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ABSTRACT
The goal of this paper is to study a stochastic game connected to a system of forward-backward stochastic differential equations (FBSDEs) involving delay and noisy memory. We derive sufficient and necessary maximum principles for a set of controls for the players to be a Nash equilibrium in the game. Furthermore, we study a corresponding FBSDE involving Malliavin derivatives. This kind of equation has not been studied before. The maximum principles give conditions for determining the Nash equilibrium of the game. We use this to derive a closed form Nash equilibrium for an economic model where the players maximize their consumption with respect to recursive utility.

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1. Introduction

The aim of this paper is to study a stochastic game between two players. The game is based on a forward stochastic differential equation (SDE) for the process X. In applications to economy, this process can be thought of as the market situation, e.g. the financial market, the housing market or the oil market. This SDE includes two kinds of memory of the past; regular memory and noisy memory. Regular memory (also called delay, see f. ex. the survey paper by Ivanov et al. [1]) means that the SDE can depend on previous values of the process X. That is, for some given $\delta > 0$, $X(t)$ depends on $X(t-\delta)$. For more on stochastic delay differential equations and optimal control with delay, see Øksendal et al. [2] and Agram and Øksendal [3]. In contrast, noisy memory means that the SDE may involve an Itô integral over previous values of the process, so for $\delta > 0$, $X(t)$ depends on $\int_{t-\delta}^{t} X(s)dB(s)$ where $\{B(s)\}_{s\in[0,T]}$ is a Brownian motion. For more on noisy memory, see Dahl et al. [4].

Connected to this SDE are two backward stochastic differential equations (BSDEs). These BSDEs are connected to the SDE in the sense that they depend on $\{X(t)\}_{t\in[0,T]}$, as well as the delay and noisy memory of this process. Hence, this forms an FBSDE system. Each of these BSDEs corresponds to one of the players in the stochastic game; corresponding to player $i=1, 2$ is a BSDE in the process $\{W_i(t)\}_{t\in[0,T]}$. The length of memory can be different for the two players, so for $i=1, 2$, player i has memory span

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δ_i. The players may also have different levels of information, which is included in the model by having (potentially) different filtrations $\{\mathcal{E}_t^{(i)}\}_{t\in[0,T]}$, $i=1, 2$.

Each of the players aim to find an optimal control u_i which maximizes their personal performance (objective) function, J_i. Seminal work in stochastic optimal control has been done by Krylov and his students, see e.g. Krylov [5, 6]. The performance function of each of the agents will be defined in such a way that it depends on the player's profit rate, the market process X and the process W_i coming from the player's BSDE (more on this in Section 2, Equation (11)). This kind of problem, where both players maximize their performance which depends on an FBSDE, is called an FBSDE stochastic game, and has been studied by e.g. Øksendal and Sulem [7]. However, they do not include memory in their model. We study conditions for a pair of controls (u_1, u_2) to be a Nash equilibrium for such a stochastic game. That is, we would like to determine controls such that the players cannot benefit by changing their actions. In order to do so, we derive sufficient and necessary maximum principles giving conditions for a control to be Nash optimal. This is done in Sections 3 and 4. Maximum principles for forward-backward stochastic differential equations (FBSDEs) have been studied by Øksendal and Sulem [7], Wang and Wu [8], Wu [9] and Wang et al. [10], but these papers do not consider delay and noisy memory.

In connection with these maximum principles, there are adjoint equations (see e.g. Øksendal [11] for an introduction to stochastic maximum principles and adjoint equations, or Øksendal and Sulem [12] for maximum principles and adjoint equations where delay is involved). In our case, these adjoint equations are a system of coupled forward-backward stochastic differential equations involving Malliavin derivatives (see Di Nunno et al. [13] for more on Malliavin derivatives). To the best of our knowledge, such equations have not been studied before. In Section 5 we study a slightly simplified version of these adjoint FBSDEs, and establish a connection between these equations and a system of FBSDEs without Malliavin derivatives. Finally, in Section 6, we apply our results to a specific example in order to determine the optimal consumption with respect to recursive utility.

2. The problem

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $B(t)$, $t \in [0, T]$ be a Brownian motion in this space. Let $(N([0,t],\mathcal{B}), 0 \le t \le T, \mathcal{B} \subseteq \mathbb{R} - \{0\})$ be an independent Poisson random measure. Denote by $\nu(\mathcal{B})$ the associated Lèvy measure such that $E[N([0,t],\mathcal{B})] = \nu(\mathcal{B})t$. Also, let $\tilde{N}(t,\cdot)$ be the corresponding compensated Poisson random measure, i.e.,

$$\tilde{N}(dt, d\mu) := N(dt, d\mu) - \nu(d\mu)dt.$$

Let $(\mathcal{F}_t)_{t\in[0,T]}$ be the P-augmented filtration generated by $B(t)$ and $\tilde{N}(t,\cdot)$.

We will consider a game between two players: player 1 and player 2. Let $u_i(t)$ be the control process chosen by player $i=1, 2$, and denote $\mathrm{u}(t) = (u_1(t), u_2(t))$. Let $\mathcal{A}_i$, $i=1, 2$, denote the set of admissible controls for player i. It is contained in a given set of càdlàg processes in $L^2(\Omega \times [0,T])$, with values in a subset $\mathcal{V}_i$ of $\mathbb{R}$. Let $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ be the combined controls for both players, and denote by $\mathcal{V} := \mathcal{V}_1 \times \mathcal{V}_2$. Let $\delta_i \ge 0$, $i=1,$

2 be the memory span of player 1 and 2, respectively. We define $\delta := \max \delta_1, \delta_2$ to be the longest memory span of the two agents.

We consider a controlled forward stochastic differential equation for a process $X(t) = X_u(t, \omega), \omega \in \mathbf{\Omega}, t \in [0, T]$ determining the market situation (in the following, we omit the ω for notational ease unless it is important to highlight its dependence):

$$\begin{array}{rcl} dX(t) & = & b(t, X(t), \boldsymbol{Y}(t), \boldsymbol{\Lambda}(t), \mathrm{u}(t), \omega)dt \\ & & +\sigma(t, X(t), \boldsymbol{Y}(t), \boldsymbol{\Lambda}(t), \mathrm{u}(t), \omega)dB(t) \\ & & +\int_{\mathbb{R}} \gamma(t^-, X(t^-), \boldsymbol{Y}(t^-), \boldsymbol{\Lambda}(t^-), \boldsymbol{u}(t^-), \zeta, \omega)\tilde{N}(dt, d\zeta), \ \ t \in [0, T], \\ X(t) & = & \xi(t), \ \ t \in [-\delta, 0), \end{array} \tag{1}$$

where $\xi(t)$ is some (given) initial process, $\boldsymbol{Y}(t) = (Y_1(t), Y_2(t)), \boldsymbol{\Lambda}(t) = (\Lambda_1(t), \Lambda_2(t))$, and $Y_i(t) := X(t - \delta_i), \Lambda_i(t) := \int_{t-\delta_i}^t X(s)dB(s)$, and $\delta_i \geq 0$ for $i = 1$, 2. The superscript t^- means that we are taking the left limit of the process is question (that is, the value before a potential jump at time t), see Øksendal and Sulem [14] for more on this.

Remark 2.1. N ote that $\xi(t)$ is a given initial process which can not be controlled (i.e., there is no dependency on $\boldsymbol{u}$ in ξ). Hence, we do not need to define the filtration $(\mathcal{F}_t)$ for $t \in [-\delta, 0)$.

Here, the *delay processes* Y_i, and the *noisy memory processes* Λ_i correspond to player $i = 1$, 2 respectively. Hence, the two players may have memories for different time intervals, depending on the values of δ_i. Also, on the coefficient functions

$$b : [0, T] \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathcal{V} \to \mathbb{R} \times \mathbf{\Omega}, \tag{2}$$

$$\sigma : [0, T] \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathcal{V} \to \mathbb{R} \times \mathbf{\Omega}, \tag{3}$$

$$\gamma : [0, T] \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathcal{V} \times \mathbb{R} \times \mathbf{\Omega} \to \mathbb{R}, \tag{4}$$

we impose the following set of assumptions.

Assumption 2.2.

(1) *The functions $b(\omega, t, \cdot), \sigma(\omega, t\cdot)$ and $\gamma(\omega, t, \zeta, \cdot)$ are assumed to be C^1 for each fixed ω, t, ζ.*

(2) *The functions $b(\cdot, x, \mathbf{y}, \mathbf{z}, \mathbf{u})$ and $\sigma(\cdot, x, \mathbf{y}, \mathbf{z}, \mathbf{u})$, and $\gamma(\cdot, x, \mathbf{y}, \mathbf{z}, \mathbf{u}, \zeta)$ are predictable for each $x, \mathbf{y}, \mathbf{z}, \mathbf{u}$.*

(3) *Lipschitz condition: The functions b, σ are Lipschitz continuous in the variables $x, \mathbf{y}, \mathbf{z}$, with the Lipschitz constant independent of the variables $t, \mathbf{u}, \omega$. Also, there exists a function $\mathcal{L} \in L^2(\nu)$, independent of $t, \mathbf{u}, \omega$, such that*

$$|\gamma(\omega, t, x_1, \mathbf{y}_1, \mathbf{z}_1, \mathbf{u}, \zeta) - \gamma(\omega, t, x_2, \mathbf{y}_2, \mathbf{z}_2, \mathbf{u}, \zeta)| \tag{5}$$

$$\leq \mathcal{L}(\zeta)\{|x_1 - x_2| + ||\mathbf{y}_1 - \mathbf{y}_2|| + ||\mathbf{z}_1 - \mathbf{z}_2||\}, \quad \nu - a.e.\zeta. \tag{6}$$

(4) Linear growth: *The functions b, σ, γ satisfy the linear growth condition in the variables $x, \mathbf{y}, \mathbf{z}$, with the linear growth constant independent of the variables $t, \mathbf{u}, \omega$ Also, there exists a non-negative function $\mathcal{K} \in L^2(\nu)$, independent of $t, \mathbf{u}, \omega$, such that*

$$|\gamma(\omega, t, x, \mathbf{y}, \mathbf{z}, \mathbf{u}, \zeta)| \tag{7}$$

$$\leq \mathcal{K}(\zeta)\{1 + |x| + ||\mathbf{y}|| + ||\mathbf{z}||\}, \quad \nu - a.e.\zeta. \tag{8}$$

Assumptions 1 and 2 are sufficient to ensure the integrands in Equation (1) have predictable versions, whenever X is càdlàg and adapted. It is always assumed that the $\tilde{N}$-integral is taken with respect to the predictable version of $\gamma(t, X(t), \mathbf{Y}(t), \mathbf{Z}(t), \mathbf{u}(t), \zeta)$. Together with the Lipschitz and linear growth conditions, this ensures that for every $\mathbf{u} \in \mathcal{A}$, there exists a unique càdlàg adapted solution $X = X^{\mathbf{u}}$ to the Equation (1) satisfying

$$E\left[\sup_{t\in[-\delta, T]} |X(t)|^2\right] < \infty. \tag{9}$$

This can be seen, for example, by regarding Equation (1) as a stochastic functional differential equation. See Dahl et al. [4] for more on this.

In addition to this, the players (potentially) have different levels of information, represented by different subfiltrations $(\mathcal{E}_t^{(i)})_{0\leq t\leq T}$ where $\mathcal{E}_t^{(i)} \subseteq \mathcal{F}_t$ for all $t \in [0, T]$, $i = 1, 2$.

For $i = 1, 2$, let $g_i(\cdot, x, y, \Lambda, w_i, z_i, k_i(\cdot), u, \omega)$ be a given predictable process w.r.t. $(\mathcal{E}_t^{(i)})_{0\leq t\leq T}$, $i = 1, 2$, and let $h_i(x, \omega)$ be an $\mathcal{F}_T$-measurable function. Associated to the FSDE (1), we have a pair of backward stochastic differential equations (BSDEs) in the unknown stochastic processes (W_i, Z_i, K_i), $i = 1, 2$:

$$\begin{aligned} dW_i(t) &= -g_i(t, X(t), \mathbf{Y}(t), \mathbf{\Lambda}(t), W_i(t), Z_i(t), K_i(t, \cdot), \boldsymbol{u}(t), \omega)dt \\ &\quad + Z_i(t)dB(t) + \int_{\mathbb{R}} K_i(t, \zeta)\tilde{N}(dt, d\zeta), \quad t \in [0, T], \\ W_i(T) &= h_i(X(T), \omega). \end{aligned} \tag{10}$$

Note that these BSDEs are coupled to the SDE (1) due to the dependency on X. Also, the BSDEs depend on the memory of the market process X, due to the dependency on the processes $\boldsymbol{Y}$ and $\mathbf{\Lambda}$. However, Equation (10) is a standard BSDE with jumps, hence the conditions for existence and uniqueness of solution are well known, see e.g. Theorem 1.5 in Øksendal and Sulem [15]. Essentially, we require that g is square integrable w.r.t. t when all other inputs are 0 and that g is Lipschitz in $\mathbf{W}, Z$ and K.

For $i = 1, 2$, let $f_i : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{A} \times \mathbf{\Omega} \to \mathbb{R}, \varphi_i : \mathbb{R} \to \mathbb{R}, \psi_i : \mathbb{R} \to \mathbb{R}$ be functions representing a profit rate, bequest function and risk evaluation, respectively. Then, the performance function of each player $i = 1, 2$ is defined by:

$$J_i(u) = E\left[\int_0^T f_i(t, X^u(t), Y_i^u(t), \Lambda_i^u(t), u_i(t))dt + \varphi_i(X^u(T)) + \psi_i(W_i^u(0))\right], \tag{11}$$

where we must assume all conditions necessary for the integrals and the expectation to exist.

Also, note that the performance J_i of player i is a function of the control $\mathrm{u}(t) = (u_1(t), u_2(t))$, which is determined by both players. Therefore, this problem setting specifies a stochastic game.

A pair of controls $(\hat{u}_1, \hat{u}_2)$ is called a *Nash equilibrium* for this stochastic game if the following holds:

$$\begin{aligned} J_1(u_1, \hat{u}_2) &\leq J_1(\hat{u}_1, \hat{u}_2) \text{ for all } u_1 \in \mathcal{A}_1, \\ J_2(\hat{u}_1, u_2) &\leq J_2(\hat{u}_1, \hat{u}_2) \text{ for all } u_2 \in \mathcal{A}_2. \end{aligned} \tag{12}$$

In words, this means that in the Nash equilibrium, neither player would like to change their control.

Assume there exists a Nash equilibrium for this forward-backward stochastic differential (FBSDE) game with delay and noisy memory. We would like to find this Nash equilibrium, and we will do so by proving sufficient and necessary maximum principles for this problem. Therefore, we define a Hamiltonian function for each player $i=1,\ 2$ as follows:

$$\begin{aligned} H_i(t,x,\boldsymbol{y},\boldsymbol{\Lambda},w_i,z_i,k_i,u_1,u_2,\lambda_i,p_i,q_i,r_i) &= f_i(t,x,y_i,\Lambda_i,u_i) \\ &+ \lambda_i g_i(t,x,\boldsymbol{y},\boldsymbol{\Lambda},w_i,z_i,k_i,u_1,u_2) + p_i b(t,x,\mathrm{y},\boldsymbol{\Lambda},u_1,u_2) \\ &+ q_i\sigma(t,x,\boldsymbol{y},\boldsymbol{\Lambda},u_1,u_2) + \int_{\mathbb{R}} r_i(\zeta)\gamma(t,x,\boldsymbol{y},\boldsymbol{\Lambda},u_1,u_2,\zeta)\nu(d\zeta). \end{aligned} \tag{13}$$

Assume H_i is C^1 in $x,y_1,y_2,\Lambda_1,\Lambda_2,w_i,z_i,k_i,u_1,u_2$ for $i=1,\ 2$. In the following, for ease of notation, we will use the abbreviation

$$H_i(t) = H_i(t,x(t),\boldsymbol{y}(t),\boldsymbol{\Lambda}(t),w_i(t),z_i(t),k_i(t),u_1(t),u_2(t),\lambda_i(t),p_i(t),q_i(t),r_i(t)).$$

For $i=1,\ 2$, we define a system of FBSDEs associated to these Hamiltonians in the unknown adjoint processes (λ_i,p_i,q_i,r_i) :

FSDE in λ_i (which depends on p_i,q_i,r_i):

$$\begin{aligned} d\lambda_i(t) &= \frac{\partial H_i}{\partial w_i}(t)dt + \frac{\partial H_i}{\partial z_i}(t)dB(t) + \int_{\mathbb{R}} \nabla_{k_i}(H_i(t,\zeta))\tilde{N}(dt,d\zeta), \\ \lambda_i(0) &= \psi_i'(W_i(0)), \end{aligned} \tag{14}$$

where $\nabla_{k_i}(H_i(t,\zeta))$ is the Fréchet derivative of H_i at k_i, see the appendix in Øksendal and Sulem [7] for a closer explanation of this gradient.

We also define a BSDE in p_i,q_i,r_i, which depends on λ_i:

$$\begin{aligned} dp_i(t) &= E[\mu_i(t)|\mathcal{F}_t]dt + q_i(t)dB(t) + \int_{\mathbb{R}} r_i(t,\zeta)\tilde{N}(dt,d\zeta), \\ p_i(T) &= \varphi_i'(X(T)) + h_i'(X(T))\lambda_i(T), \end{aligned} \tag{15}$$

where

$$\mu_i(t) = -\frac{\partial H_i}{\partial x}(t) - \frac{\partial H_i}{\partial y_i}(t+\delta_i)1_{[0,T-\delta_i]}(t) - \int_t^{t+\delta_i} D_t\left[\frac{\partial H_i}{\partial \Lambda_i}(s)1_{[0,T]}(s)ds\right]$$

and $D_t[\cdot]$ denotes the Malliavin derivative (see Remark 2.3). Note that the conditional expectation in (15) is well defined by the extension of the Malliavin derivative introduced by Aase et al. [16], see Remark 2.3. Equations (14) and (15) form an FBSDE-system involving Malliavin derivatives. To the best of our knowledge, such systems have not been studied before.

Remark 2.3. We refer to Nualart [17], Sanz-Solè [18] and Di Nunno et al. [13] for information about the Malliavin derivative $\mathrm{D_t}$ for Brownian motion $\mathrm{B(t)}$ and, more generally, Lévy processes. In Aase et al. [16], $\mathrm{D_t}$ was extended from the space $\mathbb{D}_{1,2}$ to $L^2(P)$, where $\mathbb{D}_{1,2}$ denotes the classical space of Malliavin differentiable $\mathcal{F}_T$-measurable random variables. The extension is such that for all $F \in L^2(\mathcal{F}_T,P)$, the following holds:

(i) $D_tF \in (\mathcal{S})^*$, where $(\mathcal{S})^* \supseteq L^2(P)$ denotes the Hida space of stochastic distributions,

(ii) the map $(t,\omega) \to E[D_t F|\mathcal{F}_t]$ belongs to $L^2(\mathcal{F}_T, \lambda \times P)$, where λ denotes the Lebesgue measure on $[0,T]$.

(iii) Moreover, the following generalized Clark–Ocone theorem holds:

$$F = E[F] + \int_0^T E[D_t F|\mathcal{F}_t]dB(t). \tag{16}$$

See [16], Theorem 3.11, and also [13], Theorem 6.35.

Notice that by combining Itô's isometry with the Clark–Ocone theorem, we obtain

$$E\left[\int_0^T E[D_t F|\mathcal{F}_t]^2 dt\right] = E\left[\left(\int_0^T E[D_t F|\mathcal{F}_t]dB(t)\right)^2\right] = E\left[\left(F^2 - E[F]^2\right)\right]. \tag{17}$$

(iv) As observed in Agram et al. [19], we can also apply the Clark–Ocone theorem to show the following generalized duality formula:

Let $F \in L^2(\mathcal{F}_T, P)$ and let $\varphi(t) \in L^2(\lambda \times P)$ be adapted. Then,

$$E\left[F\int_0^T \varphi(t)dB(t)\right] = E\left[\int_0^T E[D_t F|\mathcal{F}_t]\varphi(t)dt\right]. \tag{18}$$

Remark 2.4. N ote that Equation (14) is linear in λi, and hence, if p_i, q_i, r_i were given, it could be solved by using the Itô formula. However, this solution will depend on the processes X, Y_i, Λ_i and Wi, so in order to find an explicit solution for λi, we must also solve the coupled FBSDE system (1)–(10).

The BSDE (15) is linear in $\mathrm{p_i}$, and hence, if λ_i was given, it would be possible to find a unique solution to this equation by using e.g. Proposition 6.2.1 in Pham [20] or Theorem 1.7 in Øksendal and Sulem [15]. However, as for the adjoint SDE (14), this solution will depend on the coupled FBSDE system (1)–(10).

In the remaining part of the paper, we will prove a sufficient (Section 3) and a necessary maximum principle (Section 4) for this kind of FBSDE game with delay and noisy memory. Then, we will study existence and uniqueness of solutions of the FBSDE system (14) and (15) (Section 5). Finally, we will present an example which illustrates our results: optimal consumption rate with respect to recursive utility (see Section 6).

3. Sufficient maximum principle for FBSDE games with delay and noisy memory

We prove a sufficient maximum principle which roughly states that under concavity conditions, a control $(\hat{u}_1, \hat{u}_2)$ satisfying a conditional maximum principle and an ℓ^2-condition is a Nash equilibrium for the stochastic game.

Theorem 3.1. Let $\hat{u}_1 \in \mathcal{A}_1$ and $\hat{u}_2 \in \mathcal{A}_2$ with corresponding solutions $\hat{X}(t), \hat{Y}_i(t), \hat{\Lambda}_i(t), \hat{W}_i(t), \hat{Z}_i(t),$ $\hat{K}_i(t), \hat{\lambda}_i(t),$ $\hat{p}_i(t), \hat{q}_i(t), \hat{r}_i(t,\zeta)$ of the FSDE (1), the BSDE (10), and the FBSDE system (14) and (15) for i = 1, 2. Also, assume that:

- *(Concavity I) The functions* $x \to h_i(x), x \to \varphi_i(x), x \to \psi_i(x)$ *are concave for* $i =$ *1, 2.*
- *(The conditional maximum principle)*

$$\begin{aligned}&ess\ sup\ _{v \in \mathcal{A}_1} E[H_1(t, \hat{X}(t), \hat{\mathbf{Y}}(t), \hat{\mathbf{\Lambda}}(t), \hat{W}_1(t), \hat{Z}_1(t), \hat{K}_1(t, \cdot),\\ &\quad v, \hat{u}_2(t), \hat{\lambda}_1(t), \hat{p}_1(t), \hat{q}_1(t), \hat{r}_1(t, \cdot))|\mathcal{E}_t^{(1)}]\\ &\quad = E[H_1(t, \hat{X}(t), \hat{\mathbf{Y}}(t), \hat{\mathbf{\Lambda}}(t), \hat{W}_1(t), \hat{Z}_1(t), \hat{K}_1(t, \cdot),\\ &\qquad \hat{u}_1(t), \hat{u}_2(t), \hat{\lambda}_1(t), \hat{p}_1(t), \hat{q}_1(t), \hat{r}_1(t, \cdot))|\mathcal{E}_t^{(1)}],\end{aligned}$$

and similarly

$$\begin{aligned}&ess\ sup\ _{v \in \mathcal{A}_2} E[H_2(t, \hat{X}(t), \hat{\Upsilon}(t), \hat{\mathbf{\Lambda}}(t), \hat{W}_2(t), \hat{Z}_2(t), \hat{K}_2(t, \cdot),\\ &\quad \hat{u}_1, v, \hat{\lambda}_2(t), \hat{p}_2(t), \hat{q}_2(t), \hat{r}_2(t, \cdot))|\mathcal{E}_t^{(2)}]\\ &\quad = E[H_2(t, \hat{X}(t), \hat{\Upsilon}(t), \hat{\mathbf{\Lambda}}(t), \hat{W}_2(t), \hat{Z}_2(t), \hat{K}_2(t, \cdot),\\ &\quad \hat{u}_1(t), \hat{u}_2(t), \hat{\lambda}_2(t), \hat{p}_2(t), \hat{q}_2(t), \hat{r}_2(t, \cdot))|\mathcal{E}_t^{(2)}].\end{aligned}$$

- *(Concavity II) The functions*

$$\begin{aligned}&\hat{\mathcal{H}}_1(t, x, y_1, \Lambda_1, w_1, z_1, k_1)\\ &:= ess\ sup\ _{v \in \mathcal{A}_1} E\Big[H_1(t, x, y_1, \hat{y}_2, \Lambda_1, \hat{\Lambda}_2, w_1, z_1, k_1, v, \hat{u}_2, \hat{\lambda}_1, \hat{p}_1, \hat{q}_1, \hat{r}_1)|\mathcal{E}_t^{(1)}\Big]\end{aligned}$$

and

$$\begin{aligned}&\hat{\mathcal{H}}_2(t, x, y_2, \Lambda_2, w_2, z_2, k_2)\\ &:= ess\ sup\ _{v \in \mathcal{A}_2} E\Big[H_2(t, x, \hat{y}_1, y_2, \hat{\Lambda}_1, \Lambda_2, w_2, z_2, k_2, \hat{u}_1, v, \hat{\lambda}_2, \hat{p}_2, \hat{q}_2, \hat{r}_2)|\mathcal{E}_t^{(2)}\Big]\end{aligned}$$

are concave for all t a.s.

- *Finally, assume that the following* ℓ^2 *conditions hold:*

$$\begin{aligned}&E\Big[\int_0^T \{\hat{p}_i^2(t)[(\sigma(t) - \hat{\sigma}(t))^2 + \int_{\mathbb{R}} (r_i(t,\zeta) - \hat{r}_i(t,\zeta))^2 \nu(d\zeta)\Big]\\ &+(X(t) - \hat{X}(t))^2[\hat{q}_i^2(t) + \int_{\mathbb{R}} \hat{r}_i^2(t,\zeta)\nu(d\zeta)]\\ &+(Y_i(t) - \hat{Y}_i(t))^2[(\tfrac{\partial \hat{H}_i}{\partial z})^2(t) + \int_{\mathbb{R}} ||\nabla_k \hat{H}_i(t,\zeta)||^2 \nu(d\zeta)]\\ &+\hat{\lambda}_i^2(t)[(\Lambda_i(t) - \hat{\Lambda}_i(t))^2 + \int_{\mathbb{R}} (K_i(t,\zeta) - \hat{K}_i(t,\zeta))^2 \nu(d\zeta)]\}] < \infty\end{aligned}$$

for $i = 1, 2$.

Then, $(\hat{u}_1, \hat{u}_2)$ *is a Nash equilibrium.*

Proof. We would like to show that $J_1(u_1, \hat{u}_2) \leq J_1(\hat{u}_1, \hat{u}_2)$ for all $u_1 \in \mathcal{A}_1$. Choose $u_1 \in \mathcal{A}_1$. By the definition of the performance function J_1,

$$\Delta := J_1(u_1, \hat{u}_2) - J_1(\hat{u}_1, \hat{u}_2) = I_1 + I_2 + I_3,$$

where

$$
\begin{aligned}
I_1 &= E\left[\int_0^T \{f_1(t,x,y,\Lambda,\boldsymbol{u}) - f_1(t,\hat{x},\hat{y},\hat{\Lambda},\hat{\boldsymbol{u}})\}dt\right],\\
I_2 &= E\left[\varphi_1(X(T)) - \varphi_1(\hat{X}(T))\right],\\
I_3 &= E\left[\psi_1(W_1(0)) - \psi_1(\hat{W}_1(0))\right].
\end{aligned}
$$

Note that from the definition of the Hamiltonian,

$$
\begin{aligned}
I_1 \quad = \quad & E[\int_0^T \{H_1(t) - \hat{H}_1(t) - \hat{\lambda}_1(t)(g_1(t) - \hat{g}_1(t)) - \hat{p}_1(t)(b(t) - \hat{b}(t))\\
& -\hat{q}_1(t)(\sigma(t) - \hat{\sigma}(t)) - \int_{\mathbb{R}} \hat{r}_1(t,\zeta)(\gamma(t,\zeta) - \hat{\gamma}(t,\zeta)\nu(d\zeta))\}dt],
\end{aligned}
\tag{19}
$$

where we have used the abbreviation

$$
\hat{H}_1(t) := H_1(t,\hat{X}(t),\hat{\boldsymbol{Y}}(t),\hat{\boldsymbol{\Lambda}}(t),\hat{W}_1(t),\hat{Z}_1(t),\hat{K}_1(t,\cdot),\hat{\boldsymbol{u}}(t),\hat{\lambda}_1(t),\hat{p}_1(t),\hat{q}_1(t),\hat{r}_1(t),\omega)
$$

and corresponding abbreviations for $H_1(t), b(t), \hat{b}(t), \sigma, \hat{\sigma}(t), \gamma(t)$ and $\hat{\gamma}(t)$.

Also,

$$
\begin{aligned}
I_2 \quad = \quad & E[\varphi_1(X(T)) - \varphi_1(\hat{X}(T))]\\
\leq \quad & E[\varphi_1'(\hat{X}(T))(X(T) - \hat{X}(T))]\\
= \quad & E[(\hat{p}_1(T) - h_1'(\hat{X}(T))\hat{\lambda}_1(T))(X(T) - \hat{X}(T))]\\
= \quad & E[\hat{p}_1(T)(X(T) - \hat{X}(T))] - E[\hat{\lambda}_1(T)h_1'(\hat{X}(T))(X(T) - \hat{X}(T))]\\
= \quad & E[\textstyle\int_0^T \hat{p}_1(t)(dX(t) - d\hat{X}(t)) + \displaystyle\int_0^T (X(t) - \hat{X}(t))d\hat{p}_1(t)\\
& + \int_0^T \hat{q}_1(t)(\sigma(t) - \hat{\sigma}(t))dt + \int_0^T \int_{\mathbb{R}} \hat{r}_1(t,\zeta)(\gamma(t,\zeta) - \hat{\gamma}(t,\zeta))\nu(d\zeta)dt]\\
& -E[\hat{\lambda}_1(T)h_1'(\hat{X}(T))(X(T) - \hat{X}(T))]\\
= \quad & E[\textstyle\int_0^T \hat{p}_1(t)(b(t) - \hat{b}(t))dt + \displaystyle\int_0^T (X(t) - \hat{X}(t))(-\frac{\partial \hat{H}_1}{\partial x}(t)\\
& -\frac{\partial \hat{H}_1}{\partial y_1}(t+\delta_1)1_{[0,T-\delta_1]}(t) + \int_t^{t+\delta_1} D_t\left[-\frac{\partial \hat{H}_1}{\partial \Lambda_1}(s)\right]1_{[0,T]}(s)ds)dt\\
& + \int_0^T \hat{q}_1(t)(\sigma(t) - \hat{\sigma}(t))dt + \int_0^T \int_{\mathbb{R}} \hat{r}_1(t,\zeta)(\gamma(t,\zeta) - \hat{\gamma}(t,\zeta))\nu(d\zeta)dt\Big]\\
& -E[\hat{\lambda}_1(T)h_1'(\hat{X}(T))(X(T) - \hat{X}(T))],
\end{aligned}
\tag{20}
$$

where the first inequality follows from the concavity of φ_1, the second equality follows from Equation (15), the fourth equality from Itô's product rule applied to $\hat{p}_1 X$ and $\hat{p}_1 \hat{X}$, the fifth equality follows from Equation (15), the double expectation rule and Equation (1).

Also, note that

$$
\begin{aligned}
I_3 &= E[\psi_1(W_1(0)) - \psi_1(\hat{W}_1(0))] \\
&\leq E[\psi_1'(\hat{W}_1(0))(W_1(0) - \hat{W}_1(0))] \\
&= E[\hat{\lambda}_1(T)(W_1(T) - \hat{W}_1(T))] - \Big\{E\Big[\int_0^T (W_1(t) - \hat{W}_1(t))d\hat{\lambda}_1(t) \\
&\quad + \int_0^T \hat{\lambda}_1(t)(dW_1(t) - d\hat{W}_1(t)) + \int_0^T \frac{\partial \hat{H}_1}{\partial z_1}(t)(Z_1(t) - \hat{Z}_1(t))dt \\
&\quad + \int_0^T \int_{\mathbb{R}} \nabla_{k_1}\hat{H}_1(t)(K_1(t) - \hat{K}_1(t))\nu(d\zeta)dt\Big]\Big\} \\
&= E[\hat{\lambda}_1(T)(h_1(X(T)) - h_1(\hat{X}(T)))] - \Big\{E\Big[\int_0^T \frac{\partial \hat{H}_1}{\partial w_1}(t)(W_1(t) - \hat{W}_1(t))dt \\
&\quad + \int_0^T \hat{\lambda}_1(t)(-g_1(t) + \hat{g}_1(t))dt + \int_0^T \frac{\partial \hat{H}_1}{\partial z_1}(t)(Z_1(t) - \hat{Z}_1(t))dt \\
&\quad + \int_0^T \int_{\mathbb{R}} \nabla_k\hat{H}_1(t)(K_1(t) - \hat{K}_1(t))\nu(d\zeta)dt\Big]\Big\} \\
&\leq E[\hat{\lambda}_1(T)h_1'(\hat{X}(T))(X(T) - \hat{X}(T))] - \Big\{E\Big[\int_0^T \frac{\partial \hat{H}_1}{\partial w_1}(t)(W_1(t) - \hat{W}_1(t))dt \\
&\quad + \int_0^T \hat{\lambda}_1(t)(-g_1(t) + \hat{g}_1(t))dt + \int_0^T \frac{\partial \hat{H}_1}{\partial z_1}(t)(Z_1(t) - \hat{Z}_1(t))dt \\
&\quad + \int_0^T \int_{\mathbb{R}} \nabla_{k_1}\hat{H}_1(t)(K_1(t) - \hat{K}_1(t))\nu(d\zeta)dt\Big]\Big\},
\end{aligned}
\tag{21}
$$

where the first inequality follows from the concavity of ψ_1, the second equality follows from Equation (14), the third equality follows from Itô's product rule applied to $\hat{\lambda}_1 Y_1$ and $\hat{\lambda}_1\hat{Y}_1$, the fourth equality follows from Equation (10) as well as Equation (14). The final inequality follows from the concavity of h_1 and that $\hat{\lambda}_1(T) \geq 0$.

Hence,

$$
\begin{aligned}
\Delta &= I_1 + I_2 + I_3 \\
&\leq E\Big[\int_0^T \Big\{H_1(t) - \hat{H}_1(t) - \Big(\frac{\partial \hat{H}_1}{\partial x}(t) + \frac{\partial \hat{H}_1}{\partial y_1}(t+\delta_1)1_{[0,T-\delta_1]}(t) \\
&\quad + \int_t^{t+\delta_1} D_t\Big[\frac{\partial \hat{H}_1}{\partial \Lambda_1}(s)\Big]1_{[0,T]}(s)ds)(X(t) - \hat{X}(t))dt\Big\} \\
&\quad - \int_0^T \Big\{\frac{\partial \hat{H}_1}{\partial w_1}(t)(W_1(t) - \hat{W}_1(t)) + \frac{\partial \hat{H}_1}{\partial z_1}(t)(Z_1(t) - \hat{Z}_1(t)) \\
&\quad + \int_{\mathbb{R}} \nabla_{k_1}\hat{H}_1(t)(K_1(t,\zeta) - \hat{K}_1(t,\zeta))\nu(d\zeta)\Big\}dt\Big].
\end{aligned}
\tag{22}
$$

Note that by changing the order of integration and using the duality formula for Malliavin derivatives (see Di Nunno et al. [13]), we get:

$$\begin{aligned}
&E\Big[\int_0^T \frac{\partial \hat{H}_1}{\partial \Lambda_1}(s)(\Lambda_1(s)-\hat{\Lambda}_1(s))ds\Big]\\
&\quad= E\Big[\int_0^T \frac{\partial \hat{H}_1}{\partial \Lambda_1}(s)\int_{s-\delta_1}^{s}(X(t)-\hat{X}(t))dB(t)ds\Big]\\
&\quad= \int_0^T E\Big[\frac{\partial \hat{H}_1}{\partial \Lambda_1}(s)\int_{s-\delta_1}^{s}(X(t)-\hat{X}(t))dB(t)\Big]ds\\
&\quad= \int_0^T E\Big[\int_{s-\delta_1}^{s} E\Big[D_t\Big(\frac{\partial \hat{H}_1}{\partial \Lambda_1}(s)\Big)|\mathcal{F}_t\Big](X(t)-\hat{X}(t))dt\Big]ds\\
&\quad= E\Big[\int_0^T\int_t^{t+\delta_1} E\Big[D_t\Big(\frac{\partial \hat{H}_1}{\partial \Lambda_1}(s)\Big)|\mathcal{F}_t\Big]1_{[0,T]}(s)ds(X(t)-\hat{X}(t))dt\Big]\\
&\quad= E\Big[\int_0^T\int_t^{t+\delta_1} D_t\Big(\frac{\partial \hat{H}_1}{\partial \Lambda_1}(s)\Big)1_{[0,T]}(s)ds(X(t)-\hat{X}(t))dt\Big].
\end{aligned}\tag{23}$$

Also, note that

$$\begin{aligned}
&E\left[\int_0^T \frac{\partial \hat{H}}{\partial y_1}(t)(Y_1(t)-\widehat{Y}_1(t))dt\right]\\
&\quad= E\left[\int_0^T \frac{\partial \hat{H}}{\partial y_1}(t)(X(t-\delta)-\hat{X}(t-\delta_1))dt\right]\\
&\quad= E\left[\int_0^T \frac{\partial \hat{H}}{\partial y_1}(t+\delta_1)1_{[0,T-\delta_1]}(t)(X(t)-\hat{X}(t))dt\right].
\end{aligned}\tag{24}$$

Hence, by the inequality (22) combined with Equations (23) and (24),

$$\begin{aligned}
\Delta \quad &\le E\Big[\int_0^T\Big\{H_1(t)-\hat{H}_1(t)-\frac{\partial \hat{H}_1}{\partial x}(t)(X(t)-\hat{X}(t))-\frac{\partial \hat{H}_1}{\partial y_1}(t)(Y_1(t)-\hat{Y}_1(t))\\
&-\frac{\partial \hat{H}_1}{\partial \Lambda_1}(t)(\Lambda_1(t)-\hat{\Lambda}_1(t))dt-\frac{\partial \hat{H}_1}{\partial w_1}(t)(W_1(t)-\hat{W}_1(t))-\frac{\partial \hat{H}_1}{\partial z_1}(t)(Z_1(t)-\hat{Z}_1(t))\\
&+\int_{\mathbb{R}}\nabla_{k_1}\hat{H}_1(t)(K_1(t,\zeta)-\hat{K}_1(t,\zeta))\nu(d\zeta)\}dt\Big].
\end{aligned}\tag{25}$$

Fix some $t\in[0,T]$. By assumption, $\hat{\mathcal{H}}_1(\vec{x}):=\hat{\mathcal{H}}_1(t,\vec{x})$ is concave, so it is superdifferentiable[1] (see Rockafellar [21]) at the point $\vec{x}:=(\hat{X},\hat{Y}_1,\hat{\Lambda}_1,\hat{W}_1,\hat{Z}_1,\hat{K}_1)$. Thus, there exists a supergradient $\vec{a}:=(a_0,a_1,a_2,a_3,a_4,a_5(\cdot))$ such that for all $\vec{y}:=(x,y,\Lambda,w,z,k)$, the following holds:

$$\hat{\mathcal{H}}_1(\vec{x})+\vec{a}\cdot(\vec{y}-\vec{x})\ge\hat{\mathcal{H}}_1(\vec{y}).\tag{26}$$

Define

$$\begin{aligned}
\phi_1(t,x,y,\Lambda,w,z,k)&:=\hat{\mathcal{H}}_1(t,x,y,\Lambda,w,z,k)-\hat{\mathcal{H}}_1(t,\hat{X},\hat{Y}_1,\hat{\Lambda}_1,\hat{W}_1,\hat{Z}_1,\hat{K}_1)\\
&-\Big\{a_0(x-\hat{X})+a_1(y-\hat{Y}_1)+a_2(\Lambda-\Lambda_1)+a_3(w-\hat{W}_1)+a_4(z-\hat{Z}_1)\\
&+\int_{\mathbb{R}}a_5(\zeta)(k-\hat{K}_1)\nu(d\zeta))\Big\}.
\end{aligned}\tag{27}$$

Then, by Equation (26),

$$\begin{array}{lll} \phi_1(t,x,y,\Lambda,w,z,k) & \leq & 0 \text{ for all } x,y,\Lambda,w,z,k, \\ \phi_1(t,\hat{X},\hat{Y}_1,\hat{\Lambda}_1,\hat{W}_1,\hat{Z}_1,\hat{K}_1) & = & 0 \text{ (by definition)}. \end{array} \tag{28}$$

Therefore, by differentiating Equation (27) and using Equation (28), we find that

$$\begin{array}{lllll} a_0 & = & \frac{\partial \hat{\mathcal{H}}_1}{\partial x}(t,\hat{X},\hat{Y}_1,\hat{\Lambda}_1,\hat{W}_1,\hat{Z}_1,\hat{K}_1) & = & \frac{\partial \hat{H}_1}{\partial x}, \\ a_1 & = & \frac{\partial \hat{\mathcal{H}}_1}{\partial y_1}(t,\hat{X},\hat{Y}_1,\hat{\Lambda}_1,\hat{W}_1,\hat{Z}_1,\hat{K}_1) & = & \frac{\partial \hat{H}_1}{\partial y_1}, \\ a_2 & = & \frac{\partial \hat{\mathcal{H}}_1}{\partial \Lambda_1}(t,\hat{X},\hat{Y}_1,\hat{\Lambda}_1,\hat{W}_1,\hat{Z}_1,\hat{K}_1) & = & \frac{\partial \hat{H}_1}{\partial \Lambda_1}, \\ a_3 & = & \frac{\partial \hat{\mathcal{H}}_1}{\partial w_1}(t,\hat{X},\hat{Y}_1,\hat{\Lambda}_1,\hat{W}_1,\hat{Z}_1,\hat{K}_1) & = & \frac{\partial \hat{H}_1}{\partial w_1}, \\ a_4 & = & \frac{\partial \hat{\mathcal{H}}_1}{\partial z_1}(t,\hat{X},\hat{Y}_1,\hat{\Lambda}_1,\hat{W}_1,\hat{Z}_1,\hat{K}_1) & = & \frac{\partial \hat{H}_1}{\partial z_1}, \\ a_5 & = & \nabla_{k_1}\hat{\mathcal{H}}_1(t,\hat{X},\hat{Y}_1,\hat{\Lambda}_1,\hat{W}_1,\hat{Z}_1,\hat{K}_1) & = & \nabla_{k_1}\hat{H}_1. \end{array}$$

Therefore, it follows from this, Equations (25) and (28) that

$$\Delta = \phi_1(t,X(t),Y_1(t),\Lambda_1(t),W_1(t),Z_1(t),K_1(t,\cdot)) \leq 0 \ \forall \ t \in [0,T]$$

(where the final inequality follows since $\hat{\mathcal{H}}_1$ is concave). This means that $J_1(u_1,\hat{u}_2) \leq J_1(\hat{u}_1,\hat{u}_2)$ for all $u_1 \in \mathcal{A}_1$. In a similar way, one can prove that $J_2(\hat{u}_1,u_2) \leq J_2(\hat{u}_1,\hat{u}_2)$ for all $u_2 \in \mathcal{A}_2$. This completes the proof that $(\hat{u}_1,\hat{u}_2)$ is a Nash-equilibrium. □

4. Necessary maximum principle for FBSDE games with delay and noisy memory

In the following, we need some additional assumptions and notation:

- For all $t_0 \in [0,T]$ and all bounded $\mathcal{E}_i(t)$-measurable random variables $\alpha_i(\omega)$, the control

$$\beta_i(t) := 1_{(t_0,T)}(t)\alpha_i(\omega) \text{ is in } \mathcal{A}_i \text{ for } i = 1,2. \tag{29}$$

- For all $u_i, \beta_i \in \mathcal{A}_i$ with β_i bounded, there exists $\kappa_i > 0$ such that the control

$$u_i(t) + s\beta_i(t) \text{ for } t \in [0,T], \tag{30}$$

belongs to $\mathcal{A}_i$ for all $s \in (-\kappa_i,\kappa_i)$, $i = 1, 2$.

- Also, assume that the following derivative processes exist and belong to $L^2([0,T] \times \Omega)$:

$$\begin{array}{lll} x_1(t) & = & \left.\frac{d}{ds}X^{(u_1+s\beta_1,u_2)}(t)\right|_{s=0}, \\ y_1(t) & = & \left.\frac{d}{ds}Y_1^{(u_1+s\beta_1,u_2)}(t)\right|_{s=0}, \end{array} \tag{31}$$

$$\begin{aligned}\tilde{\Lambda}_1(t) &= \frac{d}{ds}\Lambda_1^{(u_1+s\beta_1,u_2)}(t)\Big|_{s=0},\\ w_1(t) &= \frac{d}{ds}W_1^{(u_1+s\beta_1,u_2)}(t)\Big|_{s=0},\\ z_1(t) &= \frac{d}{ds}Z_1^{(u_1+s\beta_1,u_2)}(t)\Big|_{s=0},\\ k_1(t) &= \frac{d}{ds}K_1^{(u_1+s\beta_1,u_2)}(t)\Big|_{s=0},\end{aligned}$$

and similarly for $x_2(t) = \frac{d}{ds}X^{(u_1,u_2+s\beta_2)}(t)|_{s=0}$ etc. Here, the derivative processes are directional derivatives, defined in the following way:

$$\begin{aligned}x_1(t) &= \frac{d}{ds}X^{(u_1+s\beta_1,u_2)}(t)\\ &:= \lim_{\Delta s\to 0}\frac{X^{(u_1+(s+\Delta s)\beta_1,u_2)}(t) - X^{(u_1+s\beta_1,u_2)}(t)}{\Delta s}.\end{aligned} \tag{32}$$

For more on this, see (4.11) in Di Nunno et al. [22] and Appendix A in Øksendal and Sulem [7]. Note also that $x_i(0) = 0$ for $i = 1, 2$ since $X(0) = x$.

If these assumptions hold, we can prove a necessary maximum principle for our noisy memory FBSDE game. The proof of the following theorem is based on the same idea as the proof of Theorem 2.2 in Øksendal and Sulem [7], however the presence of noisy memory in our problem requires some extra care.

Theorem 4.1. *Suppose that $u \in \mathcal{A}$ with corresponding solutions $X(t), Y_i(t), \Lambda_i(t), W_i(t), Z_i(t), K_i(t,\zeta), \lambda_i(t), p_i(t), q_i(t), r_i(t,\zeta)$, $i = 1, 2$, of* Equations (1), (10), (14) and (15). *Also, assume that conditions* (29)–(31) *hold. Then, the following are equivalent:*

(i) $\frac{\partial}{\partial s}J_1(u_1+s\beta_1,u_2)|_{s=0} = \frac{\partial}{\partial s}J_2(u_1,u_2+s\beta_2)|_{s=0} = 0$ *for all bounded* $\beta_1 \in \mathcal{A}_1$, $\beta_2 \in \mathcal{A}_2$.

(ii) $E[\frac{\partial H_1(t,X(t),Y(t),\mathbf{\Lambda}(t),W_1(t),Z_1(t),K_1(t,\cdot),v_1,u_2(t),\lambda_1(t),p_1(t),q_1(t),r_1(t,\cdot))}{\partial v_1}]|_{v_1=u_1(t)}$
$= E[\frac{\partial H_2(t,X(t),Y(t),\mathbf{\Lambda}(t),W_2(t),Z_2(t),K_2(t,\cdot),u_1(t),v_2,\lambda_2(t),p_2(t),q_2(t),r_2(t,\cdot))}{\partial v_2}]|_{v_2=u_2(t)} = 0.$

Proof. We only prove that $\frac{\partial}{\partial s}J_1(u_1+s\beta_1,u_2)|_{s=0} = 0$ for all bounded $\beta_1 \in \mathcal{A}_1$ is equivalent to

$$E\left[\frac{\partial H_1(t,X(t),Y(\boldsymbol{t}),\mathbf{\Lambda}(t),W_1(t),Z_1(t),K_1(t,\cdot),v_1,u_2(t),\lambda_1(t),p_1(t),q_1(t),r_1(t,\cdot))}{\partial v_1}\right]\Bigg|_{v_1=u_1(t)} = 0.$$

The remaining part of the theorem (i.e., the same statement for J_2andH_2) is proved in a similar way.

Note that, by the definition of J_1 and by interchanging differentiation and integration,

$$\begin{aligned}D_1 \quad &:= \quad \frac{\partial}{\partial s} J_1(u_1 + s\beta_1, u_2)|_{s=0} \\ &= \quad E\left[\int_0^T \left\{ \frac{\partial f_1}{\partial x}(t)x_1(t) + \frac{\partial f_1}{\partial y}(t)y_1(t) + \frac{\partial f_1}{\partial \Lambda}(t)\tilde{\Lambda}_1(t)\frac{\partial f_1}{\partial u_1}(t)\beta_1(t)\right\}dt \right. \\ &\quad \left. +\varphi_1'(X(T))x_1(T) + \phi_1'(W_1(0))w_1(0)\right].\end{aligned} \tag{33}$$

Note that the interchange of differentiation and integration is justified since everything in Equation (33) is well defined and square integrable by assumption and $P \times [0, T]$ is a finite measure space. Hence, we can apply Theorem 11.5 in Shilling [23] to change the order of the expectation/integral and the differentiation. Also, note that D_1 is a directional derivative of J_1, defined similarly as in Equation (32). For more details on directional (also called *Gâteaux derivative*), see Appendix A in Øksendal and Sulem [7]. Furthermore, note that $\frac{\partial f}{\partial x}$ is the partial derivative of the function f wrt. x inserted the corresponding processes at time t. For proofs of the differentiability of the performance functional in a similar context, see Dahl et al. [4].

We study the different parts of D_1 separately. First, by the Itô product rule, the adjoint BSDE (15) and the definition of $x_1(t)$,

$$\begin{aligned}I_1 &:= E[\varphi_1'(X(T))x_1(T)] \\ &= E[p_1(T)x_1(T)] - E[h_1'(X(T))\lambda_1(T)x_1(T)] \\ &= E[p_1(0)x_1(0)] + E\left[\int_0^T p_1(t)dx_1(t) + \int_0^T x_1(t)dp_1(t)\right. \\ &\quad \left. + \int_0^T d[p_1, x_1](t)\right] - E[h_1'(X(T))\lambda_1(T)x_1(T)] \\ &= E\left[\int_0^T p_1(t)\left(\frac{\partial b}{\partial x}(t)x_1(t) + \frac{\partial b}{\partial y_1}(t)y_1(t) + \frac{\partial b}{\partial \Lambda_1}(t)\tilde{\Lambda}_1(t) + \frac{\partial b}{\partial u_1}(t)\beta_1(t)\right)dt\right] \\ &\quad +E\left[\int_0^T x_1(t)E[\mu_1(t)|\mathcal{F}_t]dt\right] \\ &\quad +E\left[\int_0^T q_1(t)\left(\frac{\partial \sigma}{\partial x}(t)x_1(t) + \frac{\partial \sigma}{\partial y_1}(t)y_1(t) + \frac{\partial \sigma}{\partial \Lambda_1}\tilde{\Lambda}_1(t) + \frac{\partial \sigma}{\partial u_1}(t)\beta_1(t)\right)dt\right] \\ &\quad +E\left[\int_0^T \int_{\mathbb{R}} r_1(t,\zeta)\left(\frac{\partial \gamma}{\partial x}(t)x_1(t) + \frac{\partial \gamma}{\partial y_1}(t)y_1(t) + \frac{\partial \gamma}{\partial \Lambda_1}\tilde{\Lambda}_1(t) + \frac{\partial \gamma}{\partial u_1}(t)\beta_1(t)\right)d\nu(\zeta)dt\right] \\ &\quad -E[h_1'(X(T))\lambda_1(T)x_1(T)].\end{aligned} \tag{34}$$

Also, by the FSDE (14), the BSDE (10), the definition of $x_1(t)$ and the Itô product rule,

$$\begin{aligned}I_2 &:= E[\phi_1'(W_1(0))w_1(0)] \\ &= E[\lambda_1(0)w_1(0)] \\ &= E[\lambda_1(T)w_1(T)] - E\left[\int_0^T \lambda_1(t)dw_1(t) + \int_0^T w_1(t)d\lambda_1(t)\right.\end{aligned} \tag{35}$$

$$
\begin{aligned}
&+\int_0^T z_1(t)\frac{\partial H_1}{\partial z_1}(t)dt+\int_0^T\int_{\mathbb{R}}\nabla_{k_1}H_1(t,\zeta)k_1(t,\zeta)\nu(d\zeta)dt\Big]\\
&=E[\lambda_1(T)h_1'(X(T))x_1(T)]+E\Big[\int_0^T\lambda_1(t)\Big(\frac{\partial g_1}{\partial x}(t)x_1(t)+\frac{\partial g_1}{\partial y_1}(t)y_1(t)\\
&\quad+\frac{\partial g_1}{\partial \Lambda_1}(t)\tilde{\Lambda}(t)+\frac{\partial g_1}{\partial w_1}(t)w_1(t)+\frac{\partial g_1}{\partial z_1}(t)z_1(t)+\nabla_{k_1}g_1(t)k_1(t)\\
&\quad+\frac{\partial g_1}{\partial u_1}(t)\beta_1(t)\Big)dt]-E\Big[\int_0^T\frac{\partial H_1}{\partial w_1}(t)w_1(t)dt\Big]\\
&\quad-E\Big[\int_0^T z_1(t)\frac{\partial H_1}{\partial z_1}(t)dt+\int_0^T\int_{\mathbb{R}}\nabla_k H_1(t,\zeta)k_1(t,\zeta)\nu(d\zeta)dt\Big].
\end{aligned}
$$

By the definition of D_1 as well as Equations (34) and (35),

$$
\begin{aligned}
D_1 \quad = \quad & A+E\Big[\int_0^T\beta_1(t)(\frac{\partial f_1}{\partial u_1}(t)+\frac{\partial b}{\partial u_1}(t)p_1(t)+\frac{\partial \sigma}{\partial u_1}(t)q_1(t)+\frac{\partial \gamma}{\partial u_1}(t)r_1(t)\\
&+\frac{\partial g_1}{\partial u_1}(t)\lambda_1(t))dt\Big]+E\Big[\int_0^T w_1(t)\Big\{-\frac{\partial H_1}{\partial w_1}(t)+\frac{\partial g_1}{\partial w_1}(t)\lambda_1(t)\Big\}dt\\
&+\int_0^T z_1(t)\Big\{-\frac{\partial H_1}{\partial z_1}(t)+\frac{\partial g_1}{\partial x}(t)\lambda_1(t)\Big\}dt\\
&+\int_0^T k_1(t)\{-\nabla_{k_1}H_1(t)+\nabla_k g_1(t)\lambda_1(t)\}dt],
\end{aligned}
\tag{36}
$$

where

$$
\begin{aligned}
A \quad := \quad & E\Big[\int_0^T x_1(t)\Big\{\frac{\partial f_1}{\partial x}(t)+\frac{\partial b}{\partial x}(t)p_1(t)+E[\mu_1(t)|\mathcal{F}_t\Big]+\frac{\partial \sigma}{\partial x}(t)q_1(t)\\
&+\frac{\partial \gamma}{\partial x}(t)r_1(t)+\frac{\partial g_1}{\partial x}(t)\lambda_1(t)\Big\}dt+\int_0^T y_1(t)\Big\{\frac{\partial f_1}{\partial y_1}(t)+\frac{\partial b}{\partial y_1}(t)p_1(t)\\
&+\frac{\partial \sigma}{\partial y_1}(t)q_1(t)+\frac{\partial \gamma}{\partial y_1}(t)r_1(t)+\frac{\partial g_1}{\partial y_1}(t)\lambda_1(t)\Big\}dt+\int_0^T\tilde{\Lambda}_1(t)\Big\{\frac{\partial f_1}{\partial \Lambda_1}(t)\\
&+\frac{\partial b}{\partial \Lambda_1}(t)p_1(t)+\frac{\partial \sigma}{\partial \Lambda_1}(t)q_1(t)+\frac{\partial \gamma}{\partial \Lambda_1}(t)r_1(t)+\frac{\partial g_1}{\partial \Lambda_1}(t)\lambda_1(t)\Big\}dt\Big]\\
= \quad & E\Big[\int_0^T x_1(t)\Big\{\frac{\partial H_1}{\partial x}(t)+E[\mu_1(t)|\mathcal{F}_t]\Big\}dt\Big]+E\Big[\int_0^T y_1(t)\frac{\partial H_1}{\partial y_1}(t)\Big]\\
&+E\Big[\int_0^T\tilde{\Lambda}_1(t)\frac{\partial H_1}{\partial \Lambda_1}(t)\Big].
\end{aligned}
\tag{37}
$$

Then, by using the definition of the Hamiltonian H_1, see Equation (13), we see that everything inside the curly brackets in Equation (36) is equal to zero. Hence,

$$
D_1=A+E\Big[\int_0^T\beta_1(t)\frac{\partial H_1}{\partial u_1}(t)dt\Big].
$$

Recall that from the definitions of y_1 and $\tilde{\Lambda}_1$,

$$y_1(t) = x_1(t-\delta_1) \text{ and } \tilde{\Lambda}_1(t) = \int_{t-\delta_1}^{t} x_1(u)dB(u).$$

This implies, by change of variables

$$\begin{aligned} E\left[\int_0^T y_1(t)\frac{\partial H_1}{\partial y_1}(t)\right] &= E\left[\int_0^T x_1(t-\delta_1)\frac{\partial H_1}{\partial y_1}(t)dt\right] \\ &= \int_{-\delta_1}^{T-\delta_1} x_1(u)\frac{\partial H_1}{\partial y_1}(u+\delta_1)du\Big] \\ &= E\left[\int_0^T x_1(u)1_{[0,T-\delta_1]}(u)\frac{\partial H_1}{\partial y_1}(u+\delta_1)du\right]. \end{aligned}$$

Also, by the duality formula for Malliavin derivatives (see Di Nunno et al. [13]) and changing the order of integration

$$\begin{aligned} E\left[\int_0^T \tilde{\Lambda}_1(t)\frac{\partial H_1}{\partial \Lambda_1}(t)\right] &= E\left[\int_0^T\int_{t-\delta_1}^{t} x_1(u)dB(u)\frac{\partial H_1}{\partial \Lambda_1}(t)dt\right] \\ &= E\left[\int_0^T\int_{t-\delta_1}^{t} E\left[D_u\left(\frac{\partial H_1}{\partial \Lambda_1}(t)\right)|\mathcal{F}_u\right]x_1(u)du\ dt\right] \\ &= E\left[\int_0^T\int_{u}^{u+\delta_1} E\left[D_u\left(\frac{\partial H_1}{\partial \Lambda_1}(t)\right)|\mathcal{F}_u\right]1_{[0,T]}(t)dt\ x_1(u)\ du\right]. \end{aligned}$$

But, from the definition of μ_1,

$$\begin{aligned} E\left[\int_0^T x_1(t)E[\mu_1(t)|\mathcal{F}_t]dt\right] &= E\left[\int_0^T E[x_1(t)\mu_1(t)|\mathcal{F}_t]\ dt\right] \\ &= E\left[\int_0^T E[x_1(t)\Big\{-\frac{\partial H_1}{\partial x}(t) - \frac{\partial H_1}{\partial y_1}(t+\delta_1)1_{[0,T-\delta_1]}\right. \\ &\quad \left. - \int_t^{t+\delta_1} D_t\left[\frac{\partial H_1}{\partial \Lambda_1}(s)\right]1_{[0,T]}(s)ds\Big\}|\mathcal{F}_t]dt\right]. \end{aligned}$$

So, by the rule of double expectation and the calculations above, $A = 0$. This implies that $D_1 = E[\int_0^T \beta_1(t)\frac{\partial H_1}{\partial u_1}(t)dt]$, so

$$\frac{\partial}{\partial s}J_1(u_1+s\beta_1,u_2)|_{s=0} = E\left[\int_0^T \beta_1(t)\frac{\partial H_1}{\partial u_1}(t)dt\right],$$

which was what we wanted to prove. □

5. Solution of the noisy memory FBSDE

In this section, we consider a slightly simplified version of the system of noisy memory FBSDEs in Equations (14) and (15). Instead, consider the following *noisy memory FBSDE*:

FSDE in λ,

$$\begin{aligned} d\lambda(t) &= \frac{\partial H}{\partial w}(t)dt + \frac{\partial H}{\partial z}(t)dB(t) + \int_{\mathbb{R}} \nabla_k H(t,\zeta)\tilde{N}(dt,d\zeta), \\ \lambda(0) &= \phi'(W(0)). \end{aligned} \tag{38}$$

BSDE in p, qandr,

$$\begin{aligned} dp(t) &= -E[\mu(t)|\mathcal{F}_t]dt + q(t)dB(t) + \int_{\mathbb{R}} r(t,\zeta)\tilde{N}(dt,d\zeta), \\ p(T) &= \varphi'(X(T)) + h'(X(T))\lambda(T) \end{aligned} \tag{39}$$

where

$$\begin{aligned} &H(t,x,y_1,y_2,\Lambda_1,\Lambda_2,w,z,k,u_1,u_2,\lambda,p,q,r) \\ &\quad = f(t,x,y,\Lambda,u_1,u_2) + \lambda g(t,x,y_1,y_2,\Lambda_1,\Lambda_2,w,z,k,u_1,u_2) \\ &\quad + pb(t,x,y_1,y_2,\Lambda_1,\Lambda_2,u_1,u_2) + q\sigma(t,x,y_1,y_2,\Lambda_1,\Lambda_2,u_1,u_2) \\ &\quad + \int_{\mathbb{R}} r(\zeta)\gamma(t,x,y_1,y_2,\Lambda_1,\Lambda_2,u_1,u_2,\zeta)\nu(d\zeta) \end{aligned}$$

and

$$\mu(t) = \frac{\partial H}{\partial x}(t) + \frac{\partial H}{\partial y}(t+\delta)1_{[0,T-\delta]}(t) + \int_t^{t+\delta} E\left[D_t\left[\frac{\partial H}{\partial \Lambda}(s)\right]|\mathcal{F}_t\right]1_{[0,T]}(s)ds.$$

Note that the set of Equations (14) and (15) are two such systems such as (38) and (39) involving the same X process as well as the same controls u_1, u_2.

Also, consider the following system consisting of an FSDE and two BSDEs:

FSDE in λ,

$$\begin{aligned} d\tilde{\lambda}(t) &= \frac{\partial \mathcal{H}}{\partial w}(t)dt + \frac{\partial \mathcal{H}}{\partial z}(t)dB(t) + \int_{\mathbb{R}} \nabla_k \mathcal{H}(t,\zeta)\tilde{N}(dt,d\zeta), \\ \tilde{\lambda}(0) &= \phi'(W(0)). \end{aligned} \tag{40}$$

BSDE in p_1, q_1andr_1,

$$\begin{aligned} dp_1(t) &= -E[\mu_1(t)|\mathcal{F}_t]dt + q_1(t)dB(t) + \int_{\mathbb{R}} r_1(t,\zeta)\tilde{N}(dt,d\zeta), \\ p_1(T) &= \varphi'(X(T)) + h'(X(T))\tilde{\lambda}(T). \end{aligned} \tag{41}$$

BSDE in p_2, q_2 and r_2,

$$\begin{aligned} dp_2(t) &= -E[\mu_2(t)|\mathcal{F}_t]dt + q_2(t)dB(t) + \int_{\mathbb{R}} r_2(t,\zeta)\tilde{N}(dt,d\zeta), \\ p_2(T) &= 0 \end{aligned} \tag{42}$$

where

$$\begin{aligned} &\mathcal{H}(t,x,y_1,y_2,\Lambda_1,\Lambda_2,w,z,k,u_1,u_2,\tilde{\lambda},p_1,p_2,q_1,q_2,r_1,r_2) \\ &\quad = q_2(t)x + H(t,x,y_1,y_2,\Lambda_1,\Lambda_2,w,z,k,u_1,u_2,\tilde{\lambda},p_1,q_1,r_1), \\ &\qquad \mu_1(t) = q_2(t) + \frac{\partial H}{\partial x}(t) + \frac{\partial H}{\partial y}(t+\delta)1_{[0,T-\delta]}(t) \end{aligned} \tag{43}$$

and

$$\mu_2(t) = \frac{\partial H}{\partial \Lambda}(t) - \frac{\partial H}{\partial \Lambda}(t+\delta)1_{[0,T-\delta]}(t).$$

Note that $\frac{\partial \mathcal{H}}{\partial \Lambda}(t) = \frac{\partial H}{\partial \Lambda}(t)$, $\frac{\partial \mathcal{H}}{\partial \Lambda}(t) = q_2(t) + \frac{\partial H}{\partial \Lambda}(t)$ and $\frac{\partial \mathcal{H}}{\partial y}(t) = \frac{\partial H}{\partial y}(t)$. Hence, Equations (38) and (40) are structurally equal.

Then, by similar techniques as in Dahl et al. [4], we can show the following theorem:

Theorem 5.1. *Assume that* (p_i, q_i, r_i) *for* $i = 1, 2$ *and* $\tilde{\lambda}$ *solve the FBSDE system* (40)–(42). *Define* $\lambda = \tilde{\lambda}, p(t) = p_1(t), q(t) = q_1(t)$ *and* $r(t, \cdot) = r_1(t, \cdot)$ *and assume that* $E[\int_0^T (\frac{\partial H(t)}{\partial z})^2]dt < \infty$. *Then,* (p, q, r, λ) *solves the noisy memory FBSDE* (38) and (39) *and*

$$q_2(t) = \int_t^{t+\delta} E\left[D_t\left[\frac{\partial H}{\partial \Lambda}(s)\right]|\mathcal{F}_t\right]ds.$$

Proof. The jump terms do not make a difference here, so assume for simplicity that $r = r_1 = r_2 = 0$ everywhere.

In general, we know that if $dp_2(t) = -\theta(t, p_2, q_2)dt + q_2(t)dB(t), p_2(T) = F$, then

$$q_2(t) = D_t p_2(t). \tag{44}$$

Now, note that the solution p_2 of the BSDE (42) can be written

$$\begin{aligned}
p_2(t) &= -E\left[\int_t^T E[\mu_2(s)|\mathcal{F}_s]ds|\mathcal{F}_t\right] \\
&= -\int_t^T E[\mu_2(s)|\mathcal{F}_t]ds \\
&= -\int_t^T E\left[\frac{\partial H}{\partial \Lambda}(t) - \frac{\partial H}{\partial \Lambda}(t+\delta)1_{[0,T-\delta]}(t)|\mathcal{F}_t\right]ds \\
&= -\int_t^{t+\delta} E\left[\frac{\partial H(s)}{\partial \Lambda}\middle|\mathcal{F}_t\right]1_{[0,T]}(s)ds,
\end{aligned}$$

where the equalities follow from Fubini's theorem, the rule of double expectation, the definition of μ_2 and a change of variables. Hence, by Equation (44):

$$\begin{aligned}
q_2(t) &= D_t p_2(t) \\
&= D_t\left[\int_t^{t+\delta} E\left[\frac{\partial H(s)}{\partial \Lambda}\middle|\mathcal{F}_t\right]1_{[0,T]}(s)\right]ds \\
&= \int_t^{t+\delta} E\left[D_t\left(\frac{\partial H(s)}{\partial \Lambda}\right)\middle|\mathcal{F}_t\right]1_{[0,T]}(s)ds,
\end{aligned}$$

which is part of what we wanted to prove.

By inserting this expression for q_2 into the definition of μ_1, we see that

$$\mu_1(t) = \int_t^{t+\delta} E\left[D_t\left[\frac{\partial H(s)}{\partial \Lambda}\right]\middle|\mathcal{F}_t\right]1_{[0,T]}(s)ds + \frac{\partial H(t)}{\partial x} + \frac{\partial H(t+\delta)}{\partial y}1_{[0,T]}(t+\delta).$$

Hence, we see that the BSDE (41) is the same as (39), so they have the same solution. This completes the proof of the theorem. □

We can also prove the following converse result.

Theorem 5.2. *If p, q, r, λ solve the FBSDE* (38) and (39) *and we define $\tilde{\lambda} = \lambda, p_1 = p, q_1 = q, r_1 = r$ and*

$$
\begin{aligned}
p_2(t) &= \int_t^{t+\delta} E\left[\frac{\partial H}{\partial \Lambda}(s)|\mathcal{F}_t\right] 1_{[0,\,T-\delta]}(s)ds, \\
q_2(t) &= \int_t^{t+\delta} E\left[D_t\left[\frac{\partial H}{\partial \Lambda}(s)\right]|\mathcal{F}_t\right] 1_{[0,\,T-\delta]}(s)ds, \\
r_2(t,\cdot) &= 0.
\end{aligned}
$$

Then, (p_i, q_i, r_i) for $i = 1, 2$ and $\tilde{\lambda}$ solve the system of Equations (40)–(42).

Proof. A gain, the jump parts make no crucial difference, so we consider the no-jump situation for simplicity.

It is clear that Equation (40) holds from the assumptions above (from the definition of $\mathcal{H}$, see (43)). Also, the BSDE (41) holds: Clearly, the terminal condition holds, and by the computations in the proof of Theorem 5.1, the remaining part of Equation (41) also holds. Therefore, it only remains to prove that the BSDE (42) holds.

By the Itô isometry and the Clark–Ocone formula,

$$
\begin{aligned}
E\left[\int_0^T E\left[D_s\left(\frac{\partial H(r)}{\partial \Lambda}\right)\middle|\mathcal{F}_s\right]^2 ds\right] &= E\left[\left(\int_0^T E\left[D_s\frac{\partial H(r)}{\partial \Lambda}\middle|\mathcal{F}_s\right]dB_s\right)^2\right] \\
&= E\left[\left(\frac{\partial H}{\partial \Lambda}(r)\right)^2 - E\left[\frac{\partial H}{\partial \Lambda}(r)\right]^2\right].
\end{aligned}
$$

Hence,

$$
\int_0^T E\left[\int_0^T E\left[D_s(\frac{\partial H(r)}{\partial \Lambda})|\mathcal{F}_s\right]^2 ds\right]^{\frac{1}{2}} dr = \int_0^T \left(E\left[\frac{\partial H}{\partial \Lambda}(r)^2\right] - E\left[\frac{\partial H}{\partial \Lambda}(r)\right]^2\right)^{\frac{1}{2}} dt < \infty.
$$

Note that from the Clark–Ocone theorem,

$$
\frac{\partial H(r)}{\partial \Lambda} = E\left[\frac{\partial H(r)}{\partial \Lambda}\middle|\mathcal{F}_t\right] + \int_t^r E\left[D_s\left(\frac{\partial H(r)}{\partial \Lambda}\right)\middle|\mathcal{F}_s\right]dB(s).
$$

Therefore, by the definition of q_2 in the theorem and the Fubini theorem

$$
\begin{aligned}
\int_t^T q_2(s)dB(s) &= \int_t^T \int_t^T E\left[D_s\left(\frac{\partial H(r)}{\partial \Lambda}\right)\middle|\mathcal{F}_s\right] 1_{[s,\,s+\delta]}(r)drdB(s) \\
&= \int_t^T \int_t^T E\left[D_s\left(\frac{\partial H(r)}{\partial \Lambda}\right)\middle|\mathcal{F}_s\right] 1_{[r-\delta,\,r]}(s)dB(s)dr.
\end{aligned}
$$

By some algebra and the Clark–Ocone theorem (16),

$$
\begin{aligned}
\int_t^T \int_t^T E\left[D_s\left(\frac{\partial H(r)}{\partial \Lambda}\right)\middle|\mathcal{F}_s\right] 1_{[r-\delta,\,r]}(s)dB(s)dr &= \int_t^T \int_{r-\delta}^r E\left[D_s\left(\frac{\partial H(r)}{\partial \Lambda}\right)\middle|\mathcal{F}_s\right]dB(s)dr \\
&= \int_t^T \left(\frac{\partial H(r)}{\partial \Lambda} - E\left[\frac{\partial H(r)}{\partial \Lambda}\middle|\mathcal{F}_{r-\delta}\right]\right)dr.
\end{aligned}
$$

By splitting the integrals and using change of variables (twice) as well as some algebra,

$$
\begin{aligned}
&= \int_t^T \frac{\partial H(s)}{\partial \Lambda} ds - \int_{t-\delta}^{T-\delta} E\left[\frac{\partial H(s+\delta)}{\partial \Lambda}\middle|\mathcal{F}_s\right] ds \\
&= \int_t^T \frac{\partial H(s)}{\partial \Lambda} ds - \int_t^T E\left[\frac{\partial H(s+\delta)}{\partial \Lambda}\middle|\mathcal{F}_s\right] 1_{[0,T-\delta]}(s) ds \\
&\quad - \int_t^{t+\delta} E\left[\frac{\partial H(s)}{\partial \Lambda}|\mathcal{F}_t\right] 1_{[0,T-\delta]}(s) ds \\
&= \int_t^T E\left[\frac{\partial H(s)}{\partial \Lambda} - \frac{\partial H(s+\delta)}{\partial \Lambda} 1_{[0,T-\delta]}(s)\middle|\mathcal{F}_s\right] ds - p_2(t).
\end{aligned}
$$

This proves that the BSDE (42) holds as well. □

Now, we have expressed the solution of the Malliavin FBSDE via the solution of the "double" FBSDE system (40)–(42). What kind of system of equations is this? The system consists of two connected BSDEs in $(p_1, q_1, r_1)$and(p_2, q_2, r_2) respectively, and these are again connected to a FBSDE in λ. However, from Equation (42) and the definition of μ_2, we see that the right hand side of (42) does not depend on p_2. Hence, the BSDE (42) can be rewritten

$$
\begin{aligned}
dp_2(t) &= h(t, \lambda, p_1, q_1, r_1(\cdot))dt + q_2(t)dB(t) + \int_{\mathbb{R}} r_2(t,\zeta)\tilde{N}(dt, d\zeta) \\
p_2(T) &= 0.
\end{aligned}
$$

This can be solved to express $p_2$using$\lambda, p_1, q_1$and$r_1(\cdot)$ by letting $q_2(t) = r_2(t,\cdot) = 0$ for all t and

$$
p_2(t) = E\left[\int_t^T h(t, \lambda, p_1, q_1, r_1(\cdot))dt|\mathcal{F}_t\right].
$$

Now, we can substitute this solution for $p_2(t)$ into the FBSDE system (40) and (41). The resulting set of equations is a regular system of time advanced FBSDEs with jumps. There are to the best of our knowledge, no general results on existence and uniqueness of such systems of FBSDEs. However, if we simplify by removing the jumps and there was no time-advanced part (i.e., no delay process Y_i in the original FSDE (1)), there are some results by Ma et al. [24].

6. Optimal consumption rate with respect to recursive utility

In this section, we apply the previous results to the problem of determining an optimal consumption rate with respect to recursive utility (see also Øksendal and Sulem [25] and Dahl and Øksendal [26]). Let $X(t) = X^c(t)$, where the consumption rate $c(t)$ is our control, and assume that

$$
\begin{aligned}
dX(t) &= X(t)\left[\mu(t)dt + \sigma(t)dB(t) + \int_{\mathbb{R}} \gamma(t,\zeta)\tilde{N}(dt, d\zeta)\right] \\
&\quad -[c_1(t) + c_2(t)]X(t)dt, \\
X(0) &= x > 0,
\end{aligned}
\tag{45}
$$

and $W_i(t)$ is given by

$$\begin{aligned} dW_i(t) &= -[\alpha_i(t)W_i(t) + \eta_i(t)\ln(Y_i(t)) + \kappa_i(t)\ln(\Lambda_i(t)) + \ln(c_i(t)X(t))] \\ &\quad + Z_i(t)dB(t) + \textstyle\int_{\mathbb{R}} K_i(t,\zeta)\tilde{N}(dt,d\zeta), \\ W_i(T) &= 0. \end{aligned}$$

Let the performance functional be defined by $J_i(c_1, c_2) := W_i(0)$, i.e., J_i is the recursive utility for player i. Also, assume that both players have full information, so $(\mathcal{E}_t^{(i)})_t = (\mathcal{F}_t)_t fori = 1, 2$.

We would like to find a Nash equilibrium for this FBSDE game with delay. To do so we will use the maximum principle Theorem 3.1. Note that $f_i = \varphi_i = h_i = 0$ and that $\psi_i(w) = w \; for \; i = 1, 2$. The Hamiltonians are:

$$\begin{aligned} &H_i(t, x, y_1, y_2, \Lambda_1, \Lambda_2, w_i, z_i, k_i, c_1, c_2, \lambda_i, p_i, q_i, r_i(\zeta)) \\ &= \lambda_i(\alpha_i(t)w_i + \eta_i(t)\ln(y_i) + \ln(c_i x)) \\ &+ p_i(x\mu(t) - (c_1 + c_2)x) + q_i\sigma(t)x + \int_{\mathbb{R}} x r_i(\zeta)\gamma(t,\zeta)\nu(d\zeta) \text{ for } i = 1, 2. \end{aligned}$$

The adjoint BSDEs are

$$\begin{aligned} dp_i(t) &= E[\mu_i(t)|\mathcal{F}_t]dt + q_i(t)dB(t) + \textstyle\int_{\mathbb{R}} r_i(t,\zeta)\tilde{N}(dt,d\zeta), \\ p_i(T) &= 0, \end{aligned}$$

where

$$\begin{aligned} \mu_i(t) &= -\frac{\lambda_i(t)}{X(t)} - \frac{\lambda_i(t+\delta_i)\eta_i(t+\delta_i)}{Y_i(t+\delta_i)} 1_{[0,T-\delta_i]}(t) - p_i(t)(\mu(t) - (c_1(t) + c_2(t))) \\ &\quad + q_i(t)\sigma(t) + \int_{\mathbb{R}} r_i(t,\zeta)\gamma(t,\zeta)\nu(d\zeta) \end{aligned}$$

for $i = 1, 2$. Note that by the definition of Y_i, $Y_i(t + \delta_i) = X(\{t + \delta_i\} - \delta_i) = X(t)$.

The adjoint BSDEs are linear, and the solutions are given by (see Øksendal and Sulem [15])

$$\begin{aligned} \Gamma_i(t)p_i(t) &= E\left[\int_t^T \left(\frac{\lambda_i(s)}{X(s)} + \frac{\lambda_i(s+\delta_i)\eta_i(s+\delta_i)}{Y_i(s+\delta_i)} 1_{[0,T-\delta_i]}(s)\right)\Gamma_i(s)ds \Big| \mathcal{F}_t\right] \\ &= E\left[\int_t^T \left(\frac{\lambda_i(s)}{X(s)} + \frac{\lambda_i(s+\delta_i)\eta_i(s+\delta_i)}{X(s)} 1_{[0,T-\delta_i]}(s)\right)\Gamma_i(s)ds \Big| \mathcal{F}_t\right], \end{aligned} \tag{46}$$

where

$$\begin{aligned} d\Gamma_i(t) &= \Gamma_i(t)\left[(\mu(t) - (c_1(t) + c_2(t)))dt + \sigma(t)dB(t) + \int_{\mathbb{R}} \gamma(t,\zeta)\tilde{N}(dt,d\zeta)\right], \\ \Gamma_i(0) &= 1 \text{ for } i = 1, 2. \end{aligned}$$

Note that by the SDE (45),

$$x\Gamma_i(t) = X(t). \tag{47}$$

Hence, by combining Equations (46) and (47), we see that

$$X(t)p_i(t) = E\left[\textstyle\int_t^T (\lambda_i(s) + \lambda_i(s+\delta_i)\eta_i(s+\delta_i)1_{[0,T-\delta_i]}(s))ds \Big| \mathcal{F}_t\right]. \tag{48}$$

The adjoint FSDEs are

$$\begin{aligned} d\lambda_i(t) &= \lambda_i(t)\alpha_i(t)dt, \\ \lambda_i(0) &= 1, \text{ for } i = 1, 2. \end{aligned}$$

These are (non-stochastic) differential equation with solution $\lambda_i(t) = \exp(\int_0^t \alpha_i(s)ds)$ *for* $i = 1, 2$.

We maximize H_i with respect to c_i. For $i = 1, 2,$ the first order condition is:

$$\hat{c}_i(t) = \frac{\lambda_i(t)}{p_i(t)X(t)}.$$

By substituting Equation (48) into this, we find (by the sufficient maximum principle, Theorem 3.1) that the consumption rates leading to a Nash equilibrium for the recursive utility problem are given by:

$$c_i^*(t) = \frac{\lambda_i(t)}{E\left[\int_t^T (\lambda_i(s) + \lambda_i(s + \delta_i)\eta_i(s + \delta_i)1_{[0, T-\delta_i]}(t))ds|\mathcal{F}_t\right]}.$$

where $\lambda_i(t) = \exp(\int_0^t \alpha_i(s)ds)fori = 1, 2$.

7. Conclusion

In this paper, we have analyzed a two-player stochastic game connected to a set of FBSDEs involving delay and noisy memory of the market process. We have derived sufficient and necessary maximum principles for a set of controls for the two players to be a Nash equilibrium in this game. We have also studied the associated FBSDE involving Malliavin derivatives, and connected this to a system of FBSDEs not involving Malliavin derivatives. Finally, we were able to derive a closed form Nash equilibrium solution to a game where the aim is to find the optimal consumption with respect to recursive utility.

Note

1. Defined similarly as subdifferentiability for convex functions.

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