Einstein-Klein-Gordon system at $\tau = \tau_0$ such that*

$$(g_0, k_0, \phi_0, \dot{\phi}_0) \in \mathscr{B}_\varepsilon^{5,4,5,4}\left(\gamma, \frac{1}{3}\gamma, 0, 0\right). \tag{1.6}$$

Then, for ε sufficiently small the corresponding future development under the Einstein-Klein-Gordon system is future complete and the rescaled metric and second fundamental form converge to

$$(g, k) \to \left(\gamma, \frac{1}{3}\gamma\right) \text{ as } \tau \nearrow 0. \tag{1.7}$$

1.5. Overview on the paper

The paper is organized as follows. Section 2 introduces notations and the fundamental equations. Section 3 discusses the L^2-energies for the Klein-Gordon field. Section 4 states the bootstrap assumptions and introduces the L^2-energies for the perturbation of the geometry. Section 5 discusses the continuity equation and its modification for the Klein-Gordon field. In Section 6 the energy estimates for the Klein-Gordon field are performed. Section 7 recalls the elliptic estimates for lapse and shift. Section 8 discusses the hierarchy of decay for the lapse function and the Klein-Gordon field. Finally, Section 9 closes estimates for the shift and the perturbation of the geometry and ends the proof.

2. Preliminaries

2.1. The Einstein-Klein-Gordon system

We consider the Einstein-Klein-Gordon system (EKGS) consisting of the Einstein equations

$$R[\bar{g}]_{\mu\nu} - \frac{1}{2}R[\bar{g}]\bar{g}_{\mu\nu} = 2\tilde{T}_{\mu\nu}[\tilde{\phi}] \tag{2.1}$$

where the stress-energy tensor is given by

$$\tilde{T}_{\mu\nu}[\tilde{\phi}] = \bar{\nabla}_\mu\tilde{\phi}\bar{\nabla}_\nu\tilde{\phi} - \frac{1}{2}\bar{g}_{\mu\nu}\left(\bar{g}^{\rho\sigma}\bar{\nabla}_\rho\tilde{\phi}\bar{\nabla}_\sigma\tilde{\phi} + m^2\tilde{\phi}^2\right). \tag{2.2}$$

Note here $\bar{\nabla}$ is the Levi-Civita with respect to $\bar{g}$ and for $m > 0$ the Klein-Gordon equation is

$$\bar{\nabla}^\mu\bar{\nabla}_\mu\tilde{\phi} = m^2\tilde{\phi}. \tag{2.3}$$

2.2. Negative Einstein metrics, gauge choice and variables

The following setup is similar to earlier papers on the vacuum case or different matter models. We recall it briefly for the sake of completeness. Throughout the paper let γ denote a fixed negative Einstein metric such that $R_{ij}[\gamma] = -\frac{2}{9}\gamma_{ij}$. We choose the constant for convenience, but any negative Einstein metric can be treated in the same way. Roman letters will always range over spatial indices 1, 2, 3 and Greek letters will range over spacetime indices 0,1,2,3.

To model the dynamic spacetime we consider the $3+1$-dimensional metric in ADM form

$$\bar{g} = -\tilde{N}^2dt^2 + \tilde{g}_{ab}(dx^a + \tilde{X}^a dt)(dx^b + \tilde{X}^b dt) \tag{2.4}$$

where $\tilde{N}, \tilde{g}$ and $\tilde{X}$ denote the lapse function, the induced Riemannian metric on M and the shift vector field respectively. Let $\tilde{n}^\mu$ denote the the future-directed unit normal to hypersurfaces of constant t. Recall that $\bar{g}_{ab} = \tilde{g}_{ab}$ but in general $\bar{g}^{ab} \neq \tilde{g}^{ab}$. We denote by τ the trace of the second fundamental form k with respect to $\tilde{g}$ and decompose $\tilde{k} = \tilde{\Sigma} + \frac{1}{3}\tau\tilde{g}$. We then impose the CMCSH gauge via

$$t=\tau, \qquad \tilde{g}^{ij}(\tilde{\Gamma}^a_{ij}-\widehat{\Gamma}^a_{ij}) \;=0, \tag{2.5}$$

where $\tilde{\Gamma}$ and $\widehat{\Gamma}$ denote the Christoffel symbols of $\tilde{g}$ and γ, respectively.

Rescaling. We rescale the variables $(\tilde{g},\tilde{\Sigma},\tilde{N},\tilde{X},\tilde{\phi})$ with respect to mean curvature time $t=\tau$, calling the rescaled variables (g,Σ,N,X,ϕ). This coincides with the earlier works [2, 12, 22], except for the Klein-Gordon field, which is rescaled here as follows. The rescaling is done according to

$$\begin{aligned} g_{ij} &= \tau^2\tilde{g}_{ij}, & N &= \tau^2\tilde{N}, \\ g^{ij} &= \tau^{-2}\tilde{g}^{ij}, & \Sigma_{ij} &= \tau\tilde{\Sigma}_{ij}, \\ \phi &= -|\tau|^{-3/2}\tilde{\phi}, & X^i &= \tau\tilde{X}^i. \end{aligned} \tag{2.6}$$

On top of this we introduce the logarithmic time $T=-\ln(\tau/\tau_0)$, $(\leftrightarrow \tau=\tau_0\exp(-T))$ with $\partial_T=-\tau\partial_\tau$. We have the following ranges $\tau_0\le\tau\nearrow 0$ and $0\le T\nearrow\infty$. We also define $\widehat{N}:=\frac{N}{3}-1$. We let $\nabla=\nabla[g]$ denote the Levi-Civita connection for the rescaled 3-metric g, μ_g the volume form with respect to g, and $\Delta=g^{ab}\nabla_a\nabla_b$ the Laplacian with respect to this metric.

The rescaled Einstein equations in CMCSH gauge take the following form

$$R[g]-|\Sigma|_g^2+\frac{2}{3}=4\tau\rho, \tag{2.7a}$$

$$\nabla^a\Sigma_{ab}=2\tau^2\tilde{\jmath}_b, \tag{2.7b}$$

$$\left(\Delta-\frac{1}{3}\right)N=N\left(|\Sigma|_g^2-\tau\eta\right)-1, \tag{2.7c}$$

$$\begin{aligned}\Delta X^a+R[g]^a_mX^m &= 2\nabla_bN\Sigma^{ba}-\nabla^a\hat{N}+2N\tau^2\jmath^a \\ &\quad -2(N\Sigma^{mn}-\nabla^mX^n)(\Gamma^a_{mn}-\widehat{\Gamma}^a_{mn}),\end{aligned} \tag{2.7d}$$

$$\partial_Tg_{ab}=2N\Sigma_{ab}+2\widehat{N}g_{ab}-\mathcal{L}_Xg_{ab}, \tag{2.7e}$$

$$\begin{aligned}\partial_T\Sigma_{ab} &= -2\Sigma_{ab}-N\left(R[g]_{ab}+\frac{2}{9}g_{ab}\right)+\nabla_a\nabla_bN+2N\Sigma_{ai}\Sigma^i_b \\ &\quad -\frac{1}{3}\widehat{N}g_{ab}-\widehat{N}\Sigma_{ab}-\mathcal{L}_X\Sigma_{ab}+N\tau S_{ab}.\end{aligned} \tag{2.7f}$$

The energy density and energy current are defined by

$$\tilde{\rho}=\tilde{N}^2\tilde{T}^{00}, \tilde{\jmath}_a=\tilde{N}\tilde{T}^0_a, \tag{2.8}$$

respectively. Furthermore $\tilde{\jmath}^a$ is defined by raising the index using $\tilde{g}^{ab}$, and in general we raise and lower indices on un-rescaled quantities (i.e. those with tildes) using $\tilde{g}$. We define the rescaled matter quantities as

$$\begin{aligned}\rho &:= \tilde{\rho}(-\tau)^{-3}, & \eta &:= (\tilde{\rho}+\tilde{g}^{ab}\tilde{T}_{ab})(-\tau)^{-3}, \\ \jmath^b &:= \tilde{\jmath}^b(-\tau)^{-5}, & S_{ab} &:= (\tilde{T}-\frac{1}{2}\tilde{g}_{ab}\tilde{T})(-\tau)^{-1}\end{aligned} \tag{2.9}$$

Here and throughout, spatial indices for rescaled quantities will be raised and lowered using the rescaled metric g. Thus $\jmath_a$ is defined using the rescaled metric g_{ab}, and moreover $\jmath_a$ would scale as $\tilde{\jmath}_a(-\tau)^{-3}$. For the Klein-Gordon field these matter quantities are evaluated as

$$\rho = \frac{1}{2}m^2\phi^2 + \frac{\tau^2}{2}\left(\frac{3}{2}N^{-1}\phi - \phi'\right)^2 + \frac{1}{2}\tau^2 g^{ab}\nabla_a\phi\nabla_b\phi \tag{2.10}$$

$$\jmath^a = \tau\left(\tfrac{3}{2}N^{-1}\phi - \phi'\right)g^{ab}\nabla_b\phi \tag{2.11}$$

$$\eta = -\frac{1}{2}m^2\phi^2 + 2\tau^2\left(\frac{3}{2}N^{-1}\phi - \phi'\right)^2 \tag{2.12}$$

$$S_{ab} = \frac{1}{2}m^2\phi^2 g_{ab} + \tau^2\nabla_a\phi\nabla_b\phi \tag{2.13}$$

$$g_{ab}T^{ab} = -\frac{3}{2}\tau^{-2}m^2\phi^2 + \frac{3}{2}\left(\frac{3}{2}N^{-1}\phi - \phi'\right)^2 - \frac{1}{2}g^{ab}\nabla_a\phi\nabla_b\phi, \tag{2.14}$$

where we have used the following notation

$$\widehat{\partial}_0 := \partial_T + \mathcal{L}_X \ , \quad \phi' := N^{-1}\widehat{\partial}_0\phi. \tag{2.15}$$

For completeness note that the change of $\tilde{\phi}$ in the direction of the unit normal becomes, in the rescaled variables,

$$\tilde{n}^\mu\partial_\mu\tilde{\phi} = \tilde{N}^{-1}(\partial_\tau - \tilde{X}^a\partial_a)\tilde{\phi} = \tau^2N^{-1}\cdot\tau^{-1}\widehat{\partial}_0((-\tau)^{3/2}\phi) = (-\tau)^{5/2}\left(\frac{3}{2}N^{-1}\phi - \phi'\right). \tag{2.16}$$

Finally, the rescaled Klein-Gordon equation takes the form

$$\widehat{\partial}_0\phi' = \nabla^a(N\nabla_a\phi) + (4-N)\phi' + \frac{3}{2}\phi - \frac{15}{4}N^{-1}\phi - \frac{3}{2}N^{-2}\phi\widehat{\partial}_0 N - \tau^{-2}m^2N\phi. \tag{2.17}$$

Note to derive (2.17) it was convenient to move to a 'Cauchy adapted frame', see for example [24, VI§3]. This ends the setup of the EKGS in the CMCSH gauge with appropriate rescaling. In the following we will work solely with these equations.

3. Energy functionals for the Klein-Gordon field

In this section we define the L^2-energy of the Klein-Gordon field in two steps. First, we define the natural L^2-norm of a massive scalar field. In the second step we modify this energy with two non-definite terms to obtain the corrected energy, which turns out to fulfill the desired energy estimate, which is derived later.

3.1. Natural energy

The following energy is the natural L^2-energy expressed in the rescaled variables.

Definition 1.

$$
\begin{aligned}
E_k(\phi) &:= \int_M \tau^2(-1)^k\left(\phi'\Delta^k\phi' - \phi\Delta^{k+1}\phi\right)\mu_g + \int_M m^2(-1)^k\phi\Delta^k\phi\mu_g,\\
\mathcal{E}_\ell(\phi) &:= \sum_{k=1}^{\ell} E_k(\phi).
\end{aligned} \tag{3.1}
$$

We need the following lemma further below.

Lemma 1. *The following equivalence (denoted $\cong$) holds*

$$
||\phi||_{H^{k+2}} \cong ||\Delta\phi||_{H^k} + ||\phi||_{L^2} \tag{3.2}
$$

for a sufficiently regular function ϕ. This implies

$$
||\phi||_{H^k} \cong ||\Delta^{\lfloor k/2 \rfloor}\phi||_{L^2} + ||\nabla^{k^\circ}\Delta^{\lfloor k/2 \rfloor}\phi||_{L^2} + ||\phi||_{L^2}, \tag{3.3}
$$

where $k^\circ = \lceil k/2 \rceil - \lfloor k/2 \rfloor$. Thus

$$
\mathcal{E}_k(\phi)^{1/2} \cong ||\tau\phi'||_{H^k} + ||\tau\phi||_{H^{k+1}} + ||m\phi||_{H^k}. \tag{3.4}
$$

Proof. This follows from [25, Appendix H, Theorem 27]. □

It is important to note the appearance of $|\tau|$ weights in (3.4).

3.2. Modified energy

We now introduce the modified energy, which contains two indefinite terms. This modified energy is equivalent, up to an overall scaling of τ^2, with the modified energy considered in [19, (3.27)]. Since the modified energy arises by using the unit normal vector field (instead of merely ∂_τ) as a multiplier vector field it yields a better energy estimate, see Proposition 2 and (6.6), than the standard energy. Note furthermore that this procedure of adding additional off-diagonal terms to produce a modified energy with improved estimates is very similar in spirit to the corrected geometric energies given in Definition 3, see in particular (4.10). The equivalence of the modified energy to the standard energy is also shown below.

Definition 2.

$$
\begin{aligned}
\tilde{E}_k(\phi) &:= \int_M \tau^2(-1)^k\Big[\phi'\Delta^k\phi' - \phi\Delta^{k+1}\phi + 3\phi'\Delta^k(N^{-1}\phi\widehat{N})\\
&\quad + \frac{3}{2}N^{-1}\phi\Delta^k\Big(\Big(\frac{3}{2}-N\Big)N^{-1}\phi\Big)\Big]\mu_g + \int_\Sigma m^2(-1)^k\phi\Delta^k\phi\mu_g,\\
\tilde{\mathcal{E}}_\ell(\phi) &:= \sum_{k=0}^{\ell}\tilde{E}_k(\phi).
\end{aligned} \tag{3.5}
$$

Lemma 2. *Assume that there exists a constant $C>0$ such that $||N||_{L^\infty} + ||N^{-1}||_{L^\infty} + ||g-\gamma||_{H^N} < C$ and $||\widehat{N}||_{H^{N+1}} \leq Ce^{-\frac{1}{2}T}$. Then there exists a τ_0 such that for all $\tau > \tau_0$ the following equivalence holds.*

$$\tilde{\mathcal{E}}_\ell(\phi) \cong \mathcal{E}_\ell(\phi), \quad \ell \le N. \tag{3.6}$$

Proof. We first write the difference between the two energies (note without summing in k).

$$\begin{aligned} E_k(\phi) - \tilde{E}_k(\phi) &= \textstyle\int_M (-1)^k \tau^2 (3\phi' \Delta^k (N^{-1}\phi) - \frac{9}{4} N^{-1}\phi \Delta^k (N^{-1}\phi))\mu_g \\ &\quad + \textstyle\int_M (-1)^k \tau^2 \left(\frac{3}{2} N^{-1}\phi\Delta^k\phi - \phi'\Delta^k\phi\right)\mu_g. \end{aligned} \tag{3.7}$$

Examine the first term on the R.H.S. of (3.7). The claim for $\ell = 0$ is easily seen from:

$$\left|\int_M \tau^2\left(-3(N^{-1}\phi)\phi'\right)\mu_g\right| \le C\delta \int_M \tau^2 (\phi')^2 \mu_g + \frac{C\tau_0^2}{\delta m^2}\int_M m^2\phi^2\mu_g$$

where we used $||N^{-1}||_{L^\infty} \le C$ and δ is a constant we are free to choose. For sufficiently small δ and $\tau_0 = \mathrm{tr}\tilde{k}_0$ one can ensure all coefficients are strictly less than 1. Note that $0 > \tau > \tau_0$ implies $|\tau| < |\tau_0|$. Thus this term can be absorbed by the clearly positive terms of $E_0(\phi)$.

A similar argument holds for $\ell \ge 1$. For some smooth functions v, w

$$\Delta^i(vw) = (\Delta^i v)w + \sum_{|I|+|J|+1=2i} c_{IJ}\nabla^I v \nabla^{J+1} w$$

for some coefficients c_{IJ} depending on g. So for a fixed value of $k \ge 1$, integration by parts (where there are $2\lfloor k/2\rfloor + k^\circ = k$ derivatives distributed) gives

$$\begin{aligned} &\left|\int_M \tau^2(-1)^k(3\Delta^k(N^{-1}\phi)\phi')\mu_g\right| \\ &= 3\left|\int_M \tau^2\left((\nabla_a)^{k^\circ}\Delta^{\lfloor k/2\rfloor}(N^{-1}\phi)\right)\left((\nabla^a)^{k^\circ}\Delta^{\lfloor k/2\rfloor}\phi'\right)\mu_g\right| \\ &\le C\int \tau^2 N^{-1}|\nabla^{k^\circ}\Delta^{\lfloor k/2\rfloor}\phi\nabla^{k^\circ}\Delta^{\lfloor k/2\rfloor}\phi'|\mu_g + C\sum_{|I|+|J|+1=k}\int \tau^2|\nabla^I\phi\nabla^{J+1}N^{-1}\nabla^{k^\circ}\Delta^{\lfloor k/2\rfloor}|\mu_g \\ &\le C||N^{-1}||_{L^\infty}||\tau\nabla^{k^\circ}\Delta^{\lfloor k/2\rfloor}\phi||_{L^2}||\tau\nabla^{k^\circ}\Delta^{\lfloor k/2\rfloor}\phi'||_{L^2} \\ &\quad + C\sum_{|I|+|J|+1=k}||\tau\nabla^I\phi||_{L^4}||\nabla^{J+1}N^{-1}||_{L^4}||\tau\nabla^{k^\circ}\Delta^{\lfloor k/2\rfloor}\phi'||_{L^2}^2 \\ &\le C||\tau\phi||_{H^k}||\tau\phi'||_{H^k} + C\sum_{|I|\le k-1}||\tau\nabla^I\phi||_{H^1}\cdot\sum_{1\le|J|\le k-1}||\nabla^J N^{-1}||_{H^1}\cdot||\tau\phi'||_{H^k} \\ &\le C||\tau\phi||_{H^k}||\tau\phi'||_{H^k} + C||\tau\phi||_{H^k}\cdot||\widehat{N}||_{H^{k+1}}||\tau\phi'||_{H^k} \\ &\le C\delta||\tau\phi'||_{H^k}^2 + \frac{C\tau_0^2}{\delta m^2}||m\phi||_{H^k}^2. \end{aligned}$$

In the third to last line we used the following estimate

$$\begin{aligned} \sum_{1\le|J|\le k}||\nabla^J N^{-1}||_{L^2} &\le ||N^{-2}||_{L^\infty}||\widehat{N}||_{H^k} + C(||N^{-1}||_{L^\infty})\sum_{|I|\le\lfloor k/2\rfloor}||\nabla^I\widehat{N}||_{L^\infty}||\widehat{N}||_{H^k} \\ &\le C||\widehat{N}||_{H^k} + C||\widehat{N}||_{H^{k+1}}||\widehat{N}||_{H^k} \le C||\widehat{N}||_{H^k}. \end{aligned} \tag{3.8}$$

In this estimate we used $\lfloor k/2 \rfloor + 2 \le k+1$ for $k \ge 1$ in order to take the terms with low derivatives on $\widehat{N}$ out in L^∞ and embed using Sobolev, recalling also that N is controlled at one order of regularity higher than ϕ. Note we also used the Sobolev-embedding $H^1 \hookrightarrow L^4$.

In a similar way we can show

$$\begin{aligned}
&\left|\int_M \tau^2 N^{-1}\phi\Delta^k(N^{-1}\phi)\mu_g\right| + \left|\int_M \tau^2\left(\phi'\Delta^k\phi - \frac{3}{2}N^{-1}\phi\Delta^k\phi\right)\mu_g\right| \\
&\le \int_M |\tau\nabla^{k^\circ}\Delta^{\lfloor k/2\rfloor}(N^{-1}\phi)|^2\mu_g + C\int_M \tau^2\left|\nabla^{k^\circ}\Delta^{\lfloor k/2\rfloor}(N^{-1}\phi)\nabla^{k^\circ}\Delta^{\lfloor k/2\rfloor}\phi\right|\mu_g \\
&\quad + C\delta\int_M \tau^2\left|\nabla^{k^\circ}\Delta^{\lfloor k/2\rfloor}\phi'|^2\mu_g + \frac{C\tau_0^2}{\delta m^2}\int_M m^2\right|\nabla^{k^\circ}\Delta^{\lfloor k/2\rfloor}\phi\Big|^2\mu_g \\
&\le \frac{C\tau_0^2}{m^2}||m\phi||^2_{H^k} + ||\widehat{N}||^2_{H^k}||\tau\phi||^2_{H^{k+1}} + C\delta||\tau\phi'||^2_{H^k} + \frac{C\tau_0^2}{\delta m^2}||m\phi||^2_{H^k} \\
&\le \frac{C\tau_0^2}{m^2}\mathcal{E}_k(\phi) + |\tau_0|\mathcal{E}_k(\phi) + C\delta\mathcal{E}_k(\phi) + \frac{C\tau_0^2}{\delta m^2}\mathcal{E}_k(\phi).
\end{aligned}$$

Thus the claim holds by summing in k from 0 to ℓ and reducing δ and $|\tau_0|$ to be sufficiently small so that all coefficients can be made to be strictly smaller than 1. □

4. Energy norms and smallness assumptions

In this section we state the global bootstrap assumptions which allows for a cleaner presentation in the subsequent estimates. Also, we recall the definition of the corrected L^2-energy to control the perturbation of the geometry from the earlier works [2, 12].

4.1. Bootstrap assumptions

Fix the regularity $N \ge 4$ and some constant $0 < \kappa \ll 1$. We assume the following bootstrap conditions hold.

$$||g-\gamma||^2_{H^{N+1}} + ||\Sigma||^2_{H^N} \le C_I\varepsilon^2 e^{-\frac{3}{2}T}, \tag{4.1a}$$

$$||\widehat{N}||_{H^{N+1}} \le C_I\varepsilon e^{(-1+\kappa)T}, \tag{4.1b}$$

$$||X||_{H^{N+1}} \le C_I\varepsilon e^{(-1+\kappa)T}, \tag{4.1c}$$

$$\mathcal{E}_N(\phi) \le C_I\varepsilon^2 e^{2\kappa T}. \tag{4.1d}$$

A simple check shows the assumptions of Lemma 2 are consistent with (4.1). Note also that, by the bootstrap assumptions, Sobolev norms with respect to the metric g are equivalent to $||\cdot||_{H^k}$, i.e. those with respect to γ.

Remark 1. The growth in $\mathcal{E}_N(\phi)$ reflects the fact that, due to certain critical terms (see (6.8)) the energy for the Klein-Gordon field cannot be controlled uniformly. Without improved bounds on the Klein-Gordon field, which we obtain from the modified continuity equation in the next section, the bootstrap argument could not be closed.

4.2. Energy for the perturbation of the geometry

The discussion in this subsection closely follows [2, 12], however we give a brief summary here. Recall that (M, γ) is a fixed closed 3-manifold admitting an Einstein metric γ with Einstein constant $\mu = -2/9$, i.e.

$$R_{ab}[\gamma] = -\frac{2}{9}\gamma_{ab}. \tag{4.2}$$

Let $\nabla[\gamma]$ and $R[\gamma]_{ikjl}$ respectively denote the covariant derivative and the Riemann curvature components with respect to γ. The Einstein operator associated with γ is defined to act on symmetric two tensors h_{ab} by

$$\mathcal{L}h_{ab} = -\gamma^{kl}\nabla[\gamma]_k\nabla[\gamma]_l h_{ab} - R[\gamma]_{akbl}h^{kl}. \tag{4.3}$$

The lowest positive eigenvalue of $\mathcal{L}$ is denoted λ_0. Under our assumptions on (M, γ) a result of Kröncke [26] implies, as argued in [12, §2.1], that $\lambda_0 \geq 1/9$ and $\ker\mathcal{L} = \{0\}$. This condition guarantees that the energy to control the perturbation of the geometry, given below in Definition 3, is coercive and allows us to avoid introducing a shadow-gauge analog to [2]. From another perspective, see for example the discussion in [2, §1.4], if a compact 3-manifold M admits a negative Einstein metric then it is necessarily hyperbolic and hence, by Mostow rigidity, cannot be deformed since it is the only such negative Einstein structure possible on M.

We define the correction constant $\alpha = \alpha(\lambda_0, \delta_\alpha)$ by

$$\alpha = \begin{cases} 1 & \lambda_0 > 1/9 \\ 1 - \delta_\alpha & \lambda_0 = 1/9, \end{cases} \tag{4.4}$$

where $\delta_\alpha = \sqrt{1 - 9(\lambda_0 - \varepsilon')}$ with $1 \gg \varepsilon' > 0$ remains a variable to be determined in the course of the argument to follow. By fixing ε' once and for all, δ_α can be made suitable small when necessary.

The corresponding correction constant, relevant for defining the corrected energies is defined by

$$c_E = \begin{cases} 1 & \lambda_0 > 1/9 \\ 9(\lambda_0 - \varepsilon') & \lambda_0 = 1/9. \end{cases} \tag{4.5}$$

Define the following operator

$$\mathcal{L}_{g,\gamma}h_{ab} = -\frac{1}{\mu_g}\nabla[\gamma]_k(g^{kl}\mu_g\nabla[\gamma]_l h_{ab}) - 2R[\gamma]_{akbl}h^{kl} \tag{4.6}$$

which acts on symmetric two-tensors h_{ij}. Note that $\mathcal{L}_{\gamma,\gamma} = \mathcal{L}$. We recall also the decomposition of the curvature term in the spatial harmonic gauge (c.f. [23])

$$R[g]_{ab} + \frac{2}{9}g_{ab} = \frac{1}{2}\mathcal{L}_{g,\gamma}(g - \gamma)_{ab} + J_{ab}, \tag{4.7}$$

where, for $m \geq 1$,

$$||J||_{H^{m-1}} \leq C||g - \gamma||^2_{H^m}. \tag{4.8}$$

We are now ready to define the energy for the geometric perturbation. For $m \geq 1$ let

$$\mathcal{E}_{(m)} = \frac{1}{2}\int_M \langle 6\Sigma, \mathcal{L}_{g,\gamma}^{m-1} 6\Sigma\rangle \mu_g + \frac{9}{2}\int_M \langle (g-\gamma), \mathcal{L}_{g,\gamma}^{m}(g-\gamma)\rangle \mu_g \tag{4.9}$$

$$\Gamma_{(m)} = \int_M \langle 6\Sigma, \mathcal{L}_{g,\gamma}^{m-1}(g-\gamma)\rangle \mu_g. \tag{4.10}$$

Then, the following corrected energy for the geometric perturbation is defined by

Definition 3.

$$E^g{}_k(g,\Sigma) = \sum_{1\leq m\leq k} \mathcal{E}_{(m)} + c_E \Gamma_{(m)}. \tag{4.11}$$

Under the imposed conditions, the energy is coercive and equivalent

$$E^g{}_k(g,\Sigma) \cong ||g-\gamma||^2_{H^{k+1}} + ||\Sigma||^2_{H^k}.$$

4.3. Local well-posedness

Local existence theory is a prerequisite for addressing the global existence and stability problem for any Einstein-matter system. The local existence problem for the vacuum Einstein equations in CMCSH gauge was proven in [22]. We provide the corresponding result for the Einstein-Klein-Gordon system below. As it differs from the vacuum system only by coupling an additional nonlinear wave equation to the elliptic-hyperbolic system there is essentially no difference in the proof. One issue is however important to remark, which concerns the elliptic system. To preserve the crucial feature that the elliptic operators are isomorphisms we need to impose a smallness condition on the matter variables. This has been observed already in [27] for collisionless matter and, for simplicity, turned into a smallness assumption for the full perturbation. Following the strategy of proof in [22] and making an additional smallness assumption analogous to that in [27] yields the following local-existence theorem for the EKGS.

Lemma 3. *There exists a $\delta > 0$ such that for any initial data set at time T_0 for the rescaled Einstein-Klein-Gordon system in CMCSH gauge $(g_0, k_0, \phi_0, \dot{\phi}_0)$ with*

$$||g_0-\gamma||_5 + ||\Sigma_0||_4 + \sqrt{|\tau|}(||\phi_0||_5 + ||\,\dot{\phi}_0\,||_4) \leq \delta, \tag{4.12}$$

there exists a local-in-time solution to the rescaled EKGS in CMCSH gauge with this initial data. Moreover, let T_+ be the supremum of all $T > T_0$ such that the corresponding solution exists up until T. Then either $T_+ = \infty$ or

$$\limsup_{T\to T_+} ||g-\gamma||_5 + ||\Sigma||_4 + \sqrt{|\tau|}(||\phi||_5 + ||\dot{\phi}||_4) \geq 2\delta. \tag{4.13}$$

Remark 2. The mean-curvature factor in front of the Klein-Gordon field terms results from the lapse equation, where this condition assures smallness of the corresponding matter term $-\tau\eta$. As η is quadratic in the rescaled Klein-Gordon field, each term obtains a factor of $\sqrt{|\tau|}$.

Remark 3. We use the previous lemma to establish global existence of the solution corresponding to the considered perturbations by proving decay estimates that assure, in particular, that the necessary bounds above hold.

5. Modified continuity equation

In this section we cast the rescaled Klein-Gordon equation in the particular form (5.1) and derive a modified continuity equation (5.4). This is essential to prove an improved bound for the pointwise norm of the energy-density, which in turn allow for an initialization of the hierarchy that improves bounds on the Klein-Gordon field and the lapse function.

The rescaled Klein-Gordon equation (2.17) when written in terms of the small quantity $\widehat{N}$ reads

$$\widehat{\partial}_0\phi' = \nabla^a(N\nabla_a\phi) - 3\widehat{N}\phi' + \phi' + \frac{9}{2}N^{-1}\phi\widehat{N} + \frac{3}{4}N^{-1}\phi + \frac{3}{2}\phi\widehat{\partial}_0 N^{-1} - \tau^{-2}m^2N\phi. \tag{5.1}$$

The standard continuity equation, see for example [28], is

$$\begin{aligned}\partial_T\rho &= (3-N)\rho - X^a\nabla_a\rho + \tau N^{-1}\nabla_a(N^2 j^a) - \tau^2\frac{N}{3}g_{ab}T^{ab} - \tau^2 N\Sigma_{ab}T^{ab}\\ &= -3\widehat{N}\rho - X^a\nabla_a\rho + \tau^2 N^{-1}\nabla_a\left(N\nabla^a\phi\left(\frac{3}{2}\phi - \widehat{\partial}_0\phi\right)\right) + \frac{1}{2}Nm^2\phi^2\\ &\quad + \tau^2\frac{N}{6}\nabla^a\phi\nabla_a\phi - \tau^2\frac{N^{-1}}{2}\left(\frac{3}{2}\phi - \widehat{\partial}_0\phi\right)^2 + \tau^2\frac{N}{2}\Sigma_{ab}\nabla^a\phi\nabla^b\phi.\end{aligned}$$

The problematic terms in this expression are $Nm^2\phi^2$ and $N(\tau\phi')^2$. This is because naively estimating such terms using the standard Sobolev embedding $L^\infty \hookrightarrow H^2$ and the bootstrap assumptions (4.1) leads to a problematic $e^{\kappa T}$ growth for ρ. Nonetheless, motivated by the notion of a 'modified' energy, we consider a modified energy density. Consider the quantity

$$P := \phi\left(\frac{3}{2}N^{-1}\phi - \phi'\right). \tag{5.2}$$

Using the KG equation (2.17) its evolution equation is the following.

$$\begin{aligned}\partial_T P &= -\mathcal{L}_X P + \widehat{\partial}_0 P\\ &= -\mathcal{L}_X P + 3\phi\phi' + \frac{3}{2}\phi^2\widehat{\partial}_0 N^{-1} - N(\phi')^2 - \phi\widehat{\partial}_0(\phi')\\ &= -\mathcal{L}_X P + 3\phi\phi' - \phi\nabla^a(N\nabla_a\phi) + N\phi'\phi - \frac{3}{2}\phi^2 - 4\phi'\phi + \frac{15}{4}N^{-1}\phi^2\\ &\quad - N(\phi')^2 + \tau^{-2}m^2N\phi^2.\end{aligned}$$

For some constant λ define

$$\widehat{\rho} := \rho + \lambda\tau^2 P. \tag{5.3}$$

We will be interested in the modified continuity equation for $\lambda = -1/2$.

$$\partial_T \widehat{\rho} = -3\widehat{N}\rho - X^a\nabla_a\widehat{\rho} + \tau^2\frac{1}{2}(1+3/N)\phi\nabla_a N\nabla^a\phi + \tau^2\frac{1}{2}(3+N)\phi\Delta\phi$$
$$-\tau^2 N\phi'\Delta\phi + \tau^2\frac{3}{2}(1+N/9)\nabla_a\phi\nabla^a\phi - \tau^2 N\nabla^a\phi\nabla_a\phi' - 2\tau^2\phi'\nabla^a\phi\nabla_a N \quad (5.4)$$
$$+\tau^2\frac{3}{4}(1-2/N)\phi^2 + \tau^2\frac{1}{2}(1-N/2)\phi\phi' + \tau^2\frac{N}{2}\Sigma_{ab}\nabla^a\phi\nabla^b\phi.$$

Proposition 1. *Assume the bootstrap assumptions (4.1) hold. If $\lambda = -1/2$ the following equivalence holds $\rho \cong \widehat{\rho}$ and also the estimates*

$$|\partial_T\widehat{\rho}| \lesssim (||\widehat{N}||_{H^3} + ||X||_{H^2} + ||\Sigma||_{H^2} + |\tau|)\mathcal{E}_4(\phi), \quad (5.5)$$

and thus

$$\rho|_T \lesssim \rho|_{T_0} + C\varepsilon^3\int_{T_0}^{T} e^{(-1+2\kappa)s}ds \le C\varepsilon^2. \quad (5.6)$$

Proof.

$$\partial_T\widehat{\rho} = \partial_T\rho + \lambda\tau^2\partial_T P - 2\lambda\tau^2 P$$
$$= -3\widehat{N}\rho - X^a\nabla_a\widehat{\rho} + Nm^2\phi^2\left(\frac{1}{2}+\lambda\right) - \tau^2 N(\phi')^2\left(\frac{1}{2}+\lambda\right)$$
$$+\tau^2 N^{-1}\nabla_a N\nabla^a\phi\left(\frac{3}{2}\phi - \widehat{\partial}_0\phi\right) + \tau^2\Delta\phi\left(\frac{3}{2}\phi - \widehat{\partial}_0\phi\right) + \tau^2\nabla^a\phi\nabla_a\left(\frac{3}{2}\phi - \widehat{\partial}_0\phi\right)$$
$$+\tau^2\frac{N}{6}\nabla_a\phi\nabla^a\phi - \tau^2\frac{9}{8}N^{-1}\phi^2 + \tau^2\frac{3}{2}\phi\phi' + \tau^2\frac{N}{2}\Sigma_{ab}\nabla^a\phi\nabla^b\phi$$
$$+\lambda\tau^2\left(-\phi\phi' - \phi\nabla^a(N\nabla_a\phi) + N\phi\phi' - \frac{3}{2}\phi^2 + \frac{15}{4}N^{-1}\phi^2 - 3N^{-1}\phi^2 + 2\phi\phi'\right).$$

Choosing $\lambda = -1/2$ we can remove the problematic terms. Indeed the evolution equation is now (5.4). To show the equivalence, note

$$|\rho - \widehat{\rho}| = \frac{\tau^2}{2}\left|\phi\left(\frac{3}{2}N^{-1}\phi - \phi'\right)\right| \le \frac{\tau_0^2}{\delta m^2}m^2\phi^2 + \delta\tau^2\left(\frac{3}{2}N^{-1}\phi - \phi'\right)^2. \quad (5.7)$$

Thus for sufficiently small δ and $\tau \ge \tau_0$ equivalency holds. Finally, and recalling (3.4) and the standard Sobolev embedding $H^2 \hookrightarrow L^\infty$, we have

$$|\partial_T\widehat{\rho}| \lesssim ||\widehat{N}||_{L^\infty}(||m\phi||^2_{L^\infty} + ||\tau N^{-1}\phi||^2_{L^\infty} + ||\tau\phi'||^2_{L^\infty}) + ||X||_{L^\infty}||m^2\phi\nabla\phi||_{L^\infty}$$
$$+ ||X||_{L^\infty}(||\tau N^{-1}\phi||_{L^\infty} + ||\tau\phi'||_{L^\infty})(||\tau\nabla(N^{-1}\phi)||_{L^\infty} + ||\tau\nabla\phi'||_{L^\infty}) + ||\nabla\widehat{N}||_{L^\infty}||\tau^2\phi\nabla\phi||_{L^\infty}$$
$$+ |\tau|||\tau\phi'||_{L^\infty}||\Delta\phi||_{L^\infty} + \tau^2||\nabla\phi||^2_{L^\infty} + |\tau|||\nabla\phi||_{L^\infty}||\tau\nabla\phi'||_{L^\infty} + ||\tau\phi'||_{L^\infty}||\tau\nabla\phi||_{L^\infty}||\nabla\widehat{N}||_{L^\infty}$$
$$+ \tau^2||\phi||^2_{L^\infty} + |\tau|||\tau\phi'||_{L^\infty}||\phi||_{L^\infty} + ||\Sigma||_{L^\infty}||\tau\nabla\phi||^2_{L^\infty}$$
$$\lesssim ||\widehat{N}||_{H^3}\mathcal{E}_3(\phi) + ||X||_{H^2}\mathcal{E}_3(\phi) + |\tau|\mathcal{E}_4(\phi) + ||\Sigma||_{H^2}\mathcal{E}_2(\phi)\ .$$

Lastly we estimate the initial value of ρ

$$\rho|_{T_0} \lesssim (||m\phi||^2_{L^\infty} + ||\tau N^{-1}\phi||^2_{L^\infty} + ||\tau\phi'||^2_{L^\infty})|_{T_0} \lesssim \mathcal{E}_2(\phi)|_{T_0} \lesssim \varepsilon^2 e^{4\kappa T_0} \ .$$

□

Remark 4. The additional indefinite terms in $\widehat{\rho}$, just like the energy density (2.10), are chosen to be quadratic in the Klein-Gordon field. Up to constants and an overall curvature factor of τ, they are an off-diagonal combination of the terms $(m\phi)^2$ and $\tau^2\left(\frac{3}{2}N^{-1}\phi - \phi'\right)^2$ appearing in (2.10). The idea to use off-diagonal terms is inspired by the corrected geometric energies given in Definition 3, see in particular (4.10). Alternatively we can consider the physical energy density $\tilde{\rho}$. By using (2.6), (2.9) and (2.16), the additional correction terms added to $\tilde{\rho}$, up to constants, take the form $\tau\tilde{\phi}\tilde{n}^\mu\partial_\mu\tilde{\phi}$. This is one of the simplest corrections involving the Klein-Gordon field and the curvature that one can consider.

6. Energy inequalities

In this section we derive decay inequalities for the time derivative ∂_T of the modified L^2-energy norm of the Klein-Gordon field defined in Section 3.2. The main results in this section, Propositions 2 and 3, will then be combined with later estimates for the Lapse in Lemma 10 in order to close the bootstrap argument.

6.1. Zeroth order Klein-Gordon energy

Recall the definition

$$\tilde{\mathcal{E}}_0(\phi) = \int_M \tau^2\left[(\phi')^2 + g^{ab}\nabla_a\phi\nabla_b\phi + 3N^{-1}\phi\phi'\widehat{N} + \frac{3}{2}N^{-2}\phi^2\left(\frac{3}{2} - N\right)\right]\mu_g + \int_M m^2\phi^2\mu_g.$$

Proposition 2. *Assume the bootstrap assumptions (4.1) hold. Then the zeroth-order estimate holds*

$$\partial_T\tilde{\mathcal{E}}_0(\phi) \lesssim (||\Sigma||_{H^2} + ||\widehat{N}||_{H^3} + |\tau|)\tilde{\mathcal{E}}_0(\phi) \tag{6.1}$$

and thus

$$\tilde{\mathcal{E}}_0(\phi)|_T \lesssim \tilde{\mathcal{E}}_0(\phi)|_{T_0} \cdot \exp\left(C\int_{T_0}^T e^{(-1+\kappa)s}ds\right). \tag{6.2}$$

Proof. The modified energy takes the form

$$\tilde{\mathcal{E}}_0(\phi) =: \int_M (\tau^2 f_0(\phi) + m^2\phi^2)\mu_g$$

where $f_0(\phi)$ is the expression between the square brackets above. An identity taken from [12, (6.5)], valid for some function u on M, is the following

$$\partial_T \int_M u\mu_g = 3\int_M \widehat{N} u\mu_g + \int_M \widehat{\partial}_0(u)\mu_g. \tag{6.3}$$

Thus we find

$$\begin{aligned}\partial_T \tilde{\mathcal{E}}_0(\phi) &= -\int_M (3-N)(\tau^2 f_0(\phi) + m^2\phi^2)\mu_g + \int_M \left(\tau^2\widehat{\partial}_0(f_0) - 2\tau^2 f_0 + m^2\widehat{\partial}_0(\phi^2)\right)\mu_g \\ &\le ||\widehat{N}||_{L^\infty}\tilde{\mathcal{E}}_0(\phi) + \left|\int_M \tau^2\widehat{\partial}_0(f) - 2\tau^2 f + m^2\widehat{\partial}_0(\phi^2)\mu_g\right|.\end{aligned} \tag{6.4}$$

Another identity taken from [12, (6.4)] and relevant for this and later calculations is

$$\widehat{\partial}_0 g^{ab} = -2N\Sigma^{ab} - 2\widehat{N}g^{ab}. \tag{6.5}$$

Using these and the Klein-Gordon equation in the form (5.1) we find

$$\begin{aligned}&\tau^2\widehat{\partial}_0(f_0) - 2\tau^2 f_0 + m^2\widehat{\partial}_0(\phi^2) \\ &= 2\tau^2\nabla^a(\widehat{\partial}_0\phi\nabla_a\phi) + 3\tau^2\nabla^a(\phi\widehat{N}\nabla_a\phi) + \tau^2\nabla_a N(3\widehat{N}N^{-1}\phi\nabla_a\phi + 2\phi'\nabla_a\phi - \phi\nabla_a\phi) \\ &\quad + \tau^2\widehat{N}\left(-3(\phi')^2 - 5(\nabla\phi)^2 + 6N^{-1}\phi\phi' - 9\widehat{N}(N^{-1}\phi)\phi' + \frac{9}{2}\left(N - \frac{3}{2}\right)(N^{-1}\phi)^2\right) \\ &\quad - \tau^2\left(3N^{-2}\phi^2\left(\frac{3}{2} - N\right) + 2(\nabla\phi)^2\right) + 3\tau^2 N^{-1}\phi\phi'(2-N) - 2N\tau^2\Sigma^{ab}\nabla_a\phi\nabla_b\phi - 3m^2\widehat{N}\phi^2 \\ &\quad + \tau^2\widehat{\partial}_0 N^{-1}\left(3\phi\phi' + \frac{9}{2}N^{-1}\phi^2\widehat{N} - 3\phi\phi' + \frac{9}{2}\phi^2 N^{-1} - \frac{3}{2}\phi^2\right).\end{aligned} \tag{6.6}$$

Note that due to our use of the modified energy, instead of the standard energy, the terms involving $\widehat{\partial}_0 N$ in the above calculation cancel and so the final line (i.e. (6.6)) vanishes. Combining this with (6.4) we find

$$\begin{aligned}\partial_T\tilde{\mathcal{E}}_0(\phi) &\lesssim (||\Sigma||_{H^2} + ||\widehat{N}||_{H^3})\tilde{\mathcal{E}}_0(\phi) + \tau^2\int \phi^2\mu_g + \int ||\tau|^{1/2}\phi|\tau|^{3/2}\phi'|\mu_g \\ &\lesssim (||\Sigma||_{H^2} + ||\widehat{N}||_{H^3} + |\tau|)\tilde{\mathcal{E}}_0(\phi).\end{aligned}$$

Applying Grönwall's inequality yields the result. □

6.2. Higher order modified Klein-Gordon energies

Now we calculate the time derivatives of higher-order energy norms for the Klein-Gordon field. Recall the definition (3.5) of the modified L^2-energy, which we repeat again here for convenience.

$$\begin{aligned}\tilde{E}_k(\phi) &:= \int_M \tau^2(-1)^k\left[\phi'\Delta^k\phi\prime - \phi\Delta^{k+1}\phi + 3\phi'\Delta^k(N^{-1}\phi\widehat{N}) + \frac{3}{2}N^{-1}\phi\Delta^k\left(\left(\frac{3}{2}-N\right)N^{-1}\phi\right)\right]\mu_g \\ &\quad + \int_M m^2(-1)^k\phi\Delta^k\phi\mu_g, \\ \tilde{\mathcal{E}}_\ell(\phi) &:= \sum_{k=0}^{\ell}\tilde{E}_k(\phi).\end{aligned}$$

The main energy estimate for the modified higher order energies is given in the following proposition.

Proposition 3. *Assume the bootstrap assumptions (4.1) hold, then the higher order energies for $1 \le \ell \le N$ satisfy*

$$\begin{aligned}\partial_T\tilde{\mathcal{E}}_\ell(\phi) \quad &\lesssim (\|\widehat{N}\|_{H^3} + \|\Sigma\|_{H^3} + \|\Sigma\|_{H^\ell} + \|\widehat{N}\|_{H^{\ell+1}} + |\tau|)\tilde{\mathcal{E}}_\ell(\phi) \\ &\quad + (\|\widehat{N}\|_{H^{\ell+1}} + \|\Sigma\|_{H^\ell})\tilde{\mathcal{E}}_3(\phi) + |\tau|\varepsilon^4 + \sum_{k=1}^{\ell}\left|\int_M B_k\mu_g\right|,\end{aligned} \tag{6.7}$$

where B_k denote the border-line terms, which for $k \ge 1$ are defined by

$$B_k := m^2\phi'\left[N, \Delta^k\right]\phi. \tag{6.8}$$

Proof. In a similar way to Proposition 2, we have

$$\partial_T\tilde{E}_k(\phi) \le \|\widehat{N}\|_{L^\infty}\tilde{E}_k(\phi) + \left|\int_M (-1)^k(\tau^2\widehat{\partial}_0(f_k) - 2\tau^2 f_k + m^2\widehat{\partial}_0(\phi\Delta^k\phi))\mu_g\right|$$

where f_k is the integrand inside the square brackets of $\tilde{E}_k(\phi)$ above. Hence we calculate the final term above and use the Klein-Gordon equation (5.1) to simplify. For some function u we have, by repeated applications of (6.5),

$$\begin{aligned}&\widehat{\partial}_0(\Delta^k u) = \widehat{\partial}_0(g^{a_1a_2}\cdots g^{a_{2k-1}a_{2k}}\nabla_{a_1}\nabla_{a_2}\cdots\nabla_{a_{2k-1}}\nabla_{a_{2k}}u) \\ &= \Delta^k(\widehat{\partial}_0 u) - 2N\sum_{i=1}^{k} g^{a_1a_2}\cdots\Sigma^{a_{2i-1}a_{2i}}\cdots g^{a_{2k}a_{2k-1}}\nabla_{a_1}\nabla_{a_2}\cdots\nabla_{a_{2i}}\nabla_{a_{2i-1}}\cdots\nabla_{a_{2k-1}}\nabla_{a_{2k}}u \\ &\quad - 2\widehat{N}k\Delta^k u + g^{a_1a_2}\cdots g^{a_{2k-1}a_{2k}}\left[\widehat{\partial}_0, \nabla_{a_1}\nabla_{a_2}\cdots\nabla_{a_{2k-1}}\nabla_{a_{2k}}\right]u\ .\end{aligned}$$

We introduce the following compact notation for the second and last terms.

$$\begin{aligned}\Sigma^I\Delta_I^k u &:= \sum_{i=1}^{k} g^{a_1a_2}\cdots\Sigma^{a_{2i-1}a_{2i}}\cdots g^{a_{2k}a_{2k-1}}\nabla_{a_1}\nabla_{a_2}\cdots\nabla_{a_{2i}}\nabla_{a_{2i-1}}\cdots\nabla_{a_{2k-1}}\nabla_{a_{2k}}u \\ \left[\widehat{\partial}_0, \Delta^k\right]u &:= g^{a_1a_2}\cdots g^{a_{2k-1}a_{2k}}\left[\widehat{\partial}_0, \nabla_{a_1}\nabla_{a_2}\cdots\nabla_{a_{2k-1}}\nabla_{a_{2k}}\right]u\ .\end{aligned} \tag{6.9}$$

Thus

$$\left|\int_M (-1)^k(\tau^2\widehat{\partial}_0(f_k) - 2\tau^2 f_k + m^2\widehat{\partial}_0(\phi\Delta^k\phi))\mu_g\right| \lesssim I_k^1 + I_k^2 + I_k^3 + C_k^1 + C_k^2 + \left|\int_M B_k\mu_g\right|$$

where we define the lower-order integrals by

$$
\begin{aligned}
I_k^1 :=& \left|\int_M \tau^2\widehat{N}(|k|f_k - \phi\Delta^{k+1}\phi)\mu_g\right| + \left|\int_M \tau^2 N\phi'\Sigma^I\Delta_I^k\phi'\mu_g\right| + \left|\int_M \tau^2 N\phi\Sigma^I\Delta_I^{k+1}\phi\mu_g\right| \\
&+ \left|\int_M \tau^2 N\phi'\Sigma^I\Delta_I^k(N^{-1}\phi\widehat{N})\mu_g\right| + \left|\int_M \tau^2\phi\Sigma^I\Delta_I^k\left(\left(\frac{3}{2}-N\right)N^{-1}\phi\right)\mu_g\right| \\
&+ \left|\int_M \widehat{N}m^2\phi\Delta^k\phi\mu_g\right| + \left|\int_M m^2\phi\Sigma^I\Delta_I^k\phi\mu_g\right| \\
I_k^2 :=& \left|\int_M \tau^2 N\Delta\phi\Delta^k(N^{-1}\phi\widehat{N})\mu_g\right| + \left|\int_M \tau^2\nabla^a N\nabla_a\phi\Delta^k\phi\prime\mu_g\right| + \left|\int_M \tau^2\nabla^a N\nabla_a\phi\Delta^k(N^{-1}\phi\widehat{N})\mu_g\right| \\
I_k^3 :=& \left|\int_M \tau^2\left(-3\widehat{N}\phi'\Delta^k\phi\prime + (15-3N)\phi'\Delta^k(\widehat{N}N^{-1}\phi) + \frac{9}{2}\left(N-\frac{3}{2}\right)N^{-1}\phi\Delta^k(N^{-1}\phi\widehat{N})\right)\mu_g\right| \\
&+ \left|\int_M \tau^2(2-N)N^{-1}\phi\Delta^k\phi\prime\mu_g\right| + \left|\int_M \tau^2(\frac{3}{2}-N)\phi\Delta^k(N^{-1}\phi)\mu_g\right|
\end{aligned} \tag{6.10}
$$

and the integrals involving commutators by

$$
\begin{aligned}
C_k^1 :=& \left|\int_M m^2\phi\left[\widehat{\partial}_0,\Delta^k\right]\phi\mu_g\right| + \left|\int_M \tau^2\left(\phi'\left[\widehat{\partial}_0,\Delta^k\right]\phi' - \phi\left[\widehat{\partial}_0,\Delta^{k+1}\right]\phi + 3\phi'\left[\widehat{\partial}_0,\Delta^k\right](N^{-1}\phi\widehat{N})\right.\right. \\
&\left.\left.+\frac{3}{2}(N^{-1}\phi)\left[\widehat{\partial}_0,\Delta^k\right]\left(\left(\frac{3}{2}-N\right)N^{-1}\phi\right)\right)\mu_g\right|. \\
C_k^2 :=& \left|\int_M \tau^2\left(-2\Delta\phi\left[\Delta^k,N\right]\phi' - \frac{3}{2}N^{-1}\phi\left[\Delta^k,N\right]\phi'\right)\mu_g\right|.
\end{aligned} \tag{6.11}
$$

Finally, the terms without decaying factors (for example, without factors of $|\tau|$ or $||\widehat{N}||_{L^\infty}$), and which will require additional care to control, are the following.

$$
B_k := m^2\phi'\left[N,\Delta^k\right]\phi.
$$

Note the terms involving $\widehat{\partial}_0 N$ have canceled with each other. The terms I_k will be controlled using Lemmas 5 and 6. The commutator terms C_k will be controlled using Lemmas 7 and 9 below. Summing these estimates for $k=0$ to $k=\ell$, noting that $k=0$ is covered using Proposition 2, yields the claim. □

6.3. Auxiliary lemmas

As mentioned in the foregoing proof we require a series of lemmas that are used in the proof of the main energy estimate above. We list and prove those in the following. The main strategy throughout this section is to integrate by parts on each term and distribute $k\geq 1$ derivatives while also making use of the Sobolev embeddings $H^2\hookrightarrow L^\infty$ and $H^1\hookrightarrow L^4$.

Lemma 4. *For $k\geq 1$ and general functions $v,u,$ and w we have*

$$
\left|\int_M vu\Delta^k w\mu_g\right| \lesssim ||v||_{L^\infty}||u||_{H^k}||w||_{H^k} + ||u||_{L^\infty}||v||_{H^k}||w||_{H^k} + ||v||_{H^k}||u||_{H^k}||w||_{H^k}.
$$

The sums involving $|I|, |J| \leq k-1$ do not appear if $k=1$. Also assuming the bootstrap assumptions (4.1) hold, then for $k \leq N$

$$||\tau N^{-1}\phi||_{H^k} \lesssim (|\tau| + ||\widehat{N}||_{H^{k+1}})\tilde{\mathcal{E}}_k(\phi)^{1/2}, \tag{6.12}$$

$$||\tau(3-N)N^{-1}\phi||_{H^k} \lesssim |\tau|\tilde{\mathcal{E}}_k(\phi)^{1/2} + |\tau|\varepsilon^2, \tag{6.13}$$

$$||\tau(\alpha-N)N^{-1}\phi||_{H^k} \lesssim (|\tau| + ||\widehat{N}||_{H^{k+1}})\tilde{\mathcal{E}}_k(\phi)^{1/2} + ||\widehat{N}||_{H^k}\tilde{\mathcal{E}}_1(\phi)^{1/2}, \tag{6.14}$$

where $\alpha \neq 3$.

Proof. Using the Sobolev embeddings $H^2 \hookrightarrow L^\infty$ and $H^1 \hookrightarrow L^4$ and integration by parts on general functions $v, u,$ and w gives, for $k \geq 1$,

$$\begin{aligned}
\left|\int_M vu\Delta^k w\mu_g\right| &\lesssim \int_M \left|(\nabla)^{k^\circ}\Delta^{\lfloor k/2\rfloor}(vu)(\nabla)^{k^\circ}\Delta^{\lfloor k/2\rfloor}(w)\right|\mu_g \\
&\lesssim ||v||_{L^\infty}||u||_{H^k}||w||_{H^k} + ||u||_{L^\infty}||v||_{H^k}||w||_{H^k} \\
&\quad + \sum_{1\leq|I|\leq k-1}||\nabla^I v||_{L^4}\sum_{1\leq|J|\leq k-1}||\nabla^J\phi||_{L^4}||w||_{H^k} \\
&\lesssim ||v||_{L^\infty}||u||_{H^k}||w||_{H^k} + ||u|_{L^\infty}||v||_{H^k}||w||_{H^k} + ||v||_{H^k}||u||_{H^k}||w||_{H^k}.
\end{aligned}$$

In general sums involving $|I|, |J| \leq k-1$ do not appear if $k=1$. We also have

$$\begin{aligned}
&||\tau N^{-1}\phi\widehat{N}||_{H^k} \lesssim ||\tau\phi||_{H^k} + ||\tau N^{-1}\phi||_{H^k} \\
&\lesssim \frac{|\tau|}{m^2}\tilde{\mathcal{E}}_k(\phi)^{1/2} + |\tau|||N^{-1}||_{L^\infty}\tilde{\mathcal{E}}_k(\phi)^{1/2} + ||\tau\phi||_{L^\infty}||\widehat{N}||_{H^k} + |\tau|||\widehat{N}||_{H^k}\tilde{\mathcal{E}}_k(\phi)^{1/2} \\
&\lesssim |\tau|\tilde{\mathcal{E}}_k(\phi)^{1/2} + |\tau|\varepsilon^2.
\end{aligned}$$

and also

$$\begin{aligned}
||\tau N^{-1}\phi||_{H^k} &= |\tau|||N^{-1}||_{L^\infty}||\phi||_{H^k} + \left(\sum_{|I|+|J|+1\leq k}\int_M \tau^2|\nabla^{I+1}N^{-1}|^2|\nabla^J\phi|^2\mu_g\right)^{1/2} \\
&\lesssim |\tau|\tilde{\mathcal{E}}_k(\phi)^{1/2} + \sum_{1\leq|I|\leq k}||\nabla^I N^{-1}||_{L^4}\sum_{|J|\leq k-1}||\tau\nabla^J\phi||_{L^4} \\
&\lesssim (|\tau| + ||\widehat{N}||_{H^{k+1}})\tilde{\mathcal{E}}_k(\phi)^{1/2}.
\end{aligned}$$

Finally although $\widehat{N}$ is small, there are several terms involving $N-\alpha$ where $\alpha \neq 3$. In this case we take care to extract $N-\alpha$ in L^∞ when no derivatives hit it, but when derivatives do hit this term we note that $\nabla(N-\alpha) = \nabla\widehat{N}$ and this is small.

$$\begin{aligned}
||\tau(\alpha-N)N^{-1}\phi||_{H^k} &\lesssim ||\alpha-N||_{L^\infty}||\tau N^{-1}\phi||_{H^k} + ||\tau N^{-1}\phi||_{L^\infty}||\widehat{N}||_{H^k} + ||\tau N^{-1}\phi||_{H^k}||\widehat{N}||_{H^k} \\
&\lesssim ||\tau N^{-1}\phi||_{H^k} + ||N^{-1}||_{L^\infty}||\tau\phi||_{H^2}||\widehat{N}||_{H^k} \\
&\lesssim (|\tau| + ||\widehat{N}||_{H^{k+1}})\tilde{\mathcal{E}}_k(\phi)^{1/2} + ||\widehat{N}||_{H^k}\tilde{\mathcal{E}}_1(\phi)^{1/2}.
\end{aligned}$$

The first set of lower-order integrals I_k^1 are controlled in the following Lemma.

Lemma 5. *Assume the bootstrap assumptions (4.1) hold, then*

$$I_k^1 \lesssim (||\widehat{N}||_{H^2} + ||\Sigma||_{H^2})\tilde{\mathcal{E}}_k(\phi) + (||\widehat{N}||_{H^k} + ||\Sigma||_{H^k})\tilde{\mathcal{E}}_2(\phi) + (||\widehat{N}||_{H^k} + ||\Sigma||_{H^k})\tilde{\mathcal{E}}_k(\phi). \tag{6.15}$$

Proof. The results of Lemma 4 allow us to easily obtain

$$\begin{aligned} &\left|\int_M \widehat{N} m^2 \phi \Delta^k \phi \mu_g\right| + \left|\int_M \tau^2 \widehat{N}(|k|f_k - \phi\Delta^{k+1}\phi)\mu_g\right| \\ &\lesssim ||\widehat{N}||_{H^2}\tilde{\mathcal{E}}_k(\phi) + ||\widehat{N}||_{H^k}\tilde{\mathcal{E}}_2(\phi) + ||\widehat{N}||_{H^k}\tilde{\mathcal{E}}_k(\phi). \end{aligned}$$

The remaining terms involving contractions with Σ^{ab} can similarly be estimated. For example

$$\begin{aligned} \left|\int_M m^2 \phi \Sigma^I \Delta_I^k \phi \mu_g\right| &\lesssim (||\Sigma||_{H^2}||m\phi||_{H^k} + ||m\phi||_{L^\infty}||\Sigma||_{H^k} + ||m\phi||_{H^k}||\Sigma||_{H^k})||m\phi||_{H^k} \\ &\lesssim ||\Sigma||_{H^2}\tilde{\mathcal{E}}_k(\phi) + ||\Sigma||_{H^k}\tilde{\mathcal{E}}_2(\phi) + ||\Sigma||_{H^k}\tilde{\mathcal{E}}_k(\phi). \end{aligned}$$

□

Lemma 6. *Assume the bootstrap assumptions (4.1) hold, then the following estimate holds.*

$$I_k^2 + I_k^3 \lesssim (||\widehat{N}||_{H^{k+1}} + ||\widehat{N}||_{H^3})\tilde{\mathcal{E}}_k(\phi) + ||\widehat{N}||_{H^{k+1}}\tilde{\mathcal{E}}_3(\phi) + |\tau|\varepsilon^4$$

Proof. We make frequent use of the identities from Lemma 4 and also the identity (3.8) for derivatives of N^{-1}. For the first term in I_k^2, we integrate by parts only $k - 1$ times so that we avoid terms such as $||\tau\phi||_{H^{k+2}}$ since this is only controlled by $\tilde{\mathcal{E}}_{k+1}(\phi)$.

$$\begin{aligned} &\left|\int_M (-1)^k \tau^2 N \Delta\phi \Delta^k (N^{-1}\phi\widehat{N})\mu_g\right| \lesssim ||\tau N \Delta\phi||_{H^{k-1}} ||\tau(N^{-1}\widehat{N})\phi||_{H^{k+1}} \\ &\lesssim \left(||N||_{L^\infty}||\tau\phi||_{H^{k+1}} + ||\tau\Delta\phi||_{L^\infty} \sum_{1\le|I|\le k-1} ||\nabla^I N||_{L^2} + \sum_{1\le|I|\le k-2} ||\nabla^I N||_{L^4} \sum_{1\le|J|\le k-2} ||\tau\nabla^J\Delta\phi||_{L^4} \right) \\ &\times \left(||N^{-1}\widehat{N}||_{L^\infty}||\tau\phi||_{H^{k+1}} + |\tau|||\phi||_{L^\infty} \sum_{1\le|I|\le k+1} ||\nabla^I N^{-1}||_{L^2} + \sum_{1\le|I|\le k} ||\nabla^I N^{-1}||_{L^4} \sum_{1\le|J|\le k} ||\tau\nabla^J\phi||_{L^4} \right) \\ &\lesssim (||\tau\phi||_{H^{k+1}} + ||\tau\phi||_{H^4}||\widehat{N}||_{H^{k-1}})(|\tau|||\phi||_{H^2}||\widehat{N}||_{H^{k+1}} + ||\widehat{N}||_{H^{k+1}}||\tau\phi||_{H^{k+1}}) \\ &\lesssim ||\widehat{N}||_{H^{k+1}}\tilde{\mathcal{E}}_3(\phi) + ||\widehat{N}||_{H^{k+1}}\mathcal{E}_k(\phi). \end{aligned}$$

Sums involving $|I| \le k-1$ or $|I| \le k-2$ do not exist for $k=1$ and $k=2$, respectively. The key point above is the estimate

$$||\tau N^{-1}\widehat{N}\phi||_{H^{k+1}} \lesssim ||\tau\phi||_{H^{k+1}}||\widehat{N}||_{H^{k+1}} + |\tau|\varepsilon^2 \lesssim \tilde{\mathcal{E}}_k(\phi)^{1/2}||\widehat{N}||_{H^{k+1}} + |\tau|\varepsilon^2.$$

Note the first term $\tilde{\mathcal{E}}_k(\phi)^{1/2}||\widehat{N}||_{H^{k+1}}$ above is worse than the term $\tilde{\mathcal{E}}_k(\phi)^{1/2}|\tau|$ from (6.13). This is because we have more derivatives to distribute and so we must allow for a term with both high derivatives in $\widehat{N}$ and ϕ.

For the remaining terms of I_k^1 we integrate by parts k times to obtain

$$
\begin{aligned}
&\left|\int_M \tau^2 \nabla^a N \nabla_a \phi \Delta^k \phi' \right| + \left|\int_M \tau^2 \nabla^a N \nabla_a \phi \Delta^k (N^{-1} \phi \widehat{N})\right| \\
&\quad \lesssim (||\nabla N||_{L^\infty} ||\tau \nabla \phi||_{H^k} + ||\tau \nabla \phi||_{L^\infty} ||\nabla N||_{H^k} + ||\nabla N||_{H^k} ||\tau \nabla \phi||_{H^k})(||\tau \phi'||_{H^k} + ||\tau N^{-1} \phi \widehat{N}||_{H^k}) \\
&\quad \lesssim (||\widehat{N}||_{H^3} + ||\widehat{N}||_{H^{k+1}}) \mathcal{E}_k(\phi) + |\tau| ||\widehat{N}||_{H^{k+1}} \tilde{\mathcal{E}}_3(\phi) + \varepsilon^4 |\tau| ||\widehat{N}||_{H^2}.
\end{aligned}
$$

Thus

$$
I_k^2 \lesssim (||\widehat{N}||_{H^{k+1}} + ||\widehat{N}||_{H^3}) \tilde{\mathcal{E}}_k(\phi) + ||\widehat{N}||_{H^{k+1}} \tilde{\mathcal{E}}_3(\phi) + |\tau| \varepsilon^4.
$$

Turning to I_k^3, we see that the last two terms contain no factors of $\widehat{N}$. We estimate these using (6.14) to obtain

$$
\begin{aligned}
|\textstyle\int_M \tau^2 (2-N) N^{-1} \phi \Delta^k \phi' \mu_g| &\lesssim ||\tau(2-N) N^{-1} \phi||_{H^k} ||\tau \phi'||_{H^k} \\
&\lesssim (|\tau| + ||\widehat{N}||_{H^{k+1}}) \tilde{\mathcal{E}}_k(\phi) + ||\widehat{N}||_{H^k} \tilde{\mathcal{E}}_2(\phi).
\end{aligned} \tag{6.16}
$$

The other terms of I_k^3 are estimated in a similar manner, using instead (6.13). Thus

$$
I_k^3 \lesssim (||\widehat{N}||_{H^{k+1}} + |\tau| + ||\widehat{N}||_{H^2}) \tilde{\mathcal{E}}_k(\phi) + ||\widehat{N}||_{H^k} \tilde{\mathcal{E}}_2(\phi) + |\tau| \varepsilon^4.
$$

□

We now estimate the commutator terms C_k, divided into those commutators of the form $[\widehat{\partial}_0, \Delta^k]$, see Lemma 7, or those of the form $[\Delta^k, N]$, see Lemma 9. We start with the following identity, adapted from [12, (6.6)] and [31].

Lemma 7 (Commutator identity). *For some general functions v, w and $k \geq 1$ we have*

$$
\left|\int_M v [\widehat{\partial}_0, \Delta^k] w \mu_g\right| \lesssim (||v||_{H^k} ||w||_{H^{k-1}} + ||w||_{H^k} ||v||_{H^{k-1}})(||\Sigma||_{H^3} + ||\widehat{N}||_{H^3} + ||\Sigma||_{H^k} + ||\widehat{N}||_{H^k}). \tag{6.17}
$$

Proof. From [12] we have

$$
\begin{aligned}
[\widehat{\partial}_0, \Delta^k] w &:= g^{a_1 a_2} \cdots g^{a_{2k-1}} g^{a_{2k}} [\widehat{\partial}_0, \nabla_{a_1} \nabla_{a_2} \cdots \nabla_{a_{2k-1}} \nabla_{a_{2k}}] w \\
&= g^{a_1 a_2} \cdots g^{a_{2k-1}} g^{a_{2k}} \left[\sum_{i \leq 2k-1} \sum_{i+1 \leq j \leq 2k} \nabla_{a_1} \cdots \nabla_{a_{i-1}} (\nabla_{a_{i+1}} \cdots \nabla_{a_{j-1}} \nabla_c \nabla_{a_{j+1}} \cdots \nabla_{a_{2k}}(w) \cdot K^c_{a_j a_i}) \right]
\end{aligned}
$$

where

$$
K^a_{bc} := \nabla_b (N k^a_c) + \nabla_c (N k^a_b) - \nabla^a (k_{cb}).
$$

Thus after integration by parts, we find

$$\begin{aligned}
\left|\int_M v[\widehat{\partial}_0,\Delta^k]w\mu_g\right| &= \left|\sum_{\substack{|I|+|J|=2k-1\\ |J|\geq 1}}\int_M c_{IJ} v\nabla^I(\nabla^J(w)K)\mu_g\right| \\
&\lesssim \left(\sum_{\substack{|I|+|J|=2k-1\\ 1\leq |J|\leq k}} + \sum_{\substack{|I|+|J|=2k-1\\ |I|\leq k}}\right) c_{IJ}\int_M v\nabla^I(\nabla^J(w)K)\mu_g \\
&\lesssim \sum_{\substack{|I''|+|J|=k-1\\ |I'|=k\\ |J|\geq 1}}\int_M |\nabla^{I'}v\nabla^{I''}(\nabla^J w\cdot K)|\mu_g + \sum_{\substack{|I|+|J''|=k-1\\ |J'|=k\\ |J''|\geq 1}}\int_M \left|\nabla^{J'}w\nabla^{J''}(\nabla^I v\cdot K)\right|\mu_g \\
&\lesssim (||v||_{H^k}||w||_{H^{k-1}} + ||w||_{H^k}||v||_{H^{k-1}})(||K||_{L^\infty} + ||K||_{H^{k-1}}).
\end{aligned}$$

□

Finally recall $k=\Sigma+g/3$ so that $K=\nabla(Nk)=\nabla(N\Sigma)+\frac{g}{3}\nabla N$. Thus

$$||K||_{L^\infty}+||K||_{H^{k-1}}\lesssim ||\Sigma||_{H^3}+||\widehat{N}||_{H^3}+||\Sigma||_{H^k}+||\widehat{N}||_{H^k}.$$

Lemma 8. *Assume the bootstrap assumptions (4.1) hold, then*

$$C_k^1 \lesssim (\tilde{\mathcal{E}}_k(\phi)+|\tau|\varepsilon^4)(||K||_{L^\infty}+||K||_{H^{k-1}}). \tag{6.18}$$

Proof. Using Lemma 7 the 'symmetric' terms are controlled by

$$\left|\int_M (\tau^2\phi'\left[\widehat{\partial}_0,\Delta^k\right]\phi' + m^2\phi\left[\widehat{\partial}_0,\Delta^k\right]\phi)\mu_g\right| \lesssim (||K||_{L^\infty}+||K||_{H^{k-1}})\tilde{\mathcal{E}}_k(\phi).$$

The remaining terms are

$$\begin{aligned}
&\left|\int_M \tau^2\phi[\widehat{\partial}_0,\Delta^{k+1}]\phi\mu_g\right| + \left|\int_M \tau^2\phi'[\widehat{\partial}_0,\Delta^k](N^{-1}\phi\widehat{N})\mu_g\right| + \left|\int_M \tau^2(N^{-1}\phi)[\widehat{\partial}_0,\Delta^k]\left(\left(\frac{3}{2}-N\right)N^{-1}\phi)\mu_g\right| \\
&\lesssim (|\tau|||\tau\phi||_{H^{k+1}}||\phi||_{H^k} + ||\tau\phi'||_{H^k}||\tau N^{-1}\phi\widehat{N}||_{H^k} \\
&\qquad + ||\tau N^{-1}\phi||_{H^k}||\tau\phi(1-3/2N)||_{H^k})(||K||_{L^\infty}+||K||_{H^{k-1}}) \\
&\lesssim (|\tau|\mathcal{E}_k(\phi) + |\tau|\varepsilon^4 + ||\widehat{N}||_{H^{k+1}}\mathcal{E}_k(\phi))(||K||_{L^\infty}+||K||_{H^{k-1}}).
\end{aligned}$$

In the final line we used Lemma 4. □

The final result in this section controls commutators involving N.

Lemma 9. *Assume the bootstrap assumptions (4.1) hold, then*

$$C_k^2 \lesssim ||\widehat{N}||_{H^3}\tilde{\mathcal{E}}_k(\phi) + ||\widehat{N}||_{H^{k+1}}\tilde{\mathcal{E}}_3(\phi) + ||\widehat{N}||_{H^{k+1}}\tilde{\mathcal{E}}_k(\phi). \tag{6.19}$$

Proof. First note the expansion

$$[\Delta^k, N]w = \sum_{|I|+|J|=2k-1} c_{IJ}\nabla^{I+1}N\nabla^J w$$

where the constants c_{IJ} are functions of g and so will be bounded below by some large overall constant. For general functions w, v and $k \geq 1$ we have the following

$$\begin{aligned}
&\int_M v[\Delta^k, N]w\mu_g \\
&\lesssim \sum_{\substack{|I|+|J'|=k-1\\|J''|=k}} ||\nabla^{J'}v\nabla^{I+1}N\nabla^{J''}w||_{L^1} + \sum_{\substack{|I'|+|J|=k-1\\|I''|=k+1}} ||\nabla^{I'}v\nabla^J w\nabla^{I''}N||_{L^1} \\
&\lesssim \left(||v||_{L^\infty}||\widehat{N}||_{H^k} + ||\nabla N||_{L^\infty}||v||_{H^{k-1}} + \sum_{|\alpha|\leq k-2}||\nabla^\alpha v||_{L^4}\sum_{|\beta|\leq k-1}||\widehat{N}||_{L^4}\right)||\tau w||_{H^k} \\
&\quad + \left(||w||_{L^\infty}||v||_{H^{k-1}} + ||v||_{L^\infty}||w||_{H^{k-1}} + \sum_{|\alpha|\leq k-2}||\nabla^\alpha v||_{L^4}\sum_{|\beta|\leq k-2}||w||_{L^4}\right)||\widehat{N}||_{H^{k+1}} \\
&\lesssim (||v||_{H^2} + ||v||_{H^{k-1}})||\widehat{N}||_{H^{k+1}}||w||_{H^k} + \left(||w||_{H^k}||\widehat{N}||_{H^3} + ||w||_{H^2}||\widehat{N}||_{H^{k+1}}\right)||\widehat{N}||_{H^{k-1}}||w||_{H^k}.
\end{aligned}$$

The claim follows, recalling for example the estimate (6.12). □

7. Lapse and shift estimates

We first recall the following elliptic estimates from [12, Proposition 17] for the lapse and shift.

Proposition 4. *Under appropriate smallness conditions we have the pointwise estimate $N \in (0, 3]$ and the following estimates*

$$\begin{aligned}
||\widehat{N}||_{H^\ell} &\leq C(||\Sigma||^2_{H^{\ell-2}} + |\tau|||\eta||_{H^{\ell-2}}), \\
||X||_{H^\ell} &\leq C(||\Sigma||^2_{H^{\ell-2}} + ||g-\gamma||^2_{H^{\ell-1}} + |\tau|||\eta||_{H^{\ell-3}} + \tau^2||Nj||_{H^{\ell-2}}).
\end{aligned} \tag{7.1}$$

Applied to the present case this yields the following estimate for the lapse function.

Lemma 10 (Lapse estimate). *Assume the bootstrap assumptions (4.1) hold, then for $2 \leq \ell \leq N+1$ we have*

$$||\widehat{N}||_{H^\ell} \lesssim ||\Sigma||^2_{H^{\ell-2}} + |\tau|||\rho||_{L^\infty} + |\tau|\tilde{\mathcal{E}}_{\ell-2}(\phi) \tag{7.2}$$

and furthermore

$$||\widehat{N}||_{H^\ell} \lesssim \varepsilon^2|\tau| + |\tau|\tilde{\mathcal{E}}_{\ell-2}(\phi). \tag{7.3}$$

Proof. For $k \geq 0$ and some function w we have

$$\begin{aligned}
||w^2||^2_{H^k} &\lesssim ||w||^2_{L^\infty}\sum_{|I|\leq k}\int_M |\nabla^I w|^2\mu_g + \sum_{|I|+|J|+2\leq k}\int_M \left(|\nabla^{I+1}w|^4 + |\nabla^{J+1}w|^4\right)\mu_g \\
&\lesssim ||w||^2_{L^\infty}||w||^2_{H^\ell} + \sum_{|I|+1\leq k-1}||\nabla^{I+1}w||^4_{L^4} \\
&\lesssim ||w||^2_{L^\infty}||w||^2_{H^k} + ||w||^4_{H^k}.
\end{aligned} \tag{7.4}$$

Recalling the definition of η from (2.12) and the estimate from Proposition 4 we see

$$\begin{aligned}||\widehat{N}||_{H^\ell} &\lesssim ||\Sigma||^2_{H^{\ell-2}} + |\tau|||\eta||_{H^{\ell-2}} \\ &\lesssim ||\Sigma||^2_{H^{\ell-2}} + |\tau|(||m\phi||_{L^\infty} + ||\tau N^{-1}\phi||_{L^\infty} + ||\tau\phi'||_{L^\infty})\tilde{\mathcal{E}}_{\ell-2}(\phi)^{1/2} + |\tau|\tilde{\mathcal{E}}_{k-2}(\phi) \\ &\lesssim ||\Sigma||^2_{H^{\ell-2}} + |\tau|\tilde{\mathcal{E}}_{\ell-2}(\phi) + |\tau|||\rho||_{L^\infty}.\end{aligned}$$

□

Remark 5. It is crucial in the final line of the previous proof (specifically for $\ell = 2$) to use the pointwise estimate from Proposition 1 instead of the standard Sobolev estimate invoking $\tilde{\mathcal{E}}_2(\phi)$, since the latter would have created a $e^{\kappa T}$ growth preventing the envisioned bootstrap argument.

Lemma 11 (Shift estimate). *Assume the bootstrap assumptions (4.1) hold, then for* $3 \le \ell \le N+1$

$$||X||_{H^\ell} \lesssim ||\Sigma||^2_{H^{\ell-2}} + ||g-\gamma||^2_{H^{\ell-1}} + |\tau|||\rho||_{L^\infty} + |\tau|\tilde{\mathcal{E}}_{\ell-2}(\phi)$$

and furthermore

$$||X||_{H^\ell} \lesssim \varepsilon^2|\tau| + |\tau|\tilde{\mathcal{E}}_{\ell-2}(\phi). \tag{7.5}$$

Proof. Recall the estimate from Proposition 4. We may use the estimate for $||\eta||_{H^k}$ derived in Lemma 10, and will also need an estimate for the rescaled matter current

$$\jmath^a = \tau\left(\frac{3}{2}N^{-1}\phi - \phi'\right)\nabla^a\phi. \tag{7.6}$$

For $k \ge 0$ using the standard Sobolev embeddings, we have

$$\begin{aligned}\tau||\jmath||_{H^k} &\lesssim ||\tau\nabla\phi||_{L^\infty}\tilde{\mathcal{E}}_k(\phi)^{1/2} + \left(||\tau N^{-1}\phi||_{L^\infty} + ||\tau\phi'||_{L^\infty}\right)\tilde{\mathcal{E}}_k(\phi)^{1/2} + \tilde{\mathcal{E}}_k(\phi) \\ &\lesssim ||\rho||_{L^\infty} + \tilde{\mathcal{E}}_k(\phi).\end{aligned}$$

So for $3 \le \ell \le N+1$ we have

$$\begin{aligned}||X||_{H^\ell} &\lesssim ||\Sigma||^2_{H^{\ell-2}} + ||g-\gamma||^2_{H^{\ell-1}} + |\tau|||\eta||_{H^{\ell-3}} + \tau^2||Nj||_{H^{\ell-2}} \\ &\lesssim ||\Sigma||^2_{H^{\ell-2}} + ||g-\gamma||^2_{H^{\ell-1}} + |\tau|\tilde{\mathcal{E}}_{\ell-3}(\phi) + |\tau|||\rho||_{L^\infty} \\ &\quad + |\tau|\left(||N||_{L^\infty}||\tau j||_{H^{\ell-2}} + \sum_{|I|\le\ell-2}||\nabla^I\widehat{N}||_{L^\infty}||\tau j||_{L^2} + ||\widehat{N}||_{H^{\ell-2}}||\tau j||_{H^{\ell-2}}\right) \\ &\lesssim ||\Sigma||^2_{H^{\ell-2}} + ||g-\gamma||^2_{H^{\ell-1}} + |\tau|\tilde{\mathcal{E}}_{\ell-3}(\phi) + |\tau|(||\rho||_{L^\infty} + \tilde{\mathcal{E}}_{\ell-2}(\phi) + ||\widehat{N}||_{H^\ell}\tilde{\mathcal{E}}_2(\phi)).\end{aligned}$$

□

8. Hierarchy between lapse and Klein-Gordon field

In the following Lemma we estimate the borderline terms for the Klein-Gordon energy.

Lemma 12 (Borderline terms). *Assume the bootstrap assumptions (4.1) hold, then for $1 \le \ell \le N$ we have*

$$\sum_{k=1}^{\ell}\left|\int_M B_k\mu_g\right| \lesssim |\tau|^{-1}\varepsilon\tilde{\mathcal{E}}_\ell(\phi)^{1/2}||\widehat{N}||_{H^\ell} + |\tau|^{-1}\tilde{\mathcal{E}}_\ell(\phi)||\widehat{N}||_{H^2} + |\tau|^{-1}\tilde{\mathcal{E}}_\ell(\phi)^{1/2}\tilde{\mathcal{E}}_{\ell-1}(\phi)^{1/2}||\widehat{N}||_{H^{\ell+1}}. \tag{8.1}$$

Proof.

$$\begin{aligned}
\left|\int_M B_k\mu_g\right| &= \left|m^2\int_M \phi'(N\Delta^k\phi - \Delta^k(N\phi))\mu_g\right| \\
&\lesssim \int_M \sum_{|I|+1+|J|=k}|m^2\nabla^{I+1}N\nabla^J\phi||\sum_{|I'|\le k}\nabla^{I'}\phi'|\mu_g + \int_M \sum_{|I|+1+|J|=k}|\nabla^{I+1}N\nabla^J\phi\prime|\sum_{|I'|\le k}|m^2\nabla^{I'}\phi|\mu_g \\
&\lesssim ||m\phi||_{L^\infty}|\tau|^{-1}||\tau\phi'||_{H^k}||\widehat{N}||_{H^k} + |\tau|^{-1}||\tau\phi'||_{H^k}\sum_{|I|+1+|J|=k}||\nabla^{I+1}N||_{H^1}||m\nabla^J\phi||_{H^1} \\
&\quad + |\tau|^{-1}||\tau\phi'||_{L^\infty}||m\phi||_{H^k}||\widehat{N}||_{H^k} + |\tau|^{-1}||m\phi||_{H^k}\sum_{|I|+1+|J|=k}||\nabla^{I+1}N||_{H^1}||\tau\nabla^J\phi\prime||_{H^1} \\
&\lesssim |\tau|^{-1}||\rho||_{L^\infty}^{1/2}\mathcal{E}_k(\phi)^{1/2}||\widehat{N}||_{H^k} + |\tau|^{-1}\tilde{\mathcal{E}}_k(\phi)^{1/2}(||\nabla N||_{H^1}||\phi||_{H^k} + ||\phi||_{H^{k-1}}||\widehat{N}||_{H^{k+1}}) \\
&\quad + |\tau|^{-1}\varepsilon\tilde{\mathcal{E}}_k(\phi)^{1/2}||\widehat{N}||_{H^k} + |\tau|^{-1}\tilde{\mathcal{E}}_k(\phi)^{1/2}(||\nabla N||_{H^1}||\tau\phi'||_{H^k} + ||\widehat{N}||_{H^{k+1}}||\tau\phi'||_{H^{k-1}}) \\
&\lesssim |\tau|^{-1}\varepsilon\tilde{\mathcal{E}}_k(\phi)^{1/2}||\widehat{N}||_{H^k} + |\tau|^{-1}\tilde{\mathcal{E}}_k(\phi)||\widehat{N}||_{H^2} + |\tau|^{-1}\tilde{\mathcal{E}}_k(\phi)^{1/2}\tilde{\mathcal{E}}_{k-1}(\phi)^{1/2}||\widehat{N}||_{H^{k+1}}.
\end{aligned}$$

Summing from $k=1$ to ℓ gives the required result. □

Remark 6. We now outline the key ideas behind closing the Lapse and Klein-Gordon bootstrap assumptions, as proved below in Proposition 5. The estimates for $\tilde{\mathcal{E}}_0(\phi)$ and $||\widehat{N}||_{H^2}$ are readily improved. Then, starting from $\ell = 1$, the most problematic terms needed for the $\tilde{\mathcal{E}}_\ell(\phi)$ estimate are contained in Lemma 12. Nonetheless, Lemma 12 tells us that we need information about $||\widehat{N}||_{H^2}, ||\widehat{N}||_{H^{\ell+1}}$ and $\tilde{\mathcal{E}}_{\ell-1}(\phi)$, all of which have been upgraded from the previous steps. The upgraded estimate for $\tilde{\mathcal{E}}_\ell(\phi)$ will then be used, via Lemma 10, to close the bootstrap estimate for $||\widehat{N}||_{H^{\ell+2}}$. One then moves onto improving the estimate for $\tilde{\mathcal{E}}_{\ell+1}(\phi)$ and continues until $\ell = N$.

Proposition 5 (Upgraded Lapse and Klein-Gordon estimates). *Assume the bootstrap assumptions (4.1) hold, then*

$$||\widehat{N}||_{H^2} \lesssim \varepsilon^2 e^{-T}, \tag{8.2}$$

$$\tilde{\mathcal{E}}_0(\phi) \lesssim \varepsilon^2, \tag{8.3}$$

and for higher orders $1 \le \ell \le N$

$$||\widehat{N}||_{H^{k+1}} \lesssim \varepsilon^2 e^{(-1+C\varepsilon)T}, \tag{8.4}$$

$$\tilde{\mathcal{E}}_k(\phi) \lesssim \varepsilon^2 e^{C\varepsilon T}. \tag{8.5}$$

Proof. From Proposition 2

$$\tilde{E}_0(\phi)|_T \lesssim \tilde{E}_0(\phi)|_{T_0}. \tag{8.6}$$

The lapse estimate Lemma 10 with $\ell = 2$ then implies

$$||\widehat{N}||_{H^2} \lesssim \varepsilon^2 e^{-T} + e^{-T}\tilde{\mathcal{E}}_0(\phi) \lesssim \varepsilon^2 e^{-T}. \tag{8.7}$$

The Klein-Gordon estimate of Proposition 3 combined with the borderline estimate of Lemma 12 for $\ell = 2 - 1 = 1$ together imply

$$\begin{aligned}\partial_T \tilde{\mathcal{E}}_1(\phi) &\lesssim \varepsilon e^{(-1+\kappa)T}\tilde{\mathcal{E}}_1(\phi) + |\tau|\tilde{\mathcal{E}}_1(\phi) + \varepsilon^3 e^{(-1+\kappa)T} + \varepsilon^4|\tau| + |\tau|^{-1}\varepsilon\tilde{\mathcal{E}}_1(\phi)^{1/2}||\widehat{N}||_{H^2}\\ &\quad + |\tau|^{-1}\tilde{\mathcal{E}}_1(\phi)||\widehat{N}||_{H^2} + |\tau|^{-1}\tilde{\mathcal{E}}_1(\phi)^{1/2}\tilde{\mathcal{E}}_0(\phi)^{1/2}||\widehat{N}||_{H^2}\\ &\lesssim \varepsilon^3 e^{(-1+\kappa)T} + \varepsilon^4 + (\varepsilon e^{(-1+\kappa)T} + |\tau| + \varepsilon^2)\tilde{\mathcal{E}}_1(\phi).\end{aligned}$$

Thus Grönwall's inequality implies

$$\begin{aligned}\tilde{\mathcal{E}}_1(\phi)|_T &\leq \left(\tilde{\mathcal{E}}_1(\phi)|_{T_0} + C\int_{T_0}^T (\varepsilon^3 e^{(-1+\kappa)s} + \varepsilon^4)ds\right)\exp\left(C\int_{T_0}^T \left(e^{-s} + \varepsilon e^{(-1+\kappa)s} + \varepsilon^2\right)ds\right)\\ &\lesssim (\tilde{\mathcal{E}}_1(\phi)|_{T_0} + \varepsilon^3 e^{C\varepsilon T})\exp(C\varepsilon T).\end{aligned} \tag{8.8}$$

Note we used the identity: $x \leq 1 + x \leq e^x$ for $x \geq 0$.

Returning to the lapse estimate from Lemma 10 with $\ell = 3$ now implies

$$||\widehat{N}||_{H^3} \lesssim \varepsilon^2 e^{-T} + e^{-T}\tilde{\mathcal{E}}_1(\phi) \lesssim \varepsilon^2 e^{(-1+2C\varepsilon)T}. \tag{8.9}$$

Now we use this result to improve the $\ell = 2$ estimate for the Klein-Gordon field. From Lemma 12, Proposition 3 and the upgraded estimates obtained so far, we have

$$\begin{aligned}\partial_T \tilde{\mathcal{E}}_2(\phi) &\lesssim \varepsilon e^{(-1+\kappa)T}\tilde{\mathcal{E}}_2(\phi) + |\tau|\tilde{\mathcal{E}}_2(\phi) + \varepsilon^3 e^{(-1+\kappa)T} + \varepsilon^4|\tau| + |\tau|^{-1}\varepsilon\tilde{\mathcal{E}}_2(\phi)^{1/2}||\widehat{N}||_{H^2}\\ &\quad + |\tau|^{-1}\tilde{\mathcal{E}}_2(\phi)||\widehat{N}||_{H^2} + |\tau|^{-1}\tilde{\mathcal{E}}_2(\phi)^{1/2}\tilde{\mathcal{E}}_1(\phi)^{1/2}||\widehat{N}||_{H^3}\\ &\lesssim \varepsilon e^{(-1+\kappa)T}\tilde{\mathcal{E}}_2(\phi) + |\tau|\tilde{\mathcal{E}}_2(\phi) + \varepsilon^3 e^{(-1+\kappa)T} + \varepsilon^3\tilde{\mathcal{E}}_2(\phi)^{1/2} + \varepsilon^2\tilde{\mathcal{E}}_2(\phi)\\ &\quad + \varepsilon^3 e^{C\varepsilon T}\tilde{\mathcal{E}}_2(\phi)^{1/2}\\ &\lesssim \varepsilon^3 e^{(-1+\kappa)T} + \varepsilon^4 e^{2C\varepsilon T} + (\varepsilon e^{(-1+\kappa)T} + |\tau| + \varepsilon^2)\tilde{\mathcal{E}}_2(\phi).\end{aligned}$$

Thus by Grönwall:

$$\begin{aligned}\tilde{\mathcal{E}}_2(\phi)|_T &\lesssim (\tilde{\mathcal{E}}_2(\phi)|_{T_0} + \textstyle\int_{T_0}^T(\varepsilon^3 e^{(-1+\kappa)s} + \varepsilon^4 e^{C\varepsilon s}ds))\exp(C\textstyle\int_{T_0}^T(e^{-s}+\varepsilon)ds)\\ &\lesssim (\tilde{\mathcal{E}}_2(\phi)|_{T_0} + \varepsilon^3 e^{C\varepsilon T})e^{C\varepsilon T}.\end{aligned}$$

Continuing in this way we stop at $\ell = N + 1$ for the lapse estimate.

$$||\widehat{N}||_{H^{N+1}} \lesssim \varepsilon^2 e^{-T} + e^{-T}\tilde{\mathcal{E}}_{N-1}(\phi) \lesssim \varepsilon^2 e^{(-1+C\varepsilon)T}. \tag{8.10}$$

Using this to estimate the final $\ell = N$ energy (recall $N \geq 4$) for the Klein-Gordon field gives

$$\begin{aligned}\partial_T \tilde{\mathcal{E}}_N(\phi) &\lesssim \varepsilon e^{(-1+C\varepsilon)T}\tilde{\mathcal{E}}_N(\phi) + |\tau|\tilde{\mathcal{E}}_N(\phi) + \varepsilon^3 e^{(-1+C\varepsilon)T} + |\tau|\varepsilon^4 + |\tau|^{-1}\varepsilon\tilde{\mathcal{E}}_N(\phi)^{1/2}||\widehat{N}||_{H^N}\\ &\quad + |\tau|^{-1}\tilde{\mathcal{E}}_N(\phi)||\widehat{N}||_{H^2} + |\tau|^{-1}\tilde{\mathcal{E}}_N(\phi)^{1/2}\tilde{\mathcal{E}}_{N-1}(\phi)^{1/2}||\widehat{N}||_{H^{N+1}}\\ &\lesssim \varepsilon^3 e^{(-1+C\varepsilon)T} + |\tau|\tilde{\mathcal{E}}_N(\phi) + \varepsilon^3 e^{C\varepsilon T}\tilde{\mathcal{E}}_N(\phi)^{1/2} + \varepsilon^2\tilde{\mathcal{E}}_N(\phi)\\ &\lesssim \varepsilon^3 e^{(-1+C\varepsilon)T} + \varepsilon^4 e^{2C\varepsilon T} + (\varepsilon e^{(-1+C\varepsilon)T} + |\tau| + \varepsilon^2)\tilde{\mathcal{E}}_N(\phi).\end{aligned}$$

Thus by Grönwall we have

$$\begin{aligned}\tilde{\mathcal{E}}_N(\phi)|_T &\lesssim (\tilde{\mathcal{E}}_N(\phi)|_{T_0} + \int_{T_0}^{T}\left(\varepsilon^2 e^{(-1+C\varepsilon)s} + \varepsilon^4 e^{C\varepsilon s})ds\right)\exp\left(C\int_{T_0}^{T}(\varepsilon + e^{-s})ds\right)\\ &\lesssim (\tilde{\mathcal{E}}_N(\phi)|_{T_0} + \varepsilon^3 e^{C\varepsilon T})e^{C\varepsilon T}.\end{aligned}$$

9. Energy estimate - geometry

In this final section we obtain improved estimates for the shift vector field and the second fundamental form and metric perturbation.

Corollary 1 (Improved Shift estimate). *Assume the bootstrap assumptions (4.1) hold, then for* $3 \le \ell \le N+1$

$$||X||_{H^\ell} \lesssim \varepsilon^2 e^{(-1+C\varepsilon)T}. \tag{9.1}$$

Proof. This follows clearly from Lemma 11 and Proposition 5. □

Recall the definitions from Section 4.2. Using the energy estimate given in [12, Lemma 20], itself adapted from [2] we have the following estimate for the second fundamental form and metric perturbation.

Corollary 2 (Improved geometry estimate). *Assume the bootstrap assumptions (4.1) hold, then for* $1 \le \ell \le N+1$

$$\partial_T E^g{}_\ell \le -2\alpha E^g{}_\ell + 6E^g{}_\ell^{1/2}|\tau|||NS||_{H^{\ell-1}} + CE^g{}_\ell^{3/2} + CE^g{}_\ell^{1/2}(|\tau|||\eta||_{H^{\ell-1}} + \tau^2||Nj||_{H^{\ell-2}}). \tag{9.2}$$

Furthermore

$$\partial_T E^g{}_\ell \le -2\alpha E^g{}_\ell + CE^g{}_\ell^{1/2}\varepsilon^2 e^{(-1+C\varepsilon)T} + CE^g{}_\ell^{3/2},$$

Finally

$$E^g{}_\ell|_T \le C\varepsilon^2 e^{-2\alpha\zeta T}. \tag{9.3}$$

where for sufficiently small ε *we may choose* ζ *arbitrarily close to 1, in particular* $\zeta \le 1 - \frac{C\varepsilon}{\alpha}$.

Proof. The estimate (9.2) comes from [12, Lemma 20]. Recall S_{ij} from (2.13). So for $2 \le \ell \le N+1$ we find

$$\begin{aligned}&||NS||_{H^{\ell-1}}\\ &\quad\lesssim ||N||_{L^\infty}(||\phi||_{L^\infty}||\phi||_{H^{\ell-1}} + ||\phi||^2_{H^{\ell-1}} + ||\tau\nabla\phi||_{L^\infty}||\tau\nabla\phi||_{H^{\ell-1}} + ||\tau\nabla\phi||^2_{H^{\ell-1}})\\ &\qquad + ||\widehat{N}||_{H^{\ell-1}}(||\phi||^2_{L^\infty} + ||\phi||^2_{L^\infty}||\phi||_{H^{\ell-1}} + ||\phi||^2_{H^{\ell-1}}\\ &\qquad + ||\tau\nabla\phi||^2_{L^\infty} + ||\tau\nabla\phi||_{L^\infty}||\tau\nabla\phi||_{H^{\ell-1}} + ||\tau\nabla\phi||^2_{H^{\ell-1}})\\ &\quad\lesssim ||\rho||_{L^\infty} + \tilde{\mathcal{E}}_{\ell-1}(\phi) + ||\widehat{N}||_{H^{\ell-1}}(||\rho||_{L^\infty} + \tilde{\mathcal{E}}_{\ell-1}(\phi))\end{aligned}$$

where we took note of the product identity (7.4). Also following the method of Lemmas 10 and 11 we have

$$|\tau|||\eta||_{H^{\ell-1}} + \tau^2||Nj||_{H^{\ell-2}} \lesssim |\tau|\tilde{\mathcal{E}}_{\ell-1}(\phi) + |\tau|||\rho||_{L^\infty} \\ +|\tau|(||\rho||_{L^\infty} + \tilde{\mathcal{E}}_{\ell-2}(\phi))(||N||_{L^\infty} + ||\widehat{N}||_{H^\ell}).$$

Putting these together gives

$$\begin{aligned}\partial_T E^g{}_\ell &\leq -2\alpha E^g{}_\ell + 6E^g{}_\ell^{1/2}\tau||NS||_{H^{\ell-1}} + CE^g{}_\ell^{3/2} + CE^g{}_\ell^{1/2}(|\tau|||\eta||_{H^{\ell-1}} + \tau^2||Nj||_{H^{\ell-2}}) \\ &\leq -2\alpha E^g{}_\ell + 6E^g{}_\ell^{1/2}|\tau|(||\rho||_{L^\infty} + \tilde{\mathcal{E}}_{\ell-1}(\phi)) + CE^g{}_\ell^{3/2} \\ &\quad + CE^g{}_\ell^{1/2}(|\tau|\tilde{\mathcal{E}}_{\ell-1}(\phi) + |\tau|||\rho||_{L^\infty} + |\tau|||\rho||_{L^\infty}||\widehat{N}||_{H^\ell} + |\tau|\tilde{\mathcal{E}}_{\ell-2}(\phi)||\widehat{N}||_{H^\ell}).\end{aligned}$$

Thus

$$\partial_T(E^g{}_\ell^{1/2}) \leq -\alpha E^g{}_\ell^{1/2} + C\varepsilon^2 e^{-T} + CE^g{}_\ell.$$

Now recall $\alpha \in [1-\delta_\alpha, 1]$ where δ_α can be made suitably small. Given α, pick ζ such that $\frac{3}{4} < \zeta < 1$ and $-\left(\alpha\zeta - \frac{3}{4}\right) < 0$ (i.e. $\alpha\zeta > \frac{3}{4}$). Indeed we can guarantee $\alpha\zeta > \frac{3}{4}$ holds by choosing ε sufficiently small such that $1 - \delta_\alpha(\varepsilon) > \frac{3}{4\zeta}$. Then we have

$$\begin{aligned}\partial_T\left(e^{\frac{3}{4}T}E^g{}_\ell^{1/2}\right) &\leq -\left(\alpha\zeta - \frac{3}{4}\right)e^{\frac{3}{4}T}E^g{}_\ell^{1/2} + C\varepsilon^2 e^{(-1+\frac{3}{4})T} - e^{\frac{3}{4}T}E^g{}_\ell^{1/2}(\alpha(1-\zeta) - CE^g{}_\ell^{1/2}) \\ &\leq -\left(\alpha\zeta - \frac{3}{4}\right)e^{\frac{3}{4}T}E^g{}_\ell^{1/2} + C\varepsilon^2 e^{(-1+\frac{3}{4})T} - e^{\frac{3}{4}T}E^g{}_\ell^{1/2}(\alpha(1-\zeta) - C\varepsilon) \\ &\leq -\left(\alpha\zeta - \frac{3}{4}\right)e^{\frac{3}{4}T}E^g{}_\ell^{1/2} + C\varepsilon^2 e^{(-1+\frac{3}{4})T}\end{aligned}$$

where we dropped the final term by picking ε small enough so that $\alpha(1-\zeta) - C\varepsilon \geq 0$. Then by Grönwall we have

$$e^{\delta T}E^g{}_\ell^{1/2}|_T \leq \left(e^{\frac{3}{4}T_0}E^g{}_\ell^{1/2}|_{T_0} + C\varepsilon^2\int_{T_0}^T e^{(-1+\frac{3}{4})s}ds\right)\Bigg)\exp\left(-\int_{T_0}^T\left(\alpha\zeta - \frac{3}{4}\right)ds\right). \tag{9.4}$$

This implies

$$E^g{}_\ell^{1/2}|_T \leq \left(e^{\frac{3}{4}T_0}E^g{}_\ell^{1/2}|_{T_0} + C\varepsilon^2\right)e^{-\frac{3}{4}T}e^{-(\alpha\zeta-\frac{3}{4})(T-T_0)} \tag{9.5}$$

and thus

$$E^g{}_\ell|_T \leq (E^g{}_\ell|_{T_0} + C\varepsilon^4)e^{-2\alpha\zeta T} < C\varepsilon^2 e^{-\frac{3}{2}T}. \tag{9.6}$$

□

Proof of the main theorem. The main theorem is now a consequence of the foregoing lemmas. Considering a sufficiently small perturbation of the Milne initial data it follows along the lines of the corresponding argument in [30] that there exists a CMC surface in the maximal development with initial data close to the Milne geometry. This initial data set is now evolved by the rescaled CMCSH EKGS. The local existence theory then implies the existence of a solution, which according to our previous analysis obeys the decay estimates for the perturbation given in (9.3). In particular, the solution exists for $T \to \infty$ and moreover is future complete, which follows analogous to [29]. □

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