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A Lipschitz metric for the Hunter–Saxton equation

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ABSTRACT

We analyze stability of conservative solutions of the Cauchy problem on the line for the (integrated) Hunter–Saxton (HS) equation. Generically, the solutions of the HS equation develop singularities with steep gradients while preserving continuity of the solution itself. In order to obtain uniqueness, one is required to augment the equation itself by a measure that represents the associated energy, and the breakdown of the solution is associated with a complicated interplay where the measure becomes singular. The main result in this article is the construction of a Lipschitz metric that compares two solutions of the HS equation with the respective initial data. The Lipschitz metric is based on the use of the Wasserstein metric.

1. Introduction

In this article, we consider the Cauchy problem for conservative solutions of the (integrated) Hunter–Saxton (HS) Equation [1]

$$u_t + uu_x = \frac{1}{4}\int_{-\infty}^{x} u_x^2(y)\mathrm{d}y - \frac{1}{4}\int_{x}^{\infty} u_x^2(y)\mathrm{d}y, \quad u|_{t=0} = u_0. \tag{1.1}$$

The equation has been extensively studied, starting with studies of Hunter and Zheng [2,3]. The initial value problem is not well-posed without further constraints: Consider the trivial case $u_0 = 0$ which clearly has as one solution $u(t, x) = 0$. However, as can be easily verified, also

$$u(t, x) = -\frac{\alpha}{4}t\,\mathbb{I}_{\left(-\infty, -\frac{\alpha}{8}t^2\right)}(x) + \frac{2x}{t}\,\mathbb{I}_{\left(-\frac{\alpha}{8}t^2, \frac{\alpha}{8}t^2\right)}(x) + \frac{\alpha}{4}t\,\mathbb{I}_{\left(\frac{\alpha}{8}t^2, \infty\right)}(x), \tag{1.2}$$

is a solution for any $\alpha \geq 0$. Here $\mathbb{I}_A$ is the indicator (characteristic) function of the set A.

Furthermore, it turns out that the solution u of the HS equation may develop singularities in finite time in the following sense: Unless the initial data is monotone increasing, we find

$$\inf(u_x) \to -\infty \quad \text{as} \quad t \uparrow t^* = 2/\sup\left(-u_0'\right). \tag{1.3}$$

Past wave breaking there are at least two different classes of solutions, denoted conservative (energy is conserved) and dissipative (where energy is removed locally) solutions, respectively, and this dichotomy is the source of the interesting behavior of solutions of the equation. We will consider in this article the so-called conservative case where an associated energy is preserved.

Zhang and Zheng [4–6] gave the first proof of global solutions of the HS equation on the half-line using Young measures and mollifications with compactly supported initial data. Their proof covered both the conservative case and the dissipative case. Subsequently, Bressan and Constantin [7] using a clever rewrite of the equation in terms of new variables, showed global existence of conservative solutions without the assumption of compactly supported initial data. The novel variables turned the partial differential equation into a system of linear ordinary differential equations taking values in a Banach space, and where the singularities were removed. A similar, but considerably more complicated, transformation can be used to study the very closely related Camassa–Holm equation, see Bressan and Constantin [8] and Holden and Raynaud [9]. Bressan and Fonte [10] introduce a metric based on optimal transport for the Camassa–Holm equation. However, the approach is very different from the present, and is based on approximations by multi-peakons. Furthermore, Bressan et al. [11] analyze the more general equation $u_t + f(u)_x = \frac{1}{2}\int_0^x f''(u)u_x^2 dx$ (the standard HS equation corresponds to $f(u) = u^2/2$), and derive an explicit solution [11, Eqs. (32) and (33)] that is equivalent to (1.8). The convergence of a numerical method to compute the solution of the HS equation can be found in [12].

We note in passing that the original form of the HS equation is

$$(u_t + uu_x)_x = \frac{1}{2}u_x^2,$$

and like most other researchers working on the HS equation, we prefer to work with an integrated version. However, in addition to (1.1), one may study, for instance,

$$u_t + uu_x = \frac{1}{2}\int_0^x u_x^2(y)dy,$$

and while the properties are mostly the same, the explicit solutions differ.

Our aim here is to determine a Lipschitz metric d that compares two solutions $u_1(t), u_2(t)$ at time t with the corresponding initial data, i.e.,

$$d(u_1(t), u_2(t)) \leq C(t)d(u_1(0), u_2(0)),$$

where $C(t)$ denotes some increasing function of time. The existence of such a metric is clearly intrinsically connected with the uniqueness question, and as we could see from the example where (1.2) as well as the trivial solution both satisfy the equation, this is not a trivial matter. Unfortunately, none of the standard norms in H^s or L^p will work. A Lipschitz metric was derived by Bressan et al. [13], and we here offer an alternative metric that also provides a simpler and more efficient way to solve the initial value problem.

Let us be now more precise about the notion of solution. We consider the Cauchy problem for the integrated and augmented HS equation, which, in the conservative case, is given by

$$u_t + uu_x = \frac{1}{4}\int_{-\infty}^x d\mu - \frac{1}{4}\int_x^\infty d\mu, \tag{1.4a}$$

$$\mu_t + (u\mu)_x = 0. \tag{1.4b}$$

In order to study conservative solution, the HS Eq. (1.4a) is augmented by the second Eq. (1.4b) that keeps track of the energy. A short computation reveals that if the solution u is smooth and $\mu = u_x^2$, then Eq. (1.4b) is clearly satisfied. In particular, it shows that the energy $\mu(t, \mathbb{R}) = \mu(0, \mathbb{R})$ is constant in time. However, the challenge is to treat the case without this regularity, and the proper way to do that is to let μ be a non-negative and finite Radon measure. When there is a blow-up in the spatial derivative of the solution (cf. (1.3)), energy is transferred from the absolutely continuous part of the measure to the singular part, and, after the blow-up, the energy is transferred back to the absolutely continuous part of the measure. Thus, we will consider the solution space consisting of all pairs (u, μ) such that

$$u(t, \cdot) \in L^\infty(\mathbb{R}), \quad u_x(t, \cdot) \in L^2(\mathbb{R}), \quad \mu(t, \cdot) \in \mathcal{M}_+(\mathbb{R}) \quad \text{and} \quad d\mu_{ac} = u_x^2 dx,$$

where $\mathcal{M}_+(\mathbb{R})$ denotes the set of all non-negative and finite Radon measures on $\mathbb{R}$.

We would like to identify a natural Lipschitz metric, which measures the distance between pairs (u_i, μ_i), $i = 1, 2$, of solutions. The Lipschitz metric constructed by Bressan et al. [13] (and extended to the two-component HS equation by Nordli [14,15]) is based on the reformulation of the HS equation in Lagrangian coordinates which at the same time linearizes the equation. However, there is an intrinsic non-uniqueness in Lagrangian coordinates as there are several distinct ways to parametrize the particle trajectories for one and the same solution in the original, or Eulerian, coordinates. This has to be accounted for when one measures the distance between solutions in Lagrangian coordinates, as one has to identify different elements belonging to one and the same equivalence class. We denote this as relabeling. In addition, for this construction one not only needs to know the solution in Eulerian coordinates but also in Lagrangian coordinates for all t.

The present approach is based on the fact that a natural metric for measuring distances between Radon measures (with the same total mass) is given through the Wasserstein (or Monge–Kantorovich) distance d_W, which in one dimension is defined with the help of pseudo inverses, see Villani [16]. This tool has been used extensively in the field of kinetic equations [17,18], conservation laws [19,20] and non-linear diffusion equations [21–23]. To be more precise, given two positive and finite Radon measures μ_1 and μ_2, where we for simplicity assume that $\mu_1(\mathbb{R}) = \mu_2(\mathbb{R}) = C$, let

$$F_i(x) = \mu_i((-\infty, x)) \quad i = 1, 2, \tag{1.5}$$

and define their pseudo inverses $\chi_i : [0, C] \to \mathbb{R}$ as follows

$$\chi_i(\xi) = \sup\{x \mid F_i(x) < \xi\}.$$

Then, we define

$$d_W(\mu_1, \mu_2) = \|\chi_1 - \chi_2\|_{L^1([0,C])}.$$

As far as the distance between u_1 and u_2 is concerned, we are only interested in measuring the "distance in the L^∞ norm". Thus we introduce the distance d as follows:

$$d((u_1, \mu_1), (u_2, \mu_2)) = \|u_1(\chi_1(\cdot)) - u_2(\chi_2(\cdot))\|_{L^\infty([0,C])} + d_W(\mu_1, \mu_2).$$

For this to work, it is necessary that this metric behaves nicely with the time evolution. Thus as a first step, we are interested in determining the time evolution of both $\chi(t, x)$, the pseudo inverse of $\mu(t, x)$, and $u(t, \chi(t, x))$.

Let $(u(t), \mu(t))$ be a weak conservative solution to the HS equation with total energy $\mu(t, \mathbb{R}) = C$. To begin with, we assume that $F(t, x)$ is strictly increasing and smooth, which greatly simplifies the analysis. Recall that $\chi(t, \cdot) : [0, C] \to \mathbb{R}$ is given by

$$\chi(t, \eta) = \sup\{x \mid \mu(t, (-\infty, x)) < \eta\} = \sup\{x \mid F(t, x) < \eta\}.$$

According to the assumptions on $F(t, x)$, we have that $F(t, \chi(t, \eta)) = \eta$ for all $\eta \in [0, C]$ and $\chi(t, F(t, x)) = x$ for all $x \in \mathbb{R}$. Direct formal calculations yield that

$$\chi_t(t, F(t, x)) + \chi_\eta(t, F(t, x))F_t(t, x) = 0, \tag{1.6a}$$

$$\chi_\eta(t, F(t, x))F_x(t, x) = 1. \tag{1.6b}$$

Recalling (1.4b) and the definition of $F(t, x)$, we have

$$F_x(t, x) = \mu(t, x), \tag{1.7a}$$

$$F_t(t, x) = \int_{-\infty}^x d\mu_t(t) = -\int_{-\infty}^x d(u(t)\mu(t))_x = -u(t, x)\mu(t, x). \tag{1.7b}$$

Thus combining (1.6) and (1.7), we obtain

$$\chi_t(t, F(t, x)) = -\chi_\eta(t, F(t, x))F_t(t, x) = \chi_\eta(t, F(t, x))u(t, x)\mu(t, x)$$
$$= \chi_\eta(t, F(t, x))F_x(t, x)u(t, x) = u(t, x).$$

Introducing $\eta = F(t, x)$, we end up with

$$\chi_t(t, \eta) = u(t, \chi(t, \eta)),$$

where we again have used that $\chi(t, F(t, x)) = x$ for all $x \in \mathbb{R}$. As far as the time evolution of $\mathcal{U}(t, \eta) = u(t, \chi(t, \eta))$ is concerned, we have

$$\begin{aligned}
\mathcal{U}_t(t, \eta) &= u_t(t, \chi(t, \eta)) + u_x(t, \chi(t, \eta))\chi_t(t, \eta) \\
&= u_t(t, \chi(t, \eta)) + uu_x(t, \chi(t, \eta)) \\
&= \frac{1}{4}\int_{-\infty}^{\chi(t,\eta)} d\mu(t, \sigma) - \frac{1}{4}\int_{\chi(t,\eta)}^\infty d\mu(t, \sigma) \\
&= \frac{1}{2}\int_{-\infty}^{\chi(t,\eta)} d\mu(t, \sigma) - \frac{1}{4}C \\
&= \frac{1}{2}F(t, \chi(t, \eta)) - \frac{1}{4}C \\
&= \frac{1}{2}\eta - \frac{1}{4}C.
\end{aligned}$$

Thus we get the very simple system of ordinary differential equations

$$\chi_t(t, \eta) = \mathcal{U}(t, \eta), \tag{1.8a}$$

$$\mathcal{U}_t(t, \eta) = \frac{1}{2}\eta - \frac{1}{4}C. \tag{1.8b}$$

The global solution of the initial value problem is simply given by

$$\{(\chi(t, \eta), t, \mathcal{U}(t, \eta)) \in \mathbb{R}^3 \mid t \in (0, \infty), \eta \in [0, C]\}. \tag{1.9}$$

The above derivation is only of formal character, and this derivation is but valid if $F(t, x)$ is strictly increasing and smooth. However, it turns out that the simple result (1.8) also persists in the general case, but the proof is considerably more difficult, and is the main result of this article.

We prove two results. The first result, Theorem 2.9, describes a simple and explicit formula for conservative solutions of the Cauchy problem. Let $u_0 \in H^1(\mathbb{R})$ and μ_0 be a non-negative, finite Radon measure with $C = \mu_0(\mathbb{R})$. Define

$$\chi_0(\eta) = \sup\{x \mid \mu_0((-\infty, x)) < \eta\}, \tag{1.10a}$$

$$\mathcal{U}_0(\eta) = u_0(\chi_0(\eta)). \tag{1.10b}$$

If $\lim_{\eta \to 0} \chi_0(\eta) = -\lim_{\eta \to C} \chi_0(\eta) = -\infty$ (all other cases are treated in Theorem 2.9), we define

$$\chi(t, \eta) = \frac{t^2}{4}\left(\eta - \frac{C}{2}\right) + t\mathcal{U}_0(\eta) + \chi_0(\eta), \quad \text{if } \eta \in (0, C), \tag{1.11a}$$

$$\mathcal{U}(t, \xi) = \frac{t}{2}\left(\eta - \frac{C}{2}\right) + \mathcal{U}_0(\eta), \quad \text{if } \eta \in (0, C). \tag{1.11b}$$

Then we have

$$\begin{aligned}
&\{(x, t, u(t, x)) \in \mathbb{R}^3 \mid t \in [0, \infty), x \in \mathbb{R}\} \\
&= \{(\chi(t, \eta), t, \mathcal{U}(t, \eta)) \in \mathbb{R}^3 \mid t \in [0, \infty), \eta \in (0, C)\},
\end{aligned}$$

where $u = u(t, x)$ denotes the conservative solution of the HS Eq. (1.4).

The second result, Theorem 2.11, describes the Lipschitz metric. Let $u_{0,j} \in H^1(\mathbb{R})$ and $\mu_{0,j}$ be a non-negative, finite Radon measure with $C_j = \mu_{0,j}(\mathbb{R})$ for $j = 1, 2$, and define $\chi_j(t, \eta)$ and $\mathcal{U}_j(t, \eta)$ by (1.10) and (1.11) for $j = 1, 2$ where u_0 is replaced by $u_{0,j}$ and μ_0 is replaced by $\mu_{0,j}$, respectively. Next introduce $\hat{\chi}_j(t, \eta) = \chi_j(t, C_j\eta)$ and $\hat{\mathcal{U}}_j(t, \eta) = \mathcal{U}_j(t, C_j\eta)$ for $j = 1, 2$. Define

$$\begin{aligned}
&d\big((u_1(t), \mu_1(t)), (u_2(t), \mu_2(t))\big) \\
&\quad = \|\hat{\mathcal{U}}_1(t, \cdot) - \hat{\mathcal{U}}_2(t, \cdot)\|_{L^\infty([0,1])} \\
&\quad\quad + \|\hat{\chi}_1(t, \cdot) - \hat{\chi}_2(t, \cdot)\|_{L^1([0,1])} + |C_1 - C_2|.
\end{aligned}$$

Then we have, see Theorem 2.11, that

$$\begin{aligned}
&d\big((u_1(t), \mu_1(t)), (u_2(t), \mu_2(t))\big) \\
&\quad \leq \left(1 + t + \frac{1}{8}t^2\right) d\big((u_1(0), \mu_1(0)), (u_2(0), \mu_2(0))\big).
\end{aligned}$$

2. The Lipschitz metric for the Hunter–Saxton equation

Let us study the calculations (1.5)–(1.9) on two explicit examples.

Example 2.1. (i) Let

$$u_0(x) = \left(\frac{\pi}{2}\right)^{1/2} \text{erf}\left(\frac{x}{\sqrt{2}}\right), \quad \mu_0(x) = u_{0,x}^2(x)dx = e^{-x^2}dx,$$

where $\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}dt$ is the error function. We find that
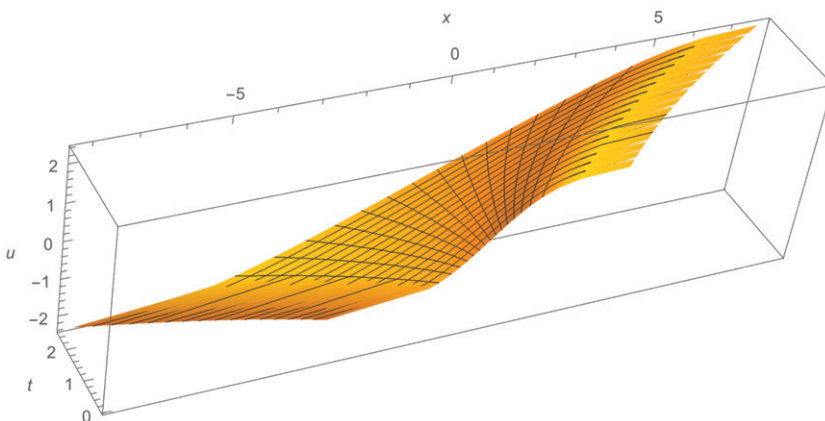
Figure 1. The surface $\{(\chi, t, \mathcal{U}) \mid t \in [0, 2.5], \eta \in (0, \sqrt{\pi})\}$ discussed in Example 2.1 (i).

$$F_0(x) = \mu_0((-\infty, x)) = \frac{\sqrt{\pi}}{2}(1 + \text{erf}(x))$$

as well as $C = F_0(\infty) = \sqrt{\pi}$. This implies that

$$\chi_0(\eta) = \text{erf}^{-1}\left(\frac{2}{\sqrt{\pi}}\eta - 1\right), \quad \eta \in (0, \sqrt{\pi}),$$

$$\mathcal{U}_0(\eta) = \left(\frac{\pi}{2}\right)^{1/2}\text{erf}\left(\frac{1}{\sqrt{2}}\text{erf}^{-1}\left(\frac{2}{\sqrt{\pi}}\eta - 1\right)\right), \quad \eta \in (0, \sqrt{\pi}).$$

Considering the system of ordinary differential Equation (1.8) with initial data $(\chi, \mathcal{U})|_{t=0} = (\chi_0, \mathcal{U}_0)$, we find

$$\chi(t, \eta) = \frac{t^2}{4}\left(\eta - \frac{1}{2}C\right) + \mathcal{U}_0(\eta)t + \chi_0(\eta),$$

$$\mathcal{U}(t, \eta) = \frac{t}{2}\left(\eta - \frac{1}{2}C\right) + \mathcal{U}_0(\eta).$$

See Figure 1. Observe that it is not easy to transform this solution explicitly back to the original variable u.

(ii) Let

$$u_0(x) = \text{arcsinh}(x), \quad \mu_0(x) = u_{0,x}^2(x)\text{d}x = \frac{\text{d}x}{1 + x^2}.$$

Note that u_0 is not bounded, yet the same transformations apply. We find that

$$F_0(x) = \mu_0((-\infty, x)) = \arctan(x) + \frac{\pi}{2}$$

as well as $C = F_0(\infty) = \pi$. This implies that

$$\chi_0(\eta) = \tan\left(\eta - \frac{\pi}{2}\right) \quad \text{and} \quad \mathcal{U}_0(\eta) = \text{arcsinh}\left(\tan\left(\eta - \frac{\pi}{2}\right)\right) \quad \text{for} \quad \eta \in (0, \pi).$$
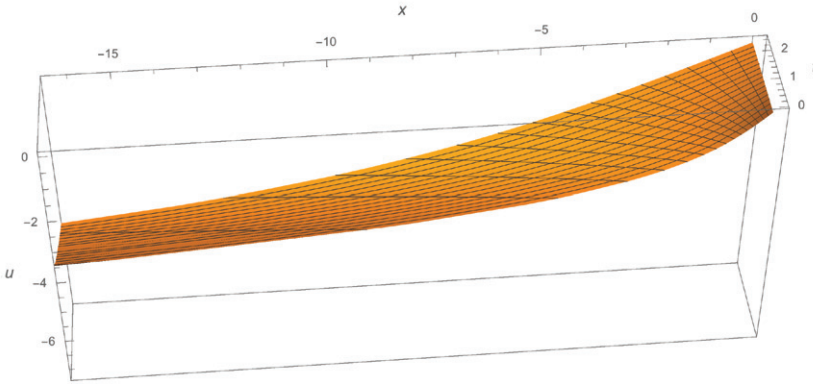
Figure 2. The surface $\{(\chi, t, \mathcal{U}) \mid t \in [0, 2.5], \eta \in (0, \pi)\}$ discussed in Example 2.1 (ii).

Here we find

$$\chi(t, \eta) = \frac{t^2}{4}\left(\eta - \frac{1}{2}C\right) + \mathcal{U}_0(\eta)t + \chi_0(\eta),$$

$$\mathcal{U}(t, \eta) = \frac{t}{2}\left(\eta - \frac{1}{2}C\right) + \mathcal{U}_0(\eta).$$

See Figure 2. Again it is not easy to transform this solution explicitly back to the original variable u.

Let us next consider an example where the initial measure is a pure point measure.

Example 2.2. This simple singular example shows the interplay between measures μ and their pseudo inverses $\chi(x)$ better.[1] Consider the example $u_0 = 0$ and $\mu_0 = \alpha\delta_0$, where δ_0 is the Dirac delta function at the origin, and $\alpha \geq 0$. Then $F_0 : \mathbb{R} \to [0, \alpha]$ reads

$$F_0(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ \alpha, & \text{if } x > 0. \end{cases}$$

The corresponding pseudo inverse $\chi_0 : [0, \alpha] \to \mathbb{R}$ is then given by

$$\chi_0(\eta) = \begin{cases} -\infty, & \text{if } \eta = 0, \\ 0, & \text{if } \eta \in (0, \alpha]. \end{cases}$$

Thus[2]

$$\chi_0(F_0(x)) = \begin{cases} -\infty, & \text{if } x \leq 0, \\ 0, & \text{if } x > 0, \end{cases} \quad \text{and} \quad F_0(\chi_0(\eta)) = 0 \quad \text{for all } \eta \in [0, \alpha].$$

In general, one observes that jumps in $F_0(x)$ are mapped to intervals where $\chi_0(\eta)$ is constant and vice versa. This means in particular that intervals where $F_0(x)$ is constant shrink to single points. Moreover, if $F_0(x)$ is constant on some interval, then $u_0(x)$ is also constant on the same interval.

---

[1]The solution in (1.2) comes from this example.
[2]Note that in the smooth case both $\chi(F)$ and $F(\chi)$ are the identity function!

Next, we compute the time evolution of both $\chi(t, \eta)$ and $\mathcal{U}(t, \eta) = u(t, \chi(t, \eta))$. Following the approach by Bressan et al. [13], we obtain that the corresponding solution in Eulerian coordinates reads for t positive

$$u(t, x) = \begin{cases} -\dfrac{\alpha}{4} t, & \text{if } x \leq -\dfrac{\alpha}{8} t^2, \\ \dfrac{2x}{t}, & \text{if } -\dfrac{\alpha}{8} t^2 \leq x \leq \dfrac{\alpha}{8} t^2, \\ \dfrac{\alpha}{4} t, & \text{if } x \geq \dfrac{\alpha}{8} t^2, \end{cases} \tag{2.1a}$$

$$\mu(t, x) = u_x^2(t, x) dx = \frac{4}{t^2} \mathbb{I}_{[-\alpha t^2/8, \alpha t^2/8]}(x) dx, \tag{2.1b}$$

$$F(t, x) = \begin{cases} 0, & \text{if } x \leq -\dfrac{\alpha}{8} t^2, \\ \dfrac{4x}{t^2} + \dfrac{\alpha}{2}, & \text{if } -\dfrac{\alpha}{8} t^2 \leq x \leq \dfrac{\alpha}{8} t^2, \\ \alpha, & \text{if } x \geq \dfrac{\alpha}{8} t^2. \end{cases} \tag{2.1c}$$

Calculating the pseudo inverse $\chi(t, \eta)$ and $\mathcal{U}(t, \eta) = u(t, \chi(t, \eta))$ for each t then yields

$$\chi(t, \eta) = \frac{t^2}{4} \left(\eta - \frac{\alpha}{2} \right), \quad \eta \in (0, \alpha], \tag{2.2a}$$

$$\mathcal{U}(t, \eta) = \frac{t}{2} \left(\eta - \frac{\alpha}{2} \right), \quad \eta \in [0, \alpha], \tag{2.2b}$$

and, in particular, that

$$\mathcal{U}_t(t, \eta) = \frac{1}{2} \left(\eta - \frac{\alpha}{2} \right), \quad \eta \in [0, \alpha].$$

Thus we still obtain the same ordinary differential Equation (1.8) as in the smooth case! In addition, note that $\chi_t(0, \eta) = 0$ for all $\eta \in (0, \alpha]$, and hence the important information is encoded in $\mathcal{U}_t(t, \eta)$.

We can of course also solve $\chi_t = \mathcal{U}$ and $\mathcal{U}_t = \frac{\eta}{2} - \frac{\alpha}{4}$ directly with initial data $\chi_0 = \mathcal{U}_0 = 0$, which again yields (2.2). To return to the Eulerian variables u and μ we have in the smooth region that

$$u(t, x) = \mathcal{U}(t, \eta), \quad x = \chi(t, \eta), \quad \eta \in (0, \alpha],$$

and we need to extend $\mathcal{U}$ and χ to all of $\mathbb{R}$ by continuity:

$$\chi(t, \eta) = \begin{cases} -\dfrac{\alpha}{8} t^2 + \eta, & \text{if } \eta \leq 0, \\ \dfrac{t^2}{4} \left(\eta - \dfrac{\alpha}{2} \right), & \text{if } \eta \in [0, \alpha], \quad t \geq 0, \\ \dfrac{\alpha}{8} t^2 + \eta - \alpha, & \text{if } \eta \geq \alpha, \end{cases}$$

$$\mathcal{U}(t,\eta) = \begin{cases} -\dfrac{\alpha}{4}t = \mathcal{U}(t,0+), & \text{if } \eta \leq 0, \\[2mm] \dfrac{t}{2}\left(\eta - \dfrac{\alpha}{2}\right), & \text{if } \eta \in [0,\alpha], \quad t \geq 0. \\[2mm] \dfrac{\alpha}{4}t = \mathcal{U}(t,\alpha-), & \text{if } \eta \geq \alpha, \end{cases}$$

Returning to the Eulerian variables we recover (2.1). We can also depict the full solution in the (x, t) plane in the new variables: The full solution reads

$$\left\{ (\chi(t,\eta), t, \mathcal{U}(t,\eta)) \in \mathbb{R}^3 \mid t \in (0,\infty), \eta \in \mathbb{R} \right\}.$$

See Figure 3.

The next example shows the difficulties that one has to face in the general case where the solution encounters a break down in the sense of steep gradients.

Example 2.3. Let

$$u_0(x) = -x\mathbb{I}_{[0,1]}(x) - \mathbb{I}_{[1,\infty)}(x), \quad \mu_0(x) = u_{0,x}^2(x)dx = \mathbb{I}_{[0,1]}(x)dx.$$

Next, we find

$$F_0(x) = x\mathbb{I}_{[0,1]}(x) + \mathbb{I}_{[1,\infty)}(x),$$
$$\chi_0(\eta) = \begin{cases} -\infty, & \text{if } \eta = 0, \\ \eta, & \text{if } \eta \in (0,1], \end{cases}$$
$$\mathcal{U}_0(\eta) = -\eta, \quad \eta \in [0,1].$$

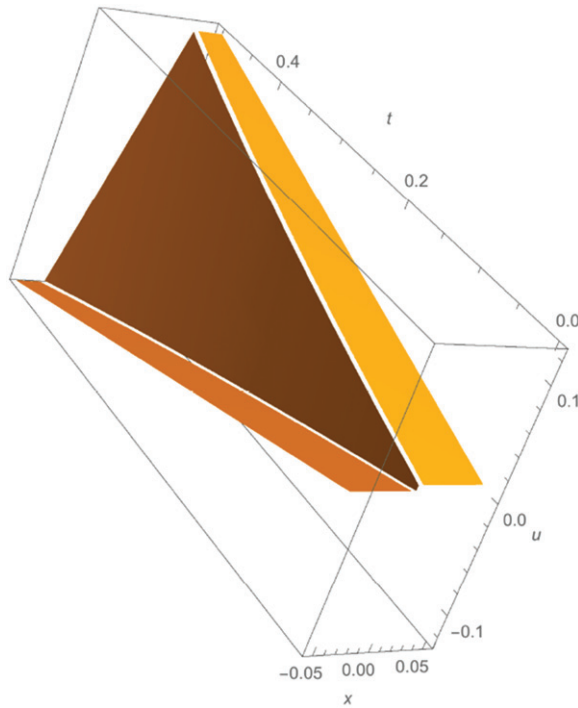**Figure 3.** The surface $\{(x,t,u) \mid t \in [0,.5], x \in [-.05,.05]\}$ discussed in Example 2.2.

Assuming (1.8) holds also in this case, we find

$$\chi(t, \eta) = \frac{t^2}{4}\left(\eta - \frac{1}{2}\right) - t\eta + \eta, \quad \eta \in (0, 1],$$

$$\mathcal{U}(t, \eta) = \frac{t}{2}\left(\eta - \frac{1}{2}\right) - \eta, \quad \eta \in [0, 1].$$

We extend the functions by continuity

$$\chi(t, \eta) = \begin{cases} -\dfrac{t^2}{8} + \eta, & \text{if } \eta \le 0, \\ \dfrac{t^2}{4}\left(\eta - \dfrac{1}{2}\right) - t\eta + \eta, & \text{if } \eta \in [0, 1], \\ \dfrac{t^2}{8} - t + \eta, & \text{if } \eta \ge 1, \end{cases}$$

$$\mathcal{U}(t, \eta) = \begin{cases} -\dfrac{t}{4}, & \text{if } \eta \le 0, \\ \dfrac{t}{2}\left(\eta - \dfrac{1}{2}\right) - \eta, & \text{if } \eta \in [0, 1], \\ \dfrac{t}{4} - 1, & \text{if } \eta \ge 1, \end{cases}$$

which gives a well-defined global solution given by $\{(\chi(t, \eta), t, \mathcal{U}(t, \eta)) | t \ge 0, \eta \in \mathbb{R}\}$. However, as we return to Eulerian variables, the time-development is more dramatic. Solving the equation $x = \chi(t, \eta)$ *for $\eta \in [0, 1]$* yields

$$\eta = \frac{4x + t^2/2}{(t-2)^2} \in [0, 1],$$

which leads to the solution

$$u(t, x) = \mathcal{U}\left(t, \frac{4x + t^2/2}{(t-2)^2}\right) = \frac{2x + t/2}{t - 2}$$

whenever

$$-\frac{t^2}{8} < x < \frac{1}{4}\left((t-2)^2 - \frac{t^2}{2}\right).$$

For $t \to 2-$, we have that $u_x \to -\infty$ at $x = -1/2$. The solution on the full line reads

$$u(t, x) = \begin{cases} -\dfrac{1}{4}t, & \text{if } x \le -\dfrac{1}{8}t^2, \\ \dfrac{2x + t/2}{t - 2}, & \text{if } -\dfrac{1}{8}t^2 \le x \le \dfrac{1}{4}\left((t-2)^2 - \dfrac{t^2}{2}\right), \\ \dfrac{1}{4}t - 1, & \text{if } x \ge \dfrac{1}{4}\left((t-2)^2 - \dfrac{t^2}{2}\right). \end{cases}$$

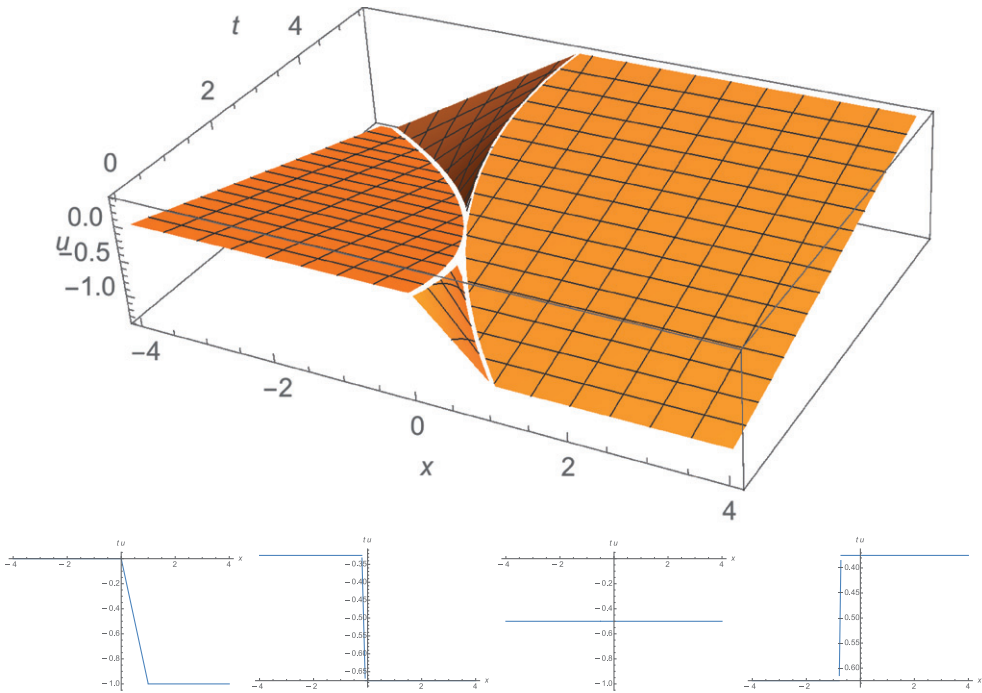The solution is illustrated in Figure 4.

Figure 4. The solution discussed in Example 2.3.

The above examples already hint that the interplay between Eulerian and Lagrangian coordinates is going to play a major role in our further considerations. We assume a smooth solution of

$$u_t + u u_x = \frac{1}{4}\left(\int_{-\infty}^x u_x^2(y)\,\mathrm{d}y - \int_x^\infty u_x^2(y)\,\mathrm{d}y\right), \tag{2.3a}$$

$$\left(u_x^2\right)_t + \left(u u_x^2\right)_x = 0. \tag{2.3b}$$

Next, we rewrite the equation in Lagrangian coordinates. Introduce the characteristics

$$y_t(t,\xi) = u\big(t, y(t,\xi)\big).$$

The Lagrangian velocity U reads

$$U(t,\xi) = u\big(t, y(t,\xi)\big).$$

Furthermore, we define the Lagrangian cumulative energy by

$$H(t,\xi) = \int_{-\infty}^{y(t,\xi)} u_x^2(t,x)\,dx.$$

From (2.3a), we get that

$$U_t = u_t \circ y + y_t u_x \circ y = \frac{1}{4}\left(\int_{-\infty}^y u_x^2\,dx - \int_y^\infty u_x^2\,dx\right) = \frac{1}{2}H - \frac{1}{4}C$$

where $C = H(t,\infty)$ is time independent, and

$$H_t = \int_{-\infty}^{y(t,\xi)} (u_x^2(t,x))_t \, dx + y_t u_x^2(t,y) = \int_{-\infty}^{y(t,\xi)} ((u_x^2)_t + (u u_x^2)_x)(t,x) \, dx = 0$$

by (2.3b). In this formal computation, we require that u and u_x are smooth and decay rapidly at infinity. Hence, the HS equation formally is equivalent to the following system of ordinary differential equations:

$$y_t = U, \tag{2.4a}$$

$$U_t = \frac{1}{2}H - \frac{1}{4}C, \tag{2.4b}$$

$$H_t = 0. \tag{2.4c}$$

Global existence of solutions to (2.4) follows from the linear nature of the system. There is no exchange of energy across the characteristics and the system (2.4) can be solved explicitly. This is in contrast to the Camassa–Holm equation where energy is exchanged across characteristics. We have

$$y(t,\xi) = \left(\frac{1}{4}H(0,\xi) - \frac{1}{8}C\right)t^2 + U(0,\xi)t + y(0,\xi),$$

$$U(t,\xi) = \left(\frac{1}{2}H(0,\xi) - \frac{1}{4}C\right)t + U(0,\xi),$$

$$H(t,\xi) = H(0,\xi).$$

We next focus on the general case without assuming regularity of the solution. It turns out that in addition to the variable u we will need a measure μ that in smooth regions coincides with the energy density $u_x^2 dx$. At wave breaking, the energy at the point where the wave breaking takes place, is transformed into a point measure. It is this dynamics that is encoded in the measure μ that allows us to treat general initial data. An important complication stems from the fact that the original solution in two variables (u, μ) is transformed into Lagrangian coordinates with three variables (y, U, H). This is a well-known consequence of the fact that one can parametrize a particle path in several different ways, corresponding to the same motion. This poses technical complications when we want to measure the distance between two distinct solutions in Lagrangian coordinates that correspond to the same Eulerian solution, and we denote this as relabeling of the solution.

We will employ the notation and the results by Bressan et al. [13] and Nordli [14]. Define the Banach spaces

$$E_1 = \left\{ f \in L^\infty(\mathbb{R}) \,|\, f' \in L^2(\mathbb{R}), \lim_{\xi \to -\infty} f(\xi) = 0 \right\},$$

$$E_2 = \left\{ f \in L^\infty(\mathbb{R}) \,|\, f' \in L^2(\mathbb{R}) \right\},$$

with norms

$$\|f\|_{E_j} = \|f\|_{L^\infty} + \|f'\|_{L^2}, \quad f \in E_j, j = 1, 2.$$

Let

$$B = E_2 \times E_2 \times E_1,$$

with norm

$$\|(f_1, f_2, f_3)\|_B = \|f_1\|_{E_2} + \|f_2\|_{E_2} + \|f_3\|_{E_1}, \quad (f_1, f_2, f_3) \in B.$$

We are given some initial data $(u_0, \mu_0) \in \mathcal{D}$, where the set $\mathcal{D}$ is defined as follows.

Definition 2.4. The set $\mathcal{D}$ consists of all pairs (u, μ) such that

- (i) $u \in E_2$;
- (ii) μ is a non-negative and finite Radon measure such that $\mu_{ac} = u_x^2 dx$, where μ_{ac} denotes the absolute continuous part of μ with respect to the Lebesgue measure.

The Lagrangian variables are given by (ζ, U, H) (with $\zeta = y - \mathrm{Id}$), and the appropriate space is defined as follows.

Definition 2.5. The set $\mathcal{F}$ consists of the elements $(\zeta, U, H) \in B = E_2 \times E_2 \times E_1$ such that

- (i) $(\zeta, U, H) \in (W^{1,\infty}(\mathbb{R}))^3$, where $\zeta(\xi) = y(\xi) - \xi$;
- (ii) $y_\xi \geq 0$, $H_\xi \geq 0$ and $y_\xi + H_\xi \geq c$, almost everywhere, where c is a strictly positive constant;
- (iii) $y_\xi H_\xi = U_\xi^2$ almost everywhere.

The key subspace $\mathcal{F}_0 \subset \mathcal{F}$ is defined by

$$\mathcal{F}_0 = \big\{ X = (y, U, H) \in \mathcal{F} \,|\, y + H = \mathrm{Id} \big\}.$$

We need to clarify the relation between the Eulerian variables (u, μ) and the Lagrangian variables (ζ, U, H). The transformation

$$L : \mathcal{D} \to \mathcal{F}_0, \quad X = L(u, \mu)$$

is defined as follows.

Definition 2.6. For any (u, μ) in $\mathcal{D}$, let

$$y(\xi) = \sup\big\{ x \,|\, \mu((-\infty, x)) + x < \xi \big\}, \tag{2.5a}$$

$$H(\xi) = \xi - y(\xi), \tag{2.5b}$$

$$U(\xi) = u \circ y(\xi). \tag{2.5c}$$

Then $X = (\zeta, U, H) \in \mathcal{F}_0$ and we denote by $L : \mathcal{D} \to \mathcal{F}_0$ the map which to any $(u, \mu) \in \mathcal{D}$ associates $(\zeta, U, H) \in \mathcal{F}_0$ as given by (2.5).

From the Lagrangian variables, we can return to Eulerian variables using the following transformation.

Definition 2.7. Given any element X in $\mathcal{F}$. Then, the pair (u, μ) defined as follows:

$$u(x) = U(\xi) \text{ for any } \xi \text{ such that } x = y(\xi), \tag{2.6a}$$

$$\mu = y_\#(H_\xi \, d\xi) \tag{2.6b}$$

belongs to $\mathcal{D}$. Here, the push-forward of a measure ν by a measurable function f is the measure $f_{\#}\nu$ defined by $f_{\#}\nu(B) = \nu(f^{-1}(B))$ for all Borel sets B. We denote by $M : \mathcal{F} \to \mathcal{D}$ the map which to any X in $\mathcal{F}$ associates (u, μ) as given by (2.6).

The key properties of these transformations are (cf. [13, Prop. 2.11])

$$L \circ M|_{\mathcal{F}_0} = \mathrm{Id}_{\mathcal{F}_0}, \quad M \circ L = \mathrm{Id}_{\mathcal{D}}. \tag{2.7}$$

The formalism up to this point has been stationary, transforming back and forth between Eulerian and Lagrangian variables. Next, we can take into consideration the time-evolution of the solution of the HS equation.

The evolution of the HS equation in Lagrangian variables is determined by the system (cf. (2.4))

$$S_t : \mathcal{F} \to \mathcal{F}, \quad X(t) = S_t(X_0), \quad X_t = S(X), \quad X|_{t=0} = X_0. \tag{2.8}$$

of ordinary differential equations. Here

$$S(X) = \begin{pmatrix} U \\ \frac{1}{2}H - \frac{1}{4}C \\ 0 \end{pmatrix},$$

Next, we address the question about relabeling. We need to identify Lagrangian solutions that correspond to one and the same solution in Eulerian coordinates. Let G be the subgroup of the group of homeomorphisms on $\mathbb{R}$ such that

$$f - \mathrm{Id} \text{ and } f^{-1} - \mathrm{Id} \text{ both belong to } W^{1,\infty}(\mathbb{R}),$$
$$f_{\xi} - 1 \text{ belongs to } L^2(\mathbb{R}).$$

By default, the HS equation is invariant under relabeling, which is given by equivalence classes

$$[X] = \{\tilde{X} \in \mathcal{F} \mid \text{ there exists } g \in G \text{ such that } X = \tilde{X} \circ g\},$$
$$\mathcal{F}/G = \{[X] \mid X \in \mathcal{F}\}.$$

The key subspace of $\mathcal{F}$ is denoted $\mathcal{F}_0$ and is defined by

$$\mathcal{F}_0 = \{X = (y, U, H) \in \mathcal{F} \mid y + H = \mathrm{Id}\}.$$

The map into the critical space $\mathcal{F}_0$ is taken care of by (cf. [13, Def. 2.9])

$$\Pi : \mathcal{F} \to \mathcal{F}_0, \quad \Pi(X) = X \circ (y + H)^{-1},$$

with the property that $\Pi(\mathcal{F}) = \mathcal{F}_0$. We note that the map $X \mapsto [X]$ from $\mathcal{F}_0$ to $\mathcal{F}/G$ is a bijection. Then we have that (cf. [13, Prop. 2.12])

$$\Pi \circ S_t \circ \Pi = \Pi \circ S_t,$$

and hence we can define the semigroup

$$\tilde{S}_t = \Pi \circ S_t : \mathcal{F}_0 \to \mathcal{F}_0. \tag{2.9}$$

We can now provide the solution of the HS equation. Consider initial data $(u_0, \mu_0) \in \mathcal{D}$, and define $\bar{X}_0 = (\bar{y}_0, \bar{U}_0, \bar{H}_0) = L(u_0, \mu_0) \in \mathcal{F}_0$ given by

$$\bar{y}_0(\xi) = \sup\{x \mid x + F(0, x) < \xi\},$$
$$\bar{U}_0(\xi) = u_0(\bar{y}_0(\xi)),$$
$$\bar{H}_0(\xi) = \xi - \bar{y}_0(\xi),$$

with $F(0, x) = \mu_0((-\infty, x))$. Next, we want to determine the solution $(u(t), \mu(t)) \in \mathcal{D}$ (we suppress the dependence in the notation on the spatial variable x when convenient) for arbitrary time t.

Define

$$\bar{X}(t) = S_t \bar{X}_0 \in \mathcal{F}, \quad X(t) = \tilde{S}_t \bar{X}_0 \in \mathcal{F}_0. \tag{2.10}$$

The advantage of $\bar{X}(t)$ is that it obeys the differential Eq. (2.8), while $X(t)$ keeps the relation $y + H = \mathrm{Id}$ for all times. From (2.9) we have that

$$X(t) = \Pi(\bar{X}(t)).$$

We know that $\bar{X}(t, \xi) = (\bar{y}(t, \xi), \bar{U}(t, \xi), \bar{H}(t, \xi)) \in \mathcal{F}$ is the solution of

$$\bar{y}_t(t, \xi) = \bar{U}(t, \xi), \tag{2.11a}$$

$$\bar{U}_t(t, \xi) = \frac{1}{2}\bar{H}(t, \xi) - \frac{1}{4}C, \tag{2.11b}$$

$$\bar{H}_t(t, \xi) = 0, \tag{2.11c}$$

where $C = \mu_0(\mathbb{R})$ and $\bar{X}(0) = \bar{X}_0$. Straightforward integration yields

$$\bar{y}(t, \xi) = \frac{1}{4}\left(\bar{H}_0(\xi) - \frac{1}{2}C\right)t^2 + \bar{U}_0(\xi)t + \bar{y}_0(\xi),$$

$$\bar{U}(t, \xi) = \frac{1}{2}\left(\bar{H}_0(\xi) - \frac{1}{2}C\right)t + \bar{U}_0(\xi),$$

$$\bar{H}(t, \xi) = \bar{H}_0(\xi) = \xi - \bar{y}_0(\xi).$$

The solution $(u(t), \mu(t)) = M(\bar{X}(t))$ in Eulerian variables reads

$$u(t, x) = \bar{U}(t, \xi), \quad \bar{y}(t, \xi) = x,$$
$$\mu(t, x) = \bar{y}_{\#}(\bar{H}_{\xi}(t, \xi)d\xi),$$

with $F(t, x) = \mu(t, (-\infty, x)) = \int_{\bar{y}(t,\xi)<x}(1 - \bar{y}_{0,\xi}(\xi))d\xi$.

However, for $X(t, \xi) = (y(t, \xi), U(t, \xi), H(t, \xi)) \in \mathcal{F}_0$, which satisfies

$$X(t) = \bar{X}(t) \circ (\bar{y} + \bar{H})^{-1} \in \mathcal{F}_0,$$

we see, using (2.7), that

$$y(t, \xi) = \sup\{x \mid x + F(t, x) < \xi\}$$
$$= \sup\{x \mid x + F(t, x) < y(t, \xi) + H(t, \xi)\},$$

where we in the second equality use that $X(t) \in \mathcal{F}_0$. Note that we still have $(u(t), \mu(t)) = M(\bar{X}(t)) = M(X(t))$, and thus

$$u(t, x) = U(t, \xi), \quad y(t, \xi) = x,$$
$$\mu(t, x) = y_{\#}\big(H_\xi(t, \xi)d\xi\big),$$

with $F(t, x) = \mu(t, (-\infty, x)) = \int_{y(t,\xi) < x} H_\xi(t, \xi)d\xi$. Since $\bar{X}(t) = X(t) \circ (\bar{y} + \bar{H})$, we find that

$$\bar{y}(t, \xi) = y\big(t, \bar{y}(t, \xi) + \bar{H}(t, \xi)\big) = \sup\{x \mid x + F(t, x) < \bar{y}(t, \xi) + \bar{H}(t, \xi)\}. \tag{2.12}$$

This is the only place in this construction where we use the quantity $X(t)$.

Define now

$$\chi(t, \eta) = \sup\{x \mid \mu(t, (-\infty, x)) < \eta\} = \sup\{x \mid F(t, x) < \eta\}.$$

We claim that

$$\chi(t, \eta) = \bar{y}\big(t, l(t, \eta)\big),$$

where we have introduced $l(t, \cdot) : [0, C] \to \mathbb{R}$ by

$$l(t, \eta) = \sup\{\xi \mid \bar{H}(t, \xi) < \eta\}. \tag{2.13}$$

Note that since $\bar{H}_t = 0$, we have that

$$l(t, \eta) = l(0, \eta) \quad \text{and} \quad l_t(t, \eta) = 0.$$

Recall that for each time t, we have (cf. (2.12))

$$\bar{y}(t, \xi) = \sup\{x \mid x + F(t, x) < \bar{y}(t, \xi) + \bar{H}(t, \xi)\},$$

which implies that

$$\bar{y}(t, \xi) + F\big(t, \bar{y}(t, \xi)\big) \leq \bar{y}(t, \xi) + \bar{H}(t, \xi) \leq \bar{y}(t, \xi) + F\big(t, \bar{y}(t, \xi)+\big).$$

Subtracting $\bar{y}(t, \xi)$ in the above inequality, we end up with

$$F\big(t, \bar{y}(t, \xi)\big) \leq \bar{H}(t, \xi) \leq F\big(t, \bar{y}(t, \xi)+\big) \quad \text{for all } \xi \in \mathbb{R}.$$

Comparing the last equation and (2.13), we have

$$F\big(t, \bar{y}\big(t, l(t, \eta)\big)\big) \leq \bar{H}\big(t, l(t, \eta)\big) = \eta \leq F\big(t, \bar{y}\big(t, l(t, \eta)\big)+\big).$$

Since $\bar{y}(t, \cdot)$ is surjective and non-decreasing, we end up with

$$\chi(t, \eta) = \sup\{x \mid F(t, x) < \eta\} = \bar{y}\big(t, l(t, \eta)\big).$$

Introduce the new function

$$\mathcal{U}(t, \eta) = \bar{U}\big(t, l(t, \eta)\big).$$

We are now ready to derive the system of ordinary differential equations for $\chi(t, \eta)$ and $\mathcal{U}(t, \eta)$. Therefore recall that

$$\bar{H}\big(t, l(t, \eta)\big) = \bar{H}\big(0, l(t, \eta)\big) = \bar{H}\big(0, l(0, \eta)\big) = \eta \quad \text{for all } \eta \in [0, C],$$

since $\bar{H}(0, \xi)$ is continuous. Direct calculations yield

$$\chi_t(t,\eta) = \frac{d}{dt}\bar{y}(t,l(0,\eta)) = \bar{y}_t(t,l(0,\eta)) = \bar{U}(t,l(0,\eta)) = \bar{U}\left(t,l(t,\eta)\right)$$

$$= \mathcal{U}(t,\eta),$$

$$\mathcal{U}_t(t,\eta) = \frac{d}{dt}\bar{U}(t,l(0,\eta)) = \bar{U}_t(t,l(0,\eta)) = \frac{1}{2}\bar{H}(t,l(0,\eta)) - \frac{1}{4}C$$

$$= \frac{1}{2}\eta - \frac{1}{4}C.$$

Thus we established rigorously the linear system

$$\chi_t(t,\eta) = \mathcal{U}(t,\eta), \tag{2.14a}$$

$$\mathcal{U}_t(t,\eta) = \frac{1}{2}\eta - \frac{1}{4}C \tag{2.14b}$$

of ordinary differential equations, with solution

$$\chi(t,\eta) = \left(\frac{1}{4}\eta - \frac{1}{8}C\right)t^2 + \mathcal{U}(0,\eta)t + \chi(0,\eta),$$

$$\mathcal{U}(t,\eta) = \left(\frac{1}{2}\eta - \frac{1}{4}C\right)t + \mathcal{U}(0,\eta).$$

Example 2.8. Recall Example 2.2. Let us compute the corresponding quantities in the case with $u_0 = 0$ and $\mu_0 = \delta_0$. Here we find that

$$\bar{y}_0(\xi) = \begin{cases} \xi, & \text{for } \xi \leq 0, \\ 0, & \text{for } \xi \in (0,1), \\ \xi - 1, & \text{for } \xi \geq 1, \end{cases}$$

$$\bar{U}_0(\xi) = 0,$$

$$\bar{H}_0(\xi) = \begin{cases} 0, & \text{for } \xi \leq 0, \\ \xi, & \text{for } \xi \in (0,1), \\ 1, & \text{for } \xi \geq 1. \end{cases}$$

The solution of $\bar{X} = S_t\bar{X}_0$ (cf. (2.10)) reads

$$\bar{y}(t,\xi) = \begin{cases} \xi - \dfrac{t^2}{8}, & \text{for } \xi \leq 0, \\ \dfrac{t^2}{4}\xi - \dfrac{t^2}{8}, & \text{for } \xi \in (0,1), \\ \dfrac{t^2}{8} + \xi - 1, & \text{for } \xi \geq 1, \end{cases}$$

$$\bar{U}(t,\xi) = \begin{cases} -\dfrac{t}{4}, & \text{for } \xi \leq 0, \\ \dfrac{t}{2}\xi - \dfrac{t}{4}, & \text{for } \xi \in (0,1), \\ \dfrac{t}{4}, & \text{for } \xi \geq 1, \end{cases}$$

$$\bar{H}(t,\xi) = \begin{cases} 0, & \text{for } \xi \le 0, \\ \xi, & \text{for } \xi \in (0,1), \\ 1, & \text{for } \xi \ge 1. \end{cases}$$

If we compute the corresponding quantities $X(t) = \bar{X}(t) \circ (\bar{y} + \bar{H})^{-1}$, we find

$$y(t,\xi) = \begin{cases} \xi, & \text{for } \xi \le -\dfrac{t^2}{8}, \\ \dfrac{t^2}{t^2+4}\xi - \dfrac{t^2}{2t^2+8}, & \text{for } \xi \in \left(-\dfrac{t^2}{8}, \dfrac{t^2}{8}+1\right), \\ \xi - 1, & \text{for } \xi \ge \dfrac{t^2}{8}+1, \end{cases}$$

$$U(t,\xi) = \begin{cases} -\dfrac{t}{4}, & \text{for } \xi \le -\dfrac{t^2}{8}, \\ \dfrac{2t}{t^2+4}\left(\xi + \dfrac{t^2}{8}\right) - \dfrac{t}{4}, & \text{for } \xi \in \left(-\dfrac{t^2}{8}, \dfrac{t^2}{8}+1\right), \\ \dfrac{t}{4}, & \text{for } \xi \ge \dfrac{t^2}{8}+1, \end{cases}$$

$$H(t,\xi) = \begin{cases} 0, & \text{for } \xi \le -\dfrac{t^2}{8}, \\ \dfrac{4}{t^2+4}\left(\xi + \dfrac{t^2}{8}\right), & \text{for } \xi \in \left(-\dfrac{t^2}{8}, \dfrac{t^2}{8}+1\right), \\ 1, & \text{for } \xi \ge \dfrac{t^2}{8}+1, \end{cases}$$

with the property that $y + H = \mathrm{Id}$. Finally, we find

$$l(t,\eta) = \begin{cases} -\infty, & \text{for } \eta = 0, \\ \eta, & \text{for } \eta \in (0,1], \end{cases}$$

$$\chi(t,\eta) = \bar{y}\big(t, l(t,\eta)\big) = \frac{t^2}{4}\left(\eta - \frac{1}{2}\right), \quad \eta \in (0,1],$$

$$\mathcal{U}(t,\eta) = \bar{U}\big(t, l(t,\eta)\big) = \frac{t}{2}\left(\eta - \frac{1}{2}\right), \quad \eta \in (0,1],$$

as expected.

Theorem 2.9. *Let $u_0 \in E_2$ and μ_0 be a non-negative, finite Radon measure with $C = \mu_0(\mathbb{R})$. Let $(u(t), \mu(t))$ denote the conservative solution of the HS equation. Define*

$$\chi_0(\eta) = \sup\{x \mid \mu_0((-\infty, x)) < \eta\},$$
$$\mathcal{U}_0(\eta) = u_0(\chi_0(\eta)).$$

If $\lim_{\eta \to 0} \chi_0(\eta)$ and $\lim_{\eta \to C} \chi_0(\eta)$ are finite, we define

$$\chi(t, \eta) = \begin{cases} -\dfrac{C}{8}t^2 + t\,\mathcal{U}_0(0) + \chi_0(0) + \eta, & \text{if } \eta < 0, \\[2mm] \dfrac{t^2}{4}\left(\eta - \dfrac{C}{2}\right) + t\,\mathcal{U}_0(\eta) + \chi_0(\eta), & \text{if } \eta \in (0, C], \\[2mm] \dfrac{C}{8}t^2 + t\,\mathcal{U}_0(C) + \chi_0(C) + \eta - C, & \text{if } \eta > C, \end{cases}$$

$$\mathcal{U}(t, \eta) = \begin{cases} -\dfrac{C}{4}t + \mathcal{U}_0(0), & \text{if } \eta < 0, \\[2mm] \dfrac{t}{2}\left(\eta - \dfrac{C}{2}\right) + \mathcal{U}_0(\eta), & \text{if } \eta \in (0, C], \\[2mm] \dfrac{C}{4}t + \mathcal{U}_0(C), & \text{if } \eta \geq C. \end{cases}$$

Then we have

$$\big\{(x, t, u(t, x)) \in \mathbb{R}^3 \mid t \in [0, \infty), \ x \in \mathbb{R}\big\}$$
$$= \big\{(\chi(t, \eta), t, \mathcal{U}(t, \eta)) \in \mathbb{R}^3 \mid t \in [0, \infty), \ \eta \in \mathbb{R}\big\}.$$

If $\lim_{\eta \to 0} \chi_0(\eta) = -\lim_{\eta \to C} \chi_0(\eta) = -\infty$, we define

$$\chi(t, \eta) = \frac{t^2}{4}\left(\eta - \frac{C}{2}\right) + t\,\mathcal{U}_0(\eta) + \chi_0(\eta), \quad \text{if } \eta \in (0, C),$$

$$\mathcal{U}(t, \eta) = \frac{t}{2}\left(\eta - \frac{C}{2}\right) + \mathcal{U}_0(\eta), \quad \text{if } \eta \in (0, C).$$

Then we have

$$\big\{(x, t, u(t, x)) \in \mathbb{R}^3 \mid t \in [0, \infty), \ x \in \mathbb{R}\big\}$$
$$= \big\{(\chi(t, \eta), t, \mathcal{U}(t, \eta)) \in \mathbb{R}^3 \mid t \in [0, \infty), \ \eta \in (0, C)\big\}.$$

Similar results hold if one of $\lim_{\eta \to 0} \chi_0(\eta)$ and $\lim_{\eta \to C} \chi_0(\eta)$ is finite.

We can now introduce the new Lipschitz metric. Define

$$d\big((u_1(t), \mu_1(t)), (u_2(t), \mu_2(t))\big) = \|\mathcal{U}_1(t) - \mathcal{U}_2(t)\|_{L^\infty([0,C])} + \|\chi_1(t) - \chi_2(t)\|_{L^1([0,C])},$$

which implies that

$$d\big((u_1(t), \mu_1(t)), (u_2(t), \mu_2(t))\big)$$
$$= \|\mathcal{U}_1(t) - \mathcal{U}_2(t)\|_{L^\infty([0,C])} + \|\chi_1(t) - \chi_2(t)\|_{L^1([0,C])}$$
$$\leq (1 + Ct)\|\mathcal{U}_1(0) - \mathcal{U}_2(0)\|_{L^\infty([0,C])} + \|\chi_1(0) - \chi_2(0)\|_{L^1([0,C])}$$
$$\leq (1 + Ct)d\big((u_1(0), \mu_1(0)), (u_2(0), \mu_2(0))\big).$$

A drawback of the above construction is the fact that we are only able to compare solutions (u_1, μ_1) and (u_2, μ_2) with the same energy, viz. $\mu_1(\mathbb{R}) = \mu_2(\mathbb{R}) = C$. The rest of this section is therefore devoted to overcoming this limitation.

A closer look at the system (2.14) of ordinary differential equations reveals that we can rescale $\chi(t, \eta)$ and $\mathcal{U}(t, \eta)$ in the following way. Let

$$\hat{\chi}(t, \eta) = \chi(t, C\eta),$$
$$\hat{\mathcal{U}}(t, \eta) = \mathcal{U}(t, C\eta).$$

Then $\hat{\chi}(t, \cdot) : [0, 1] \to \mathbb{R}$, $\hat{\mathcal{U}}(t, \cdot) : [0, 1] \to \mathbb{R}$ for all t and

$$\hat{\chi}_t(t, \eta) = \hat{\mathcal{U}}(t, \eta),$$
$$\hat{\mathcal{U}}_t(t, \eta) = \frac{1}{2} C \left(\eta - \frac{1}{2} \right).$$

Direct computations then yield

$$\|\hat{\mathcal{U}}_1(t, \cdot) - \hat{\mathcal{U}}_2(t, \cdot)\|_{L^\infty([0,1])} \le \|\hat{\mathcal{U}}_1(0, \cdot) - \hat{\mathcal{U}}_2(0, \cdot)\|_{L^\infty([0,1])} + \frac{1}{2} t \, |\, C_1 - C_2 \,| \, \|\eta - \frac{1}{2}\|_{L^\infty([0,1])}$$
$$\le \|\hat{\mathcal{U}}_1(0, \cdot) - \hat{\mathcal{U}}_2(0, \cdot)\|_{L^\infty([0,1])} + \frac{1}{4} t \, |\, C_1 - C_2 \,|$$

and

$$\|\hat{\chi}_1(t, \cdot) - \hat{\chi}_2(t, \cdot)\|_{L^1([0,1])} \le \|\, \hat{\chi}_1(0, \cdot) - \hat{\chi}_2(0, \cdot)\|_{L^1([0,1])} + t\|\hat{\mathcal{U}}_1(0, \cdot) - \hat{\mathcal{U}}_2(0, \cdot)\|_{L^\infty([0,1])}$$
$$+ \frac{1}{8} t^2 \, |\, C_1 - C_2 \,|.$$

Thus introducing the redefined distance by

$$d((u_1, \mu_1), (u_2, \mu_2)) = \|\chi_1(C_1 \cdot) - \chi_2(C_2 \cdot)\|_{L^1([0,1])}$$
$$+ \|u_1(\chi_1(C_1 \cdot)) - u_2(\chi_2(C_2 \cdot))\|_{L^\infty([0,1])}$$
$$+ |\, C_1 - C_2 \,|,$$

we end up with

$$d\big((u_1(t), \mu_1(t)), (u_2(t), \mu_2(t))\big)$$
$$\le \left(1 + t + \frac{1}{8} t^2 \right) d\big((u_1(0), \mu_1(0)), (u_2(0), \mu_2(0))\big).$$

In particular, $d((u_1, \mu_1), (u_2, \mu_2)) = 0$ immediately implies that $C_1 = \mu_1(\mathbb{R}) = \mu_2(\mathbb{R}) = C_2$, which then implies $\chi_1(t, x) = \chi_2(t, x)$ and thus $u_1(t, x) = u_2(t, x)$.

Example 2.10. Recall Example 2.2. Consider initial data $u_0 = 0$ and $\mu_{0,i} = \alpha_i \delta_0$ yielding solutions $(u_0(t), \mu_i(t))$. Here we find

$$d((u_0, \mu_{0,1}), (u_0, \mu_{0,2})) = d((0, \alpha_1 \delta_0), (0, \alpha_2 \delta_0)) = |\, \alpha_1 - \alpha_2 \,|.$$

Thus

$$d\big((u_1(t), \mu_1(t)), (u_2(t), \mu_2(t))\big) \leq \left(1 + t + \frac{1}{8}t^2\right) |\alpha_1 - \alpha_2|.$$

Theorem 2.11. *Consider $u_{0,j}$ and $\mu_{0,j}$ as in Theorem 2.9 for j= 1, 2 with $C_j = \mu_{0,j}(\mathbb{R})$. Assume in addition that*

$$\int_{-\infty}^0 F_{0,j}(x)dx + \int_0^\infty \big(C_j - F_{j,0}(x)\big)dx < \infty, \quad j = 1, 2. \tag{2.15}$$

Define the metric

$$d\big((u_1(t), \mu_1(t)), (u_2(t), \mu_2(t))\big)$$
$$= \|\mathcal{U}_1(t, C_1 \cdot) - \mathcal{U}_2(t, C_2 \cdot)\|_{L^\infty([0,1])}$$
$$+ \|\chi_1(t, C_1 \cdot) - \chi_2(t, C_2 \cdot)\|_{L^1([0,1])} + |C_1 - C_2|.$$

Then we have

$$d\big((u_1(t), \mu_1(t)), (u_2(t), \mu_2(t))\big) \leq \left(1 + t + \frac{1}{8}t^2\right) d\big((u_{0,1}, \mu_{0,1}), (u_{0,2}, \mu_{0,2})\big).$$

Proof. It is left to show that $\chi_j(t) \in L^1([0, C_j])$ for all t positive and $j=$ 1, 2. For the remainder of this proof, fix j and drop it in the notation.

Note that (2.15) is equivalent to $\chi(0, \eta) \in L^1([0, C])$. Indeed, denote by η_1 the point at which $\chi(0, \eta)$ changes from negative to positive, then by definition

$$\chi(0, \eta) = \sup\{x \mid F(0, x) < \eta\},$$

and thus

$$\|\chi\|_{L^1([0,C])} = \int_0^C |\chi(\eta)| \, d\eta = -\int_0^{\eta_1} \chi(\eta)d\eta + \int_{\eta_1}^C \chi(\eta)d\eta$$
$$\leq \int_{-\infty}^0 F(0, x)dx + \int_0^\infty (C - F(x))dx.$$

It remains to show that $\chi(t)$ remains integrable, i.e., $\chi(t) \in L^1([0, C])$ for all t positive. Translating the condition (2.15), we find

$$\int_{-\infty}^{\xi_1} \bar{H}\bar{y}_\xi(\xi)d\xi + \int_{\xi_1}^\infty (C - \bar{H})\bar{y}_\xi(\xi)d\xi < \infty, \tag{2.16}$$

where ξ_1 is chosen such that $\bar{y}(\xi_1) = 0$. Note that it does not matter if there exists a single point or a whole interval such that $\bar{y}(\xi) = 0$, since in the latter case $\bar{y}_\xi(\xi) = 0$. Denote by $\xi(t)$ the time-dependent function such that $\bar{y}(t, \xi(t)) = 0$ for all t, which is not unique. Then the first term can be rewritten as

$$\int_{-\infty}^{\xi(t)} \bar{H}(t,\xi)\bar{y}_\xi(t,\xi)\mathrm{d}\xi = \int_{-\infty}^{\xi(t)} \bar{H}(0,\xi)\bar{y}_\xi(t,\xi)\mathrm{d}\xi$$

$$= \int_{-\infty}^{\xi(t)} \bar{H}(0,\xi)\left(\bar{y}_\xi(0,\xi) + t\bar{U}_\xi(0,\xi) + \frac{1}{4}t^2\bar{H}_\xi(0,\xi)\right)\mathrm{d}\xi,$$

$$\leq \int_{-\infty}^{\xi(t)} (1+t)\bar{H}\bar{y}_\xi(0,\xi)\mathrm{d}\xi + \int_{-\infty}^{\xi(t)} \left(t+\frac{1}{4}t^2\right)\bar{H}\bar{H}_\xi(0,\xi)\mathrm{d}\xi$$

$$= (1+t)\int_{-\infty}^{\xi(0)} \bar{H}\bar{y}_\xi(0,\xi)\mathrm{d}\xi + (1+t)\int_{\xi(0)}^{\xi(t)} \bar{H}\bar{y}_\xi(0,\xi)\mathrm{d}\xi + \left(\frac{1}{2}t+\frac{1}{8}t^2\right)H^2(0,\xi(t))$$

$$\leq (1+t)\int_{-\infty}^{\xi(0)} \bar{H}\bar{y}_\xi(0,\xi)\mathrm{d}\xi + (1+t)\int_{\xi(0)}^{\xi(t)} \bar{H}\bar{y}_\xi(0,\xi)\mathrm{d}\xi + \left(\frac{1}{2}t+\frac{1}{8}t^2\right)C^2,$$

where we used (2.11) and that $\bar{U}_\xi^2(t,\xi) = \bar{H}_\xi\bar{y}_\xi(t,\xi)$. The term on the right-hand side will be finite if we can show that the second integral on the right-hand side is finite. Therefore observe that

$$\int_{\xi(0)}^{\xi(t)} \bar{H}\bar{y}_\xi(0,\xi)\mathrm{d}\xi = \int_{\bar{y}(0,\xi(0))}^{\bar{y}(0,\xi(t))} F(x)\mathrm{d}x = \int_0^{\bar{y}(0,\xi(t))} F(x)\mathrm{d}x = \int_{\bar{y}(t,\xi(t))}^{\bar{y}(0,\xi(t))} F(x)\mathrm{d}x$$

$$\leq C\,|\bar{y}(t,\xi(t)) - \bar{y}(0,\xi(t))\,| \leq C\left(t\,|\,U(0,\xi(t))\,| + \frac{t^2}{8}C\right)$$

$$\leq C\left(t\|u_0\|_{L^\infty(\mathbb{R})} + \frac{t^2}{8}C\right) < \infty.$$

Similar considerations yield that the second integral in (2.16) remains finite as time evolves. □

Remark 2.12. Observe that the distance introduced in Theorem 2.11 gives at most a quadratic growth in time, while the distance in [13] has at most an exponential growth in time.

We make a comparison with the more complicated Camassa–Holm equation in the next remark.

Remark 2.13. Consider an interval $[\xi_1, \xi_2]$ such that $U_0(\xi) = U_0(\xi_1)$ and $H(\xi_1) = H(\xi)$ for all $\xi \in [\xi_1, \xi_2]$. This property will remain true for all later times. In particular, this means that these intervals do not show up in our metric, and the function $\chi(t,\eta)$ always has a constant jump at the corresponding point η. This is in big contrast to the Camassa–Holm equation where jumps in $\chi(t,\eta)$ may be created and then subsequently disappear immediately again. Thus the construction for the Camassa–Holm equation is much more involved than the HS construction.

This is illustrated in the next examples.

Example 2.14. Given the initial data $(u_0, \mu_0) = (0, \delta_0 + 2\delta_1)$, direct calculations yield

$$
u(t, x) = \begin{cases}
-\dfrac{3}{4}t, & x \le -\dfrac{3}{8}t^2, \\[2mm]
\dfrac{2}{t}x, & -\dfrac{3}{8}t^2 \le x \le -\dfrac{1}{8}t^2, \\[2mm]
-\dfrac{1}{4}t, & -\dfrac{1}{8}t^2 \le x \le 1-\dfrac{1}{8}t^2, \\[2mm]
\dfrac{2}{t}(x-1), & 1-\dfrac{1}{8}t^2 \le x \le 1+\dfrac{3}{8}t^2, \\[2mm]
\dfrac{3}{4}t, & 1+\dfrac{3}{8}t^2 \le x,
\end{cases}
$$

$$
F(t, x) = \begin{cases}
0, & x \le -\dfrac{3}{8}t^2, \\[2mm]
\dfrac{3}{2}+\dfrac{4}{t^2}x, & -\dfrac{3}{8}t^2 \le x \le -\dfrac{1}{8}t^2, \\[2mm]
1, & -\dfrac{1}{8}t^2 \le x \le 1-\dfrac{1}{8}t^2, \\[2mm]
\dfrac{3}{2}+\dfrac{4}{t^2}(x-1), & 1-\dfrac{1}{8}t^2 \le x \le 1+\dfrac{3}{8}t^2, \\[2mm]
3, & 1+\dfrac{3}{8}t^2 \le x.
\end{cases}
$$

Calculating the pseudo inverse $\chi(t, \eta)$ and $\mathcal{U}(t, \eta) = u(t, \chi(t, \eta))$ for each t, then yields

$$
\chi(t, \eta) = \begin{cases}
-\infty, & \eta = 0, \\[2mm]
\dfrac{t^2}{4}\left(\eta-\dfrac{3}{2}\right), & 0 < \eta \le 1, \\[2mm]
1+\dfrac{t^2}{4}\left(\eta-\dfrac{3}{2}\right), & 1 < \eta \le 3,
\end{cases}
$$

$$
\mathcal{U}(t, \eta) = \dfrac{t}{2}\left(\eta-\dfrac{3}{2}\right), \quad \text{for all } \eta \in [0, 3].
$$

Here two observations are important. Note that $\mathcal{U}(t, \eta)$ is continuous and differentiable with respect to η, while $\chi(t, \eta)$ on the other hand has at each time t a discontinuity at $\eta = 1$ (and of course at $\eta = 0$). In particular, one has

$$
\lim_{\eta \to 1-} \chi(t, \eta) = -\dfrac{1}{8}t^2 \quad \text{and} \quad \lim_{\eta \to 1+} \chi(t, \eta) = 1-\dfrac{1}{8}t^2.
$$

Thus the jump in function value remains unchanged even if the limit from the left and the right are time dependent.

In order to understand the behavior of $l(t, \eta)$, let us have a look at the solution in Lagrangian coordinates, which is given by

$$y(t, \xi) = \begin{cases} \xi - \dfrac{3}{8} t^2, & \xi \leq 0, \\[2mm] \dfrac{1}{4} \left(\xi - \dfrac{3}{2} \right) t^2, & 0 \leq \xi \leq 1, \\[2mm] \xi - 1 - \dfrac{1}{8} t^2, & 1 \leq \xi \leq 2, \\[2mm] 1 + \dfrac{1}{4} \left(\xi - \dfrac{5}{2} \right) t^2, & 2 \leq \xi \leq 4, \\[2mm] \xi - 3 + \dfrac{3}{8} t^2, & 4 \leq \xi, \end{cases}$$

$$U(t, \xi) = \begin{cases} -\dfrac{3}{4} t, & \xi \leq 0, \\[2mm] \dfrac{1}{2} \left(\xi - \dfrac{3}{2} \right) t, & 0 \leq \xi \leq 1, \\[2mm] -\dfrac{1}{4} t, & 1 \leq \xi \leq 2, \\[2mm] \dfrac{1}{2} \left(\xi - \dfrac{5}{2} \right) t, & 2 \leq \xi \leq 4, \\[2mm] \dfrac{3}{4} t, & 4 \leq \xi, \end{cases}$$

$$H(t, \xi) = \begin{cases} 0, & \xi \leq 0, \\ \xi, & 0 \leq \xi \leq 1, \\ 1, & 1 \leq \xi \leq 2, \\ \xi - 1, & 2 \leq \xi \leq 4, \\ 3, & 4 \leq \xi. \end{cases}$$

Hence direct computations yield that

$$l(t, \eta) = l(0, \eta) = \begin{cases} -\infty, & \eta = 0, \\ \eta, & 0 < \eta \leq 1, \\ \eta + 1, & 1 < \eta \leq 3. \end{cases}$$

Note that $l(0, \eta)$ is non-constant on the intervals where both $\mathcal{U}(0, \eta)$ and $\chi(0, \eta)$ are constant.

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