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Testing the homogeneous marginal utility of income assumption

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ABSTRACT

We develop a test for the hypothesis that every agent from a population of heterogeneous consumers has the same marginal utility of income function. This homogeneous marginal utility of income (HMUI) assumption is often (implicitly) used in applied demand studies because it has nice aggregation properties and facilitates welfare analysis. If the HMUI assumption holds, we can also identify the common marginal utility of income function. We apply our results using a U.S. cross sectional dataset on food consumption.

KEYWORDS

Aggregation; marginal utility of income; nonparametric

JEL CLASSIFICATION

C12; C14; D12; D60

1. Introduction

In empirical demand analysis, it is often (implicitly) assumed that consumers have identical marginal utility of income functions. We call this the homogeneous marginal utility of income assumption (HMUI). The main contribution of this article is to provide a statistical test for the HMUI assumption. We provide two groups of tests. The first is based on the conditional quantiles of the Engel share function. The second is based on the values of the conditional moments of this function. The tests are obtained using an overidentification result on the marginal utility of income function (given that the HMUI assumption holds). As such, if the HMUI assumption holds, it becomes possible to identify this common marginal utility of income function.

The HMUI assumption is frequently imposed on many functional forms in the parametric demand literature, like the Gorman polar form (Gorman, 1961) or the Almost Ideal Demand System (Deaton and Muellbauer, 1980), because it has nice aggregation properties. In particular, if HMUI assumption holds, then individual demand functions can be expressed as the sum of the mean demand function, which satisfies all regularity properties of an individual demand function (i.e., homogeneity, adding up, Slutsky symmetry, and negativity), and an individual heterogeneity term that has mean zero.¹ As such, the representative demand function can easily be estimated using a parametric or nonparametric conditional mean estimator. Given that the mean demand function satisfies all regularity assumptions, it can also be used to conduct welfare analysis, e.g. computation of the deadweight loss of proposed tax changes or the computation of (average) equivalent and compensating variation. We refer to Hausman and Newey (1995), Blundell et al. (2003), and Blundell et al. (2012) for such applications. Schlee (2007) showed that if the HMUI assumption is satisfied and if the marginal utility of income decreases with income (which is a reasonable assumption), then the compensating variation obtained from using the mean demand function will be a lower bound for the mean of the agents' compensating variation.

The problem of measuring the marginal utility of income is an old problem which dates back to the contributions of Frisch (1932), Samuelson (1937), Vickrey (1945), and Morgan (1945). More recent

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¹Various recent studies analyze demand behavior and test rationality restrictions for settings where the HMUI assumption does not necessarily hold. See, for example Hausman and Newey (2016) and Dette et al. (2016).

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studies try to measure the marginal utility of income using data from happiness studies (Layard et al., 2008). In this article, we show that the marginal utility of income function is easily identified if the HMUI assumption holds. This allows us to make interpersonal comparison of utility. For example, if we redistribute one dollar from an individual with income x towards an individual with income y , then aggregate utility changes by $\beta(y) - \beta(x)$ where $\beta(\cdot)$ is the common marginal utility of income function. In this sense, the HMUI assumption and our identification result greatly simplify cost benefit analysis.

Interestingly, our test and identification results only require data on the consumption of a single good or group of goods and the expenditure levels for a single cross-section of consumers. In this sense, there is no need for multiple cross-sections, panel data, or consumption data on multiple goods. On the other hand, if observations on multiple goods are available, additional testable restrictions can be obtained.

We implement our test using the 2007 wave of the Consumer Expenditure Survey (CEX), a U.S. consumer budget survey. We focus our analysis on the consumption of food. Based on this data set, we find that we cannot reject the HMUI assumption. Next, we estimate the marginal utility of income function. According to our estimates, aggregate welfare increases equally when we either give one dollar to an agent at the median of the income distribution, 0.61 dollars to someone at the 1st decile, or 1.45 dollars to someone at the 9th decile.

Overview. Section 2 contains the framework and the main testability and identification results. Section 3 presents our estimation procedure and describes our statistical test for the HMUI assumption. Section 4 contains the application, and Section 5 concludes.

2. The HMUI assumption

We consider a probability space (J, Ω, P) of agents where agent $j \in J$ has a twice-continuously differentiable and measurable, indirect utility function $v_j(\mathbf{p}, x) : \mathbb{R}_{++}^n \times I \rightarrow \mathbb{R}$. The indirect utility function $v_j(\mathbf{p}, x)$ gives the maximal utility for agent j obtainable at prices \mathbf{p} and income x . This function is strictly quasi-convex, homogeneous of degree zero in prices and income, strictly increasing in x , and decreasing in \mathbf{p} . We assume that the domain of income levels $I \subseteq \mathbb{R}_{++}$ is convex and open. Denote by $V(\mathbf{p}, x)$ the mean indirect utility function over all agents for a vector of prices and income,

$$V(\mathbf{p}, x) = \mathbb{E}_j[v_j(\mathbf{p}, x)].$$

The fact that we aggregate conditional on some level of income differs from how aggregation is typically performed in representative consumer settings.² If $q_j(\mathbf{p}, x)$ denotes the Marshallian demand of agent j for good q , then the mean demand function for good q is defined by

$$\bar{q}(\mathbf{p}, x) = \mathbb{E}_j[q_j(\mathbf{p}, x)].$$

Lewbel (2001) calls this the statistical demand function, but we follow Schlee (2007) and use the term “mean demand function.” Using Roy’s identity, we can express this demand function in terms of derivatives of the indirect utility functions of the agents

$$\bar{q}(\mathbf{p}, x) = \mathbb{E}_j \left[- \frac{\frac{\partial v_j(\mathbf{p}, x)}{\partial p_q}}{\frac{\partial v_j(\mathbf{p}, x)}{\partial x}} \right].$$

On the other hand, the demand function for the representative agent with utility function $V(\mathbf{p}, x)$, is given by

$$Q(\mathbf{p}, x) = - \frac{\mathbb{E}_j \left[\frac{\partial v_j(\mathbf{p}, x)}{\partial p_q} \right]}{\mathbb{E}_j \left[\frac{\partial v_j(\mathbf{p}, x)}{\partial x} \right]}.$$

²In representative consumer (macroeconomic) settings representative indirect utility function depend on prices and aggregate income. See for example Lewbel (1989) for aggregation in such settings.

The two demand functions $Q(\mathbf{p}, x)$ and $\bar{q}(\mathbf{p}, x)$ will coincide if and only if the marginal utility of income functions, $\partial v_j(\mathbf{p}, x)/\partial x$, are identical for all agents $j = 1, \dots, n$.³ We call this the *HMUI*. By integrating out the indirect utility function, we obtain that the HMUI assumption requires the existence of agent specific functions A_j and a function B such that

$$v_j(\mathbf{p}, x) = A_j(\mathbf{p}) + B(\mathbf{p}, x). \tag{1}$$

The key feature of this utility function is that income only enters via the function B which is the same for all consumers. We refer to Lewbel (2001) and Schlee (2007) for a detailed discussion of this family of indirect utility functions.

Consider a setting where all agents' indirect utility function satisfies the HMUI assumption, i.e., Eq. (1). Let \mathbf{q}_R be a subset of goods with price vector \mathbf{p}_R . The share of these goods as a fraction of the total expenditure x is given by $s = \frac{\mathbf{p}'_R \mathbf{q}_R}{x}$. Using Roy's identity, we see that the share function for the goods in R for agent j has the functional structure

$$s_j(\mathbf{p}, x) = -\frac{a_j(\mathbf{p}) + b(\mathbf{p}, x)}{x\beta(\mathbf{p}, x)}, \tag{2}$$

where $a_j(\mathbf{p}) = \mathbf{p}'_R \nabla_{\mathbf{p}_R} A_j(\mathbf{p})$, $b(\mathbf{p}, x) = \mathbf{p}'_R \nabla_{\mathbf{p}_R} B(\mathbf{p}, x)$, and $\beta(\mathbf{p}, x) = \partial B(\mathbf{p}, x)/\partial x$. We take this functional form as a starting point to develop our test. Observe that $\beta(\mathbf{p}, x)$, being the marginal utility of income, is strictly positive.

We consider the setting where we have a single cross-section with fixed prices \mathbf{p} for all goods, and we observe the consumption share of the goods in R , together with the disposable income for a large number of agents. This allows us to identify the joint distribution of shares and expenditure levels (w_j, x_j) where $w_j = s_j(\mathbf{p}, x_j)$, i.e., w_j is the consumption share of the goods in R for some agent j with income x_j .⁴ Given that we restrict ourselves to a single cross-section, with identical prices for all consumers, we omit from now on the dependency on prices and simply write $s_j(x)$ instead of $s_j(\mathbf{p}, x)$, i.e., $s_j(x)$ is the Engel share curve of consumer j . Similarly, we write a_j and $b(x)$ instead of $a_j(\mathbf{p})$ and $b(\mathbf{p}, x)$. If, in addition to household income, we also observe some individual characteristics, like household size, education level, marital status, age, etc., it is possible to take these into account by performing the entire analysis conditional on some value of these characteristics. In addition, this would allow us to make both functions A_j and B dependent on these observable characteristics. For ease of notation, however, we omit such observable characteristics although we will control for household composition and some other characteristics in the empirical application.

Let $w_j(x)$ be cumulative distribution function (cdf) w_j conditional on $x_j = x$, i.e., the random variable with cdf, $\Pr(w_j \leq w | x_j = x)$.

Assumption 1. *For all $x \in I$, the random variable $w_j(x)$ is continuously distributed on its domain and has finite mean.*

If the HMUI specification (2) holds, then Assumption 1 implies that a_j is also continuously distributed and has a finite mean.

Assumption 2. *For all values $w \in [0, 1]$ and any level of income $x \in I$,*

$$\Pr(w_j \leq z | x_j = x) = \Pr(s_j(x_j) \leq z | x_j = x) = \Pr(s_j(x) \leq z).$$

Assumption 2 states that the distribution of agents' types (i.e., unobserved preference heterogeneity) is independently distributed from the expenditure levels. This is not an innocuous assumption, but

³The necessary part of this result requires that $Q(\mathbf{p}, x)$ equals $\bar{q}(\mathbf{p}, x)$ for all distributions of agents and agents have different demand functions.

⁴We use w_j instead of s_j in order to make clear the distinction between what is observed, w_j , and the underlying data generating process $s_j(\mathbf{p}, x)$.

similar assumptions are frequently used in the literature.⁵ Again, if we have observations on additional household characteristics, the independence assumption can be relaxed to independence conditional on the value of these characteristics. Assumption 2 also rules out endogeneity of the expenditure level. In principle, endogeneity of the expenditure level could be allowed for by using nonparametric IV or control function methods for regression models under endogeneity (see, for example, Imbens and Newey, 2009, for a possible approach). A particular difficulty, however, lies in finding an appropriate instrument.

If $s_j(x)$ has the HMUI specification (2), then Assumption 2 also implies that for all a in the support of a_j ,

$$\Pr(a_j \leq a | x_j = x) = \Pr(a_j \leq a).$$

Let $\bar{w}(x)$ be the mean Engel share curve, conditional on the expenditure level x , i.e., $E(w_j | x_j = x)$; then, if (2) holds, we have

$$\begin{aligned} \bar{w}(x) &= \mathbb{E}_j[w_j | x_j = x], \\ &= \mathbb{E}_j[s_j(x_j) | x_j = x] = \mathbb{E}_j[s_j(x)], \\ &= -\frac{\mathbb{E}_j[a_j] + b(x)}{x\beta(x)} \equiv -\frac{\bar{a} + b(x)}{x\beta(x)}. \end{aligned}$$

2.1. Testable implications from conditional quantiles

Let $z_\pi(x)$ be the π th quantile of the distribution of the random variable $(w_j - \bar{w}(x_j))$ conditional on the expenditure level $x_j = x$ (i.e., the π th conditional quantile of the error $w_j - \bar{w}(x_j)$)

$$\Pr(w_j - \bar{w}(x_j) \leq z_\pi(x) | x_j = x) = \pi.$$

The value of $z_\pi(x)$ is identified from the joint distribution of (w_j, x_j) . The following proposition shows that for the HMUI assumption to hold, the ratio of $z_\pi(x)$ x and $z_\pi(y)$ y , for two different incomes x and y , should not depend on the value of π .

Proposition 1. *If Assumptions 1 and 2 hold and the share functions $s_j(\cdot)$ are of the form (2), then for all income levels $x, y \in I$,*

$$z_\pi(x) x \beta(x) = z_\pi(y) y \beta(y). \tag{3}$$

In particular, if $z_\pi(y) \neq 0$, then

$$\frac{z_\pi(x) x}{z_\pi(y) y} = \frac{\beta(y)}{\beta(x)}. \tag{4}$$

Proof. If the demand functions $s_j(x)$ are of the form (2), then we have that

$$\begin{aligned} \Pr(w_j - \bar{w}(x_j) \leq z_\pi(x) | x_j = x) &= \pi, \\ \iff \Pr(-a_j + \bar{a} \leq z_\pi(x)\beta(x_j)x_j | x_j = x) &= \pi, \\ \iff \Pr(-a_j \leq z_\pi(x)\beta(x)x - \bar{a}) &= \pi. \end{aligned}$$

Here the last line follows from Assumption 2. Performing the same derivation at the income level y gives

$$\begin{aligned} \Pr(w_j - \bar{w}(x_j) \leq z_\pi(y) | x_j = y) &= \pi, \\ \iff \Pr(-a_j \leq z_\pi(y)\beta(y)y - \bar{a}) &= \pi, \end{aligned}$$

Equating these two expressions and using Assumption 1, we obtain that $z_\pi(x)\beta(x)x = z_\pi(y)\beta(y)y$ or equivalently (if $z_\pi(y) \neq 0$)

$$\frac{z_\pi(x) x}{z_\pi(y) y} = \frac{\beta(y)}{\beta(x)}.$$

⁵See for example, Lewbel (2001, Eq. 4), Hausman and Newey (2016, Assumption 1), Dette et al. (2016), Blundell et al. (2014, Assumption A.1), and Bhattacharya (2015).

As the right-hand side is independent of π , so must be the left-hand side. □

Conditions (3) and (4) show two things. First of all, given that $\frac{z_\pi(x)x}{z_\pi(y)y}$ is independent of π , it provides us with a test for the hypothesis that the HMUI assumption is satisfied. In fact, this ratio is also independent of the particular subset of goods under consideration. Given this, if we have observations on the consumption of more than one good or group of goods, this would yield an additional testable implication. Second, if it is indeed the case that the left-hand side is independent of π , then condition (3) shows how to identify the value of $\beta(\cdot)$, up to a normalization, i.e., the marginal utility of income function is identified. In particular, if we normalize for a fixed level of income \bar{y} , $\beta(\bar{y}) = 1$, then $\beta(x) = \frac{z_\pi(\bar{y})\bar{y}}{z_\pi(x)x}$.

Proposition 1 shows that condition (3) is a necessary condition if every individual has a share function of the form (2). The following proposition shows that condition (3) is in fact the strongest testable implication of the HMUI specification (2). In particular, it shows that (i) condition (3) can not be further strengthened and (ii) that there is no subclasses of distributions over shares and expenditures (w_j, x_j) for which (3) becomes both a necessary and sufficient condition for (2).⁶ The proof is given in Appendix B.

Proposition 2. *Consider a joint distribution of shares and expenditure levels (w_j, x_j) . Let $\bar{w}(x)$ be the conditional mean*

$$\bar{w}(x) = \mathbb{E}_j(w_j|x_j = x),$$

and let $z_\pi(x)$ be the π th conditional quantile of $w_j - \bar{w}(x_j)$ given $x_j = x$, i.e.,

$$\Pr(w_j - \bar{w}(x) \leq z_\pi(x)|x_j = x) = \pi.$$

In addition, assume that for all $x \in I$, $z_\pi(x)$ is strictly increasing in $\pi \in [0, 1]$ and that there exists a function $\beta(x) : \mathbb{R}_{++} \rightarrow \mathbb{R}_{++}$ such that (3) is satisfied, i.e., for all $\pi \in [0, 1]$ and all $x, y \in I$,

$$z_\pi(x)x\beta(x) = z_\pi(y)y\beta(y).$$

Then, we have as follows:

- (i) *There exists a distribution of share functions $s_j(x)$ that satisfies the HMUI specification (2) and, for all $x \in I$ and $\pi \in [0, 1]$,*

$$\Pr(s_j(x) - \bar{w}(x) \leq z_\pi(x)) = \pi \text{ and } \mathbb{E}_j(s_j(x)) = \bar{w}(x).$$

- (ii) *There exists a distribution of share functions $s_j(x)$ that violates the specification (2) and, for all $x \in I$ and $\pi \in [0, 1]$,*

$$\Pr(s_j(x) - \bar{w}(x) \leq z_\pi(x)) = \pi \text{ and } \mathbb{E}_j(s_j(x)) = \bar{w}(x).$$

The first part of Proposition 2 demonstrates that (3) is optimal in the sense that any other feature of the distribution of (w_j, x_j) cannot further contribute to finding validations of (2). Additionally, the second part shows the limitation that accepting (3) never allows us to confirm that the demand share functions are of the HMUI form (2), i.e., it is impossible to find sufficient conditions to test (2). As such, the HMUI specification (2) is refutable but not verifiable.

If we are willing to impose additional assumptions on the data generating process, it becomes possible to obtain a necessary and sufficient testable implication of (2). In particular, consider the following assumption.

Assumption 3. *There exists a function $h : J \rightarrow \mathbb{R}$ such that for all $x \in I$ and all $j, j' \in J$, $h(j) \geq h(j')$ if and only if $s_j(x) \geq s_{j'}(x)$.*

Assumption 3 states that the heterogeneity component can be brought back to a single dimension $h(j)$ that perfectly orders the Engel share curves. Similar assumptions have been used in the literature

⁶See Theorem 1 of Mourifie and Wan (2015) and Proposition 1.1 of Kitagawa (2015) for similar type results for treatment response models.

(e.g., Blundell et al., 2014; Hoderlein and Vanhems, 2013). Imposing it identifies the individual Engel share curves $s_j(x)$ by the conditional quantile functions of $w_j(x)$. As such, Assumption 3, together with Assumptions 1 and 2, gives (3) as a necessary and sufficient condition for specification (2). The proof follows immediately from the first part of the proof of Proposition 2.

Corollary 1. *If Assumptions 1, 2, and 3 hold, then for all $j \in J$, $s_j(x)$ is of the form (2) if and only if (3) holds.*

We base our test of the HMUI assumption on condition (4). However, one caveat is in order. Notice that the identification of $\beta(x)$ fails if $z_\pi(y) = 0$. This would occur if, for example, $\pi = 0.5$ and the conditional distribution of w_j given $x_j = y$ is symmetric around $\bar{w}(y)$. As such, we should focus only on values of π for which $z_\pi(y)$ is significantly different from zero.

2.2. Testable implications from conditional moments

The procedure above developed testable implications by using the conditional quantiles of the variable $w_j - \bar{w}(x_j)$. A similar result can be obtained by using the conditional moments of the variable $w_j - \bar{w}(x_j)$. In order to use this test, one must strengthen Assumption 1.

Assumption 4. *For all $x \in I$, the random variable $w_j - \bar{w}(x_j)$ has finite conditional moments up to order m ; i.e.,*

$$\mu^m(x) = \mathbb{E}_j[(w_j - \bar{w}(x_j))^m | x_j = x],$$

exists and is finite.

If $m \geq 2$ and (2) holds, then the variance of $w_j - \bar{w}(x_j)$ conditional on $x_j = x$ is defined by

$$\begin{aligned} \mu^2(x) &= \mathbb{E}_j[(w_j - \bar{w}(x_j))^2 | x_j = x], \\ &= \mathbb{E}_j \left[\left(\frac{-a_j + \bar{a}}{\beta(x_j)x_j} \right)^2 | x_j = x \right], \\ &= \frac{1}{\beta(x)^2 x^2} \mathbb{E}_j[(a_j - \bar{a})^2]. \end{aligned}$$

From this, we obtain that for two different income levels x and y ,

$$\sqrt{\frac{\mu^2(x)}{\mu^2(y)} \frac{x}{y}} = \frac{\beta(y)}{\beta(x)}.$$

More general, we have the following result.

Proposition 3. *If Assumptions 1, 2, and 4 hold and if the share functions $s_j(\cdot)$ are of the form (2), then for all income levels x and $y \in I$,*

$$(\mu^m(x))^{1/m} x \beta(x) = (\mu^m(y))^{1/m} y \beta(y). \tag{5}$$

In particular, if $\mu^m(y) \neq 0$, then

$$\left(\frac{\mu^m(x)}{\mu^m(y)} \right)^{1/m} \frac{x}{y} = \frac{\beta(y)}{\beta(x)}. \tag{6}$$

The right-hand side of Eq. (6) is independent of m , which gives another set of testable implications. Moreover, it gives us another means to identify the marginal utility of income function. Caution should

be taken when $\mu^m(y)$ is (close to) zero, which can happen if the conditional distribution is symmetric and m is odd. Given this, we focus in our empirical analysis on the even moments.

3. Implementation

We estimate the conditional quantiles and moments in two steps. Consider an iid sample $i = 1, \dots, n$ of shares and incomes (w_i, x_i) . First, we nonparametrically estimate the mean consumption share function $\bar{w}(x)$. We do this using a (leave one out) local linear kernel regression of the consumption shares w_i on the log of expenditure, $\ln(x_i)$. In particular, we define, $\hat{w}_n(x_i)$ as the optimal value of a_0 in the minimization problem

$$\min_{a_0, b_0} \sum_{j \neq i} [w_j - a_0 - b_0(\ln(x_i) - \ln(x_j))]^2 \kappa \left(\frac{\ln(x_j) - \ln(x_i)}{h_n} \right),$$

where $\kappa(\cdot)$ is a symmetric kernel and h_n is the bandwidth. In practice, we use the Gaussian kernel and choose h_n by cross-validation. If $h_n \rightarrow 0$ and $nh_n \rightarrow \infty$ for $n \rightarrow \infty$, we know that the estimator $\hat{w}_n(x_i)$ consistently estimates $\bar{w}(x_i)$. Under suitable conditions (see Li and Racine, 2007, Section 2.4), the random variable $\sqrt{nh_n}(\hat{w}_n(x_i) - \bar{w}(x_i))$ has an asymptotic normal distribution.

Next, we estimate $\Pr(w_i - \bar{w}(x_i) \leq z | x_i = x)$ by using a smoothed nonparametric conditional cdf estimator, i.e., the solution of a_1 in the minimization problem

$$\min_{a_1, b_1} \sum_j \left[K \left(\frac{z - w_j + \hat{w}_n(x_j)}{g_n} \right) - a_1 - b_1(\ln(x_j) - \ln(x)) \right]^2 \kappa \left(\frac{\ln(x_j) - \ln(x)}{h_n} \right), \tag{7}$$

where $K(y) = \int_{-\infty}^y \kappa(\psi) d\psi$ and g_n is a suitable chosen bandwidth of the order $o(h_n)$. Finally, we estimate $z_\pi(x)$ as the nonparametric quantile estimator, $\hat{z}_{n,\pi}(x)$ of this local linear kernel estimator (see Li and Racine, 2007, Section 6.3). Other estimators for the conditional cdf and quantiles are possible but provide very similar results. The derivation of the asymptotic properties of $\hat{z}_{n,\pi}(x)$ is made difficult by the fact that the mean share function $\bar{w}(x_i)$ is replaced by the estimate $\hat{w}_n(x_i)$. However, using a proof which is very similar to the one of Zhou and Zhu (2015, Theorems 1 and 2), it can be shown that the estimator satisfies the so called adaptiveness property which means that substituting the true mean share function $\bar{w}(x_i)$ by its estimate $\hat{w}_n(x_i)$ does not change the consistency of $\hat{z}_{n,\pi}(x)$ or asymptotic normality of $\sqrt{nh_n}(\hat{z}_{n,\pi}(x) - z_\pi(x))$.

The conditional moments $\mu(x)^m$ are estimated as the solution of a_2 in the following minimization problem:

$$\min_{a_2, b_2} \sum_j [(w_j - \hat{w}_n(x_j))^m - a_2 - b_2(\ln(x) - \ln(x_j))]^2 k \left(\frac{\ln(x_j) - \ln(x)}{h_n} \right). \tag{8}$$

This gives us the estimator $\hat{\mu}_n^m(x)$. Using a proof similar to the one of Fan and Yao (1998, Theorem 1), it can be shown that the substitution of $\bar{w}(x_i)$ by the estimator $\hat{w}_n(x_i)$ in (8) does not influence the consistency of $\hat{\mu}_n^m(x)$ nor the asymptotic normality of $\sqrt{nh_n}(\hat{\mu}_n^m(x) - \mu^m(x))$.

Testing the HMUI Assumption. For fixed numbers $I, K, J \in \mathbb{N}$, consider a finite set of income levels $[y^1, \dots, y^J]$ a finite set of values $[\pi^1, \dots, \pi^K]$ from $(0, 1)$ and a finite number of integers $[m^1, \dots, m^J]$ larger than 1. Let x be some fixed income level, and define

$$\theta_{\pi^k}(y^i) = \frac{z_{\pi^k}(x) x}{z_{\pi^k}(y^i) y^i},$$

$$\tau_{m^j}(y^i) = \left(\frac{\mu^{m^j}(x)}{\mu^{m^j}(y^i)} \right)^{1/m^j} \frac{x}{y^i}.$$

Conditions (4) and (6) show that, if the HMUI assumption holds and if all moments $\mu^{m^j}(\cdot)$ exist, then the values of $\theta_{\pi^k}(y^i)$ and $\tau_{m^j}(y^i)$ should be equal for all values of π^k and m^j . In other words, for all y^i , π^ℓ , π^k , m^j , and m^o ,

$$\theta_{\pi^i}(y^i) = \theta_{\pi^k}(y^i) = \tau_{m^j}(y^i) = \tau_{m^o}(y^i),$$

let

$$Z = \max_{\{y^i\}_{i \leq I}} \left\{ \max_{\{m^j\}_{j \leq J}, \{\pi^k\}_{k \leq K}} \{\theta_{\pi^k}(y^i), \tau_{m^j}(y^i)\} - \min_{\{m^j\}_{j \leq J}, \{\pi^k\}_{k \leq K}} \{\theta_{\pi^k}(y^i), \tau_{m^j}(y^i)\} \right\}.$$

Observe that by definition, $Z \geq 0$. However, if the HMUI assumption holds, then $Z = 0$. As such, we can test the HMUI by developing a test for the null hypothesis $Z = 0$ against the alternative hypothesis that $Z > 0$. Consider the estimate \widehat{Z}_n of Z obtained by replacing the quantiles $z_\pi(\cdot)$ and moments $\mu^m(\cdot)$ by their estimates $\widehat{z}_{n,\pi}(\cdot)$ and $\widehat{\mu}_n^m(\cdot)$ using a sample of size n and a bandwidth of size h_n . Given the finiteness of I, K , and J , we know that the random variable $\sqrt{nh_n}(\widehat{Z}_n - Z)$ has a nondegenerate limiting distribution. A one-sided $(1 - \alpha)\%$ confidence interval for this random variable at $Z = 0$ is determined by the value c_α for which,

$$\begin{aligned} & \Pr\left(\sqrt{nh_n}(\widehat{Z}_n - Z) \leq c_\alpha\right) = 1 - \alpha, \\ \iff & \Pr\left(\sqrt{nh_n}\widehat{Z}_n \leq c_\alpha\right) = 1 - \alpha, \\ \iff & \Pr\left(0 \geq -\frac{c_\alpha}{\sqrt{nh_n}} + \widehat{Z}_n\right) = 1 - \alpha. \end{aligned}$$

In order to test the null hypothesis, we should determine the number c_α , compute the value of $\frac{-c_\alpha}{\sqrt{nh_n}} + \widehat{Z}_n$, and verify if this value is larger than zero. We reject the HMUI assumption at the $(1 - \alpha)\%$ confidence level if it is.

The value of c_α is determined by the (asymptotic) distribution of $\sqrt{nh_n}(\widehat{Z}_n - Z)$, which is not known. In addition, a bootstrap approximation would not be valid given that the asymptotic distribution of $\sqrt{nh_n}(\widehat{Z}_n - Z)$ is discontinuous in Z at $Z = 0$. Appendix A contains a discussion and some simulation results, which clearly shows the inconsistency of the bootstrap. Moreover, kernel estimators have an asymptotic bias, which further complicates the bootstrap procedure (see, for example Horowitz, 2001). As a solution, we resort to a subsampling procedure. Although subsampling underperforms the bootstrap in settings where both are applicable, it has the advantage that it is valid under very weak conditions. We refer to Politis et al. (1999) for a detailed discussion of this procedure.

The idea of subsampling is to take subsamples from the observed sample of size $v \ll n$ without replacement. Assume that the corresponding bandwidth is given by h_v with $h_v \rightarrow 0$ and $vh_v \rightarrow \infty$, and let $Z_{v,n}^*$ be the estimate of Z obtained from such a subsample. If we take a large number of subsamples, we can compute the value of \widehat{c}_α such that for $(1 - \alpha)100\%$ of the subsamples,

$$\sqrt{vh_v}Z_{v,n}^* \leq \widehat{c}_\alpha.$$

Then asymptotically, under the null hypothesis $Z = 0$ and for $vh_v \rightarrow \infty$ and $nh_n/(vh_v) \rightarrow \infty$,

$$\Pr\left(\sqrt{nh_n}\widehat{Z}_n \leq \widehat{c}_\alpha\right) \rightarrow (1 - \alpha).$$

This gives the following procedure. First, we take a large number of subsamples of size v and compute the value of \widehat{c}_α .⁷ Next, we compute $\frac{-\widehat{c}_\alpha}{\sqrt{nh_n}} + \widehat{Z}_n$ and look whether this value is larger than zero. If it is, we reject the HMUI assumption at the $(1 - \alpha)\%$ confidence level.

⁷We draw 10,000 subsamples of size $v = n^{0.7}$. Results are not sensitive to variations in this subsample size.

A possible disadvantage of subsampling is that it may not give uniformly valid inference, although subsampling gives asymptotic valid pointwise inference (see Andrews and Guggenberger, 2009a,b).⁸ The uniformity problem occurs because the asymptotic distribution of $\sqrt{nh_n}(\widehat{Z}_n - Z)$ is discontinuous at $Z = 0$. In particular, it can be shown that the subsampling procedure does not converge uniformly to the limiting distribution for values of $Z > 0$ close to 0.⁹ As such, the subsampling procedure does not provide correct inference if one wishes to derive confidence intervals for Z when Z is very close to 0. In our setup, however, we only care about testing the null hypothesis $Z = 0$ (i.e., establishing a confidence interval around Z when $Z = 0$). When we restrict ourselves to data generating processes for which the true parameter value has $Z = 0$, subsampling does give asymptotically valid inference.

4. Application

Data Description. We use data from the CEX, which is a U.S. consumer cross-sectional budget survey. We focus on the 2007 diary survey, and we base our analysis on the consumption share of food.¹⁰ Taking food as the commodity of interest has the advantage that it is nondurable. As the diary survey reports expenditures on a two-week basis, we first convert these to yearly equivalents. Next, we deseasonalize using a dummy regression approach.¹¹ In order to take into account that variation in expenditures can be driven by the household composition, e.g., the number of adults or the number of kids living in the family we deflate total expenditures by the Organisation for Economic Co-operation and Development (OECD) equivalence scale of the household. Given that we use the diary data, the expenditure level used should be interpreted as expenditures on nondurable consumption. The restriction to nondurables is valid if nondurable consumption is separable in the utility function.

Next, we control for several observable characteristics by restricting our sample to (i) households who have completed the two-week diary, (ii) households who are not living in student housing, (iii) households who are vehicle owners, (iv) households where both members work at least 17 hours, (v) households in which both members are not self-employed, (vi) households in which the age of the reference person is at least 21, and finally we restrict attention to (vii) households that consist of a husband, a wife, and possibly children. Finally, we also remove some outlier observations.¹² This leaves us with a sample size of 2,163 observations.

Results. Figure 1 plots the values of $\hat{\theta}_\pi(y) = \frac{\hat{z}_\pi(x) x}{\hat{z}_\pi(y) y}$ and $\hat{t}_m(y) = \left(\frac{\hat{\mu}^m(x)}{\hat{\mu}^m(y)}\right)^{1/m} \frac{x}{y}$, where x is the median expenditure level in 2007 and y ranges over the different quantiles of the expenditure level in the data set. We focus on the values of $\pi = 0.1, 0.15, 0.25, 0.75, 0.85,$ and 0.9 , which are sufficiently far away from the average in the sample such that $z_\pi(y) \neq 0$ and on the even moments $m = 2, 4, 6, 8$. Tables 2 and 3 in Appendix C contain the exact figures. If the HMUI hypothesis holds, then these graphs should trace out the marginal utility of income function $\beta(y)/\beta(x)$ for varying value of y . We see that in most cases the graphs are downward sloping, which is consistent with the widely accepted idea that the marginal utility of income is decreasing, i.e., one additional dollar is worth more to a poor person than to a rich person. The HMUI hypothesis holds if these various graphs are identical for varying values of π and m . Visual inspection shows that the graphs are indeed quite close to each other. Implementing our test gives

⁸I would like to thank a referee for pointing this out.

⁹This can be modelled by considering Z as a drifting sequence that converges to zero.

¹⁰Food is an aggregate of cereals, bakery products, beef, pork, poultry, seafood, other meat, eggs, milk products, other dairy products, fresh fruit, fresh vegetables, processed fruit, processed vegetables, sweets, fat and oils, non-alcoholic beverages, prepared food, snacks, and condiments.

¹¹Specifically, the expenditures on each category (reported for two weeks) are regressed on month dummies. Residuals from this regression (which can be interpreted as the variation in expenditures that cannot be explained by seasonality or by months) are added to the mean expenditures for each category in order to construct deseasonalized expenditures.

¹²In particular, we removed observations for which rescaled total expenditures or expenditure shares are not within three standard deviations from the mean and observations for which total rescaled expenditures are among the 5% lowest or 5% highest expenditures or for which the expenditure share is close to zero.

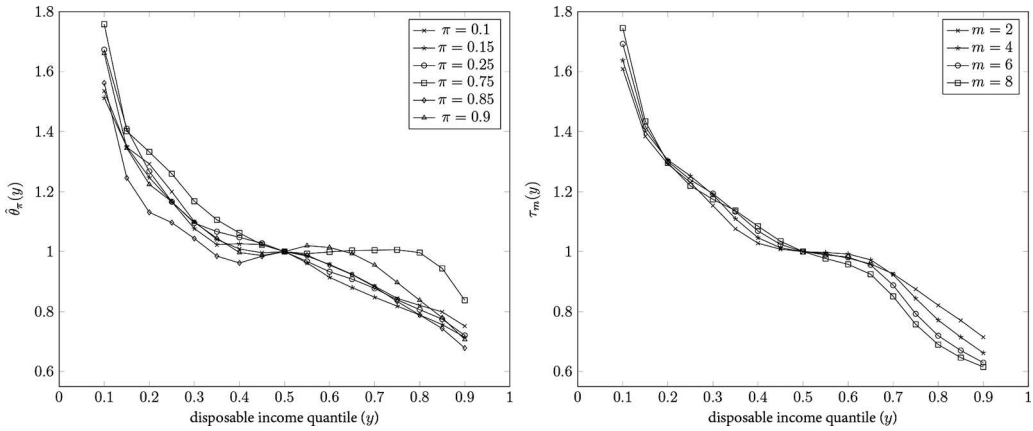


Figure 1. Estimates of the marginal utility of income ($\hat{\theta}_\pi(y^j)$ left and $\tau_m(y^j)$ right).

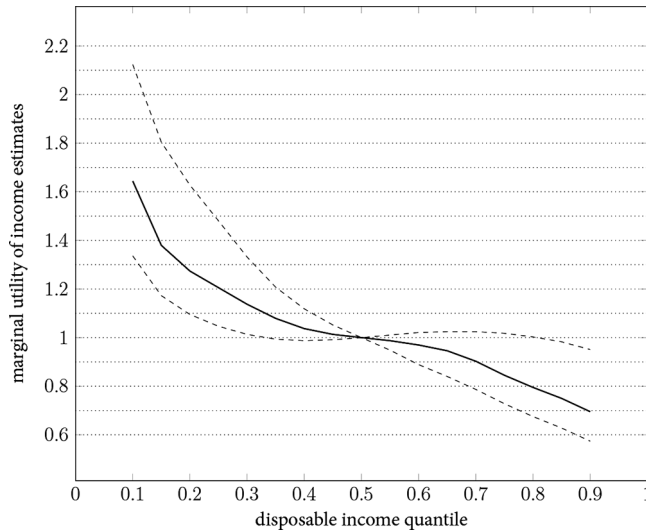


Figure 2. Estimates of the marginal utility of income with 90% confidence interval.

for a significance level of $\alpha = 0.05$ a critical value (i.e., $-\hat{c}_\alpha/\sqrt{nh_n} + \hat{Z}_n$) of -0.227 and a critical value of -0.1553 for a significance level of $\alpha = 0.10$. The p -value for the hypothesis that $Z = 0$ is equal to 0.4058 . As such, on the basis of our test, we do not reject the hypothesis that $Z = 0$.

Figure 2 and Table 4 give the estimate of the marginal utility of income function $\frac{\beta(y)}{\beta(x)}$ obtained as the average over all considered quantiles and moments together with the 95% pointwise CI based on the same subsampling procedure. The marginal utility of income function allows us to make interpersonal welfare comparisons. For example, redistributing one dollar from an individual at the median income level towards an individual at the first decile improves aggregate utility by $1.64 - 1 = 0.64$. In other words, in order to increase aggregate utility by one unit, one can either allocate \$1 to somebody at the median income level or \$0.61 (i.e., $1/1.64$) to somebody at the first decile. Giving the same dollar to someone at the 9th decile only increases aggregate utility by 0.69. Alternatively, in order to increase utility by one unit, one should allocate as much as 1.45\$ (i.e., $1/0.69$) to somebody at the 9th decile.

5. Conclusion

We developed a test to verify whether a population of individuals have the same marginal utility of income function. The homogeneous marginal utility of income assumption is frequently used in applied demand and welfare analyses. In addition, if the test is not rejected, it allows us to identify the marginal utility of income function. Interestingly, our test and identification results are modest in terms of data requirements. We apply our procedure to a U.S. cross-sectional data set on food consumption, and we find that we cannot reject the homogeneous marginal utility of income hypothesis. We estimate that (conditional on the HMUI assumption) one dollar is more than twice as valuable to someone at the first decile compared to someone at the ninth decile of the (disposable) income distribution.

Appendix A: Inconsistency of the bootstrap

In this appendix, we discuss why the bootstrap provides incorrect inference for testing the null hypothesis that the HMUI assumption holds. The arguments used here are very similar to the ones given by Andrews (2000).

Consider the setting where we only consider one value of θ and one value of τ that are estimated based on a finite sample of size n using estimators $\hat{\theta}_n$ and $\hat{\tau}_n$. Let

$$\sqrt{nh_n} \begin{pmatrix} \hat{\theta}_n - \theta \\ \hat{\tau}_n - \tau \end{pmatrix} \rightarrow \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} \sim J,$$

where J is a nondegenerate distribution. Usually, J will be a bivariate normal distribution with full support. Let $Z(a, b) = \max\{a, b\} - \min\{a, b\}$ be our test statistic, and define

$$T_n = \sqrt{nh_n} \left(Z(\hat{\theta}_n, \hat{\tau}_n) - Z(\theta, \tau) \right).$$

If the null hypothesis $Z(\theta, \tau) = 0$ holds, i.e., $\theta = \tau$, then by the continuous mapping theorem,

$$\begin{aligned} T_n &= \max\{\sqrt{nh_n}(\hat{\theta}_n - \theta), \sqrt{nh_n}(\hat{\tau}_n - \tau)\} - \min\{\sqrt{nh_n}(\hat{\theta}_n - \theta), \sqrt{nh_n}(\hat{\tau}_n - \tau)\}, \\ &\rightarrow \max\{W_1, W_2\} - \min\{W_1, W_2\} \end{aligned}$$

Consider now a bootstrap procedure to estimate the distribution of T_n . Denote by θ_n^* and τ_n^* the estimates of θ and τ based on a bootstrap sample of $\{(w_i, x_i), i = 1, \dots, n\}$. Assume that the bootstrap is valid in the sense that the joint distribution of $[\sqrt{nh_n}(\theta_n^* - \hat{\theta}_n), \sqrt{nh_n}(\tau_n^* - \hat{\tau}_n)]$ can be used to approximate the joint distribution of $[W_1, W_2]$, i.e., J .

Now, take the case where the null hypothesis holds, i.e., $Z(\theta, \tau) = 0$, and assume that the distribution J is such that there are values $c > 0$ and $\alpha > 0$ such that $\lim_{n \rightarrow \infty} \Pr(\sqrt{nh_n}(\hat{\theta}_n - \hat{\tau}_n) > c) > \alpha$. This will be the case, for example, if J is the bivariate normal distribution and W_1, W_2 are not perfectly correlated. Consider the event

$$A = \left\{ \omega \left| \limsup_{n \rightarrow \infty} \mathbb{1} \left[\sqrt{nh_n}(\hat{\theta}_n(\omega) - \hat{\tau}_n(\omega)) \geq c \right] = 1 \right. \right\},$$

where $\mathbb{1}[\cdot]$ is the indicator function. Then,

$$\begin{aligned} \alpha &< \limsup_{n \rightarrow \infty} \Pr(\sqrt{nh_n}(\hat{\theta}_n - \hat{\tau}_n) \geq c), \\ &\leq \Pr \left(\limsup_{n \rightarrow \infty} \mathbb{1}[\sqrt{nh_n}(\hat{\theta}_n - \hat{\tau}_n) \geq c] = 1 \right), \\ &= \Pr(A). \end{aligned}$$

From the Hewitt–Savage zero–one law, one establishes that $\Pr(A) = 1$.¹³ As such, for each $\omega \in A$, we have that $\sqrt{nh_n}(\hat{\theta}_n(\omega) - \hat{\tau}_n(\omega)) > c$ occurs infinitely often. For a fixed $\omega \in A$, consider the infinite subsequence $\{n_k | k \geq 1\}$ of $\{n | n \geq 1\}$ for which $\sqrt{n_k h_{n_k}}(\hat{\theta}_{n_k}(\omega) - \hat{\tau}_{n_k}(\omega)) \geq c$. Observe that in this sequence $\hat{\theta}_{n_k} > \hat{\tau}_{n_k}$, so $Z(\hat{\theta}_{n_k}, \hat{\tau}_{n_k}) = \sqrt{n_k h_{n_k}}(\hat{\theta}_{n_k} - \hat{\tau}_{n_k}) \geq c$. Then,

$$\begin{aligned} \sqrt{n_k h_{n_k}} \left(Z(\theta_{n_k}^*, \tau_{n_k}^*) - Z(\hat{\theta}_{n_k}, \hat{\tau}_{n_k}) \right) &= \sqrt{n_k h_{n_k}} \max\{\theta_{n_k}^* - \hat{\theta}_{n_k}, \tau_{n_k}^* - \hat{\tau}_{n_k}\}, \\ &\quad - \sqrt{n_k h_{n_k}} \min\{\theta_{n_k}^* - \hat{\tau}_{n_k}, \tau_{n_k}^* - \hat{\tau}_{n_k}\}, \\ &= \sqrt{n_k h_{n_k}} \max\{(\theta_{n_k}^* - \hat{\theta}_{n_k}), (\tau_{n_k}^* - \hat{\tau}_{n_k}) + (\hat{\tau}_{n_k} - \hat{\theta}_{n_k})\}, \\ &\quad - \sqrt{n_k h_{n_k}} \min\{(\theta_{n_k}^* - \hat{\theta}_{n_k}) + (\hat{\theta}_{n_k} - \hat{\tau}_{n_k}), (\tau_{n_k}^* - \hat{\tau}_{n_k})\}, \\ &\leq \max\{\sqrt{n_k h_{n_k}}(\theta_{n_k}^* - \hat{\theta}_{n_k}), \sqrt{n_k h_{n_k}}(\tau_{n_k}^* - \hat{\tau}_{n_k}) - c\}, \\ &\quad - \min\{\sqrt{n_k h_{n_k}}(\theta_{n_k}^* - \hat{\theta}_{n_k}) + c, \sqrt{n_k h_{n_k}}(\tau_{n_k}^* - \hat{\tau}_{n_k})\}, \\ &\rightarrow \max\{W_1, W_2 - c\} - \min\{W_1 + c, W_2\}, \\ &\leq \max\{W_1, W_2\} - \min\{W_1, W_2\}. \end{aligned}$$

Here, the last inequality is strict with positive probability. Along the subsequence n_k , the sequence $\sqrt{n_k h_{n_k}}(Z_{n_k}^* - \hat{Z}_{n_k}) \not\rightarrow \max\{W_1, W_2\} - \min\{W_1, W_2\}$. Conclude that $\sqrt{nh_n}(Z_n^* - \hat{Z}_n) \not\rightarrow \max\{W_1, W_2\} - \min\{W_1, W_2\}$ as $n \rightarrow \infty$ conditional on the empirical distribution. This is true for all $\omega \in A$, so with probability one, the bootstrap is not consistent.

Simulation Results. As an additional exercise to demonstrate the invalidity of the bootstrap, we perform a simple Monte Carlo simulation. Let

$$Z = \max_i \{\max\{\mu^i | i = 1, \dots, T\} - \min\{\mu^i | i = 1, \dots, T\}\} = 0,$$

where $\{\mu^i | i = 1, \dots, T\}$ are the means of n independent and identically distributed (i.i.d.) normally distributed variables with mean zero and variance σ^i . Consider the finite sample test statistic

$$\hat{Z}_n = \max_i \{\max\{\hat{\mu}_n^i | i = 1, \dots, T\} - \min\{\hat{\mu}_n^i | i = 1, \dots, T\}\},$$

where $\hat{\mu}_n^i$ is the sample average for a sample of size n . Clearly, $\sqrt{n}(\hat{Z}_n - Z) = \sqrt{n}\hat{Z}_n$ has a non-degenerate distribution, so we focus on testing the null hypothesis $Z = 0$. We take four scenarios with varying values of T , namely $T = 4, 8, 16$, and 32 . We set the lowest variance, say σ^1 , equal to one. The other values increase with 0.5 increments, i.e., $\sigma^2 = 1.5, \sigma^3 = 2, \dots$

For each scenario, we consider sample sizes of $n = 1,000, 2,000, 3,000$ and $4,000$, and for each of these cases, we perform 1000 Monte Carlo iterations. For each iteration, we verify whether the null hypothesis

Table 1. Simulation results.

| | Scenario I | | Scenario II | | Scenario III | | Scenario IV | |
|-------------|------------|---------|-------------|---------|--------------|---------|-------------|---------|
| | $T = 4$ | | $T = 8$ | | $T = 16$ | | $T = 32$ | |
| sample size | subs | bootstr | subs | bootstr | subs | bootstr | subs | bootstr |
| 1000 | 0.04 | 0.19 | 0.03 | 0.30 | 0.03 | 0.46 | 0.04 | 0.66 |
| 2000 | 0.04 | 0.22 | 0.03 | 0.29 | 0.04 | 0.43 | 0.03 | 0.61 |
| 3000 | 0.04 | 0.20 | 0.04 | 0.33 | 0.05 | 0.43 | 0.05 | 0.61 |
| 4000 | 0.05 | 0.21 | 0.05 | 0.32 | 0.04 | 0.43 | 0.05 | 0.60 |

¹³This follows from the fact that A is determined by the random variables $\{(w_i, x_i), i = 1, \dots\}$, which are independently distributed, and that A is invariant to finite permutations of the indices $i = 1, 2, \dots$.

($Z = 0$) is rejected at the 95% significance level based on 500 subsamples of size $n^{0.6}$ and 500 bootstrap samples. Table 1 gives the fraction of Monte Carlo iterations where the null hypothesis is rejected. As can be seen from the tables, the subsampling procedure gives rejection probabilities close to 0.05. The bootstrap on the other hand is clearly not consistent. In this setting, the rejection probability is too high, ranging from around 20% in scenario I up to 60% in scenario IV.

Appendix B: Proof of Proposition 2

Let $\bar{w}(x) = \mathbb{E}_j(w_j|x_j = x)$ and $\Pr(w_j - \bar{w}(x) \leq z_\pi(x)|x_j = x) = \pi$.

First of all, observe that $\int_0^1 z_\pi(x) d\pi = 0$ as $z_\pi(x)$ is the quantile of a conditional error term which has, by construction, mean zero. Also (by assumption), $z_\pi(x)$ is strictly increasing in $\pi \in [0, 1]$.

Next, $z_0(x) = \inf_j\{w_j - \bar{w}(x)|x_j = x\}$ and $z_1(x) = \sup_j\{w_j - \bar{w}(x)|x_j = x\}$. So,

$$z_0(x) + \bar{w}(x) = \inf_j\{w_j|x_j = x\} \geq 0,$$

$$z_1(x) + \bar{w}(x) = \sup_j\{w_j|x_j = x\} \leq 1.$$

Part 1. Let $r \sim U[0, 1]$ be a uniformly distributed random variable, distributed independently from x . Define the share functions

$$s_r(x) = \bar{w}(x) + z_r(x),$$

for $x \in I$. In other words, we define the share function of individual r with the quantiles of the observed conditional share functions.

Fact 1: $s_r(x) \in [0, 1]$. We have that

$$s_r(x) = \bar{w}(x) + z_r(x) \leq \bar{w}(x) + z_1(x) \leq 1,$$

$$s_r(x) = \bar{w}(x) + z_r(x) \geq \bar{w}(x) + z_0(x) \geq 0.$$

This shows that $s_r(x) : I \rightarrow [0, 1]$.

Fact 2: $\mathbb{E}_r(s_r(x)) = \bar{w}(x)$. Indeed,

$$\mathbb{E}_r(s_r(x)) = \int_0^1 (\bar{w}(x) + z_r(x)) dr = \bar{w}(x)$$

Fact 3: $\Pr(s_r(x) \leq z_\pi(x)) = \pi$.

$$\begin{aligned} \Pr(s_r(x) - \bar{w}(x) \leq z_\pi(x)) &= \int_0^1 I(\bar{w}(x) + z_r(x) - \bar{w}(x) \leq z_\pi(x)) dr, \\ &= \int_0^1 I(z_r(x) \leq z_\pi(x)) dr, \\ &= \int_0^1 I(r \leq \pi) dr = \pi. \end{aligned}$$

The last line is due to the strict monotonicity of $z_\pi(x)$ in π .

Fact 4: $s_r(x)$ satisfies condition (2). For all $x, y \in I$ and $\pi \in [0, 1]$, we have that

$$z_\pi(x)x\beta(x) = z_\pi(y)y\beta(y).$$

As such, the values $c_\pi \equiv z_\pi(x)x\beta(x)$ only depend on π . Substituting,

$$s_r(x) = \bar{w}(x) + \frac{c_r}{x\beta(x)} = \frac{c_r + \bar{w}(x)x\beta(x)}{x\beta(x)}.$$

Define $-a_r = c_r$ and $-b(x) = \bar{w}(x)x\beta(x)$. Then,

$$s_r(x) = -\frac{a_r + b(x)}{x\beta(x)},$$

which has the HMUI specification (2).

Part II. As before, define $c_\pi = z_\pi(x)x\beta(x)$, which is independent of $x \in I$. Observe that $\int_0^1 c_\pi d\pi = x\beta(x) \int_0^1 z_\pi(x) d\pi = 0$. In addition, c_π is strictly increasing and continuous in π (as $z_\pi(x)$ is strict continuous and increasing in π). Let $\delta = 1/N$ for some large $N \in \mathbb{N}$. We will later on choose the specific value of δ . Then, there exists a strictly increasing (continuous) function $g : [0, \delta/2] \rightarrow \mathbb{R}$ such that

$$c_\pi = g\left(\frac{\pi\delta}{2}\right).$$

Consider the function $h(r) : [0, \delta] \rightarrow \mathbb{R}$ where

$$h(r) = \begin{cases} g(\delta/2 - r) & \text{if } 0 \leq r \leq \delta/2, \\ g(r - \delta/2) & \text{if } \delta \geq r \geq \delta/2. \end{cases}$$

The function g is continuous on $[0, \delta]$ strictly decreasing on $[0, \delta/2]$ and strictly increasing on $[\delta/2, \delta]$.

For $r \in \mathbb{R}$, let $\lfloor r \rfloor$ equal $(r \bmod \delta)$ which is contained in $[0, \delta]$. Now, extend $h(r)$ to the domain \mathbb{R}_+ by defining

$$\ell(r) = h(\lfloor r \rfloor).$$

Observe that the function $\ell : \mathbb{R}_+ \rightarrow [c_0, c_1]$ is periodic in the sense that $\ell(x) = \ell(x + a\delta)$ for all $a \in \mathbb{N}$.

Consider the uniformly distributed random variable $r \sim U(0, \delta)$, independently distributed from x , and specify the stochastic functions,

$$s_r(x) = \frac{\ell(x+r)}{x\beta(x)} + \bar{w}(x) = \frac{h(\lfloor x+r \rfloor)}{x\beta(x)} + \bar{w}(x).$$

Fact 1: $s_r(x) \in [0, 1]$. First observe that for all x , $z_0(\pi)x\beta(x) = c_0$ and $z_1(\pi)x\beta(x) = c_1$. Also, for all $r \in [0, \delta]$:

$$c_1 = h(\delta) > h(r) > h(\delta/2) = c_0.$$

Given this,

$$z_0(x) = \frac{c_0}{x\beta(x)} \leq s_r(x) - \bar{w}(x) = \frac{h(\lfloor x+r \rfloor)}{x\beta(x)} \leq \frac{c_1}{x\beta(x)} = z_1(x).$$

This shows that $z_0(x) \leq s_r(x) - \bar{w}(x) \leq z_1(x)$. Then,

$$0 \leq z_0(x) + \bar{w}(x) \leq s_r(x) \leq z_1(x) + \bar{w}(x) \leq 1.$$

Fact 2: $\mathbb{E}_r(s_r(x)) = \bar{w}(x)$. We have

$$\begin{aligned} \mathbb{E}_r(\ell(x+r_j)) &= \frac{1}{\delta} \int_0^\delta \ell(x+r) dr = \frac{1}{\delta} \int_x^{x+\delta} \ell(r) dr, \\ &= \frac{1}{\delta} \int_x^0 \ell(r) dr + \frac{1}{\delta} \int_0^{x+\delta} \ell(r) dr = \frac{1}{\delta} \int_x^0 \ell(r+\delta) dr + \frac{1}{\delta} \int_0^{x+\delta} \ell(r) dr, \\ &= \frac{1}{\delta} \int_{x+\delta}^\delta \ell(r) dr + \frac{1}{\delta} \int_0^{x+\delta} \ell(r) dr = \frac{1}{\delta} \int_0^\delta \ell(r) dr = \frac{1}{\delta} \int_0^\delta h(r) dr, \\ &= \frac{1}{\delta} \int_0^{\delta/2} h(r) dr + \frac{1}{\delta} \int_{\delta/2}^\delta h(r) dr, \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{\delta} \int_0^{\delta/2} g(\delta/2 - r)dr + \frac{1}{\delta} \int_{\delta/2}^{\delta} g(r - \delta/2)dr, \\
 &= -\frac{1}{\delta} \int_{\delta/2}^0 g(r)dr + \frac{1}{\delta} \int_0^{\delta/2} g(r)dr = \frac{1}{\delta} 2 \int_0^{\delta/2} g(r)dr, \\
 &= \int_0^1 g(\pi\delta/2)d\pi = \int_0^1 c_\pi d\pi = 0.
 \end{aligned}$$

The second line uses the periodicity of ℓ . The third line uses the fact that ℓ and h are identical on $[0, \delta]$. Conclude that, $\mathbb{E}_r(s_r(x)) = \frac{\mathbb{E}_r(\ell(x+r))}{x\beta(x)} + \bar{w}(x) = \bar{w}(x)$.

Fact 3: $\Pr(s_r(x) - \bar{w}(x) \leq z_\pi(x)) = \pi$. We have

$$\Pr(s_j(x) - \bar{w}(x) \leq z_\pi(x)) = \Pr(\ell(x + r_j) \leq z_\pi(x)x\beta(x)) = \Pr(\ell(x + r_j) \leq c_\pi).$$

Then,

$$\begin{aligned}
 \Pr(\ell(x + r_j) \leq c_\pi) &= \frac{1}{\delta} \int_0^\delta I(\ell(x + r) \leq c_\pi)dr, \\
 &= \frac{1}{\delta} \int_x^{x+\delta} I(\ell(r) \leq c_\pi)dr = \frac{1}{\delta} \int_x^0 I(\ell(r) \leq c_\pi)dr + \frac{1}{\delta} \int_0^{x+\delta} I(\ell(r) \leq c_\pi)dr, \\
 &= \frac{1}{\delta} \int_0^{x+\delta} I(\ell(r) \leq c_\pi)dr + \frac{1}{\delta} \int_x^0 I(\ell(r + \delta) \leq c_\pi)dr, \\
 &= \frac{1}{\delta} \int_0^{x+\delta} I(\ell(r) \leq c_\pi)dr + \frac{1}{\delta} \int_{x+\delta}^\delta I(\ell(r) \leq c_\pi)dr, \\
 &= \frac{1}{\delta} \int_0^\delta I(h(r) \leq c_\pi)dr = \frac{1}{\delta} \int_0^{\delta/2} I(h(r) \leq c_\pi)dr + \frac{1}{\delta} \int_{\delta/2}^\delta I(h(r) \leq c_\pi)dr, \\
 &= \frac{1}{\delta} \int_0^{\delta/2} I(g(\delta/2 - r) \leq c_\pi)dr + \frac{1}{\delta} \int_{\delta/2}^\delta I(g(r - \delta/2) \leq c_\pi)dr, \\
 &= -\frac{1}{\delta} \int_{\delta/2}^0 I(g(r) \leq c_\pi)dr + \frac{1}{\delta} \int_0^{\delta/2} I(g(r) \leq c_\pi)dr, \\
 &= \frac{1}{\delta} 2 \int_0^{\delta/2} I(g(r) \leq c_\pi)dr = \frac{1}{\delta} 2 \int_0^{\delta/2} I(r \leq g^{-1}(c_\pi))dr, \\
 &= \frac{1}{\delta} 2 \int_0^{g^{-1}(c_\pi)} dr = \frac{1}{\delta} 2g^{-1}(c_\pi) = \frac{1}{\delta} 2 \frac{\pi\delta}{2} = \pi.
 \end{aligned}$$

Fact 4: the functions $s_r(x)$ do not satisfy (2). Observe that if HMUI is satisfied, then for any r and $p \in [0, 1]$ and for all $x, y > 0$, it must be that,

$$\frac{s_r(x) - \mathbb{E}_v(s_v(x))}{s_p(x) - \mathbb{E}_v(s_v(x))} = \frac{s_r(y) - \mathbb{E}_v(s_v(y))}{s_p(y) - \mathbb{E}_v(s_v(y))}.$$

In other words, we need that for all $x, y \in I$ and all $r, p \in [0, \delta]$,

$$\frac{h(\lfloor x + r \rfloor)}{h(\lfloor x + p \rfloor)} = \frac{h(\lfloor y + r \rfloor)}{h(\lfloor y + p \rfloor)}.$$

We can always find numbers $0 < r_1 < \delta/2$ and $\delta/2 < r_2 < \delta$ such that $h(r) > 0$ for all $0 < r < r_1$ and for all $\delta > r > r_2$.

Take δ small enough such that there is an $a \in \mathbb{N}$ for which $[a\delta, (a+1)\delta] \subseteq I$. Let $x, y \in I$, $r, p \in [0, \delta]$ such that $0 < \lfloor x+r \rfloor < \lfloor x+p \rfloor < r_1$ and $r_2 < \lfloor y+r \rfloor < \lfloor y+p \rfloor < \delta$ (this can always be done by taking r and p sufficiently small and $r < p$, taking x strictly above but close to $a\delta$ and choosing y strictly below but very close to $(a+1)\delta$). Observe that $h(\lfloor x+r \rfloor) > h(\lfloor x+p \rfloor) > 0$ (as h is strictly decreasing on $[0, \delta/2]$) and $0 < h(\lfloor y+r \rfloor) < h(\lfloor y+p \rfloor)$ (as h is strictly increasing on $[\delta/2, \delta]$). As such,

$$\frac{\ell(x+r)}{\ell(x+p)} > 1 > \frac{\ell(y+r)}{\ell(y+p)},$$

so the HMUI assumption is not satisfied.

Appendix C: Tables

Table 2. Estimates of $\beta(y)/\beta(x)$ using the conditional quantiles.

| Percentile | $\pi = 0.1$ | $\pi = 0.15$ | $\pi = 0.25$ | $\pi = 0.75$ | $\pi = 0.85$ | $\pi = 0.9$ |
|------------|-------------|--------------|--------------|--------------|--------------|-------------|
| 0.10 | 1.5353 | 1.5125 | 1.6733 | 1.7581 | 1.5621 | 1.6611 |
| 0.15 | 1.3481 | 1.3493 | 1.4099 | 1.4014 | 1.2456 | 1.3451 |
| 0.20 | 1.2923 | 1.2470 | 1.2673 | 1.3322 | 1.1311 | 1.2242 |
| 0.25 | 1.1996 | 1.1655 | 1.1662 | 1.2594 | 1.0964 | 1.1662 |
| 0.30 | 1.0997 | 1.0767 | 1.0951 | 1.1680 | 1.0435 | 1.0974 |
| 0.35 | 1.0402 | 1.0232 | 1.0664 | 1.1059 | 0.9848 | 1.0449 |
| 0.40 | 1.0088 | 1.0261 | 1.0475 | 1.0620 | 0.9621 | 0.9968 |
| 0.45 | 0.9960 | 1.0233 | 1.0279 | 1.0228 | 0.9843 | 0.9869 |
| 0.50 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| 0.55 | 0.9851 | 0.9622 | 0.9673 | 0.9928 | 0.9883 | 1.0197 |
| 0.60 | 0.9570 | 0.9142 | 0.9328 | 0.9989 | 0.9552 | 1.0138 |
| 0.65 | 0.9243 | 0.8800 | 0.9082 | 1.0038 | 0.9231 | 0.9933 |
| 0.70 | 0.8846 | 0.8479 | 0.8773 | 1.0046 | 0.8825 | 0.9552 |
| 0.75 | 0.8440 | 0.8182 | 0.8394 | 1.0061 | 0.8337 | 0.8970 |
| 0.80 | 0.8208 | 0.7887 | 0.8064 | 0.9965 | 0.7887 | 0.8375 |
| 0.85 | 0.7988 | 0.7566 | 0.7747 | 0.9439 | 0.7434 | 0.7795 |
| 0.90 | 0.7518 | 0.7149 | 0.7206 | 0.8377 | 0.6785 | 0.7069 |

Table 3. Estimates of $\beta(y)/\beta(x)$ using the conditional moments.

| Percentile | $m = 2$ | $m = 4$ | $m = 6$ | $m = 8$ |
|------------|---------|---------|---------|---------|
| 0.10 | 1.6090 | 1.6376 | 1.6925 | 1.7452 |
| 0.15 | 1.3842 | 1.4005 | 1.4172 | 1.4341 |
| 0.20 | 1.2917 | 1.3055 | 1.3009 | 1.2960 |
| 0.25 | 1.2304 | 1.2523 | 1.2394 | 1.2192 |
| 0.30 | 1.1530 | 1.1901 | 1.1935 | 1.1741 |
| 0.35 | 1.0758 | 1.1097 | 1.1331 | 1.1367 |
| 0.40 | 1.0279 | 1.0462 | 1.0687 | 1.0845 |
| 0.45 | 1.0077 | 1.0125 | 1.0238 | 1.0346 |
| 0.50 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| 0.55 | 0.9927 | 0.9968 | 0.9891 | 0.9768 |
| 0.60 | 0.9782 | 0.9919 | 0.9814 | 0.9573 |
| 0.65 | 0.9588 | 0.9725 | 0.9558 | 0.9244 |
| 0.70 | 0.9257 | 0.9230 | 0.8877 | 0.8506 |
| 0.75 | 0.8747 | 0.8439 | 0.7925 | 0.7574 |
| 0.80 | 0.8209 | 0.7714 | 0.7195 | 0.6899 |
| 0.85 | 0.7704 | 0.7147 | 0.6702 | 0.6464 |
| 0.90 | 0.7146 | 0.6623 | 0.6297 | 0.6154 |

Table 4. Estimates of $\beta(y)/\beta(x)$ with the 90% confidence bounds.

| Percentile | Lower bound | Estimate | Upper bound |
|------------|-------------|----------|-------------|
| 0.10 | 1.3361 | 1.6440 | 2.1235 |
| 0.15 | 1.1726 | 1.3794 | 1.8040 |
| 0.20 | 1.0952 | 1.2737 | 1.6274 |
| 0.25 | 1.0466 | 1.2054 | 1.4803 |
| 0.30 | 1.0134 | 1.1372 | 1.3313 |
| 0.35 | 0.9935 | 1.0790 | 1.2073 |
| 0.40 | 0.9875 | 1.0370 | 1.1184 |
| 0.45 | 0.9911 | 1.0133 | 1.0520 |
| 0.50 | 1.0000 | 1.0000 | 1.0000 |
| 0.55 | 0.9483 | 0.9874 | 1.0106 |
| 0.60 | 0.8888 | 0.9696 | 1.0208 |
| 0.65 | 0.8400 | 0.9458 | 1.0246 |
| 0.70 | 0.7868 | 0.9027 | 1.0245 |
| 0.75 | 0.7280 | 0.8451 | 1.0177 |
| 0.80 | 0.6752 | 0.7951 | 1.0028 |
| 0.85 | 0.6280 | 0.7499 | 0.9827 |
| 0.90 | 0.5740 | 0.6953 | 0.9509 |

References

- Andrews, D. W. K. (2000). Inconsistency of the bootstrap when a parameter is on the bound of the parameter space. *Econometrica* 68:399–405.
- Andrews, D. W. K., Guggenberger, P. (2009a). Hybrid and size-corrected subsampling methods. *Econometrica* 77:721–762.
- Andrews, D. W. K., Guggenberger, P. (2009b). Validity of subsampling and “plug-in asymptotic” inference for parameters defined by moment inequalities. *Econometric Theory* 25:669–709.
- Bhattacharya, D. (2015). Nonparametric welfare analysis for discrete choice. *Econometrica* 83:617–649.
- Blundell, R., Browning, M., Crawford, I. (2003). Nonparametric engel curves and revealed preference. *Econometrica* 71:205–240.
- Blundell, R., Horowitz, J. L., Parey, M. (2012). Measuring the price responsiveness of gasoline demand: Economic shape restrictions and nonparametric demand estimation. *Quantitative Economics* 3:29–51.
- Blundell, R., Kristensen, D., Matzkin, R. (2014). Bounding quantile demand functions using revealed preference inequalities. *Journal of Econometrics* 179:112–127.
- Deaton, A., Muellbauer, J. (1980). *Economics and Consumer Behavior*. Cambridge University Press, Cambridge, New York, Melbourne and Madrid.
- Dette, H., Hoderlein, S., Neumeier, N. (2016). Testing multivariate economic restrictions using quantiles: The example of Slutsky negative semidefiniteness. *Journal of Econometrics* 191:129–144.
- Fan, J., Yao, Q. (1998). Efficient estimation of conditional variance functions in stochastic regressions. *Biometrika* 85:645–660.
- Frisch, R. (1932). *New Methods for Measuring Marginal Utility*. Tubingen: Verlag von J.C.B. Mohr.
- Gorman, W. M. (1961). On a class of preference fields. *Metroeconomica* 13:53–56.
- Hausman, J. A., Newey, W. K. (1995). Nonparametric estimation of exact consumers surplus and deadweight loss. *Econometrica* 63:1445–1476.
- Hausman, J. A., Newey, W. K. (2016). Individual heterogeneity and average welfare. *Econometrica* 84:1225–1248.
- Hoderlein, S., Vanhems, A. (2013). Estimating the distribution of welfare effects using quantiles. Boston College Working Papers in Economics 893.
- Horowitz, J. L. (2001). The bootstrap. In: *Handbook of Econometrics*. 1st ed. Oxford: Elsevier, pp. 3159–3228.
- Imbens, G. W., Newey, W. K. (2009). Identification and estimation of triangular simultaneous equations models without additivity. *Econometrica* 77:1481–1512.
- Kitagawa, T. (2015). A test for instrument validity. *Econometrica* 83:2043–2063.
- Layard, R., Nickell, S., Mayraz, G. (2008). The marginal utility of income. *Journal of Public Economics* 92:1846–1857.
- Lewbel, A. (1989). Exact aggregation and a representative consumer. *Quarterly Journal of Economics* 104:621–633.
- Lewbel, A. (2001). Demand systems with and without errors. *American Economic Review* 91:611–618.
- Li, Q., Racine, S. (2007). *Nonparametric Econometrics*. Princeton: Princeton University Press.
- Morgan, J. N. (1945). Can we measure the marginal utility of money? *Econometrica* 13:129–152.
- Mourifié, I., Wan, Y. (2015). Testing local average treatment effect assumptions. Forthcoming in the Review of Economic Studies.
- Politis, D. N., Romano, J. P., Wolf, M. (1999). *Subsampling*. Springer, Verlag, New York.
- Samuelson, P. A. (1937). A note on measurement of utility. *The Review of Economic Studies* 4:155–161.
- Schlee, E. E. (2007). Measuring consumer welfare with mean demands. *International Economic Review* 48:869–899.
- Vickrey, W. (1945). Measuring marginal utility by reactions to risk. *Econometrica* 13:319–333.
- Zhou, T., Zhu, L. (2015). Conditional median absolute deviation. *Journal of Statistical Computation and Simulation* 85:2101–2114.