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On the invertibility of EGARCH(*p*, *q*)

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ABSTRACT

Of the two most widely estimated univariate asymmetric conditional volatility models, the exponential GARCH (or EGARCH) specification is said to be able to capture asymmetry, which refers to the different effects on conditional volatility of positive and negative effects of equal magnitude, and leverage, which refers to the negative correlation between the returns shocks and subsequent shocks to volatility. However, the statistical properties of the (quasi-)maximum likelihood estimator (QMLE) of the EGARCH(p, q) parameters are not available under general conditions, but only for special cases under highly restrictive and unverifiable sufficient conditions, such as EGARCH(1,0) or EGARCH(1,1), and possibly only under simulation. A limitation in the development of asymptotic properties of the QMLE for the EGARCH(p, q) model is the lack of an invertibility condition for the returns shocks underlying the model. It is shown in this article that the EGARCH(p, q) model can be derived from a stochastic process, for which sufficient invertibility conditions can be stated simply and explicitly when the parameters respect a simple condition.[1] This will be useful in reinterpreting the existing properties of the QMLE of the EGARCH(p, q) parameters.

KEYWORDS
Asymmetry; asymptotic properties; existence; invertibility; leverage; stochastic process; sufficient conditions

JEL CLASSIFICATION
C22; C52; C58; G32

1. Introduction

In addition to modeling and forecasting volatility, and capturing clustering, two key characteristics of univariate time-varying conditional volatility models in the GARCH class of Engle (1982) and Bollerslev (1986) are asymmetry and (possible) leverage. Asymmetry refers to the different impacts on volatility of positive and negative shocks of equal magnitude, whereas leverage, as a special case of asymmetry, captures the negative correlation between the returns shocks and subsequent shocks to volatility. Black (1976) defined leverage in terms of the debt-to-equity ratio, with increases in volatility arising from negative shocks to returns and decreases in volatility arising from positive shocks to returns.

The two most widely estimated asymmetric univariate models of conditional volatility are the exponential GARCH (or EGARCH) model of Nelson (1990, 1991), and the Glosten, Jagannathan and Runkle (GJR) (alternatively, asymmetric, or threshold) model of Glosten et al. (1992). As EGARCH is a discrete-time approximation to a continuous-time stochastic volatility process, and is expressed in logarithms, conditional volatility is guaranteed to be positive without any restrictions on the parameters.

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Color versions of one or more of the figures in the article can be found online at www.tandfonline.com/lecr.

[1]Using the notation introduced in part 2, this refers to the cases where $\alpha \geq |\gamma|$ or $\alpha \leq -|\gamma|$. The first inequality is generally assumed in the literature related to the invertibility of EGARCH. This article provides (in the Appendix) an argument for the possible lack of invertibility when these conditions are not met.

In order to capture (possible) leverage, the EGARCH model requires parametric restrictions to be satisfied. Leverage is not possible for GJR, unless the short run persistence parameter is negative, which is not consistent with the standard sufficient condition for conditional volatility to be positive, or for the process to be consistent with a random coefficient autoregressive model (see McAleer, 2014; McAleer and Hafner, 2014).

As GARCH can be obtained from random coefficient autoregressive models (see Tsay, 1987), and similarly for GJR (see McAleer et al., 2007; McAleer, 2014), the statistical properties for the (quasi-)maximum likelihood estimator (QMLE) of the GARCH and GJR parameters are straightforward to establish. However, the statistical properties for the QMLE of the EGARCH parameters are not available under general conditions. A limitation in the development of asymptotic properties of the QMLE for EGARCH is the lack of an invertibility condition for the returns shocks underlying the model.

McAleer and Hafner (2014) showed that EGARCH(1,1) could be derived from a random coefficient complex nonlinear moving average (RCCNMA) process. The reason for the lack of statistical properties of the QMLE of EGARCH(p, q) under general conditions is that the stationarity and invertibility conditions for the RCCNMA process are not known, except possibly under simulation, in part because the RCCNMA process is not in the class of random coefficient linear moving average models (for further details, see Marek (2005).

The recent literature on the asymptotic properties of the QMLE of EGARCH shows that such properties are available only for some special cases, and typically under highly restrictive and unverifiable conditions. For example, Straumann and Mikosch (2006) derive some asymptotic results for the simple EGARCH(1,1) model, but their regularity conditions are difficult to interpret or verify. Wintenberger (2013) proves consistency and asymptotic normality for the QMLE of EGARCH(1,1) under the nonverifiable assumption of invertibility of the model. Francq et al. (2013) show that the QMLE of the EGARCH(1,1) model is strongly consistent and asymptotically normal under strong assumptions. Demos and Kyriakopoulou (2014) present sufficient conditions for asymptotic normality under highly restrictive conditions that are difficult to verify. Anyfantaki and Demos (2016) derive exact likelihood-based estimators for the theoretical properties of the time-varying parameter EGARCH(1,1)-in-Mean model. However, as the expression for the likelihood function is unknown, they resort to simulation methods to estimate the parameters.

This article considers the more general EGARCH(p, q) model. It is shown that the EGARCH(p, q) model can be derived from a stochastic process, for which sufficient invertibility conditions can be stated simply and explicitly when the parameters respect a simple condition ($\alpha \geq |\gamma|$ or $\alpha \leq -|\gamma|$), so that verification is possible rather than assumed, as is typical in the literature. This will be useful in reinterpreting the existing properties of the QMLE of the EGARCH(p, q) parameters.

The remainder of the article is organized as follows. In Section 2, the EARCH(∞) model is discussed, together with notation and lemmas. Section 3 develops a key result for invertibility of the EARCH(∞) model. Section 4 analyzes the EGARCH(p, q) specification, while Section 5 develops regularity conditions for the invertibility of EGARCH(p, q). Section 6 considers the special case of the N(0,1) distribution. Some concluding comments are given in Section 7. Proofs of the lemmas and propositions are given in the Appendix.

2. EARCH(∞), notation, and lemmas

Instead of using a recursive equation for conditional volatility, which would require proofs of existence and uniqueness, we will work with a direct definition of the stochastic process that drives the so-called innovations, ε_t. The new process will define uniquely the stochastic process that drives the innovation, as follows:

$$\varepsilon_t = \eta_t . \exp\left(\frac{\omega}{2} + \sum_{i=1}^{+\infty} \beta_i \left(\frac{\alpha}{2}|\eta_{t-i}| + \frac{\gamma}{2}\eta_{t-i}\right)\right), \qquad (0)$$

where $\omega \in \Re$, $(\alpha, \gamma) \in \Re^2$, $\sum_i |\beta_i| < \infty$,$(\eta_t)$ is independently and identically distributed (i.i.d.), with $E[\eta_t] = 0$ and $E\left[\eta_t^2\right] = 1$, so that $\eta_t \in L^2$. Thus, we have the EARCH(∞) model, as introduced by

Nelson (1990, 1991):

$$\begin{cases} \log(\sigma_t^2) \equiv \omega + \sum_{i=1}^{\infty} \beta_i \left(\alpha \left|\eta_{t-i}\right| + \gamma \eta_{t-i}\right) \\ \varepsilon_t = \eta_t \sigma_t. \end{cases}$$

The primary purpose of this article is to establish invertibility of the model, where invertibility refers to the fact that the normalized shocks, η_t, may be written in terms of the previous observed values, that is, η_t is $\sigma\left(\varepsilon_t, \varepsilon_{t-1}, \ldots\right)$-adapted. Note that this definition is equivalent to that used by Wintenberger (2013), namely, that σ_t is $\sigma\left(\varepsilon_{t-1}, \varepsilon_{t-2}, \ldots\right)$-adapted. This is also equivalent, for instance, to invertibility, as defined in Tong (1990), and referred to as "global invertibility" in Sorokin (2011).

In a similar manner to proving invertibility for the Moving Average (MA) case, we will approximate recursively all the i.i.d. shocks in terms of the past observed shocks (that is, ε_t) and some arbitrary fixed initial values, and then prove that this backward recursion converges almost surely to the real value of η_t.

Consider the following notation:

$$\delta_t \equiv \frac{\alpha}{2} + \frac{\gamma}{2} sign\left(\eta_t\right),$$

so that

$$\varepsilon_t = \eta_t . \exp\left(\frac{\omega}{2} + \sum_{i=1}^{\infty} \beta_i \delta_{t-i} \left|\eta_{t-i}\right|\right). \tag{1}$$

As $\text{sign}(\eta_t) = \text{sign}(\varepsilon_t)$, δ_t is indeed $\sigma\left(\varepsilon_t\right)$-adapted. Therefore, by proving that $|\eta_t|$ is $\sigma\left(\varepsilon_t, \varepsilon_{t-1}, \ldots\right)$-adapted, it will follow automatically that the model is invertible.

By assuming that the distribution of η_t does not admit a probability mass at 0, we can take the absolute value and then the logarithm of ε_t. In order to be rigorous in the development below, we assume that $\eta_t \neq 0$,[2] almost surely. By rewriting the equation, we have

$$\log\left|\eta_t\right| = \log\left|\varepsilon_t\right| - \frac{\omega}{2} - \sum_{i=1}^{\infty} \beta_i \delta_{t-i} \left|\eta_{t-i}\right|. \tag{2}$$

Define the following function:

$$g_{\alpha,\gamma}\left(x, y\right) \equiv -\frac{\alpha + sign(y).\gamma}{2} \exp(x),$$

so that we have

$$\log\left|\eta_t\right| = \log\left|\varepsilon_t\right| - \frac{\omega}{2} + \sum_{i=1}^{\infty} \beta_i . g_{\alpha,\gamma}\left(\log\left|\eta_{t-i}\right|, \varepsilon_{t-i}\right).$$

This function is not Lipschitzian, so that we need to find some results about its variability. Lemma 1.1 gives a solution,[3] which will be used in several proofs that are given in what follows.

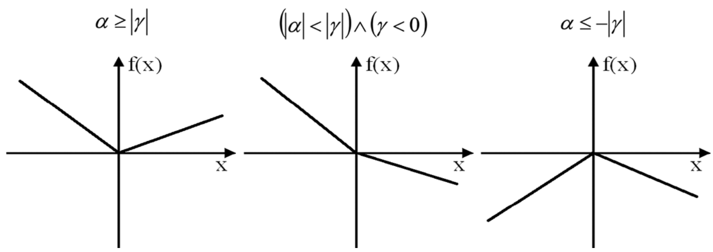
Lemma 1.1.
(1) $\left|g_{\alpha,\gamma}\left(x_1, y\right) - g_{\alpha,\gamma}\left(x_2, y\right)\right| \leq \left|\frac{\alpha + sign(y).\gamma}{2}\right| \exp\left(\max(x_1, x_2)\right) |x_1 - x_2|$.
(2) $\left|g_{\alpha,\gamma}\left(x_1, y\right) - g_{\alpha,\gamma}\left(x_2, y\right)\right| \geq \left|\frac{\alpha + sign(y).\gamma}{2}\right| \exp\left(\frac{x_1 + x_2}{2}\right) |x_1 - x_2|$.

The proof of Lemma 1.1 is given in the Appendix, part 1.

[2] This assumption is unnecessary, and is here only for rigor in the first few equations. We will discuss below the possibility of avoiding this assumption while retaining invertibility.
[3] Our method is similar to those in the literature, which are based on, for instance, finding a bound for the so-called Lyapunov exponents or Lipschitz coefficients.

The model is invertible as long as $|\eta_t|$ is $\sigma(\varepsilon_t, \varepsilon_{t-1}, \ldots)$-adapted. Obviously, we can equivalently prove invertibility by also showing that $\log|\eta_t|$ is $\sigma(\varepsilon_t, \varepsilon_{t-1}, \ldots)$-adapted. Lemma 1.1 will be very useful in the proof as the previous equation defines clearly a recursive relation among the $\log|\eta_t|$, where $g_{\alpha,\gamma}$ plays a crucial role. Knowing the variability of this function will help us in the next part to control the convergence of the recursion. Before doing so, we need to introduce some other simple conditions relating to the parameters.

By ensuring positivity, the EGARCH model allows the possibility of leverage, namely, that positive shocks lead to a decrease in volatility and negative shocks lead to an increase in volatility. Therefore, leverage occurs when $|\alpha| < |\gamma|$ and $\gamma < 0$.[4] There are also the two other cases where shocks lead to either an increase in volatility ($\alpha \geq |\gamma|$) or a decrease in volatility ($\alpha \leq -|\gamma|$). The fourth possibility is symmetric to the leverage case and hence need not be considered in detail. All of these cases allow asymmetry as there are still two coefficients. The three cases are summarized in the graphs given below, where $f(x) = \alpha|x| + \gamma x$:



Unfortunately the invertibility is difficult to prove in the case of leverage. Therefore, when it comes to invertibility of EGARCH, it is generally assumed that $\alpha \geq |\gamma|$. This article will, of course, consider the case $\alpha \geq |\gamma|$, and will also extend invertibility to $\alpha \leq -|\gamma|$, but not to the case of leverage.[5]

There is a drawback in the case $\alpha \leq -|\gamma|$: in order to prove invertibility, we need to assume the initial values to be equal to 0 for the independent shocks, η_t, before a certain range.[6] This obviously has no impact on the invertibility of the model as we only require the independent shocks to be $\sigma(\varepsilon_t, \varepsilon_{t-1}, \ldots)$-adapted, but this is an important point to consider in deriving the quasi-log likelihood function of the model. Indeed, in such a situation, one might attribute initial values for the unobserved η_t (or equivalently, σ_t) before a certain range, hoping that such values will not have a persistent impact on the recursively estimated η_t, and hence on their supposed consistency toward the "real" η_t values. Consequently, one has to be cautious in the case where $\alpha \leq -|\gamma|$, and must take null initial values. In the first part of the Appendix, we provide a simple counterexample with non-null initial values as a proof of the above statement. This also has an interesting implication on the analysis of invertibility (or, more precisely, on the possible lack of invertibility of the model) when leverage is assumed.

We have already introduced two first technical conditions, as follows:

$$\begin{aligned} &\text{(i)} \quad \eta_t \neq 0 \text{ almost surely;} \\ &\text{(ii)} \quad \alpha \geq |\gamma| \text{ or } \alpha \leq -|\gamma|. \end{aligned} \tag{3}$$

[4]This definition of leverage may not be precisely the one given in Black (1976), as this may also depend on the value of ω. Nelson divides the influence of each independent shock on the volatilities into two parts: one driven by its absolute value and the factor α, and one driven by its sign and the factor γ. Therefore, it is often stated that leverage is achieved when $\gamma < 0$. However, we use the above definition to ease the understanding of the article.

[5]When leverage is assumed, the recursions used in our invertibility proofs can be so erratic that the model might be considered as not invertible (see the counterexample in the Appendix, part 1).

[6]On the contrary, when $\alpha \geq |\gamma|$, the chosen initial values do not have any influence on the convergence of the recursions (and hence on the proof of invertibility). Therefore, the invertibility as defined in the sense of Straumann and Mikosch (2006) (different from ours) is also derived here, while for $\alpha \leq -|\gamma|$, EGARCH is not invertible in this sense as the recursion depends on the initial values.

Although the first condition is assumed only to ensure a minimum of rigor in the early proofs (as will be discussed later, this condition is not necessary), the second condition cannot be ignored. Indeed, $\alpha \geq |\gamma|$ is generally assumed in the literature and seems to be respected empirically in finance (for example, see Nelson's, 1990, study on stock index returns data, and Ball and Torous (1999), in the case of short-term interest rates).[7] Moreover, we will also analyze the case $\alpha \leq -|\gamma|$ which, to the best of our knowledge, has not yet been done in the literature.

3. Key result for the invertibility of EARCH(∞)

In this section, we derive an upper bound for the absolute difference between the true value of an independent shock and a $\sigma(\varepsilon_t, \varepsilon_{t-1}, \ldots)$-adapted series. As discussed above, this series is defined from a recurrence where proper initial and constant values are taken, which will be made explicit in what follows.

The main use of such an upper bound is to control invertibility of any EARCH(∞) model, as the convergence of this bound toward zero is a sufficient condition for invertibility to hold. This is actually one of the most important theoretical results of the article. Indeed, any EGARCH(p, q) model with proper specification may be written as an EARCH(∞) model, so that the upper bound will be useful for deriving invertibility conditions for a more general range of EGARCH models than has been proved in the literature. Furthermore, the shape of this bound also has the substantial advantage to reduce the analysis of any EGARCH(p, q) model to the simpler case of an EGARCH(1,1) model. The proper EGARCH specification and parameter transformation to choose to benefit from this advantage will be discussed in Section 4. In Section 5, we will use these results, together with those from Section 4 to derive two different invertibility conditions. In fact, both Sections 4 and 5 describe a way to use the upper bound to deduce invertibility conditions for EGARCH(p, q). Therefore, as will be discussed below, less restrictive conditions might subsequently be established. In any event, the interesting result, namely the possibility of reducing an analysis of invertibility of any EGARCH(p, q) model to that of an appropriate EGARCH(1,1) model, will be very useful.

From here on, we treat jointly the cases $\alpha \geq |\gamma|$ and $\alpha \leq -|\gamma|$, and all the β_i coefficients are also assumed to be non-negative.[8] Recall the following equation, which was derived from Eq. (2) in Section 2:

$$\log|\eta_t| = \log|\varepsilon_t| - \frac{\omega}{2} + \sum_{i=1}^{\infty} \beta_i . g_{\alpha,\gamma}\left(\log|\eta_{t-i}|, \varepsilon_{t-i}\right), \tag{4}$$

which clearly defines a recursion among the $\log|\eta_t|$. That is, for a fixed t and independent shock η_t, we can, for instance, find its exact value from the observed shocks, ε_t, when $(\eta_s)_{s \leq t-n}$ are known, for any positive integer n. Therefore, extending n steps backward this "exact" recursion gives the definition of the following $u_k^{(n)}$ series:

$$\begin{cases} u_1^{(n)} = \log|\varepsilon_{t-n+1}| - \dfrac{\omega}{2} + \displaystyle\sum_{i=0}^{\infty} \beta_{i+1} g_{\alpha,\gamma}\left(\log|\eta_{t-n-i}|, \varepsilon_{t-n-i}\right) \\ u_{k+1}^{(n)} = \log\left|\varepsilon_{t-n+k+1}\right| - \dfrac{\omega}{2} + \displaystyle\sum_{j=1}^{k} \beta_j g_{\alpha,\gamma}\left(u_{k+1-j}^{(n)}, \varepsilon_{t-n+k+1-j}\right) + \displaystyle\sum_{i=0}^{\infty} \beta_{i+1+k} g_{\alpha,\gamma} \\ \qquad \left(\log|\eta_{t-n-i}|, \varepsilon_{t-n-i}\right). \end{cases} \tag{5}$$

As it may not be entirely straightforward for these series to represent an "exact" recursion, we provide the following lemma.

[7]In this article, for some time series, the distribution of the independent shocks are estimated to follow a Student's t distribution, with degrees of freedom (df) very close to 2. If df = 2 or < 2, the assumption $E\left[\eta_t^2\right] = 1$ would not be possible.
[8]Otherwise, we would have the same issues as in the case of leverage, or in the counter-example in Appendix, part 1.

Lemma 2.1.

$$u_k^{(n)} = \log\left|\eta_{t-n+k}\right|, \qquad \forall n \in \mathrm{N}^*, \forall k \in \mathrm{N}^*.$$

The proof of Lemma 2.1 is given in the Appendix, part 2.

As the lag n increases, more observed shocks, ε_t, and relatively less independent shocks, $(\eta_s)_{s\leq t-n}$, are needed to infer the value of η_t. Hence, if we want invertibility of the model, when n is large enough, the past values $(\eta_s)_{s\leq t-n}$ should not have much influence on η_t through the above "exact" recursion. We could set all these past values equal to 0[9] and examine if, when n goes to infinity, only the knowledge of the observed shocks is needed to infer the value of η_t. This will obviously prove invertibility. According to this point, consider the following $v_k^{(n)}$ series for any n:

$$\begin{cases} v_1^{(n)} = \log|\varepsilon_{t-n+1}| - \dfrac{\omega}{2} \\ v_{k+1}^{(n)} = \log\left|\varepsilon_{t-n+k+1}\right| - \dfrac{\omega}{2} + \displaystyle\sum_{j=1}^{k} \beta_j g_{\alpha,\gamma}\left(v_{k+1-j}^{(n)}, \varepsilon_{t-n+k+1-j}\right). \end{cases} \tag{6}$$

These series are $\sigma(\varepsilon_t, \varepsilon_{t-1}, \ldots)$-adapted and are identical to the $u_k^{(n)}$, where all the past independent shocks $(\eta_s)_{s\leq t-n}$ are set to zero. From Lemma 2.1, and following the previous discussion, in order to prove invertibility, we need to show that

$$\left|v_n^{(n)} - \log|\eta_t|\right| = \left|v_n^{(n)} - u_n^{(n)}\right| \underset{n\to+\infty}{\overset{a.s.}{\rightarrow}} 0.$$

One way to show this condition is to find an upper bound to $\left|v_n^{(n)} - u_n^{(n)}\right|$ and then show that this bound converges to 0 as n goes to infinity. This part is dedicated to finding such an upper bound; convergence toward zero is deferred to Section 5, when the specific case of EGARCH(p, q) invertibility will be examined.

Applying inequality (1) of Lemma 1.1, we have

$$\left|v_n^{(n)} - u_n^{(n)}\right| \leq \sum_{i=0}^{+\infty} \beta_{i+n} |\delta_{t-n-i}| \, |\eta_{t-n-i}| + \sum_{j=1}^{n-1} \beta_j \delta_{t-j} \exp\left(\max\left(v_{n-j}^{(n)}, u_{n-j}^{(n)}\right)\right) \left|v_{n-j}^{(n)} - u_{n-j}^{(n)}\right|.$$

When $\alpha \leq -|\gamma|$, using the fact that in this case $g_{\alpha,\gamma}$ is a positive ($\delta_{t-j-i} \leq 0$) and increasing function with respect to its first parameter, we can show recursively that $v_{n-j}^{(n)} \leq \log\left|\eta_{t-j}\right|, \forall j \in [0, n[$,[10] as $v_{n-j}^{(n)} = \log\left|\eta_{t-j}\right| + \sum_{i=1}^{+\infty} \beta_i \delta_{t-j-i} \left|\eta_{t-j-i}\right| + \sum_{i=1}^{n-j-1} \beta_i g_{\alpha,\gamma}\left(v_{n-j-i}^{(n)}, \varepsilon_{t-j-i}\right)$.

When $\alpha \geq |\gamma|$, using the above equality for $v_{n-j}^{(n)}$, as $g_{\alpha,\gamma} \leq 0$, $\delta_{t-j-i} \geq 0$, and $\max\left(u_{n-j}^{(n)}, v_{n-j}^{(n)}\right) = \log\left|\eta_{t-j}\right| + \left(v_{n-j}^{(n)} - \log\left|\eta_{t-j}\right|\right)^+$, we have $v_{n-j}^{(n)} \leq \log\left|\eta_{t-j}\right| + \sum_{i=1}^{+\infty} \beta_i \delta_{t-j-i} \left|\eta_{t-j-i}\right|$.

Overall, we may introduce a new notation, ξ_{t-j}, such that

$$\max\left(u_{n-j}^{(n)}, v_{n-j}^{(n)}\right) \leq \xi_{t-j} \equiv \begin{cases} \log\left|\eta_{t-j}\right|, & \text{when } \alpha \leq -|\gamma| \\ \log\left|\eta_{t-j}\right| + \displaystyle\sum_{i=1}^{\infty} \beta_i \delta_{t-j-i} \left|\eta_{t-j-i}\right|, & \text{when } \alpha \geq |\gamma|. \end{cases}$$

[9] As stated in the previous part and in the counterexample given in the Appendix, when $\alpha \leq -|\gamma|$, we must take such null initial values. However, when $\alpha \geq |\gamma|$, we can choose any initial values in $\Re$, and we will still be able to prove invertibility.

[10] As explained in the counterexample, the reason why we should have null initial values is that it leads to such inequalities. Otherwise, the behavior of the $v_{n-j}^{(n)}$ may be explosive!

Therefore, it follows that

$$\left|u_n^{(n)} - v_n^{(n)}\right| \leq \sum_{i=0}^{\infty} \beta_{i+n} \left|\delta_{t-n-i}\right| \left|\eta_{t-n-i}\right| + \sum_{j=1}^{n-1} \beta_j \left|\delta_{t-j}\right| \exp\left(\xi_{t-j}\right) \left|u_{n-j}^{(n)} - v_{n-j}^{(n)}\right|.$$

For both the cases of $\alpha \geq |\gamma|$ and $\alpha \leq -|\gamma|$, we apply the same inequality to the $\left|u_{n-j}^{(n)} - v_{n-j}^{(n)}\right|$ and reiterate the process several times, as follows:

Define

$$a_k \equiv \sum_{i=0}^{+\infty} \left|\delta_{t-n-i}\right| \left|\eta_{t-n-i}\right| \left(\beta_{i+n} + \sum_{p=1}^{k-1} \sum_{i_1,\ldots,i_p \in A_p^{(n)}} \widehat{\Pi}_p \widehat{D}_p \exp\left(\sum_{j=1}^{p} \xi_{t-\widehat{S}_j} \right) \times \beta_{i+n-\widehat{S}_p} \right) + \sum_{i_1,\ldots,i_k \in A_k^{(n)}} \widehat{\Pi}_k \widehat{D}_k \exp\left(\sum_{i=1}^{k} \xi_{t-\widehat{S}_j} \right) \left| u_{n-\widehat{S}_k}^{(n)} - v_{n-\widehat{S}_k}^{(n)} \right|$$

Where we have as follows:

- $\widehat{S}_l = \sum_{j=1}^{l} i_j$
- $A_p^{(n)} = \left\{ i_1 \geq 1, \ldots, i_p \geq 1 : \widehat{S}_p \leq n-1 \right\}$
- $\widehat{\Pi}_l = \prod_{j=1}^{l} \beta_{i_j}$
- $\widehat{D}_l = \prod_{j=1}^{l} \left|\delta_{t-\widehat{S}_j}\right|$.

The above calculations lead to the following lemma.

Lemma 2.2.

$$\left|v_n^{(n)} - u_n^{(n)}\right| \leq a_k, \qquad \forall k \in [1, n[\,.$$

The proof of Lemma 2.2 is given in the Appendix, part 2.

By taking $k = n-1$, and using the inequality $\left|u_1^{(n)} - v_1^{(n)}\right| \leq \sum_{i=0}^{\infty} \beta_{i+1} \left|\delta_{t-n-i}\right| \left|\eta_{t-n-i}\right|$, we have the following general result for EARCH(∞), which is the appropriate upper bound:

Proposition 2.1. *If $\alpha \geq |\gamma|$ or $\alpha \leq -|\gamma|$, and $\beta_i \geq 0, \forall i$, then we have the following inequality for the defined series, u and v, for EARCH(∞):*

$$\left|u_n^{(n)} - v_n^{(n)}\right| \leq \sum_{i=0}^{+\infty} \left|\delta_{t-n-i}\right| \left|\eta_{t-n-i}\right| \left(\beta_{i+n} + \sum_{p=1}^{n-1} \sum_{i_1,\ldots,i_p \in A_p^{(n)}} \widehat{\Pi}_p \widehat{D}_p \exp\left(\sum_{j=1}^{p} \xi_{t-\widehat{S}_j} \right) \times \beta_{i+n-\widehat{S}_p} \right).$$

An examination of invertibility for a general EARCH(∞) would use this upper bound. In our case, as it could be difficult if we do not assume a minimum on the shape of the β_i coefficients, we will examine the case of EGARCH(p, q).

4. EGARCH(p, q) specification

An EGARCH(p,q) model admitting a canonical representation[11] also has an EARCH(∞) representation, so invertibility can be analyzed according to the upper bound found in Section 3. However, two conditions on the parameters of the EARCH(∞) representation were established as

$$(\alpha \geq |\gamma| \text{ or } \alpha \leq -|\gamma|) \qquad \text{and} \qquad \beta_i \geq 0, \forall i \geq 1.$$

If the condition $(\alpha \geq |\gamma|$ or $\alpha \leq -|\gamma|)$ can be easily verified, it is not automatic that the EARCH(∞) representation would have non-negative β_i coefficients. Lemma 3.1 presents two sufficient conditions on the EGARCH(p,q) specification to tackle this issue.

Finally, when an EGARCH model admits an EARCH(∞) representation, we can guess intuitively that its β_i coefficients will decay exponentially toward 0 when i goes to infinity. Hence, when these coefficients are non-negative, we might want to find for them an upper bound of the following kind: $\beta_i \leq C.\beta^{i-1}$, where $C > 0$ and $\beta \in]0,1[$. Directly substituting the $C.\beta^{i-1}$ in the upper bound[12] of Section 3 will ease considerably the derivation of the invertibility conditions. Moreover, one can easily check that the new upper bound will be identical to what can be derived for the EARCH(∞) representation of an EGARCH(1,1) model for parameters subject to the following transformations:

$$\begin{aligned} \alpha &\leftarrow C \times \alpha \\ \gamma &\leftarrow C \times \gamma \\ \beta_i &\leftarrow \beta^{i-1}. \end{aligned} \tag{7}$$

This hypothetical EGARCH(1,1) would clearly have the following specification:

$$\log \sigma_t = \frac{\omega}{2} + \beta \log \sigma_{t-1} + \delta_{t-1} |\eta_{t-1}|.$$

Before deriving the invertibility conditions for EGARCH(p,q) in Section 5, we should provide sufficient conditions on the specification in order to have the following situations:

(i) Existence of an EARCH(∞) representation;
(ii) Non-negativity of the β_i coefficients;
(iii) $C > 0$ and $\beta \in]0,1[$, such that $\beta_i \leq C.\beta^{i-1}, \forall i \geq 1$.

Define the general EGARCH(p,q) model specification, as follows:

$$\log \sigma_t = \frac{\omega}{2} + \sum_{i=1}^{p} a_i \log \sigma_{t-i} + \sum_{i=1}^{q} b_i \delta_{t-i} |\eta_{t-i}|, \qquad a_i \in \Re, b_i \in \Re.$$

By using the backward lag operator L, this model can be rewritten as

$$\left(1 - \sum_{i=1}^{p} a_i L^i\right) \log \sigma_t = \frac{\omega}{2} + \sum_{i=1}^{q} b_i \delta_{t-i} |\eta_{t-i}|, \qquad a_i \in \Re, b_i \in \Re. \tag{8}$$

If we factorize the polynomial $\left(1 - \sum_{i=1}^{p} a_i L^i\right)$, we obtain ($\theta_i \in C$, i.e., it is a complex number)

$$(1-\theta_1 L) \ldots (1-\theta_p L) \log \sigma_t = \frac{\omega}{2} + \sum_{i=1}^{q} b_i \delta_{t-i} |\eta_{t-i}|, \qquad |\theta_i| \in [0,1[, b_i \in \Re. \tag{9}$$

For the first point (i), the non-anticipative EARCH(∞) representation is obtained if we have $\forall i \in [1,p], |\theta_i| < 1$. For the second point (ii), appropriate sufficient conditions are given in the following lemma.

[11] Without roots inside or on the border of the unit circle for the "AR" part, as we will make it explicit.
[12] The positivity of all the elements in the upper bound allows the substitution.

Lemma 3.1. *Assuming $\theta_i \in C, |\theta_i| < 1, \forall i$ as in (9), if one of the following conditions is verified, the EARCH(∞) representation of the EGARCH(p,q) model admits non-negative β_i coefficients:*
(1) *All the coefficients a_i and b_i as defined in (8) are non-negative;*
(2) *All the coefficients θ_i and b_i as defined in (9) are non-negative.*

The proof of Lemma 3.1 is given in the Appendix, part 3.

Remark. As the above conditions are only sufficient, it would be possible to have the non-negativity of the β_i coefficients under less restrictive assumptions. For instance, one can easily see that a "mix" of the two conditions could also be used. Indeed, if we can rewrite the model as

$$(1-\theta_1 L)\ldots.(1-\theta_{p_1} L)\left(1-\sum_{i=1}^{p_2} a_i L^i\right)\log\sigma_t = \frac{\omega}{2} + \sum_{i=1}^{q} b_i \delta_{t-i}|\eta_{t-i}|,$$

with $p_1 + p_2 = p$ and a_i, b_i, θ_i non-negative, we would also have $\beta_i \geq 0, \forall i$. Furthermore, even though the empirical specification in Nelson (1990) does not satisfy the conditions, straightforward algebra can still lead to non-negative β_i.

From here on, the "updated" coefficients, as described in Eq. (7), will be given as α^*, γ^*, β^* below. The third point (iii) is addressed in the following lemma.

Lemma 3.2. *We reorder the θ_i from (9) such as $|\theta_1| \geq \cdots \geq |\theta_p|$, and take $\beta^* \in]0,1[$ such that $\beta^* > \max_i |\theta_i|$. We assume from the EARCH(∞) representation that $\beta_i \geq 0, \forall i$. Then*

$$\beta_i \leq C.\beta^{*i-1},$$

with

$$C \equiv \frac{\max_{1\leq m\leq q}\left|\sum_{i=1}^{m} b_i\theta_1^{1-i}\right|}{\prod_{p\geq i\geq 2}\left(1-\frac{|\theta_i|}{\beta^*}\right)}.$$

The proof of Lemma 3.2 is given in the Appendix, part 3.

Remark. In the EARCH(∞) representation, the above β^* is greater than the maximum of the absolute values of the θ_i. When all the $|\theta_i|$ are different, we could choose β^* as being the maximum value. However, the polynomial $\left(1-\sum_{i=1}^{p} a_i L^i\right)$ may have double roots, or at least, as it is a polynomial with real coefficients, may admit couples of complex roots and their conjugates, thereby having the same absolute value. In these cases, we would not be able to find an upper bound like $\beta_i \leq C.\beta^{*i-1}$ if we used $\beta^* = \max_i |\theta_i|$. Therefore, in our "general" analysis, we consider a coefficient such as $\beta^* > \max_i |\theta_i|$. This coefficient can be chosen arbitrarily as long as it is strictly less than 1 and above the absolute values of the θ_i.

If we consider the inequality in Proposition 2.1, we can also use the $\beta_i \leq C\beta^{*i-1}$ inequality from Lemma 3.2 to obtain the new upper bound

$$\left|u_n^{(n)} - v_n^{(n)}\right| \leq \sum_{i=0}^{+\infty}\beta^{*i}\left|\delta^*_{t-n-i}\right||\eta_{t-n-i}|\left(\beta^{*n-1} + \sum_{p=1}^{n-1}\beta^{*n-1-p}\sum_{1\leq s_1<\cdots<s_p\leq n-1}\exp\left(\sum_{j=1}^{p}\log\left(\left|\delta^*_{t-s_j}\right|\right)+\xi^*_{t-s_j}\right)\right) \tag{10}$$

where the previous α and β parameters are replaced by the following coefficients:

$$\alpha^* \equiv C\alpha, \qquad \gamma^* \equiv C\gamma,$$

so that δ_t and ξ_t are redefined accordingly into δ_t^* and ξ_t^* using α^* and β^*.

5. Invertibility of EGARCH(*p*, *q*)

This section is dedicated to the derivation of two invertibility conditions for EGARCH(p,q) from the previous results. Therefore, we consider an EGARCH(p,q) model whose specification can be described by Eqs. (8) and (9), and we also assume the technical conditions from Eq. (3) to hold. Furthermore, we assume that the roots of the AR part of EGARCH lie outside the unit circle, and that the parameters satisfy conditions (1) or (2)[13] from Lemma 3.1 such that, as stated in Section 4, the model admits an EARCH(∞) representation with non-negative β_i coefficients. Finally, we use the upper bounds derived in Lemma 3.2 for these coefficients, combined with the upper bound found in part 3 of Lemma 2.1, to obtain Eq. (10) in Section 4. Therefore, in order to prove invertibility of the EGARCH(p,q) model, we provide two sufficient conditions such that the upper bound from (10) converges to zero.

The results from Section 4 and this section are sufficient to use the general upper bound from Section 3 to derive the invertibility conditions of EGARCH(p,q). Hence, further studies of the points addressed in these sections may lead to more general invertibility conditions. In this section, we provide two conditions derived from Eq. (10), which satisfy the following two requirements for the invertibility set:[14] (i) invertibility should be easily verifiable, and (ii) it should be asymptotically equivalent to that in Straumann and Mikosch (2006).[15] The two conditions are, of course, not identical, and one does not imply the other. The first condition is easily verifiable, and leads to a general condition where the distribution of η_t is not involved. However, this condition is asymptotically more restrictive than that of Straumann and Mikosch (2006). The second condition is also easily verifiable and satisfies the requirement of asymptotic equivalence but is more restrictive than the first condition for high values of β^*.

We now introduce these conditions and examine how they have been derived. First, we rewrite the inequality (10) to make the proof of invertibility more straightforward. For $\alpha \geq |\gamma|$, note that

$$\begin{aligned}\sum_{j=1}^{p} \xi^*_{t-s_j} &= \sum_{j=1}^{p} \log\left|\eta_{t-s_j}\right| + \sum_{j=1}^{p}\sum_{i=1}^{+\infty} \beta^{*i-1}\delta^*_{t-s_j-i}\left|\eta_{t-s_j-i}\right| \\ &= \sum_{j=1}^{p} \log\left|\eta_{t-s_j}\right| + \sum_{l=1}^{+\infty} \sum_{\substack{1\le j\le p\\ i\ge 1\\ i+s_j=l}} \beta^{*i-1}\delta^*_{t-l}\left|\eta_{t-l}\right| \\ &= \sum_{j=1}^{p} \log\left|\eta_{t-s_j}\right| + \sum_{l=1}^{n-1} \sum_{\substack{1\le j\le p\\ i\ge 1\\ i+s_j=l}} \beta^{*i-1}\delta^*_{t-l}\left|\eta_{t-l}\right| + \sum_{l=n}^{+\infty} \sum_{\substack{1\le j\le p\\ i\ge 1\\ i+s_j=l}} \beta^{*i-1}\delta^*_{t-l}\left|\eta_{t-l}\right|.\end{aligned}$$

As $1 \le s_1 < \cdots < s_p \le n-1$, $\sum_{\substack{1\le j\le p\\ i\ge 1\\ i+s_j=l}} \beta^{*i-1} \le \frac{1}{1-\beta^*}$ if $l < n$, and $\sum_{\substack{1\le j\le p\\ i\ge 1\\ i+s_j=l}} \beta^{*i-1} \le \frac{\beta^{*l-n}}{1-\beta^*}$ if $l \ge n$,

it follows that

$$\sum_{j=1}^{p} \xi^*_{t-s_j} \le \sum_{j=1}^{p} \log\left|\eta_{t-s_j}\right| + \sum_{l=1}^{n-1} \frac{\delta^*_{t-l}\left|\eta_{t-l}\right|}{1-\beta^*} + \sum_{l=n}^{+\infty} \frac{\beta^{*l-n}}{1-\beta^*}\delta^*_{t-l}\left|\eta_{t-l}\right|. \tag{11}$$

For $\alpha \le -|\gamma|$, we have $\sum_{j=1}^{p} \xi^*_{t-s_j} = \sum_{j=1}^{p} \log\left|\eta_{t-s_j}\right|$.

[13] Or a mix of the two (see the remark after Lemma 3.1).

[14] That is, the set of parameters of the model such that we know that invertibility holds.

[15] By asymptotically equivalent, we mean that the invertibility set for $\beta = 0$ is the same as in Straumann and Mikosch (2006). Indeed, they present a sufficient invertibility condition for EGARCH(1,1) which is difficult to verify as Monte Carlo simulations are needed, so that a distribution for the independent shocks has to be assumed. However, an easily verifiable condition is provided for EARCH(1) (that is, when $\beta = 0$). We derive an invertibility condition that is identical to that of Straumann and Mikosch for EARCH(1) as a guarantee that our invertibility set is not too restrictive.

It follows that

$$\left|u_n^{(n)} - v_n^{(n)}\right| \leq \begin{cases} B_n \exp\left(\sum_{l=1}^{n-1} \frac{\delta_{t-l}^* \left|\eta_{t-l}\right|}{1-\beta^*}\right) \left(\beta^{*n-1} + \sum_{p=1}^{n-1} \beta^{*n-1-p} \sum_{1 \leq s_1 < \cdots < s_p \leq n-1} \right. \\ \left. \qquad \exp\left(\sum_{j=1}^{p} \log\left(\delta_{t-s_j}^* \left|\eta_{t-s_j}\right|\right)\right)\right), \qquad when\ \alpha \geq |\gamma| \\ B_n \left(\beta^{*n-1} + \sum_{p=1}^{n-1} \beta^{*n-1-p} \sum_{1 \leq s_1 < \cdots < s_p \leq n-1} \exp\left(\sum_{j=1}^{p} \log\left(\left|\delta_{t-s_j}^*\right| \left|\eta_{t-s_j}\right|\right)\right)\right), \\ \quad when\ \alpha \leq -|\gamma| \end{cases} \tag{12}$$

where

$$B_n = \begin{cases} \sum_{i=0}^{+\infty} \beta^{*i} \delta_{t-n-i}^* \left|\eta_{t-n-i}\right| \exp\left(\sum_{l=n}^{+\infty} \frac{\beta^{*l-n}}{1-\beta^*} \delta_{t-l}^* \left|\eta_{t-l}\right|\right), \qquad when\ \alpha \geq |\gamma| \\ \sum_{i=0}^{+\infty} \beta^{*i} \left|\delta_{t-n-i}^*\right| \left|\eta_{t-n-i}\right|, \qquad when\ \alpha \leq -|\gamma|. \end{cases}$$

We now provide the first sufficient condition for invertibility of the EGARCH(p, q) specification, the proof of which is established through Lemma 4.1 and Proposition 4.1:

$$\left| \begin{array}{l} \mathrm{E}\left[\frac{\delta_t^* \left|\eta_t\right|}{1-\beta^*}\right] + \log\left(\beta^* + E\left[\delta_t^* \left|\eta_t\right|\right]\right) < 0, \qquad when\ \alpha \geq |\gamma| \\ \log\left(\beta^* + E\left[\left|\delta_t^*\right| \left|\eta_t\right|\right]\right) < 0, \qquad when\ \alpha \leq -|\gamma|. \end{array} \right. \tag{Condition 1}$$

Asymptotically, for $\alpha \geq |\gamma|$, that is, when $\beta^* = 0$, we find the following condition for EARCH(1):

$$\mathrm{E}\left[\delta_t^* \left|\eta_t\right|\right] + \log\left(E\left[\delta_t^* \left|\eta_t\right|\right]\right) < 0.$$

This condition is more restrictive by concavity of the log(.) than the easily verifiable condition for EARCH(1) from Straumann and Mikosch (2006), namely,

$$\mathrm{E}\left[\delta_t^* \left|\eta_t\right|\right] + E\left[\log\left(\delta_t^* \left|\eta_t\right|\right)\right] < 0.$$

Fortunately, the second invertibility condition introduced below will satisfy this simple condition asymptotically. However, a general condition involving only the parameters of the model can cbe obtained using the fact that $\mathrm{E}\left[\eta_t\right] = 0$ and $\mathrm{E}\left[\left|\eta_t\right|\right] \leq 1$ (as $\mathrm{E}\left[\eta_t^2\right] = 1$) as follows:

$$\begin{cases} \frac{\alpha^*}{2\left(1-\beta^*\right)} + \log\left(\beta^* + \frac{\alpha^*}{2}\right) < 0, \qquad when\ \alpha \geq |\gamma| \\ \log\left(\beta^* + \frac{-\alpha^*}{2}\right) < 0, \qquad when\ \alpha \leq -|\gamma|. \end{cases} \tag{13}$$

We notice also that when we set β^* toward 0 for $\alpha \geq |\gamma|$, the condition $\alpha^* < 1$ proposed by Straumann and Mikosch (2006) in their Remark 3.10 is also obtained here.

Remark. We continue to assume that $\mathrm{P}\left(\eta_t = 0\right) = 0$ in order to retain rigor in the proofs. However, as can be seen from the above results, a distribution for the independent shocks admitting mass at 0 can verify the invertibility conditions. Moreover, in the next condition, as the expectation[16] will be outside the log(.), invertibility will even be easier to verify for a distribution with a probability mass at zero.

[16] More precisely, it will be the expected shortfall.

Generally, the existence of such a mass will not impede invertibility and may even be the contrary: in the recursion, the appearance of a null shock will tend to cancel the influence of the initial values.

Lemma 4.1. *For any $\nu > 1/2$, we have with probability 1*

$$B_n = \exp\left(o\left(n^{\nu}\right)\right).$$

The proof of Lemma 4.1 is given in the Appendix, part 4.

The following proposition uses Lemma 4.1 to prove that Condition 1 leads to invertibility. We will explain below how this proposition is derived. Inside the larger brackets in inequality (12), we have sums of independent variables, $1 \leq s_1 < \cdots < s_p \leq n-1$, which are more difficult to control than a sum from 1 to p, for instance. So we cannot simply use the Law of Large Number (LLN), as in the case of the EARCH(1) model. Therefore, we will simply take the expectation in the proof to return to a sum over consecutive indexes (we also take expectations in order to use Lemma 1.2 in the Appendix with the Markov inequality to obtain convergence toward zero of $\left|v_n^{(n)} - u_n^{(n)}\right|$). This operation may explain why the curves drawn in Section 6 are convex, while those in Wintenberger (2013) look more concave. The operation of taking expectations might make the invertibility set implied by (10) look more restrictive than it really is. However, we have not yet found any clearer conditions than Condition 1 or Condition 2 to show invertibility.

Proposition 4.1. *If $\alpha \geq |\gamma|$ or $\alpha \leq -|\gamma|$, if the roots of $\left(1 - \sum_{i=1}^{p} a_i L^i\right)$ lie outside the unit circle and if the EARCH(∞) representation has its β_i coefficients non-negative, then when Condition 1 is verified, EGARCH(p,q) is invertible as*

$$\left|v_n^{(n)} - u_n^{(n)}\right| = \left|v_n^{(n)} - \log|\eta_t|\right| \underset{n\to\infty}{\overset{a.s.}{\to}} 0.$$

The proof of Proposition 4.1 is given in the Appendix, part 4.

Another invertibility condition can also be deduced from inequality (12). This condition has the advantage of being "asymptotically" equivalent (for $\beta^* = 0$) to that introduced by Straumann and Mikosch (2006), despite being more restrictive than Condition 1 elsewhere (namely, for most $\beta^* \neq 0$), as will be seen in the plots of Section 6 in the case of standard normal shocks. This condition is stated as follows:

$$\left|\begin{array}{l} \textit{When} \quad \alpha \geq |\gamma|: \\ \max\limits_{k\in[0,1]} \left(\mathrm{E}\left[\dfrac{\delta_t^*|\eta_t|}{1-\beta^*}\right] + k\left(ES_k\left[\log\left(\delta_t^*|\eta_t|\right)\right] - \log(k)\right) + (1-k)\left(\log\left(\beta^*\right) - \log(1-k)\right)\right) < 0 \\ \textit{When} \quad \alpha \leq -|\gamma|: \\ \max\limits_{k\in[0,1]} \left(k\left(ES_k\left[\log\left(\left|\delta_t^*\right||\eta_t|\right)\right] - \log(k)\right) + (1-k)\left(\log\left(\beta^*\right) - \log(1-k)\right)\right) < 0, \end{array}\right.$$

(Condition 2)

where, for any random variable X, $ES_k[X]$ is the so-called Expected Shortfall or Conditional Value-at-Risk (CVaR) of X at the k level. In other words, $ES_k[X] = \frac{1}{k}\int_{1-k}^{1} VaR_u(X)\,du$, where $VaR_u(X)$ is the so-called Value-at-Risk of X, or the u-quantile of its distribution, defined as $VaR_u(X) \equiv \inf\{s \in \Re : \mathrm{P}(X \geq s) \leq 1-u\}$.

Proposition 4.2. *If $\alpha \geq |\gamma|$ or $\alpha \leq -|\gamma|$, if the roots of $\left(1 - \sum_{i=1}^{p} a_i L^i\right)$ lie outside the unit circle and if the EARCH(∞) representation has its β_i coefficients non-negative, then when Condition 2 holds,* *EGARCH(p,q) is invertible, as*

$$\left|v_n^{(n)} - u_n^{(n)}\right| = \left|v_n^{(n)} - \log|\eta_t|\right| \underset{n\to\infty}{\overset{a.s.}{\to}} 0.$$

The proof of Proposition 4.2 is given in the Appendix, part 4.

Remark. Value-at-Risk and Expected Shortfall can easily be estimated empirically. For instance, if we have a sample of n independent observations of variable X, namely, $X_1 \dots X_n$, and they are reordered as $X_{(1)} \dots X_{(n)}$, then the empirical Value-at-Risk values should be $\underset{u}{\widehat{VaR}}(X) = X_{(\lfloor u*n \rfloor)}$, and Expected Shortfall should be

$$\underset{k}{\widehat{ES}}[X] = \frac{1}{k}\int_{1-k}^{1} \underset{u}{\widehat{VaR}}(X)\, du.$$

However, for $p \in \{1, 2, \dots, n\}$, $\underset{\frac{p}{n}}{\widehat{ES}}[X] = \frac{1}{p}\sum_{i=1}^{p} X_{(i)}$.

Moreover,[17] the function $f(k) \equiv \mathrm{E}\left[\frac{\delta_t^*|\eta_t|}{1-\beta^*}\right] + k\left(ES_k\left[\log\left(\delta_t^*|\eta_t|\right)\right] - \log(k)\right) + (1-k)\left(\log(\beta^*) - \log(1-k)\right)$ is differentiable and concave, so that it admits the maximum for $k_{\max}$ that satisfies

$$f'(k_{\max}) = VaR_{k_{\max}}(\log(\delta_t|\eta_t|)) - \log(k_{\max}) + \log(1-k_{\max}) - \log(\beta^*) = 0.$$

Otherwise, if f were not continuously differentiable it would be $k_{\max}$, such as $f'\left(k_{\max}^-\right) \geq 0 \geq f'\left(k_{\max}^+\right)$. In our simulations in Section 6, we actually find p such that $f'\left(\frac{p}{n-1}\right) \geq 0 \geq f'\left(\frac{p+1}{n-1}\right)$ and check the maximum of f for these two values, and reach a conclusion on invertibility if it is negative.

It is worth noting as follows:

(i) When $\beta^* = 0$, we find the same condition as the one in Straumann and Mikosch (2006) (in their Remark 3.10).

(ii) As $\eta_t \in L^2$, we have $\mathrm{E}\left[\log(|\eta_t|)\right] \in \Re\bigcup\{-\infty\}$, so even if $\log(|\eta_t|) \notin L^1$, we can replace $E\left[\log(|\eta_t|)\right]$ in the above conditions by $-\infty$, taken as a limit case. We can also use this form if $\mathrm{P}(\eta_t = 0) \neq 0$ without changing the conditions. A comparable remark can be made also for the expected shortfall (when k is big enough).

(iii) In order to deduce the invertibility condition, we did not take expectations, as in the previous condition, but used a direct upper bound. However, this new upper bound might be too conservative, again making the invertibility set from Condition 2 more restrictive than the "global" invertibility set implied by (10).

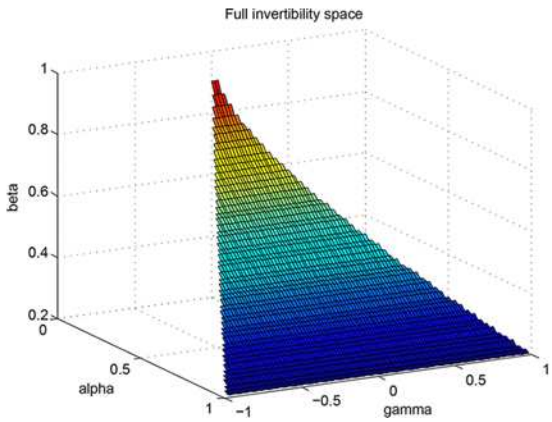
6. Special case of the N(0,1) distribution

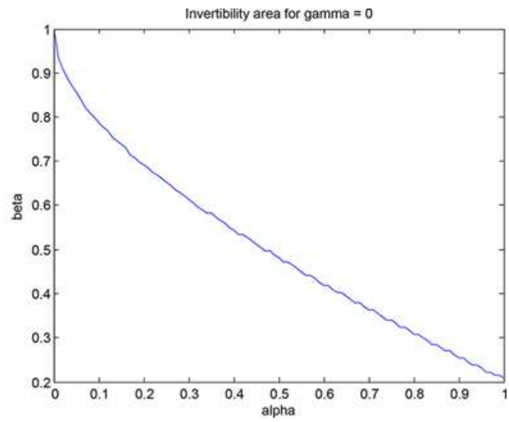
In the case of the Gaussian distribution, Condition 1 can be rewritten as

$$\begin{cases} \dfrac{\alpha^*}{\sqrt{2\pi}(1-\beta^*)} + \log\left(\beta^* + \dfrac{\alpha^*}{\sqrt{2\pi}}\right) < 0, & \textit{when } \alpha \geq |\gamma| \\ \log\left(\beta^* + \dfrac{-\alpha^*}{\sqrt{2\pi}}\right) < 0, & \textit{when } \alpha \leq -|\gamma|. \end{cases}$$

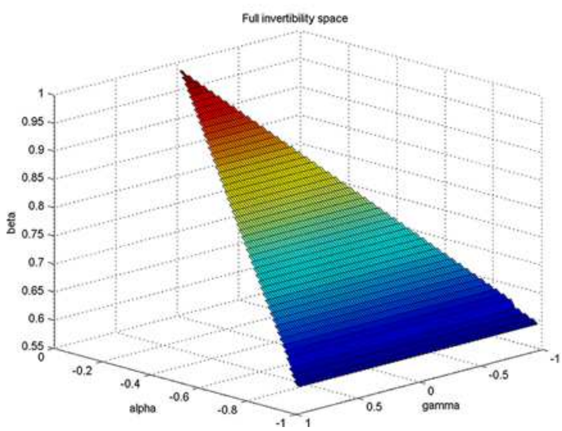
[17] For $\alpha \geq |\gamma|$, but this remark is also valid for $\alpha \leq -|\gamma|$, at the cost of a very slight change in the f function.

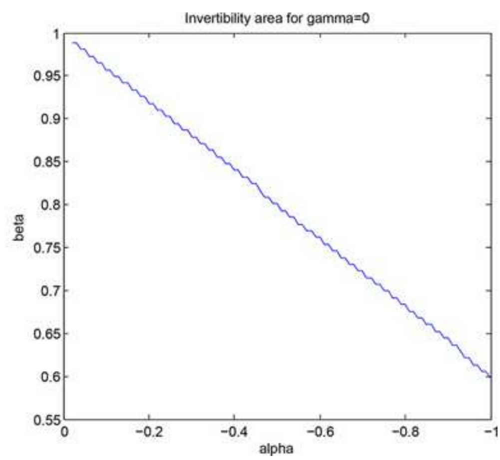
If we display the maximum β for several values of α and γ such invertibility holds, we have as follows:
For $\alpha \geq |\gamma|$:



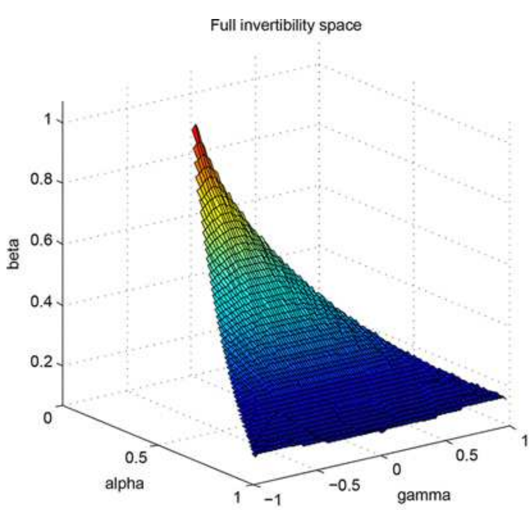


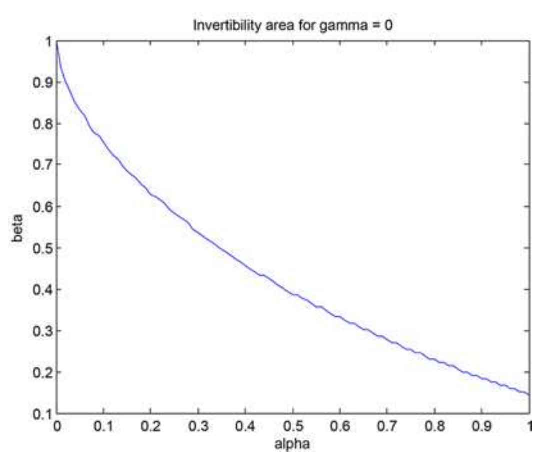
For $\alpha \leq -|\gamma|$:



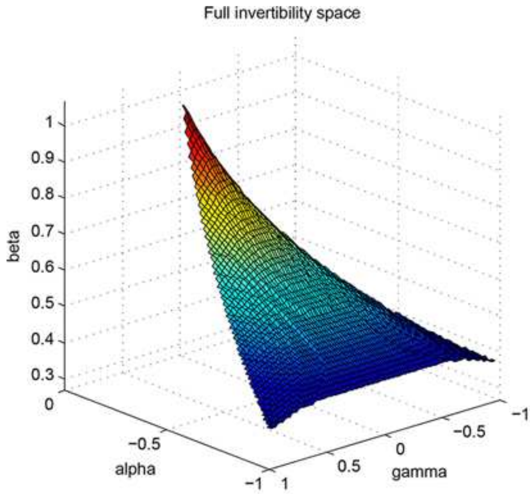


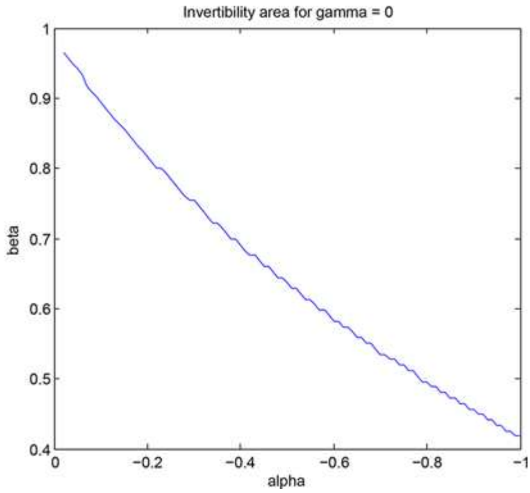
For Condition 2, we do not have an analytical formula but, as explained in part 5, the invertibility condition can be easily calculated. Some graphs for Condition 2 are the following ones:
For $\alpha \geq |\gamma|$:





For $\alpha \leq -|\gamma|$:





7. Concluding remarks

The graphs in part 6 display invertibility sets that seem to be more restrictive than that from Straumann and Mikosch (2006) in the case of EGARCH(1,1) with N(0,1) independent shocks. But it is important to note that these invertibility sets are only those implied by Condition 1 and Condition 2, which are more restrictive than that implied by (10). Indeed, inequality (10) gives the definition of the full set of invertibility found in this article, which is equivalently the set of parameters such that the right-hand side of the inequality converges toward zero. By applying the expectation in part 5, or by using a conservative upper bound, we derived Condition 1 and Condition 2, which are only sufficient conditions, such that the right-hand side of (10) converges to zero. These conditions might be improved such that they lead to a larger invertibility set by following the main arguments of the article and the "main" inequality found in part 3, and restated in part 4 under Eq. (10). In any case, we can at least consider the union of the two plotted invertibility sets (from the two conditions) as a larger set of parameters, such that invertibility can be easily verified through either one of the conditions introduced in part 5.

The present article followed an approach based on considering a stochastic process uniquely defined such that the EGARCH model can be easily derived following the definition of EGARCH given in Nelson (1991). This approach does not yet seem to have been used in the literature on the invertibility of EGARCH. By following this approach, and despite some inconveniences discussed in the previous paragraph, we are able to derive general and verifiable invertibility conditions. Moreover, an invertibility condition equivalent to that of Straumann and Mikosch was shown when it is easily verifiable (that is, asymptotically for $\beta = 0$).

Finally, the invertibility of any EGARCH(p, q) model can be studied simply through an examination of the invertibility of an EGARCH(1,1) model. While only some degenerate cases of EGARCH have been examined previously, the extension of invertibility to the general case of EGARCH(p, q) is the main contribution of the article. As shown in some other articles in the literature, such as Wintenberger (2013), invertibility is particularly helpful in the derivation of asymptotic properties of the QMLE of the parameters. Therefore, further research would be the derivation of the asymptotic properties of the QMLE of the EGARCH(p, q) specification based on the results established here.

Appendix

Part 1: Proofs of lemma 1.1 and counterexample for EARCH(1)

Proof of Lemma 1.1. The case $x_1 = x_2$ is obvious, so assume $x_1 \neq x_2$. We have

$$\left|g_{\alpha,\gamma}(x_1,y) - g_{\alpha,\gamma}(x_2,y)\right| = \left|\frac{\alpha + sign(y).\gamma}{2}\right| . \left|\frac{\exp(x_1)-\exp(x_2)}{x_1-x_2}\right| . |x_1 - x_2| .$$

If we note $x_{\min}$ and $x_{\max}$, respectively, the min and the max among x_1 and x_2, we know that $\exists c \in]x_{\min}, x_{\max}[$ such that

$$\left|\frac{\exp(x_1)-\exp(x_2)}{x_1-x_2}\right| = \exp(c).$$

The first inequality is obtained by the fact that exp is an increasing function. For the second inequality, some straightforward algebra leads to

$$c = \frac{x_{\max}+x_{\min}}{2} + \log\left(\frac{\exp(x)-\exp(-x)}{2x}\right),$$

where $x = \frac{x_{\max}-x_{\min}}{2}$. By using the Taylor expansion of the function exp(.), as x > 0, we have the terms in the log(.) function are greater than 1, and therefore, c is greater than $\frac{x_1+x_2}{2}$. This proves the second inequality.

Two well-known lemmas to be used in the proofs

The Borel–Cantelli Lemma. Consider the probabilized space, (Ω, A, P), *and* $A_n \in \mathrm{A}, \forall n \geq 0$.

(1) If $\sum_{n\geq 0} \mathrm{P}(A_n) < +\infty$, then $\mathrm{P}\left(\limsup_n A_n\right) = 0$.

(2) If $(A_n)_n$ is independent and if $\sum_{n\geq 0} \mathrm{P}(A_n) = +\infty$, then $\mathrm{P}\left(\limsup_n A_n\right) = 1$.

Lemma 1.2. *If* $\forall \varepsilon > 0$ *and* $\sum_n \mathrm{P}(|X_n - X| > \varepsilon) < +\infty$, *then* $X_n \underset{n\to\infty}{\overset{P.a.s.}{\to}} X$.

A counterexample with non-null initial value for EARCH(1) when $\alpha \leq -|\gamma|$

We give a counterexample where invertibility cannot be achieved using the usual method of proof for EARCH(1) when $\alpha \leq -|\gamma|$[18] and some non-null initial value is taken in recursions below. We assume the normality of the shocks, that is, $\eta_t \sim \mathrm{N}(0,1)$, which is not too restrictive an assumption (distributions with larger tails will surely lead to the same result). Now we introduce the following recursive series for a fixed $n \in \mathrm{N}^*$:

$$\begin{cases} u_1^{(n)} = \log|\varepsilon_{t-n+1}| - \dfrac{\omega}{2} + g_{\alpha,\gamma}\left(\log|\eta_{t-n}|, \varepsilon_{t-n}\right) \\ u_{k+1}^{(n)} = \log\left|\varepsilon_{t-n+k+1}\right| - \dfrac{\omega}{2} + g_{\alpha,\gamma}\left(u_k^{(n)}, \varepsilon_{t-n+k}\right). \end{cases}$$

We can easily check by recursion that $u_k^{(n)} = \log\left|\eta_{t-n+k}\right|, \forall n \in \mathrm{N}^*, \forall k \in \mathrm{N}^*$.

Also define, for any $c_0 \in \Re$ (where c_0 is the initial value for $\log|\eta_{t-n}|$ and, by assumption, $c_0 \neq -\infty$)

$$\begin{cases} v_1^{(n)} = \log|\varepsilon_{t-n+1}| - \dfrac{\omega}{2} + g_{\alpha,\gamma}\left(c_0, \varepsilon_{t-n}\right) \\ v_{k+1}^{(n)} = \log\left|\varepsilon_{t-n+k+1}\right| - \dfrac{\omega}{2} + g_{\alpha,\gamma}\left(v_k^{(n)}, \varepsilon_{t-n+k}\right). \end{cases}$$

[18]We assume, without loss of generality, that $-\alpha - \gamma > 0$.

These series are the EARCH(1) versions of those used for EARCH(∞). In proving invertibility of the model, our method is based on proving that $\left|v_n^{(n)} - u_n^{(n)}\right|$ converges toward zero almost surely. In the counterexample, we will prove that it is not converging.

Lemma 1.3. *If we consider the extracted series, $u_k^{(4n)}$ and $v_k^{(4n)}$, we have*

$$\mathrm{P}\left(\forall k \in \mathrm{N}, \exists n \geq k : v_4^{(4n)} \geq \log|\eta_{t-4n+4}| + n^{3/2}\right) = 1.$$

Proof. Define

$$A_n \equiv \left\{0 \leq \eta_{t-4n} \leq \frac{\exp(c_0)}{2}; \eta_{t-4n+1} \geq \sqrt{\log(n^{7/4})} - 1; \eta_{t-4n+2} \geq \frac{4}{-\alpha-\gamma}; \eta_{t-4n+3} \geq \frac{2}{-\alpha-\gamma}\right\}.$$

Under independence, we have

$$\mathrm{P}(A_n) = \mathrm{P}\left(0 \leq \eta_{t-4n} \leq \frac{\exp(c_0)}{2}\right) \times \mathrm{P}\left(\eta_{t-4n+1} \geq \sqrt{\log(n^{7/4})} - 1\right) \times \mathrm{P}\left(\eta_{t-4n+2} \geq \frac{4}{-\alpha-\gamma}\right) \times \mathrm{P}\left(\eta_{t-4n+3} \geq \frac{2}{-\alpha-\gamma}\right).$$

As all terms except for the second do not depend on n, and therefore are constant, we can rewrite the above equality as follows, where $\Phi(.)$ is the cumulative density function (CDF) of the normal distribution:

$$\mathrm{P}(A_n) = C_{\alpha,\gamma} \times \Phi\left(1 - \sqrt{\log(n^{7/4})}\right),$$

where

$$C_{\alpha,\gamma} \equiv \mathrm{P}\left(0 \leq \eta_{t-4n} \leq \frac{\exp(c_0)}{2}\right) \times \mathrm{P}\left(\eta_{t-4n+2} \geq \frac{4}{-\alpha-\gamma}\right) \times \mathrm{P}\left(\eta_{t-4n+3} \geq \frac{2}{-\alpha-\gamma}\right) \neq 0.$$

It follows that

$$\Phi\left(1 - \sqrt{\log(n^{7/4})}\right) = \int_{\sqrt{\log(n^{7/4})}-1}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \geq \int_{\sqrt{\log(n^{7/4})}-1}^{\sqrt{\log(n^{7/4})}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \geq \frac{1}{\sqrt{2\pi}} e^{-\frac{\log(n^{7/4})}{2}}$$
$$\geq \frac{1}{\sqrt{2\pi}} \frac{1}{n^{7/8}}$$

and, by direct comparison to a Bertrand sum, it follows that $\sum_n \mathrm{P}(A_n)$ diverges. Therefore, as the A_n are independent, we can apply line (2) of the Borel–Cantelli Lemma, as stated previously, and $\forall k \in \mathrm{N}, \exists n \geq k : A_n$ will occur with probability one.

Consider taking n sufficiently large such that the event A_n occurs. By straightforward calculus and a Taylor expansion, it follows that

$$v_1^{(4n)} \geq \log|\eta_{t-4n+1}| + \frac{-\alpha-\gamma}{4}\exp(c_0),$$
$$v_2^{(4n)} \geq \log(\eta_{t-4n+2}) + \sqrt{\log(n^{3/2}+1)},$$
$$v_3^{(4n)} \geq \log(\eta_{t-4n+3}) + \log(n^{3/2}+1),$$
$$v_4^{(4n)} \geq \log|\eta_{t-4n+4}| + n^{3/2}.$$

Lemma 1.3 allows us to prove Proposition 1.1.

Proposition 1.1. *If $\eta_t \overset{i.i.d.}{\sim} \mathrm{N}(0,1)$ and $\alpha < -|\gamma|$, then we cannot prove invertibility using our method, as $\left|v_n^{(n)} - u_n^{(n)}\right|$ does not converge to 0, and even almost surely admits an extracted series that diverges toward infinity.*

Proof. In order to show that $\left|v_n^{(n)} - u_n^{(n)}\right|$ diverges, we have to show that one of its extracting series diverges. Consider $\left|v_{4n}^{(4n)} - u_{4n}^{(4n)}\right|$. By applying (2) of Lemma 1.2 recursively, by taking $v_0^{(4n)} \equiv c_0$, and as $g_{\alpha,\gamma}(.,.) > 0$, and using Eq. (2), we obtain

$$\left|v_{4n}^{(4n)} - u_{4n}^{(4n)}\right| \geq \exp\left(4n\log\left|\frac{|\alpha|-|\gamma|}{2}\right| + \sum_{\substack{i=1\\ i\neq 4n-4}}^{4n-1}\left(\log|\eta_{t-i}| + \frac{\delta_{t-i-1}|\eta_{t-i-1}|}{2}\right)\right.$$
$$\left. + \frac{\log|\eta_{t-4n+4}| + v_4^{(4n)}}{2}\right)\left||\eta_{t-4n}| - \exp(c_0)\right|.$$

By the assumption on the distribution, and by using LLN, it follows that

$$4n\log\left|\frac{|\alpha|-|\gamma|}{2}\right| + \sum_{\substack{i=1\\ i\neq 4n-4}}^{4n-1}\left(\log|\eta_{t-i}| + \frac{\delta_{t-i-1}|\eta_{t-i-1}|}{2}\right) + \log|\eta_{t-4n+4}| = O(n).$$

Using Lemma 1.3, it follows that (almost surely)

$$\forall N \in \mathrm{N}, \exists n \geq N: \qquad \left|v_{4n}^{(4n)} - u_{4n}^{(4n)}\right| \geq \exp\left(O(n) + \frac{n^{3/2}}{2}\right)\left||\eta_{t-4n}| - \exp(c_0)\right|$$
$$\geq \exp\left(O(n) + \frac{n^{3/2}}{2}\right)\frac{\exp(c_0)}{2}.$$

The above result shows that if we want to have $\left|v_n^{(n)} - u_n^{(n)}\right| = \left|v_n^{(n)} - \log|\eta_t|\right| \underset{n\to\infty}{\overset{a.s.}{\to}} 0$, one should take $c_0 = -\infty$ (in this case, we can prove invertibility of the model). When this equality does not hold, the recursion above is divergent. Such divergence is obtained only because, for an infinite number of n, there exists k such that $v_k^{(n)} > \log\left|\eta_{t-n+k}\right|$. If we had, for instance, $v_1^{(n)} \leq \log\left|\eta_{t-n+k}\right|$ for all n (as in the case when the initial value is null), a simple recursion would lead to the same inequality for all k: $v_k^{(n)} \leq \log\left|\eta_{t-n+k}\right|$,[19] so that we would not be able to derive the divergence. This leads us to the case of leverage, as introduced in Section 2. In this case, if we modify properly the definition of the events A_n, as introduced in the above proofs by having a negative value[20] for η_{t-4n}, the other inequalities being unchanged, we would have $v_k^{(n)} > \log\left|\eta_{t-n+k}\right|$ for $k = 1$ and therefore a sequence of $v_k^{(n)}$ leading to a $O(n^{3/2})$ term. If this is, however, not enough to prove the divergence using inequality (2) of Lemma 1.1, as further negative values for the independent shocks may cancel out this $O(n^{3/2})$ term, the property we have found with this counterexample can be considered as a theoretical (and partial) explanation of the erratic behavior of the recursion used for deriving invertibility in the case of leverage, a phenomenon which has only been derived empirically, as in Wintenberger (2013). Therefore, in the case of leverage, we can assume the lack of invertibility to be highly likely.

[19] We are using actually this property in Section 3 for the proof of invertibility.
[20] Sufficiently large in absolute value, depending on the chosen initial value.

Part 2: Proofs of lemmas and propositions for invertibility of EARCH(∞)

Proof of Lemma 2.1. We prove the result recursively for any $n \in \mathrm{N}^*$. Fix $n > 0$ and define

$$\left(\mathrm{H}_p\right) \equiv \text{“}\forall k \in \left[1, p\right], u_k^{(n)} = \log\left|\eta_{t-n+k}\right|\text{''}.$$

According to Eqs. (4) and (5), (H_1) is true. Assume $\left(\mathrm{H}_p\right)$ and prove $\left(\mathrm{H}_{p+1}\right)$

$$\begin{aligned}
u_{p+1}^{(n)} &= \log\left|\varepsilon_{t-n+p+1}\right| - \frac{\omega}{2} + \sum_{j=1}^{p} \beta_j g_{\alpha,\gamma}\left(u_{p+1-j}^{(n)}, \varepsilon_{t-n+p+1-j}\right) \\
&\quad + \sum_{i=0}^{\infty} \beta_{i+1+p} g_{\alpha,\gamma}\left(\log\left|\eta_{t-n-i}\right|, \varepsilon_{t-n-i}\right) \\
&= \log\left|\varepsilon_{t-n+p+1}\right| - \frac{\omega}{2} + \sum_{j=1}^{p} \beta_j g_{\alpha,\gamma}\left(\log\left|\eta_{t-n+p+1-j}\right|, \varepsilon_{t-n+p+1-j}\right) \\
&\quad + \sum_{i=p+1}^{\infty} \beta_i g_{\alpha,\gamma}\left(\log\left|\eta_{t-n+p+1-i}\right|, \varepsilon_{t-n+p+1-i}\right),
\end{aligned}$$

by using $\left(\mathrm{H}_p\right)$; then we can conclude by matching the previous equality with (4), so that $\left(\mathrm{H}_{p+1}\right)$ is true.

Proof of Lemma 2.2. We will prove the lemma recursively:

$$(H_k) : \text{“}\left|u_n^{(n)} - v_n^{(n)}\right| \le a_k''.$$

We have $\left|u_n^{(n)} - v_n^{(n)}\right| \le \sum_{i=0}^{\infty} \beta_{i+n}\left|\delta_{t-n-i}\right|\left|\eta_{t-n-i}\right| + \sum_{j=1}^{n-1} \beta_j\left|\delta_{t-j}\right| \exp\left(\xi_{t-j}\right)\left|u_{n-j}^{(n)} - v_{n-j}^{(n)}\right| \equiv a_1$, and also $\left|u_{n-j}^{(n)} - v_{n-j}^{(n)}\right| \le \sum_{i=0}^{\infty} \beta_{i+n-j}\left|\delta_{t-n-i}\right|\left|\eta_{t-n-i}\right| + \sum_{l=1}^{n-j-1} \beta_l\left|\delta_{t-j-l}\right| \exp\left(\xi_{t-j-l}\right)\left|u_{n-j-l}^{(n)} - v_{n-j-l}^{(n)}\right|$, so that we can write

$$\begin{aligned}
\left|u_n^{(n)} - v_n^{(n)}\right| &\le \sum_{i=0}^{\infty} \beta_{i+n}\left|\delta_{t-n-i}\right|\left|\eta_{t-n-i}\right| + \sum_{j=1}^{n-1}\sum_{i=0}^{+\infty} \beta_j\beta_{i+n-j}\left|\delta_{t-j}\right| \exp\left(\xi_{t-j}\right)\delta_{t-n-i}\left|\eta_{t-n-i}\right| \\
&\quad + \sum_{j=1}^{n-2}\sum_{l=1}^{n-j-1} \beta_j\beta_l\left|\delta_{t-j-l}\right|\left|\delta_{t-j}\right| \exp\left(\xi_{t-j-l} + \xi_{t-j}\right)\left|u_{n-j-l}^{(n)} - v_{n-j-l}^{(n)}\right| \equiv a_2.
\end{aligned}$$

So we have (H_1) and (H_2), which are true. Assume (H_k) and prove (H_{k+1})

$$\begin{aligned}
a_k &= \sum_{i=0}^{+\infty}\left|\delta_{t-n-i}\right|\left|\eta_{t-n-i}\right|\left(\beta_{i+n} + \sum_{p=1}^{k-1}\sum_{i_1,\dots,i_p \in A_p^{(n)}} \widehat{\Pi}_p\widehat{D}_p \exp\left(\sum_{j=1}^{p}\xi_{t-\widehat{S}_j}\right) \times \beta_{i+n-\widehat{S}_p}\right) \\
&\quad + \sum_{i_1,\dots,i_k \in A_k^{(n)}} \widehat{\Pi}_k\widehat{D}_k \exp\left(\sum_{j=1}^{k}\xi_{t-\widehat{S}_j}\right)\left|u_{n-\widehat{S}_k}^{(n)} - v_{n-\widehat{S}_k}^{(n)}\right|.
\end{aligned}$$

However,

$$\left|u_{n-\widehat{S}_k}^{(n)} - v_{n-\widehat{S}_k}^{(n)}\right| \le \sum_{i=0}^{\infty} \beta_{i+n-\widehat{S}_k}\left|\delta_{t-n-i}\right|\left|\eta_{t-n-i}\right| + \sum_{l=1}^{n-\widehat{S}_k-1} \beta_l\left|\delta_{t-\widehat{S}_k-l}\right| \exp\left(\xi_{t-\widehat{S}_k-l}\right)\left|u_{n-\widehat{S}_k-l}^{(n)} - v_{n-\widehat{S}_k-l}^{(n)}\right|,$$

so that

$$\sum_{i_1,\ldots,i_k \in A_k^{(n)}} \widehat{\Pi}_k \widehat{D}_k \exp\left(\sum_{j=1}^{k} \xi_{t-\widehat{S}_j}\right) \left| u_{n-\widehat{S}_k}^{(n)} - v_{n-\widehat{S}_k}^{(n)} \right|$$

$$\leq \sum_{\substack{i_1,\ldots,i_k \in A_k^{(n)} \\ \widehat{S}_k = n-1}} \widehat{D}_k \widehat{\Pi}_k \exp\left(\sum_{j=1}^{k} \xi_{t-\widehat{S}_j}\right) \left| u_1^{(n)} - v_1^{(n)} \right|$$

$$+ \sum_{\substack{i_1,\ldots,i_k \in A_k^{(n)} \\ \widehat{S}_k < n-1}} \widehat{D}_k \widehat{\Pi}_k \exp\left(\sum_{j=1}^{k} \xi_{t-\widehat{S}_j}\right) \left(\sum_{i_{k+1}=1}^{n-\widehat{S}_k-1} \beta_l \left| \delta_{t-\widehat{S}_k-i_{k+1}} \right| \right.$$

$$\left. \exp\left(\xi_{t-\widehat{S}_k-i_{k+1}}\right) \left| u_{n-\widehat{S}_k-i_{k+1}}^{(n)} - v_{n-\widehat{S}_k-i_{k+1}}^{(n)} \right| \right)$$

$$+ \sum_{\substack{i_1,\ldots,i_k \in A_k^{(n)} \\ \widehat{S}_k < n-1}} \widehat{D}_k \widehat{\Pi}_k \exp\left(\sum_{j=1}^{k} \xi_{t-\widehat{S}_j}\right) \left(\sum_{i=0}^{\infty} \beta_{i+n-\widehat{S}_k} \left| \delta_{t-n-i} \right| \left| \eta_{t-n-i} \right| \right).$$

By using the inequality $\left| u_1^{(n)} - v_1^{(n)} \right| \leq \sum_{i=0}^{\infty} \beta_{i+1} \left| \delta_{t-n-i} \right| \left| \eta_{t-n-i} \right|$, and by recombining the sums above, we can see that

$$\sum_{i_1,\ldots,i_k \in A_k^{(n)}} \widehat{\Pi}_k \widehat{D}_k \exp\left(\sum_{j=1}^{k} \xi_{t-\widehat{S}_j}\right) \left| u_{n-\widehat{S}_k}^{(n)} - v_{n-\widehat{S}_k}^{(n)} \right|$$

$$\leq \sum_{\substack{i_1,\ldots,i_k \in A_k^{(n)} \\ \widehat{S}_k < n-1}} \widehat{D}_k \widehat{\Pi}_k \exp\left(\sum_{j=1}^{k} \xi_{t-\widehat{S}_j}\right) \left(\sum_{i_{k+1}=1}^{n-\widehat{S}_k-1} \beta_l \left| \delta_{t-\widehat{S}_k-i_{k+1}} \right| \right.$$

$$\left. \exp\left(\xi_{t-\widehat{S}_k-i_{k+1}}\right) \left| u_{n-\widehat{S}_k-i_{k+1}}^{(n)} - v_{n-\widehat{S}_k-i_{k+1}}^{(n)} \right| \right)$$

$$+ \sum_{i_1,\ldots,i_k \in A_k^{(n)}} \widehat{D}_k \widehat{\Pi}_k \exp\left(\sum_{j=1}^{k} \xi_{t-\widehat{S}_j}\right) \left(\sum_{i=0}^{\infty} \beta_{i+n-\widehat{S}_k} \left| \delta_{t-n-i} \right| \left| \eta_{t-n-i} \right| \right)$$

$$\leq \sum_{\substack{i_1,\ldots,i_k \in A_k^{(n)} \\ \widehat{S}_k < n-1}} \sum_{i_{k+1}=1}^{n-\widehat{S}_k-1} \widehat{D}_k \widehat{\Pi}_k \beta_l \left| \delta_{t-\widehat{S}_k-i_{k+1}} \right| \exp\left(\sum_{j=1}^{k} \xi_{t-\widehat{S}_j} + \xi_{t-\widehat{S}_k-i_{k+1}}\right) \left| u_{n-\widehat{S}_k-i_{k+1}}^{(n)} - v_{n-\widehat{S}_k-i_{k+1}}^{(n)} \right|$$

$$+ \sum_{i=0}^{\infty} \left| \delta_{t-n-i} \right| \left| \eta_{t-n-i} \right| \left(\sum_{i_1,\ldots,i_k \in A_k^{(n)}} \widehat{D}_k \widehat{\Pi}_k \exp\left(\sum_{j=1}^{k} \xi_{t-\widehat{S}_j}\right) \beta_{i+n-\widehat{S}_k} \right).$$

By noticing that

$$\left\{i_1, \ldots, i_k \in A_k^{(n)}, i_{k+1} \in \left[1, n - \widehat{S}_k - 1\right] : \widehat{S}_k < n - 1\right\} = A_{k+1}^{(n)},$$

we finally have the result

$$\left|u_n^{(n)} - v_n^{(n)}\right| \leq a_k \leq a_{k+1} \Rightarrow \left(H_{k+1}\right)$$

is true.

Part 3: EGARCH(p, q) specification

Proof of Lemma 3.1. (1) If we rename $y_i \equiv \delta_{t-i} |\eta_{t-i}|$, one can easily check recursively the positivity of the β_i coefficients by taking the partial differential of $\log \sigma_t$ with respect to y_t, as

$$\log \sigma_t = \frac{\omega}{2} + \sum_{i \geq 1} \beta_i y_i \Rightarrow \forall i \frac{\partial \log \sigma_t}{\partial y_i} = \beta_i.$$

We can use this equation recursively

$$\frac{\partial \log \sigma_t}{\partial y_i} = \sum_{j=1}^{p} a_j \frac{\partial \log \sigma_{t-j}}{\partial y_i} + \sum_{k=1}^{q} b_k 1_{k=i},$$

where 1 represents the index function.

(2) A straightforward recursion leads to the β_i as a mixture of sums and products of the θ_i and b_i, so they are non-negative. □

Proof of Lemma 3.2. In order to prove this lemma, we present a recursion. Starting with $(1 - \theta_1 L)^{-1} \times \left(\sum_{i=1}^{q} b_i \delta_{t-i} |\eta_{t-i}|\right)$:

$$\begin{aligned}
(1 - \theta_1 L)^{-1} \times \left(\sum_{i=1}^{q} b_i \delta_{t-i} |\eta_{t-i}|\right) &= \sum_{l=0}^{+\infty} \theta_1^l \sum_{i=1}^{q} b_i \delta_{t-l-i} \left|\eta_{t-l-i}\right| \\
&= \sum_{m=1}^{+\infty} \theta_1^{m-1} \left(\sum_{i=1}^{\min(q,m)} b_i \theta_1^{1-i}\right) \delta_{t-m} |\eta_{t-m}| .
\end{aligned}$$

By taking $m = i + l$, we can introduce $\beta_{\sup}$:

$$\begin{aligned}
\sum_{m=1}^{+\infty} \theta_1^{m-1} \left(\sum_{i=1}^{\min(q,m)} b_i \theta_1^{1-i}\right) \delta_{t-m} |\eta_{t-m}| &= \sum_{m=1}^{+\infty} \beta_{\sup}^{m-1} \left(\frac{\theta_1}{\beta_{\sup}}\right)^{m-1} \left(\sum_{i=1}^{\min(q,m)} b_i \theta_1^{1-i}\right) \delta_{t-m} |\eta_{t-m}| \\
&= \sum_{m=1}^{+\infty} \beta_{\sup}^{m-1} C_m \delta_{t-m} |\eta_{t-m}| ,
\end{aligned}$$

where

$$C_m \equiv \left(\frac{\theta_1}{\beta_{\sup}}\right)^{m-1} \left(\sum_{i=1}^{\min(q,m)} b_i \theta_1^{1-i}\right),$$

so that

$$|C_m| \leq \max_{1\leq m\leq q}\left|\sum_{i=1}^{m} b_i\theta_1^{1-i}\right|.$$

Consider

$$(1-\theta_i L)^{-1}\times\left(\sum_{m=1}^{+\infty}\beta_{\sup}^{m-1}C_m\delta_{t-m}\left|\eta_{t-m}\right|\right),\qquad \text{and for any other } \theta_i,\ i\geq 2,$$

so that

$$\begin{aligned}(1-\theta_i L)^{-1}\times\left(\sum_{m=1}^{+\infty}\beta_{\sup}^{m-1}C_m\delta_{t-m}\left|\eta_{t-m}\right|\right) &= \sum_{l=0}^{+\infty}\theta_i^l\sum_{m=1}^{+\infty}\beta_{\sup}^{m-1}C_m\delta_{t-l-m}\left|\eta_{t-l-m}\right| \\ &= \sum_{s=1}^{+\infty}\beta_{\sup}^{s-1}\left(\sum_{l=0}^{s-1}\left(\frac{\theta_i}{\beta_{\sup}}\right)^l C_{s-l}\right)\delta_{t-s}\left|\eta_{t-s}\right|.\end{aligned}$$

It follows by assumption that $\frac{|\theta_i|}{\beta_{\sup}} < 1$, and by definition that $|C_m| \leq \max_{1\leq m\leq q}\left|\sum_{i=1}^{m} b_i\theta_1^{1-i}\right|$. If we redefine recursively

$$C_s := \sum_{l=0}^{s-1}\left(\frac{|\theta_i|}{\beta_{\sup}}\right)^l C_{s-l},$$

it follows that

$$|C_s| \leq \frac{\max_{1\leq m\leq q}\left|\sum_{i=1}^{m} b_i\theta_1^{1-i}\right|}{\left(1-\frac{|\theta_i|}{\beta_{\sup}}\right)},$$

from which it follows that

$$(1-\theta_i L)^{-1}\times(1-\theta_1 L)^{-1}\times\left(\sum_{i=1}^{q} b_i\delta_{t-i}\left|\eta_{t-i}\right|\right) = \sum_{s=1}^{+\infty}\beta_{\sup}^{s-1}C_s\delta_{t-s}\left|\eta_{t-s}\right|.$$

Therefore, one can easily check from the above recursion that

$$(1-\theta_1 L)^{-1}\times\cdots\times\left(1-\theta_p L\right)^{-1}\times\left(\sum_{i=1}^{q} b_i\delta_{t-i}\left|\eta_{t-i}\right|\right) = \sum_{u=1}^{+\infty}\beta_{\sup}^{u-1}C_u\delta_{t-u}\left|\eta_{t-u}\right|,$$

where

$$|C_u| \leq \frac{\max_{1\leq m\leq q}\left|\sum_{i=1}^{m} b_i\theta_1^{1-i}\right|}{\prod_{p\geq i\geq 2}\left(1-\frac{|\theta_i|}{\beta_{\sup}}\right)} \equiv C \qquad \text{(where } C_u \text{ is a positive number).}$$

Part 4: Invertibility of EGARCH(p, q)

Proof of Lemma 4.1. We prove this lemma in the case where $\alpha \geq |\gamma|$ (the proof is identical for the case $\alpha \leq -|\gamma|$). We have

$$B_n = \sum_{i=0}^{+\infty}\beta^{*i}\delta^*_{t-n-i}\left|\eta_{t-n-i}\right|\exp\left(\sum_{l=n}^{+\infty}\frac{\beta^{*l-n}}{1-\beta^*}\delta^*_{t-l}\left|\eta_{t-l}\right|\right).$$

We know that $X_n \equiv \sum_{l=n}^{+\infty} \frac{\beta^{*l-n}}{1-\beta^*}\delta^*_{t-l}\left|\eta_{t-l}\right|$ are L^2-variables, as an absolutely convergent sum of L^2-variables (where it is assumed that $\mathrm{E}\left[\eta_t^2\right] = 1$) as L^2 is a Hilbert space). Furthermore, by using Chebychev inequality, we obtain

$$\mathrm{P}\left(|X_n| \geq n^{\nu}\right) \leq \frac{\mathrm{E}\left[|X_n|^2\right]}{n^{2\nu}} = \frac{\mathrm{E}\left[|X_1|^2\right]}{n^{2\nu}},$$

as the X_n are identically distributed. Therefore, $\sum \mathrm{P}\left(|X_n| \geq n^{\nu}\right) < \infty$ and, by using the Borel–Cantelli Lemma, we have with probability one that $X_n = O\left(n^{\nu}\right)$. As this is true $\forall \nu > 1/2$, we also have $X_n = o\left(n^{\nu}\right)$.

By using the same reasoning with $Y_n \equiv \sum_{i=0}^{+\infty} \beta^{*i}\delta^*_{t-n-i}\left|\eta_{t-n-i}\right|$, we obtain the result.

Proof of Proposition 4.1. As in the previous case, we show the proof for $\alpha \geq |\gamma|$ (as the proof is essentially identical for $\alpha \leq -|\gamma|$). According to condition 1 and by continuity, we know that $\exists \varepsilon_1, \varepsilon_2 > 0$, so that

$$\mathrm{E}\left[\frac{\delta^*_t\left|\eta_t\right|}{1-\beta^*}\right] + \log\left(\beta^* + E\left[\delta^*_t\left|\eta_t\right|\right]\right) + \varepsilon_1 < -\varepsilon_2.$$

We also have inequality (18):

$$\left|u_n^{(n)} - v_n^{(n)}\right|$$
$$\leq B_n \exp\left(\sum_{l=1}^{n-1}\frac{\delta^*_{t-l}\left|\eta_{t-l}\right|}{1-\beta^*}\right)\left(\beta^{*n-1} + \sum_{p=1}^{n-1}\beta^{*n-1-p}\sum_{1\leq s_1<\cdots<s_p\leq n-1}\exp\left(\sum_{j=1}^{p}\log\left(\delta^*_{t-s_j}\left|\eta_{t-s_j}\right|\right)\right)\right).$$

If we note

$$Z_n = \frac{\left(\beta^{*n-1} + \sum_{p=1}^{n-1}\beta^{*n-1-p}\sum_{1\leq s_1<\cdots<s_p\leq n-1}\exp\left(\sum_{j=1}^{p}\log\left(\delta^*_{t-s_j}\left|\eta_{t-s_j}\right|\right)\right)\right)}{\exp\left((n-1)\log\left(\beta^* + E\left[\delta^*_t\left|\eta_t\right|\right]\right) + (n-1)\varepsilon_1\right)},$$

we have

$$\left|u_n^{(n)} - v_n^{(n)}\right| \leq B_n \exp\left(\sum_{l=1}^{n-1}\frac{\delta^*_{t-l}\left|\eta_{t-l}\right|}{1-\beta^*} + (n-1)\log\left(\beta^* + E\left[\delta^*_t\left|\eta_t\right|\right]\right) + (n-1)\varepsilon_1\right) Z_n.$$

It can be shown that Z_n goes to zero almost surely, as follows. Let $\varepsilon > 0$ by the Markov inequality

$$\mathrm{P}\left(Z_n > \varepsilon\right) \leq \frac{\mathrm{E}\left[\beta^{*n-1} + \sum_{p=1}^{n-1}\beta^{*n-1-p}\sum_{1\leq s_1<\cdots<s_p\leq n-1}\exp\left(\sum_{j=1}^{p}\log\left(\delta^*_{t-s_j}\left|\eta_{t-s_j}\right|\right)\right)\right]}{\exp\left((n-1)\log\left(\beta^* + E\left[\delta^*_t\left|\eta_t\right|\right]\right) + (n-1)\varepsilon_1\right)\times\varepsilon}.$$

However,

$$\mathrm{E}\left[\beta^{*n-1} + \sum_{p=1}^{n-1}\beta^{*n-1-p}\sum_{1\leq s_1<\cdots<s_p\leq n-1}\exp\left(\sum_{j=1}^{p}\log\left(\delta^*_{t-s_j}\left|\eta_{t-s_j}\right|\right)\right)\right]$$
$$= \sum_{p=0}^{n-1}\binom{n-1}{p}\beta^{*n-1-p}\left(\mathrm{E}\left[\delta^*_t\left|\eta_t\right|\right]\right)^p,$$

where

$$\binom{n-1}{p} = \frac{(n-1)!}{(n-1-p)!p!},$$

as $\delta^*_{t-s_j}\left|\eta_{t-s_j}\right|$ are L^1 and i.i.d. Using Newton's formula, it can be shown that

$$\mathrm{E}\left[\beta^{*n-1}+\sum_{p=1}^{n-1}\beta^{*n-1-p}\sum_{1\le s_1<\cdots<s_p\le n-1}\exp\left(\sum_{j=1}^{p}\log\left(\delta^*_{t-s_j}\left|\eta_{t-s_j}\right|\right)\right)\right]=\left(\beta^*+\mathrm{E}\left[\delta^*_t\left|\eta_t\right|\right]\right)^{n-1}.$$

Therefore,

$$\mathrm{P}\left(Z_n>\varepsilon\right)\le\frac{\exp\left(-(n-1)\varepsilon_1\right)}{\varepsilon},$$

and, by using Lemma 1.2, we can show that

$$Z_n\underset{n\to\infty}{\overset{a.s.}{\to}}0.$$

Moreover, by LLN,

$$\sum_{l=1}^{n-1}\frac{\delta^*_{t-l}\left|\eta_{t-l}\right|}{1-\beta^*}=\mathrm{E}\left[\frac{\delta^*_t\left|\eta_t\right|}{1-\beta^*}\right]\times n+o\left(n\right).$$

Therefore,

$$\exp\left(\sum_{l=1}^{n-1}\frac{\delta^*_{t-l}\left|\eta_{t-l}\right|}{1-\beta^*}+n\log\left(\beta^*+E\left[\delta^*_t\left|\eta_t\right|\right]\right)+n\varepsilon_1\right)=\exp\left(-n\varepsilon_2+o\left(n\right)\right).$$

According to Lemma 4.1, we have

$$B_n=\exp\left(o(n)\right).$$

Therefore,

$$\left|u_n^{(n)}-v_n^{(n)}\right|\le\exp\left(-n\varepsilon_2+o\left(n\right)\right)Z_n,$$
$$\left|v_n^{(n)}-u_n^{(n)}\right|=\left|v_n^{(n)}-\log\left|\eta_t\right|\right|\underset{n\to\infty}{\overset{a.s.}{\to}}0,$$

which proves invertibility of EGARCH(p,q).

Proof of Proposition 4.2. We show the proof for $\alpha\ge|\gamma|$ (as the proof is almost identical for $\alpha\le-|\gamma|$). The proof of this proposition is also based on inequality (12):

$$\left|u_n^{(n)}-v_n^{(n)}\right|$$
$$\le B_n\exp\left(\sum_{l=1}^{n-1}\frac{\delta^*_{t-l}\left|\eta_{t-l}\right|}{1-\beta^*}\right)\left(\beta^{*n-1}+\sum_{p=1}^{n-1}\beta^{*n-1-p}\sum_{1\le s_1<\cdots<s_p\le n-1}\exp\left(\sum_{j=1}^{p}\log\left(\delta^*_{t-s_j}\left|\eta_{t-s_j}\right|\right)\right)\right).$$

Each sum, $\sum_{j=1}^{p}\log\left(\delta^*_{t-s_j}\left|\eta_{t-s_j}\right|\right)$, where $1\le s_1<\cdots<s_p\le n-1$, can be bounded from above by the sums of the p highest values from the $\log\left(\delta^*_{t-s_j}\left|\eta_{t-s_j}\right|\right)$ for each realization of the process. Therefore, using our notation, we can write

$$\left|u_n^{(n)}-v_n^{(n)}\right|\le B_n\sum_{k\in\left\{0,\frac{1}{n-1},\frac{2}{n-1},\ldots,1\right\}}\binom{n-1}{k(n-1)}\exp\left((n-1)\left(k.\underset{k}{\hat{ES}}\left[\log\left(\delta^*_t\left|\eta_{t-}\right|\right)\right]\right.\right.$$
$$\left.\left.+\frac{1}{n-1}\sum_{l=1}^{n-1}\frac{\delta^*_{t-l}\left|\eta_{t-l}\right|}{1-\beta^*}+(1-k)\log\left(\beta^*\right)\right)\right).$$

Using Stirling's formula, we have for $k \in [0, 1]$

$$\binom{n-1}{k(n-1)} \sim \exp\left(\left(-k\log(k) - (k-1)\log(k-1)\right)(n-1)\right) * \frac{1}{\sqrt{2\pi k(1-k)(n-1)}},$$

which leads to Condition 2.

However, we also have to prove uniform convergence for all $k \in [k_{\min}, 1]$ in order to conclude as follows:

(i) The convergence following convergence may not be uniform for all $k \in [0, 1]$:

$$\binom{n-1}{k(n-1)} \Big/ \left(\exp\left(\left(-k\log(k) - (k-1)\log(k-1)\right)(n-1)\right) * \frac{1}{\sqrt{2\pi k(1-k)(n-1)}}\right) \longrightarrow 1.$$

However, it is uniform for all intervals of the form $k \in [k_{\min}, 1]$, where $k_{\min} > 0$. Therefore, we can take $k_{\min} > 0$ small enough such that all the sum elements for $k < k_{\min}$ are negligible, and use the uniform convergence for the remaining part.

(ii) $k.\underset{k}{\hat{ES}}\left[\log\left(\delta_t^* \, |\eta_{t-}|\right)\right]$ (that is, empirical estimation of the expected shortfall times k) should also converge uniformly for all $k \in [k_{\min}, 1]$ toward $k.\underset{k}{ES}\left[\log\left(\delta_t^* \, |\eta_{t-}|\right)\right]$.

The uniform convergence is a consequence of the second theorem of Dini: by LLN, we have the simple convergence of $k.\underset{k}{\hat{ES}}\left[\log\left(\delta_t^* \, |\eta_{t-}|\right)\right]$ toward $k.\underset{k}{ES}\left[\log\left(\delta_t^* \, |\eta_{t-}|\right)\right]$ for all $k \in [k_{\min}, 1]$, k belongs to a compact set, $k \mapsto k.\underset{k}{\hat{ES}}\left[\log\left(\delta_t^* \, |\eta_{t-}|\right)\right]$ are increasing functions for all n, and $k \mapsto kES_k[X] = \int_{1-k}^{1} VaR_u(X)\,du$ is continuous.

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