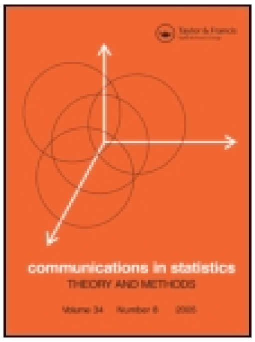

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Communications in Statistics - Theory and Methods

ISSN: 0361-0926 (Print) 1532-415X (Online) Journal homepage: https://www.tandfonline.com/loi/lsta20

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To cite this article: Qian Zhang, Xingzhong Xu & Shiyun Mi (2016) A generalized *p*-value approach to inference on the performance measures of an $M/E_k/1$ queueing system, Communications in Statistics - Theory and Methods, 45:8, 2256-2267, DOI: 10.1080/03610926.2013.879181

To link to this article: https://doi.org/10.1080/03610926.2013.879181

COMMUNICATIONS IN STATISTICS—THEORY AND METHODS
2016, VOL. 45, NO. 8, 2256–2267
http://dx.doi.org/10.1080/03610926.2013.879181

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A generalized *p*-value approach to inference on the performance measures of an $M/E_k/1$ queueing system

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ABSTRACT
The hypothesis tests of performance measures for an $M/E_k/1$ queueing system are considered. With pivotal models deduced from sufficient statistics for the unknown parameters, a generalized *p*-value approach to derive tests about parametric functions are proposed. The focus is on derivation of the *p*-values of hypothesis testing for five popular performance measures of the system in the steady state. Given a sample T, let $p(T)$ be the p values we developed. We derive a closed form expression to show that, for small samples, the probability $P(p(T) \leqslant \gamma)$ is approximately equal to γ, for $0 \leqslant \gamma \leqslant 1$.

ARTICLE HISTORY
Received 2 April 2013
Accepted 19 December 2013

KEYWORDS
$M/E_k/1$ queueing system; Performance measures; Hypothesis testing; Generalized p values; Frequentist properties.

MATHEMATICS SUBJECT CLASSIFICATION
62F03; 68M20

1. Introduction

In a queueing system, it is important to carry out a statistical analysis. When operating a queueing system, monitoring and control of the performance measures of the system are essential to ensure that the system performance is up to design standards. “A model is not of much use unless it is related with the system through empirical data analysis, parameter estimation and tests of relevant hypothesis” (Bhat and Rao, 1997). In this article, performance measures of the classical single server markovian model ($M/E_k/1$) are chosen to analyse. The pioneering paper in attempting at statistical inference in $M/M/1$ queues was made by Clarke (1957). He developed maximum likelihood estimates of arrival rate and service rate. Another notable contribution has been made by Lilliefors (1966), who presented the confidence intervals of parameters such as traffic intensity for the $M/M/1$, $M/M/2$, and $M/E_k/1$ queues. Problems of large sample estimation and tests for the parameters in a single server queue are discussed by Basawa and Prabhu (1988). For a review of statistical analysis in queueing systems, see Bhat and Rao (1997). There has been recent interest in Bayesian analysis of queueing models. Armero and Conesa (1998) made inference about the parameters in stationary $M^k/M/1$ and $M/E_k/1$ queues from a Bayesian point of view. Insua et al. (1998) analyzed $M/E_r/1$ and $M/H_k/1$ queues using Bayesian approach. Some references are Armero (1994), Armero and Conesa (2000), Ausín (2004). Recently, Bayesian inference and prediction of some popular performance measures in $M/M/1$ queue was carried out by Choudhury

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and Borthakur (2007). Xu and Zhang (2010) derived the uniformly most accurate confidence bounds and the uniformly most powerful unbiased tests for the mean sojourn time of an $M/M/1$ queueing system.

Throughout this article, we consider an experiment in which m interarrival times and n service times are measured. We use $\{X_i, i = 1, \dots, m\}$ and $\{Y_i, i = 1, \dots, n\}$ to denote the positive random variables of interarrival and service time for the i th customer of an $M/E_k/1$ First Come First Service queueing system. Hence, $X_1, X_2, \dots, X_m$ are independent and identically distributed (i.i.d.) with exponential distribution $F(t) = 1 - e^{-\lambda x}$, $t > 0$, and $Y_1, Y_2, \dots, Y_n$ are i.i.d with Erlang probability density

$$g(t) = \frac{(\mu k)^k}{(k-1)!} t^{k-1} e^{-k\mu t}, \quad (0 < t < \infty)$$

where λ and μ are unknown positive parameters, k is considered as a fixed constant. Two samples $(X_1, \dots, X_m)$ and $(Y_1, \dots, Y_n)$ are assumed to be mutually independent.

It is very frequent to assume steady-state in queueing theory. Steady-state implies traffic intensity $\rho = \frac{\lambda}{\mu} < 1$. A queue in the steady-state is called the stationary queue. The objective of this paper is to test hypotheses for five performance measures of the stationary $M/E_k/1$ queue: traffic intensity (ρ), mean system size (L), the mean queue size (L_q), the mean waiting time in system (W), and the mean waiting time in queue (W_q).

Inference for the performance measures in a stationary $M/E_k/1$ queue is complex. The restriction $\rho < 1$ in the parameter space must be incorporated. Note that the five performance measures are functions of parameters in two populations. With pivotal models deduced from sufficient statistics of the exponential population and the Erlang population, the generalized p-value approach for hypothesis testing about parametric functions are proposed. The concept of the generalized p-value was first introduced by Tsui and Weerahandi (1989) to deal with some nontrivial statistical testing problems. These problems involve nuisance parameters in such a fashion that the derivation of a standard pivot is not possible. For the five performance measures the generalized p-values of hypothesis testing are derived. The frequentist properties of these p-values with fixed sample sizes are investigated. For the former three performance measures the tests are just the classical F tests (see Cox, 1965 or Lilliefors, 1966). For the later two performance measures, given a sample T, we provide a relatively tight upper bound for $P(p(T) \leqslant \gamma)$ of the p-values we derived, for $0 \leqslant \gamma \leqslant 1$. The tests given by us are satisfactory which will be seen from Theorems 3.2 and 3.3 and simulation results.

The remainder of this article proceeds as follows. In Sec. 2, the generalized p-values of hypothesis testing problems are derived. The frequentist properties of p-values are studied in Sec. 3. Simulation examples are presented in Sec. 4.

2. The generalized *p*-values

In this section, we consider tests of certain hypotheses concerning the performance measures ρ, L, L_q, W, W_q. Expressions for the steady-state performance measures of the $M/E_k/1$ queue can be found in Gross and Harris (1997). Let $\theta_i, i = 1, \cdots, 5$ represent ρ, L, L_q, W, W_q, respectively. We will discuss tests for the parameters $\theta_i = g_i(\lambda, \mu), i = 1, \cdots, 5$, respectively, where

$$\theta_1 = g_1(\lambda, \mu) = \frac{\lambda}{\mu} \triangleq \rho, \quad \theta_2 = g_2(\lambda, \mu) = \rho + \frac{(k+1)\rho^2}{2k(1-\rho)}, \quad \theta_3 = g_3(\lambda, \mu) = \frac{(k+1)\rho^2}{2k(1-\rho)},$$

$$\theta_4 = g_4(\lambda, \mu) = \frac{1}{\mu} + \frac{(k+1)\lambda}{2k\mu(\mu - \lambda)}, \quad \theta_5 = g_5(\lambda, \mu) = \frac{(k+1)\lambda}{2k\mu(\mu - \lambda)}.$$

The five measures in a stationary $M/E_k/1$ queue are all functions of λ and μ. In general, they can be expressed as $\theta = g(\lambda, \mu)$. Consider hypotheses of the forms:

$$H : \theta \leqslant \theta_0 \text{ vs. } K : \theta > \theta_0; \tag{2.1}$$

$$H : \theta \geqslant \theta_0 \text{ vs. } K : \theta < \theta_0; \tag{2.2}$$

$$H : \theta = \theta_0 \text{ vs. } K : \theta \neq \theta_0. \tag{2.3}$$

As mentioned in Sec. 1, $\{X_i, i = 1, 2, \ldots, m\}$ and $\{Y_j, j = 1, 2 \ldots, n\}$ are independent exponentially distributed. Hence, the statistics $T_1 = 2m\overline{X} = 2\sum_{i=1}^{m} X_i$ and $T_2 = 2kn\overline{Y} = 2k\sum_{j=1}^{n} Y_j$ are sufficient for the parameters (λ, μ). They can be expressed in the following pivotal model

$$\lambda T_1 = E_1, \quad \mu T_2 = E_2, \tag{2.4}$$

where

$$E_1 \sim \chi^2(2m), \quad E_2 \sim \chi^2(2kn), \tag{2.5}$$

and E_1 and E_2 are mutually independent. The pivotal model (2.4) can be inverted into

$$\lambda = E_1/T_1, \quad \mu = E_2/T_2. \tag{2.6}$$

In order to simplify our notation, denote

$$T = (T_1, T_2), \quad E = (E_1, E_2), \quad \xi = (\lambda, \mu),$$

and Q is the distribution of E given by (2.5).

In a stationary $M/E_k/1$ queue, $\lambda < \mu$ is the condition for stationarity. However, due to the independence of E_1 and E_2, $\lambda \geqslant \mu$ will occur in the expression (2.6). This occurrence is supposed to be unreasonable. For convenience, we assume that $\lambda \leqslant \mu$. The model (2.4) can be expressed as

$$T_1 = E_1/\lambda, \quad T_2 = E_2/\mu.$$

Given (t_1, t_2), the observations of (T_1, T_2), and (e_1, e_2), the observations of (E_1, E_2), which distribution in the parameter space $0 < \lambda \leqslant \mu$ is most likely to generate t_1, t_2? It is natural to take the usual least squares method to fulfil it. Consider the following minimum value problem (Xu and Li, 2006):

$$\min_{0<\lambda\leqslant\mu} \left\{(T_1 - E_1/\lambda)^2 + (T_2 - E_2/\mu)^2\right\}.$$

The minimum value can be attained at

$$\lambda = \begin{cases} E_1/T_1, & E_1/T_1 \leqslant E_2/T_2, \\ (E_1^2 + E_2^2)/(T_1E_1 + T_2E_2), & E_1/T_1 > E_2/T_2, \end{cases}$$

$$\mu = \begin{cases} E_2/T_2, & E_1/T_1 \leqslant E_2/T_2, \\ (E_1^2 + E_2^2)/(T_1E_1 + T_2E_2), & E_1/T_1 > E_2/T_2. \end{cases}$$

Substituting this in $\theta = g(\lambda, \mu)$ gives

$$\theta = \widehat{\theta}_{g,T}(E) \triangleq \begin{cases} g\left(\dfrac{E_1}{T_1}, \dfrac{E_2}{T_2}\right), & \dfrac{E_1}{T_1} \leqslant \dfrac{E_2}{T_2}, \\ g\left(\dfrac{E_1^2 + E_2^2}{T_1E_1 + T_2E_2}, \dfrac{E_1^2 + E_2^2}{T_1E_1 + T_2E_2}\right), & \dfrac{E_1}{T_1} > \dfrac{E_2}{T_2}. \end{cases} \tag{2.7}$$

Consider the expression of $\widehat{\theta}_{g,t}(E)$ given by (2.7), here $t = (t_1, t_2)$ is the observation of the statistic T. Substituting $E = (\lambda T_1, \mu T_2)$, we obtain

$$R_g(T; t, \lambda, \mu) = \widehat{\theta}_{g,t}(\lambda T_1, \mu T_2). \tag{2.17}$$

Then:

(1) the observation of $R_g(T; t, \lambda, \mu)$ satisfies

$$R_g(t; t, \lambda, \mu) = g(\lambda, \mu) = \theta;$$

(2) given t, λ, μ, the conditional distribution of $R_g(T; t, \lambda, \mu)$ is free of parameters, that is

$$P\left(R_g(T; t, \lambda, \mu) \leqslant z | t, \lambda, \mu\right) = Q\left(\widehat{\theta}_{g,t}(E) \leqslant z\right).$$

Therefore, the random variable given by (2.17) is a generalized pivotal quantity for $\theta = g(\lambda, \mu)$ (refer to Weerahandi, 1993), and also a Fiducial generalized pivotal quantity (refer to Hannig, 2006). Let

$$S_g(T; t, \lambda, \mu) = g(\lambda, \mu) - R_g(T; t, \lambda, \mu),$$

hence $S_g(t; t, \lambda, \mu) = 0$. Moreover, $S_g(T; t, \lambda, \mu)$ is generalized test variable (see Tsui and Weerahandi, 1989). Therefore, we can get the generalized p value

$$p_g(t) = P_{g(\lambda,\mu)=\theta_0}\left(S_g(T; t, \lambda, \mu) \geqslant 0\right) = Q\left(\widehat{\theta}_{g,t}(E) \leqslant \theta_0\right).$$

In this way, the p values $p_{l,i}(t)$, $i = 1, 2, \cdots, 5$ can be obtained by substituting $g = g_i$, $i = 1, 2, \cdots, 5$ into the above expression.

Based on (2.7), the generalized p values for the tests of the hypotheses (2.1), (2.2), and (2.3) concerning the parameters θ_i, $i = 1, 2, \cdots, 5$ can be derived. The generalized p values for testing (2.1) concerning θ_i, $i = 1, 2, \cdots, 5$ can be defined by

$$p_{l,i}(T) = Q(\widehat{\theta}_{g_i,T}(E) \leqslant \theta_0), \quad i = 1, 2, \cdots, 5. \tag{2.8}$$

For testing (2.2), the generalized p values can be defined by

$$p_{r,i}(T) = Q(\widehat{\theta}_{g_i,T}(E) \geqslant \theta_0), \quad i = 1, 2, \cdots, 5. \tag{2.9}$$

In the case of testing (2.3), the p values can be defined by

$$p_{c,i}(T) = 2\min\left\{p_{l,i}(T), p_{r,i}(T)\right\}, \quad i = 1, 2, \cdots, 5. \tag{2.10}$$

It follows from the expressions (2.9), (2.10), and (2.11),

$$p_{r,i}(T) = 1 - p_{l,i}(T), \quad p_{c,i}(T) = 2\min\{p_{l,i}(T), \ 1 - p_{l,i}(T)\}, \quad i = 1, 2, \cdots, 5.$$

Hence, the performances of $p_{l,i}(T)$, $i = 1, 2, \cdots, 5$ are mainly investigated.

Notice that the equilibrium of the system requires $\rho < 1$. The constant θ_0 is constrained in the problem of testing (2.1) for various θ_i, $i = 1, \cdots, 5$. For θ_1, $0 < \theta_0 < 1$, and for θ_i, $i = 2, 3, 4, 5$, $\theta_0 > 0$. It follows from the expressions (2.5) and (2.8),

$$p_{l,1}(T) = Q\left(\frac{E_1 T_2}{E_2 T_1} \leqslant \theta_0\right) = F_{2m,2kn}\left(\theta_0 \frac{knT_1}{mT_2}\right), \tag{2.11}$$

where $F_{a,b}(\cdot)$ denotes F-distribution function with degrees of freedom a and b. Let $\theta_L = \frac{k(1+\theta_0)+k\sqrt{1+\theta_0^2+2\theta_0/k}}{k-1}$ and $\theta_{L_q} = \frac{2\theta_0}{\sqrt{\theta_0^2+2\theta_0(k+1)/k+\theta_0}}$, we have

$$p_{l,2}(T) = Q\left(\frac{E_1T_2}{E_2T_1} \leqslant \theta_L\right) = F_{2m,2kn}\left(\theta_L\frac{knT_1}{mT_2}\right). \tag{2.12}$$

$$p_{l,3}(T) = Q\left(\frac{E_1T_2}{E_2T_1} \leqslant \theta_{L_q}\right) = F_{2m,2kn}\left(\theta_{L_q}\frac{knT_1}{mT_2}\right). \tag{2.13}$$

$$\begin{aligned} p_{l,4}(T) &= Q\left(\frac{k+1}{2k}\cdot\frac{E_1/T_1}{E_2/T_2(E_2/T_2-E_1/T_1)} + \frac{1}{E_2/T_2} \leqslant \theta_0\right) \\ &= \mathsf{E}K_{2m}\left(\frac{2k(E_2\theta_0+T_2)T_1E_2}{(3k+1)T_2^2+2kE_2\theta_0T_2}\right). \end{aligned} \tag{2.14}$$

where $K_a(\cdot)$ is χ^2-distribution function with degrees of freedom a, the expectation symbol E is taken with respect to E_2:

$$p_{l,5}(T) = Q\left(\frac{E_2T_1}{E_1T_2}\left(\frac{E_2}{T_2}-\frac{E_1}{T_1}\right) \geqslant \frac{k+1}{2k\theta_0}\right) = \mathsf{E}K_{2m}\left(\frac{2k\theta_0E_2^2T_1}{(k+1)T_2^2+2k\theta_0E_2T_2}\right). \tag{2.15}$$

3. Frequentist properties

We start with investigate the frequentist properties of the first three generalized p-values.

Theorem 3.1. *For testing hypotheses (2.1) concerning θ_i, $i = 1, 2, 3$, the p values defined by (2.12), (2.13), and (2.14) have the following properties for $i = 1, 2, 3$, respectively:*

(1) the distribution of $p_{l,i}(T)$ only depends on the parameter θ_i;

(2) the p value $p_{l,i}(T)$ is stochastically decreasing in θ_i, namely when $\theta_i' < \theta_i''$,

$$P_{\theta_i'}(p_{l,i}(T) \leqslant \gamma) \leqslant P_{\theta_i''}(p_{l,i}(T) \leqslant \gamma), \quad \gamma \in (0,1); \text{ and}$$

(3) when $\theta_i = \theta_0$, $p_{l,i}(T)$ distributes according to the uniform distribution over the interval (0,1).

Proof. For each $i = 1, 2, 3$, the p value $p_{l,i}(T)$ depends on T only through T_1/T_2 and is strictly increasing in T_1/T_2. Further, the distribution of $(\rho knT_1)/(mT_2)$ is F-distribution with degrees of freedom $2m$ and $2kn$, and θ_i, $i = 1, 2, 3$ are all strictly increasing function of ρ. Hence, conclusions (1) and (2) hold. When $\theta_i = \theta_0$, $p_{l,i}(T) = F_{2m,2kn}((\rho knT_1)/(mT_2))$, $i = 1, 2, 3$, thus conclusion (3) holds. The proof is completed. □

For $0 < \gamma < 1$, let

$$D_{i,\gamma} = \{t : p_{l,i}(t) \leqslant \gamma\}, \quad i = 1, 2, 3.$$

By Theorem 3.1, the rejection region $D_{i,\gamma}$ defines an exact unbiased level-γ test, $i = 1, 2, 3$. For p values $p_{r,i}(T)$, $i = 1, 2, 3$, analogous results in Theorem 3.1 hold. For p values $p_{c,i}(T)$, $i = 1, 2, 3$, the conclusion (3) in Theorem 3.1 hold. Hence exact tests can be obtained.

For the p value $p_{l,4}(T)$, the following conclusions hold.

Theorem 3.2. *Consider the problem of testing hypotheses (2.1) concerning mean sojourn time θ_4. The p value is given by (2.15). For arbitrary γ $(0 < \gamma < 1)$, if the rejection region is taken as $\{p_{l,4}(T) \leqslant \gamma\}$, we have following results.*

(1) *The true level of this test is attained on the common boundary of H and K, namely*

$$\sup_{g_4(\lambda,\mu)\leqslant\theta_0} P_{\lambda,\mu}(p_{l,4}(T)\leqslant\gamma) = \sup_{g_4(\lambda,\mu)=\theta_0} P_{\lambda,\mu}(p_{l,4}(T)\leqslant\gamma).$$

The minimum power of the test is also reached on the common boundary, namely

$$\inf_{g_4(\lambda,\mu)\geqslant\theta_0} P_{\lambda,\mu}(p_{l,4}(T)\leqslant\gamma) = \inf_{g_4(\lambda,\mu)=\theta_0} P_{\lambda,\mu}(p_{l,4}(T)\leqslant\gamma).$$

(2) *On the common boundary, let* $\varphi_\gamma(\lambda) = P_{\lambda,\mu(\lambda)}(p_{l,4}(T)\leqslant\gamma)$*, where* $\mu(\lambda)$ *satisfies* $g_4(\lambda,\mu(\lambda)) = \theta_0$*. Then,*

$$\lim_{\lambda\to 0}\varphi_\gamma(\lambda) = \lim_{\lambda\to\infty}\varphi_\gamma(\lambda) = \gamma.$$

(3) *On the common boundary, the test satisfies*

$$\gamma \leqslant \varphi_\gamma(\lambda) \leqslant \gamma - \gamma ln\gamma, \ \ \lambda > 0.$$

Proof. Let $E^* = (E_1^*, E_2^*)$ be an independent copy of $E = (E_1, E_2)$. For no confusion, denote the distribution of E^* by Q^*. According to (2.4) and (2.15),

$$p_{l,4}(T) \overset{d}{=} p_{l,4}(\frac{E_1^*}{\lambda}, \frac{E_2^*}{\mu}) = Q\left(\frac{k+1}{2k}\cdot\frac{\lambda E_1/E_1^*}{\mu E_2/E_2^*(\mu E_2/E_2^* - \lambda E_1/E_1^*)} + \frac{1}{\mu E_2/E_2^*} \leqslant \theta_0\right), \tag{3.1}$$

where $\overset{d}{=}$ denotes identically distributed.

(1) For any (λ,μ) satisfying $g_4(\lambda,\mu) = \frac{1}{\mu} + \frac{(k+1)\lambda}{2k\mu(\mu-\lambda)} < \theta_0$, choose $\mu' < \mu$ and $g_4(\lambda,\mu') = \frac{1}{\mu'} + \frac{(k+1)\lambda}{2k\mu'(\mu'-\lambda)} = \theta_0$. By (3.1),

$$p_{l,4}\left(\frac{E_1^*}{\lambda}, \frac{E_2^*}{\mu'}\right) \leqslant p_{l,4}\left(\frac{E_1^*}{\lambda}, \frac{E_2^*}{\mu}\right).$$

Hence,

$$P_{\lambda,\mu'}\left((p_{l,4}(T)\leqslant\gamma)\right) = Q^*\left(p_{l,4}\left(\frac{E_1^*}{\lambda}, \frac{E_2^*}{\mu'}\right)\leqslant\gamma\right) \geqslant Q^*\left(p_{l,4}\left(\frac{E_1^*}{\lambda}, \frac{E_2^*}{\mu}\right)\leqslant\gamma\right)$$
$$= P_{\lambda,\mu}(p_{l,4}(T)\leqslant\gamma).$$

For any (λ,μ) satisfying $g_4(\lambda,\mu) = \frac{1}{\mu} + \frac{(k+1)\lambda}{2k\mu(\mu-\lambda)} > \theta_0$, choose $\mu'' > \mu$ satisfies $g_4(\lambda,\mu'') = \frac{1}{\mu''} + \frac{(k+1)\lambda}{2k\mu''(\mu''-\lambda)} = \theta_0$. Analogously, we have

$$P_{\lambda,\mu''}\left((p_{l,4}(T)\leqslant\gamma)\right) \leqslant P_{\lambda,\mu}(p_{l,4}(T)\leqslant\gamma).$$

It follows that conclusion (1) holds.

(2) According to (3.1),

$$\varphi_\gamma(\lambda) = Q^*\left(Q\left(\frac{k+1}{2k}\cdot\frac{\lambda E_1/E_1^*}{\mu(\lambda)E_2/E_2^*(\mu(\lambda)E_2/E_2^* - \lambda E_1/E_1^*)} + \frac{1}{\mu(\lambda)E_2/E_2^*} \leqslant \theta_0\right)\leqslant\gamma\right), \tag{3.2}$$

where $\mu(\lambda)^2 = \mu(\lambda)\lambda + \mu(\lambda)/\theta_0 + \frac{(1-k)\lambda}{2k\theta_0}$. Since E and E^* are identically distributed. Because

$$\lim_{\lambda\to 0}\mu(\lambda) = 1/\theta_0, \ \ \lim_{\lambda\to\infty}\frac{\lambda}{\mu(\lambda)} = 1, \ \ \lim_{\lambda\to\infty}\mu(\lambda) = \infty$$

(3.2) implies

$$\lim_{\lambda\to 0}\varphi_\gamma(\lambda) = Q^*\left(Q\left(\frac{E_2}{E_2^*}\geqslant 1\right)\leqslant\gamma\right) = Q^*(1 - K_{2kn}(E_2^*)\leqslant\gamma) = \gamma,$$

$$\lim_{\lambda\to\infty}\varphi_\gamma(\lambda)=Q^*\left(Q\left(\frac{E_2}{E_2^*}\geqslant\frac{E_1}{E_1^*}\right)\leqslant\gamma\right)=Q^*\left(1-F_{2kn,2m}\left(\frac{E_2^*}{E_1^*}\right)\leqslant\gamma\right)=\gamma.$$

(3) Notice that

$$Q\left(\frac{k+1}{2k}\cdot\frac{\lambda E_1/E_1^*}{\mu(\lambda)E_2/E_2^*(\mu(\lambda)E_2/E_2^*-\lambda E_1/E_1^*)}+\frac{1}{\mu(\lambda)E_2/E_2^*}\leqslant\theta_0\right)$$
$$=Q\left(\frac{1-k}{2k\theta_0}\cdot\frac{\lambda E_1}{E_1^*}+\frac{\mu(\lambda)}{\theta_0}\cdot\frac{E_2}{E_2^*}+\frac{\mu(\lambda)\lambda E_1}{E_1^*}\frac{E_2}{E_2^*}\leqslant\left(\frac{\mu(\lambda)E_2}{E_2^*}\right)^2\right) \quad (3.3)$$

For $0<a<1$, notice that $ax+(1-a)\geqslant x^a.x>0$. Taking $a=\mu(\lambda)\lambda/(\mu(\lambda)\lambda+(1-k)\lambda/2k\theta_0)$, $x=\frac{E_2}{E_2^*}$, in the above inequation, we get

$$(3.3)\leqslant Q\left((\mu(\lambda)\lambda+(1-k)\lambda/2k\theta_0)\cdot\frac{E_1}{E_1^*}\left(\frac{E_2}{E_2^*}\right)^a+\frac{\mu(\lambda)}{\theta_0}\cdot\frac{E_2}{E_2^*}\leqslant\left(\frac{\mu(\lambda)E_2}{E_2^*}\right)^2\right) \quad (3.4)$$

Apply the inequation again, let $a=(\mu(\lambda)\lambda+(1-k)\lambda/2k\theta_0)/\mu(\lambda)^2$, $x=\frac{E_1}{E_1^*}(\frac{E_2}{E_2^*})^{\frac{(k-1)\lambda/2k\theta_0}{\mu(\lambda)\lambda+(1-k)\lambda/2k\theta_0}}$. We get

$$(3.4)\leqslant Q\left(\frac{E_2}{E_2^*}\left(\frac{E_1}{E_1^*}\left(\frac{E_2}{E_2^*}\right)^{\frac{(k-1)\lambda/2k\theta_0}{\mu(\lambda)\lambda+(1-k)\lambda/2k\theta_0}}\right)^{\frac{\mu(\lambda)\lambda+(1-k)\lambda/2k\theta_0}{\mu(\lambda)^2}}\leqslant\left(\frac{E_2}{E_2^*}\right)^2\right)=\gamma \quad (3.5)$$

Notice that

$$Q\left(\frac{1-k}{2k\theta_0}\cdot\frac{\lambda E_1}{E_1^*}+\frac{\mu(\lambda)}{\theta_0}\cdot\frac{E_2}{E_2^*}+\frac{\mu(\lambda)\lambda E_1}{E_1^*}\frac{E_2}{E_2^*}\leqslant\left(\frac{\mu(\lambda)E_2}{E_2^*}\right)^2\right)$$
$$\geqslant Q\left(\frac{E_2}{E_2^*}\geqslant\frac{E_1}{E_1^*},\frac{E_2}{E_2^*}\geqslant 1,\frac{E_2^2}{E_2^{*2}}\geqslant\frac{E_1}{E_1^*}\right). \quad (3.6)$$

Let $V=E_1+E_2$, $W=E_1/(E_1+E_2)$. We know that $V\sim\chi^2(2m+2kn)$, $W\sim\beta(2m,2kn)$, and V is independent of W. If V^* and W^* are defined as V and W, then

$$Q\left(\frac{E_2}{E_2^*}\geqslant\frac{E_1}{E_1^*},\frac{E_2}{E_2^*}\geqslant 1,\frac{E_2^2}{E_2^{*2}}\geqslant\frac{E_1}{E_1^*}\right)=Q\left(V(1-W)\geqslant V^*(1-W^*),\frac{1-W}{W}\right.$$
$$\left.\geqslant\frac{1-W^*}{W^*},\frac{V^2(1-W)^2}{W}\geqslant\frac{V^{*2}(1-W^*)^2}{W^*}\right)$$
$$\geqslant Q(V\geqslant V^*,W\leqslant W^*)=(1-K_{2m+2n}(V^*))B_{2m,2n}(W^*), \quad (3.7)$$

where $B_{a,b}(\cdot)$ denotes the cumulate distribution function of $\beta(a,b)$. Because both $1-K_{2m+2n}(V^*)$ and $B_{2m,2n}(W^*)$ are uniformly distributed on (0,1), from (3.2) and (3.3),

$$\varphi_\gamma(\lambda)\leqslant Q^*\left((1-K_{2m+2n}(V^*))B_{2m,2n}(W^*)\leqslant\gamma\right)=\gamma-\gamma ln\gamma.$$

The conclusion (3) is true. Thus the proof of the theorem is completed. □

For the p value $p_{l,5}(T)$, analogous results to $p_{l,4}(T)$ are established as follows.

Theorem 3.3. *Consider the problem for testing (2.1) of the mean waiting time θ_5. The p value is given by (2.16). For arbitrary $\gamma(0 < \gamma < 1)$, if the rejection region is taken as $\{p_{l,5}(T) \leqslant \gamma\}$, then:*

(1) the true level of the test is attained on the common boundary of H and K, namely

$$\sup_{g_5(\lambda,\mu)\leqslant\theta_0} P_{\lambda,\mu}(p_{l,5}(T) \leqslant \gamma) = \sup_{g_5(\lambda,\mu)=\theta_0} P_{\lambda,\mu}(p_{l,5}(T) \leqslant \gamma),$$

and the minimum power of the test is also obtained on the common boundary, namely

$$\inf_{g_5(\lambda,\mu)\geqslant\theta_0} P_{\lambda,\mu}(p_{l,5}(T) \leqslant \gamma) = \inf_{g_5(\lambda,\mu)=\theta_0} P_{\lambda,\mu}(p_{l,5}(T) \leqslant \gamma);$$

(2) on the common boundary,

$$\lim_{\lambda\to 0} \psi_\gamma(\lambda) = \lim_{\mu\to\infty} \psi_\gamma(\lambda) = \gamma;$$

where $\psi_\gamma(\lambda) = P_{\lambda,\mu(\lambda)}(p_{l,5}(T) \leqslant \gamma)$, and $\mu(\lambda)$ satisfies $g_5(\lambda, \mu(\lambda)) = \theta_0$;

(3) on the common boundary, the test satisfies

$$\gamma \leqslant \psi_\gamma(\lambda) \leqslant \gamma - \gamma ln\gamma, \quad \lambda > 0.$$

Proof. Take notations $E^* = (E_1^*, E_2^*)$ and Q^* are same as those in the proof of Theorem 3.2. By (2.4) and (2.16),

$$p_{l,5}(T) \overset{d}{=} p_{l,5}\left(\frac{E_1^*}{\lambda}, \frac{E_2^*}{\mu}\right) = Q\left(\frac{\mu E_2 E_1^*}{E_1 E_2^*}\left(\frac{\mu E_2}{\lambda E_2^*} - \frac{E_1}{E_1^*}\right) \geqslant \frac{2k+1}{2k\theta_0}\right). \tag{3.8}$$

(1) The proof is similar as that of Theorem 3.2(1), and therefore is omitted.

(2) Let $\theta_0' = \frac{2k}{k+1}\theta_0$ From (3.4),

$$\psi_\gamma(\lambda) = Q^*\left(Q\left(\mu(\lambda)^2 \frac{E_2^2 E_1^*}{E_1(E_2^*)^2} \geqslant \mu(\lambda)\lambda \frac{E_2}{E_2^*} + \frac{\lambda}{\theta_0'}\right) \leqslant \gamma\right), \tag{3.9}$$

where $\mu(\lambda) = \lambda/2 + \sqrt{\lambda^2/4 + \lambda/\theta_0'}$. Because

$$\lim_{\lambda\to 0} \frac{\lambda}{\mu^2(\lambda)} = \theta_0', \quad \lim_{\lambda\to\infty} \frac{\lambda}{\mu(\lambda)} = 1,$$

we have

$$\lim_{\lambda\to 0} \psi_\gamma(\lambda) = Q^*\left(Q\left(\frac{E_2^2 E_1^*}{E_1(E_2^*)^2} \geqslant 1\right) \leqslant \gamma\right) = Q^*\left(Q\left(\frac{E_2^2}{E_1} \geqslant \frac{(E_2^*)^2}{E_1^*}\right) \leqslant \gamma\right) = \gamma$$

and

$$\lim_{\lambda\to\infty} \psi_\gamma(\lambda) = Q^*\left(Q\left(\frac{E_2^2 E_1^*}{E_1(E_2^*)^2} \geqslant \frac{E_2}{E_2^*}\right) \leqslant \gamma\right) = Q^*\left(Q\left(\frac{E_2}{E_1} \geqslant \frac{E_2^*}{E_1^*}\right) \leqslant \gamma\right) = \gamma.$$

(3) For $0 < a < 1$, notice that

$$\{(x, y) : y \geqslant ax + (1 - a), x > 0\} \subset \{(x, y) : y \geqslant x^a.x > 0\}.$$

Taking $a = \mu(\lambda)\lambda/\mu(\lambda)^2 = \lambda/\mu(\lambda), x = \frac{E_2^2 E_1^*}{E_1(E_2^*)^2}, y = E_2/E_2^*$ in the above relationship of two sets. We know $0 < a < 1$ because $\mu(\lambda)^2 = \mu(\lambda)\lambda + \lambda/\theta_0'$. We get

$$\psi_\gamma(\lambda) \geqslant Q^*\left(Q\left(\frac{E_2^2 E_1^*}{E_1(E_2^*)^2} \geqslant \left(\frac{E_2}{E_2^*}\right)^a\right) \leqslant \gamma\right) = Q^*\left(Q\left(\frac{E_2^{2-a}}{E_1} \geqslant \frac{(E_2^*)^{2-a}}{E_1^*}\right) \leqslant \gamma\right) = \gamma.$$

On the other hand, as the proof of Theorem 3.2(3),

$$\begin{aligned}\psi_\gamma(\lambda) &\leqslant Q^*\left(Q\left(\frac{E_2^2E_1^*}{E_1(E_2^*)^2} \geqslant \frac{E_2}{E_2^*}, \frac{E_2^2E_1^*}{E_1(E_2^*)^2} \geqslant 1\right) \leqslant \gamma\right)\\ &\leqslant Q^*\left(Q\left(\frac{E_2}{E_2^*} \geqslant \frac{E_1}{E_1^*}, \frac{E_2}{E_2^*} \geqslant 1\right) \leqslant \gamma\right)\\ &\leqslant \gamma - \gamma ln\gamma.\end{aligned}$$

The conclusion (3) is established. So we complete the proof. □

Now, we explain the results given in Theorems 3.2 and 3.3. Although the fixed level tests specified by the p values $p_{l,4}(T)$ and $p_{l,5}(T)$ are not unbiased, Theorem 3.2(1) and Theorem 3.3(1) present the properties which an unbiased test must admit. These properties also mean that the levels of the two given tests depend on the probabilities of the rejection regions under probability distributions in the common boundaries of the null hypotheses and alternative hypotheses. When nuisance parameter exists, the common boundary of the null hypotheses and alternative hypotheses is often a set of probability distributions. We call the set *boundary set* temporarily. It is shown in Theorem 3.2(2) and Theorem 3.3(2) that on the boundaries of the *boundary sets*, the levels of the two given tests tend to nominal levels. Therefore the levels of the tests depend on the probabilities of the rejection regions under probability distributions in the middles of the boundary sets. Theorem 3.2(3) and Theorem 3.3(3) establish the same upper bound and lower bound of these probabilities. The lower bound is the nominal level and the upper bound is $\gamma - \gamma ln\gamma$. From simulations in Sec. 4, we can see that this upper bound is not sharp. However it is important to give an upper bound of the difference between true and nominal levels of a test. In existing papers about generalized p values, the investigations of frequentist properties are mostly resort to simulations. Only an exception is the investigation of generalized p values for Behrens-Fisher problem given by Tang and Tsui (2007). Noting that $\gamma - \gamma ln\gamma$ is the cumulate distribution function of the product of two independent random variables with same uniformly distribution on (0,1), perhaps the upper bound $\gamma - \gamma ln\gamma$ is the bound for generalized p values in some other hypothesis testing problems.

Simply Theorems 3.2 and 3.3 can be summarized in

$$\sup_{g_i(\lambda,\mu)\leqslant\theta_0} P_{\lambda,\mu}(p_{l,i}(T) \leqslant \gamma) = \gamma - \gamma ln\gamma, \quad \inf_{g_i(\lambda,\mu)\leqslant\theta_0} P_{\lambda,\mu}(p_{l,i}(T) \leqslant \gamma) = \gamma, \quad i = 4, 5.$$

The former equalities give an same upper bound of the true levels of the two left-sided tests, and the latter equalities indicate that the powers of the tests are not lower than the nominal level. From them the frequentist properties of other p values can be obtained. For the right-sided tests, by the relationship between $p_{r,i}(T)$ and $p_{l,i}(T)$, $i = 3, 4$, we have

$$\begin{aligned}\sup_{g_i(\lambda,\mu)\geqslant\theta_0} P_{\lambda,\mu}(p_{r,i}(T) \leqslant \gamma) &= \sup_{g_i(\lambda,\mu)\geqslant\theta_0} P_{\lambda,\mu}(p_{l,i}(T) \geqslant 1-\gamma)\\ &= 1 - \inf_{g_i(\lambda,\mu)\geqslant\theta_0} P_{\lambda,\mu}(p_{l,i}(T) < 1-\gamma)\\ &= \gamma.\end{aligned}$$

Hence, for right-sided tests, true levels equal the nominal ones. For two-sided test,

$$\begin{aligned}P_{\lambda,\mu}(p_{c,i}(T) \leqslant \gamma) &= P_{\lambda,\mu}(p_{l,i}(T) \leqslant \gamma/2, p_{l,i}(T) \geqslant 1-\gamma/2)\\ &= P_{\lambda,\mu}(p_{l,i}(T) \leqslant \gamma/2) + 1 - P_{\lambda,\mu}(p_{l,i}(T) < 1-\gamma/2),\end{aligned}$$

$i = 4, 5$. If $g_i(\lambda, \mu) = \theta_0$, then for $i = 4, 5$,

$$\gamma + (1 - \gamma/2)ln(1 - \gamma/2) \leqslant P_{\lambda,\mu}(p_{c,i}(T) \leqslant \gamma) \leqslant \gamma - (\gamma/2)ln(\gamma/2).$$

The bounds of the levels of two-sided tests are obtained.

4. Simulation

Because the p values for testing hypotheses about former three parameters are just the classical F tests, the simulations are performed only for later two parameters θ_4 and θ_5. For p values given by (2.15) and (2.16), we provide another convenient computing approach by simulation. The detailed procedure is given in the following algorithm.

Algorithm 1.

(1) For given data $(x_1, \ldots, x_m; y_1, \ldots, y_n)$, choose a large simulation sample size, say $M = 10,000$.
(2) For $j = 1, \ldots, M$, produce an e_{1j} from the distribution χ^2_{2m} and an e_{2j} from the distribution χ^2_{2kn}.
(3) Compute $\widehat{\theta_{ij}} = g_i(\widehat{\lambda}, \widehat{\mu}) = g_i(\frac{e_{1j}}{t_1}, \frac{e_{2j}}{t_2})$, $i = 4$ or $5, t_1 = 2m\bar{x} = 2\sum_{i=1}^m x_i, t_2 = 2k\bar{y} = 2\sum_{i=1}^n y_i$.
(4) End j loop.
(5) The simulated p values are computed by the frequency of $\{\widehat{\theta_{ij}} \leqslant \theta_0, j = 1, \ldots, M\}$ for testing $H : \theta_i \leqslant \theta_0$ and $p_{r,i} = (1 - p_{l,i})$ for testing $H : \theta_i \geqslant \theta_0, i = 4, 5$. For testing $H : \theta_i = \theta_0$ the simulated p values are $p_{c,i} = 2\min\{p_{l,i}, p_{r,i}\}, i = 4, 5$.

We evaluate the performances of the proposed tests by their empirical Type I error rates (TIRs). The Monte Carlo method is used to obtain the empirical TIRs. By Theorems 3.2 and 3.3, the fixed level tests determined by p values for testing θ_4 and θ_5 were studied when θ_4 and θ_5 are in the boundary sets. The TIRs of the tests for different values of (λ, μ) are compared with different sample size (m, n).

The Monte Carlo method applied to obtain TIRs is as follows. Generating Monte Carlo sample with size N and computing statistic T, we have $t_1, t_2, \cdots, t_N$. For a given p value $p(t)$, define I_j as the indicator of rejection event $\{p(t_j) \leqslant \gamma\}$ for $j = 1, 2, \ldots, N$, where γ is a preassigned significant level. The p value $p(t_j)$ is computed by Algorithm 1, where the simulation sample size of (e_1, e_2) is M. Then $\widehat{\gamma} = \sum_{j=1}^N I_j/N$ is a simulated value of the predetermined size $\gamma = 0.01, 0.05, 0.1$. We calculate $\widehat{\gamma}$ for varying λ by repeating this process.

The simulation example is conducted as follows. We consider testing hypotheses (2.1), (2.2), and (2.3) concerning θ_4 and θ_5, where $\theta_0 = 2$ for testing θ_4 and $\theta_0 = 1.5$ for testing θ_5, respectively. The Monte Carlo sample size $N = 1000$ and the simulation sample size $M = 10{,}000$ in computing p value. To ensure (λ, μ) is in the boundary set, μ should satisfy $\mu = \frac{\lambda + 1/\theta_0 + \sqrt{(\lambda+1/\theta_0)^2 - 2\lambda(k-1)/(k\theta_0)}}{2}$ for testing hypotheses about θ_4, and $\mu = \frac{\lambda + \sqrt{\lambda^2 + 2(k+1)\cdot\lambda/(k\theta_0)}}{2}$ about θ_5, respectively. Table 1 contains the results for $m = 5, n = 10$ and $\lambda = 0.01, 0.25, 1, 100$. The first three parts are about θ_4 and the later three parts are about θ_5, respectively. The empirical TIRs $\widehat{\gamma}_{l,i}, \widehat{\gamma}_{r,i}$, and $\widehat{\gamma}_{c,i}$ are about testing hypotheses (2.1), (2.2), and (2.3) concerning θ_i respectively, $i = 4, 5$. The last row shows the values of the upper bound $\gamma - \gamma\ln\gamma$ for $\widehat{\gamma}_{l,i}$ and $\widehat{\gamma}_{r,i}$, and $\gamma - (\gamma/2)\ln(\gamma/2)$ for $\widehat{\gamma}_{c,i}$. From the simulation results in Table 1, we can see that the proposed tests perform well.

Table 1. Empirical TIRs for testing W and W_q for $M/E_4/1$.

I						
		$\lambda = 0.01$	$\lambda = 0.25$	$\lambda = 1$	$\lambda = 100$	$\gamma - \gamma \ln \gamma$
$\widehat{\gamma}_{l,4}$	$\gamma = 0.01$	0.009	0.011	0.011	0.009	0.056
	$\gamma = 0.05$	0.056	0.081	0.042	0.051	0.200
	$\gamma = 0.1$	0.103	0.117	0.113	0.107	0.330
II						
		$\lambda = 0.01$	$\lambda = 0.25$	$\lambda = 1$	$\lambda = 100$	$\gamma - \gamma \ln \gamma$
$\widehat{\gamma}_{r,4}$	$\gamma = 0.01$	0.018	0.006	0.010	0.008	0.056
	$\gamma = 0.05$	0.045	0.032	0.047	0.047	0.200
	$\gamma = 0.1$	0.097	0.076	0.087	0.103	0.330
III						
		$\lambda = 0.01$	$\lambda = 0.25$	$\lambda = 1$	$\lambda = 100$	$\gamma - (\gamma/2) \ln(\gamma/2)$
$\widehat{\gamma}_{c,4}$	$\gamma = 0.01$	0.017	0.007	0.010	0.006	0.036
	$\gamma = 0.05$	0.045	0.060	0.037	0.049	0.117
	$\gamma = 0.1$	0.100	0.104	0.102	0.098	0.250
IV						
		$\lambda = 0.01$	$\lambda = 0.25$	$\lambda = 1$	$\lambda = 100$	$\gamma - \gamma \ln \gamma$
$\widehat{\gamma}_{l,5}$	$\gamma = 0.01$	0.011	0.008	0.011	0.010	0.056
	$\gamma = 0.05$	0.058	0.064	0.044	0.049	0.200
	$\gamma = 0.1$	0.116	0.096	0.094	0.095	0.330
V						
		$\lambda = 0.01$	$\lambda = 0.25$	$\lambda = 1$	$\lambda = 100$	$\gamma - \gamma \ln \gamma$
$\widehat{\gamma}_{r,5}$	$\gamma = 0.01$	0.011	0.010	0.010	0.006	0.056
	$\gamma = 0.05$	0.046	0.01	0.047	0.045	0.200
	$\gamma = 0.1$	0.088	0.097	0.093	0.093	0.330
VI						
		$\lambda = 0.01$	$\lambda = 0.25$	$\lambda = 1$	$\lambda = 100$	$\gamma - (\gamma/2) \ln(\gamma/2)$
$\widehat{\gamma}_{c,5}$	$\gamma = 0.01$	0.011	0.010	0.010	0.003	0.036
	$\gamma = 0.05$	0.052	0.046	0.043	0.048	0.117
	$\gamma = 0.1$	0.097	0.094	0.086	0.112	0.250

Funding

This work was supported by NNSF of China (10771015) and National Key Scientific and Technological Project "Research on Global Petroleum Resources Data System (2011ZX05028-004)".

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