Stochastic Analysis and Applications

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/lsaa20

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STOCHASTIC ANALYSIS AND APPLICATIONS
2021, VOL. 39, NO. 2, 235–251
https://doi.org/10.1080/07362994.2020.1794893

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REVIEW

Mean-field FBSDE and optimal control

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ABSTRACT
We study optimal control for mean-field forward–backward stochastic differential equations with payoff functionals of mean-field type. Sufficient and necessary optimality conditions in terms of a stochastic maximum principle are derived. As an illustration, we solve an optimal portfolio with mean-field risk minimization problem.

ARTICLE HISTORY
Received 21 May 2019
Accepted 8 July 2020

2010 MATHEMATICS SUBJECT CLASSIFICATION
60H05; 60H20; 60J75; 93E20; 91G80; 91B70

KEYWORDS
Mean-field forward–backward SDE; risk minimization; stochastic maximum principle

1. Introduction

After the seminal work by Lasry and Lions [1], where they introduced mean-field game theory that is devoted to the analysis of differential games with infinitely many players. Mean-field games attracted a lot of attention and forward/backward stochastic differential equations of mean-field type are used, extensively, as dynamics (see e.g., Huang et al. [2], Xu and Zhang [3] and Xu and Shi [4]). In Huang [5], the author studies a linear–quadratic game with a major player and a large number of minor players. The dynamics of the major player is influenced by an aggregation of all minor players (mean-field coupling) whereas the minor players' dynamics depend on the control of the major player in addition to their individual controls as well as the mean-field coupling, i.e., a system of partially control-coupled forward stochastic differential equations (SDEs). This work (Ref. [5]) was generalized to the non-linear case in Nourian and Caines [6]. In all previously mentioned works, the authors find ϵ-Nash equilibrium for mean-field games, where each player play a game with the aggregation of the other players (the mass). In the present paper, the setting is different. We consider a mean-field type control problem where the goal is to find an optimal control via stochastic maximum principle. The mass or the laws of state processes are not freezed, they vary with the change of the control. Thus, finding an optimal control will yield optimal laws.

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Furthermore, in our control problem we consider a controlled partially coupled forward–backward SDE of mean-field type (MF-FBSDE) as dynamics, which is a novel contribution. We have also used the Sobolov space of random measures, introduced in Agram et al. [7–9], in which, the Fréchet derivative with respect to the measure can be taken directly. This is a new approach compared to what is standard in the literature, where the Wasserstein metric space for measures and the lifting technique, introduced by Lions [10], is used to differentiate a function of a measure.

Existence of a fully coupled MF-FBSDE is studied by Carmona and Delarue [11] under Lipschitz assumption on the coefficients but no uniqueness result was proven. Bensoussan et al. [12] prove existence and uniqueness of a fully coupled MF-FBSDE by assuming Lipschitz and monotonicity conditions. Recently, Djehiche and Hamadene [13] prove the same results but under weak monotonicity assumptions and without the non-degeneracy condition on the forward equation.

The purpose of our work is to derive necessary and sufficient optimality conditions in terms of a stochastic maximum principle for a set $\hat{u}$ of admissible controls which maximize a cost functional of the form

$$\begin{aligned} J(u) = E[&h(X(T), M(T)) + \phi(Y(0), N(0)) \\ &+ \int_0^T f(t, X(t), Y(t), Z(t), M(t), N(t), u(t))dt], \end{aligned}$$

with respect to admissible controls u, for some functions $f, h, \phi,$ under dynamics governed by MF-FBSDEs. More specifically, we consider the coupled system

$$\begin{cases} dX(t) &= b(t, X(t), M(t), u(t))dt + \sigma(t, X(t), M(t), u(t))dB(t), t \in [0, T], \\ X(0) &= x_0, \end{cases}$$
$$\begin{cases} dY(t) &= -g(t, X(t), Y(t), Z(t)), M(t), N(t), u(t))dt + Z(t)dB(t), t \in [0, T], \\ Y(T) &= \psi(X(T)), \end{cases}$$

for some functions b, σ and a Brownian motion $B(t)$. $M(t)$ and $N(t)$ denote the marginal laws of X and Y, respectively. As an application, we will consider a risk minimization control problem. More precisely, we want to minimize the risk given by $Y(0) = -\mathcal{E}[\varphi(X(T))]$ such that $\mathcal{E}[\varphi(X(T))]$ is the convex risk measure by means of backward stochastic differential equations of mean-field type (MF-BSDEs). Let us recall what we mean by the convex risk measure:

Definition 1.1. *A convex risk measure is a map $\mathcal{E} : L^p(\mathcal{F}_T) \to \mathbb{R},\ p \in [2, \infty]$ that satisfies the following properties:*

- *(Convexity) $\mathcal{E}(\lambda\varphi_1 + (1-\lambda)\varphi_2) \le \lambda\mathcal{E}(\varphi_1) + (1-\lambda)\mathcal{E}(\varphi_2)$ for all $\lambda \in [0, 1]$ and all $\varphi_1, \varphi_2 \in L^p(\mathcal{F}_T)$.*
- *(Monotonicity) If $\varphi_1 \le \varphi_2$, then $\mathcal{E}(\varphi_1) \ge \mathcal{E}(\varphi_2)$.*
- *(Translation invariance) $\mathcal{E}(\varphi + a) = \mathcal{E}(\varphi) - a$ for all $\varphi \in L^p(\mathcal{F}_T)$ and all constants a.*
- *$\mathcal{E}(0) = 0$.*

The construction of risk measures from solutions of BSDEs is given as follows: Assume that $M(t) := E[Y(t)]$ in the driver $g(t, y, m, z, n)$ of the above MF-BSDE and that $z \mapsto g(t, y, E[y], z)$ is convex for all t. Then

$$\mathcal{E}[\varphi(X(T))] = -Y(0)$$

defines a convex risk measure. This shows how crucial is the choice of the functional g. Through this connection, the problem of risk minimization is equivalent to stochastic optimal control of MF-FBSDEs, as shown in Øksendal and Sulem [14], for the non-mean-field case. The rest of the paper is organized as follows. In Section 2, we give some mathematical background. In Section 3, we study a stochastic optimal control of MF-FBSDE where sufficient and necessary optimality conditions are derived. In the last section, we construct a dynamic risk measure by means of MF-BSDE and then we solve an associated risk minimization problem.

2. Generalities

Let $B = B(t), t \in [0, T]$ be a one-dimensional Brownian motion defined in a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$. The filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ is assumed to be the P-augmented filtration generated by B.

Definition 2.1. Let $\mathbb{Z}$ be the set of integers.

- Let $\mathcal{M}^k$ be the space of random measures μ on $\mathbb{R}$ equipped with the norm

$$||\mu||^2_{\mathcal{L}^k} \quad := \quad E\left[\int_{\mathbb{R}}|\hat{\mu}(y)|^2(1+|y|)^k e^{-y^2}dy\right], k \in \mathbb{Z}, \tag{2.1}$$

  where $\hat{\mu}$ is the Fourier transform of the measure μ, i.e.,

$$\hat{\mu}(y) \quad := \quad \int_{\mathbb{R}} e^{ixy} d\mu(x); \quad y \in \mathbb{R}.$$

  We endow $\mathcal{M}^k$ with the inner product $\langle \mu, \eta \rangle := \int_{\mathbb{R}}|\hat{\mu}(y) - \hat{\eta}(y)|^2(1+|y|)^k e^{-y^2}$ $\mu, \eta, y \in \mathbb{R}$, $\hat{\mu}$ and $\hat{\eta}$ are the Fourier transform of the measures μ and η, respectively. Then $(\mathcal{M}^k, ||\cdot||_{\mathcal{M}^k})$ is a pre-Hilbert space, for each k. Let $\mathcal{M}$ be the union (inductive limit) of $\mathcal{M}^k, k \in \mathbb{Z}$.
- We denote by $\mathcal{M}_0$ the set of all deterministic elements of $\mathcal{M}$.

We give some examples:

Example 2.2 (Measures). Let us give some examples of measures in $\mathcal{M}_0^0$ and $\mathcal{M}^0$:

(1) Suppose that $\mu = \delta_{x_0}$, the unit point mass at $x_0 \in \mathbb{R}$. Then $\delta_{x_0} \in \mathcal{M}_0^0$ and

$$\hat{\mu}(y) \quad = \int_{\mathbb{R}} e^{ixy} d\mu(x) \quad = e^{ix_0 y},$$

and hence

$$||\mu||^2_{\mathcal{M}_0^0} \quad = \int_{\mathbb{R}} |e^{ix_0 y}|^2 e^{-y^2} dy \quad < \infty.$$

(2) Suppose $d\mu(x) = f(x)dx$, where $f \in L^1(\mathbb{R})$. Then $\mu \in \mathcal{M}_0^0$ and by Riemann–Lebesgue lemma, $\hat{\mu}(y) \in C_0(\mathbb{R})$, i.e., $\hat{\mu}$ is continuous and $\hat{\mu}(y) \to 0$ when $|y| \to \infty$. In particular, $|\hat{\mu}|$ is bounded on $\mathbb{R}$ and hence

$$||\mu||^2_{\mathcal{M}_0^0} = \int_{\mathbb{R}} |\hat{\mu}(y)|^2 e^{-y^2} dy < \infty.$$

(3) Suppose that μ is any finite positive measure on $\mathbb{R}$. Then $\mu \in \mathcal{M}_0^0$ and

$$|\hat{\mu}(y)| \leq \int_{\mathbb{R}} d\mu(y) = \mu(\mathbb{R}) < \infty, \text{ for all } y,$$

and hence

$$||\mu||^2_{\mathcal{M}_0^0} = \int_{\mathbb{R}} |\hat{\mu}(y)|^2 e^{-y^2} dy < \infty.$$

(4) Next, suppose $x_0 = x_0(\omega)$ is random. Then $\delta_{x_0(\omega)}$ is a random measure in $\mathcal{M}^0$. Similarly, if $f(x) = f(x, \omega)$ is random, then $d\mu(x, \omega) = f(x, \omega)dx$ is a random measure in $\mathcal{M}^0$.

We denote by U a nonempty convex subset of $\mathbb{R}$ and we denote by $\mathcal{U}_{\mathbb{G}}$ the set of U-valued $\mathbb{G}$-progressively measurable processes where $\mathbb{G} := \{\mathcal{G}_t\}_{t \geq 0}$ with $\mathcal{G}_t \subseteq \mathcal{F}_t$ for all $t \geq 0$; we consider them as the admissible control processes.

We will also use the following spaces:

- $\mathcal{S}^2$ is the set of $\mathbb{R}$-valued $\mathbb{F}$-adapted càdlàg processes $X = X(t), t \in [0, T]$, such that
$$||X||^2_{\mathcal{S}^2} := E\left[\sup_{t \in [0, T]} |X(t)|^2\right] < \infty,$$
- $\mathbb{L}^2$ is the set of $\mathbb{R}$-valued $\mathbb{F}$-adapted processes $Q = Q(t), t \in [0, T]$, such that
$$||Q||^2_{\mathbb{L}^2} := E\left[\int_0^T |Q(t)|^2 dt\right] < \infty.$$
- $\mathcal{K}$ denotes the set of absolutely continuous functions $m : [0, T] \to \mathcal{M}_0$.
- $\mathbb{K}$ is the set of bounded linear functionals $K : \mathcal{M}_0 \to \mathbb{R}$ equipped with the operator norm
$$||K||_{\mathbb{K}} := \sup_{m \in \mathcal{M}_0, ||m||_{\mathcal{M}_0} \leq 1} |K(m)|.$$
- $\mathcal{S}^2_{\mathbb{K}}$ is the set of $\mathbb{F}$-adapted stochastic processes $p : [0, T] \times \Omega \mapsto \mathbb{K}$, such that
$$||p||^2_{\mathcal{S}^2_{\mathbb{K}}} := E\left[\sup_{t \in [0, T]} ||p(t)||^2_{\mathbb{K}}\right] < \infty.$$
- $\mathbb{L}^2_{\mathbb{K}}$ is the set of $\mathbb{F}$-adapted stochastic processes $q : [0, T] \times \Omega \mapsto \mathbb{K}$, such that
$$||q||^2_{\mathbb{L}^2_{\mathbb{K}}} := E\left[\int_0^T ||q(t)||^2_{\mathbb{K}} dt\right] < \infty.$$

We recall now the notion of differentiability which will be used in the sequel. Let $\mathcal{X}, \mathcal{Y}$ be two Banach spaces with norms $||\cdot||_{\mathcal{X}}, ||\cdot||_{\mathcal{Y}}$, respectively, and let $F : \mathcal{X} \to \mathcal{Y}$.

- We say that F has a directional derivative (or Gateaux derivative) at $v \in \mathcal{X}$ in the direction if

$$D_w F(v) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(F(v + \varepsilon w) - F(v))$$

exists in $\mathcal{Y}$.

- We say that F is Fréchet differentiable at $v \in \mathcal{X}$ if there exists a continuous linear map $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\lim_{\substack{h \to 0 \\ h \in \mathcal{X}}} \frac{1}{||h||_{\mathcal{X}}} ||F(v + h) - F(v) - A(h)||_{\mathcal{Y}} = 0,$$

  where $A(h) = \langle A, h \rangle$ is the action of the liner operator A on h. In this case we call A the gradient (or Fréchet derivative) of F at v and we write

$$A = \nabla_v F.$$

- If F is Fréchet differentiable at v with Fréchet derivative $\nabla_v F$, then F has a directional derivative in all directions $w \in \mathcal{X}$ and

$$D_w F(v) = \nabla_v F(w) = \langle \nabla_v F, w \rangle.$$

In particular, note that if F is a linear operator, then $\nabla_v F = F$ for all v.

3. Optimal control problem

Here we denote by $M(t) := \mathcal{L}(X(t))$ the law of $X(t)$ at time t and by $N(t) := \mathcal{L}(Y(t))$ the law of $Y(t)$ at time t. We assume that our system is governed by a coupled system of MF-FBSDE as follows:

The MF-SDE $X^u(t) = X(t)$ is given by

$$\begin{cases} dX(t) &= b(t, X(t), M(t), u(t))dt + \sigma(t, X(t), M(t), u(t))dB(t), t \in [0, T], \\ X(0) &= x_0, \end{cases} \tag{3.1}$$

for functions $\sigma, b : \Omega \times [0, T] \times \mathbb{R} \times \mathcal{M}_0 \times U \to \mathbb{R}$ which are supposed to be $\mathcal{F}_t$-measurable and the initial value $x_0 \in \mathbb{R}$.

The couple MF-BSDE $(Y^u(t), Z^u(t)) = (Y(t), Z(t))$ satisfies

$$\begin{cases} dY(t) &= -g(t, X(t), Y(t), Z(t)), M(t), N(t), u(t))dt + Z(t)dB(t), t \in [0, T], \\ Y(T) &= \psi(X(T)), \end{cases} \tag{3.2}$$

where $g : \Omega \times [0, T] \times \mathbb{R}^3 \times \mathcal{M}_0^2 \times U \to \mathbb{R}$ is $\mathbb{F}$-adapted and $\psi : \Omega \times \mathbb{R} \times \mathcal{M}_0 \to \mathbb{R}$ is $\mathcal{F}_T$-measurable.

It follows from the definition of the norm (2.1) that

$$||\mathcal{L}(X^{(1)}) - \mathcal{L}(X^{(2)})||^2_{\mathcal{M}_0} \quad \leq \quad E\big[(X^{(1)} - X^{(2)})^2\big],$$

where $X^{(1)}$ and $X^{(2)}$ are random variables that follow the distributions $\mathcal{L}(X^{(1)})$ and $\mathcal{L}(X^{(2)})$, respectively.

Assume that (C is a constant that may change from line to line)

(A1) there exists $C > 0$, such that

- for all $t \in [0, T]$, for all fixed $u \in U$, $x, x' \in \mathbb{R}, m, m' \in \mathcal{M}_0$

$$|\sigma(t,x,m,u) - \sigma(t,x',m',u)| + |b(t,x,m,u) - b(t,x',m',u)| \\ \leq C\big(|x-x'| + ||m-m'||_{\mathcal{M}_0}\big).$$

- for all $t \in [0, T]$, for all fixed $u \in U$,

$$|\sigma(t,0,\delta_0,u)| + |b(t,0,\delta_0,u)| \leq C,$$

where δ_0 is the distribution law of zero, i.e., the Dirac measure with mass at zero.

(A2) there exists $C > 0$, such that, for all fixed $u \in U$ and all knowing $X(t) \in \mathcal{S}^2$ of Equation (3.1) and $M(t) := \mathcal{L}(X(t)) \in \mathcal{M}_0$, we have

- for all $t \in [0, T]$, $y, y', z, z' \in \mathbb{R}, n, n' \in \mathcal{M}_0$

$$\big|g(t,x,y,z,m,n,u) - g\big(t,x,y',z',m,n',u\big)\big| \\ \leq C\big(|y-y'| + |z-z'| + ||n-n'||_{\mathcal{M}_0}\big).$$

- for all $t \in [0, T]$,

$$|g(t,x,0,0,m,\delta_0,u)| \leq C.$$

Proposition 3.1. *Under Assumptions (A1) and (A2), the MF-FBSDE (3.1)–(3.2) admits a unique solution* $(X, Y, Z) \in \mathcal{S}^2 \times \mathcal{S}^2 \times \mathbb{L}^2$.

Since the system is partially coupled i.e., the forward equation does not depend on the solution of the backward one, we can solve the system separately as follows: we first find a solution $X(t)$ of the MF-SDE (3.1) and then we plug it into the backward Equation (3.2), then we solve it.

Our aim is to maximize the performance functional of the form

$$J(u) \quad = E\Big[h(X(T), M(T)) + \phi(Y(0), N(0)) + \int_0^T f(t, X(t), Y(t), Z(t), M(t), N(t), u(t))dt\Big],$$

over all admissible controls, for functions $f : \Omega \times [0, T] \times \mathbb{R}^3 \times \mathcal{M}_0^2 \times U \to \mathbb{R}, h : \Omega \times \mathbb{R} \times \mathcal{M}_0 \to \mathbb{R}$ and $\phi : \Omega \times \mathbb{R} \times \mathcal{M}_0 \to \mathbb{R}$.

Now, we can define the Hamiltonian

$$H : \Omega \times [0, T] \times \mathbb{R}^3 \times \mathcal{M}_0^2 \times U \times \mathbb{R}^2 \times \mathcal{K} \times \mathbb{R} \times \mathcal{K} \to \mathbb{R}$$

by

$$\begin{aligned} H(t,x,y,z,m,n,u,p^0,q^0,p^1,\lambda^0,\lambda^1) = &f(t,x,y,z,n,u) + p^0 b(t,x,m,u) \\ &+ q^0\sigma(t,x,m,u) + \lambda^0 g(t,x,y,z,m,n,u) \\ &+ \langle p^1, m'\rangle - \langle \lambda^1, n'\rangle. \end{aligned} \tag{3.3}$$

Remark 3.2. *For ease of notation we drop the dependence of all variables except for the time* $t, \forall \Phi \in \{\sigma, f, H, h, g, \phi\}$, *we write* $\Phi(t), \forall t$. *Moreover, we will use*

$$\hat{\Phi}(t) := \Phi(t, \hat{X}(t), \hat{Y}(t), \hat{Z}(t), \hat{M}(t), \hat{N}(t), \hat{u}(t))$$
$$\check{\Phi}(t) := \Phi(t, \hat{X}(t), \hat{Y}(t), \hat{Z}(t), \hat{M}(t), \hat{N}(t), u(t))].$$

We assume that

(A3) $\{\sigma, f, H, h, g, \phi, \psi\}$ are continuously differentiable with bounded partial derivatives w.r.t all the variables.

For $u \in \mathcal{U}$ with corresponding solution $X^u = X$, define, whenever solutions exist, $p\hat{u} = p = (p^0, p^1)$ and $q\hat{u} = q = (q^0, q^1)$ and $\lambda\hat{u} = \lambda = (\lambda^0, \lambda^1)$ by the adjoint equations:

The BSDE for the unknown processes $(p^0, q^0) \in \mathcal{S}^2 \times \mathbb{L}^2$

$$\begin{cases} dp^0(t) &= -\partial_x H(t)dt + q^0(t)dB(t), t \in [0,T], \\ p^0(T) &= \partial_x h(T) + \lambda^0(T)\partial_x \psi(T). \end{cases} \tag{3.4}$$

The MF-BSDE for the unknown processes $(p^1, q^1) \in \mathcal{S}^2_{\mathbb{K}} \times \mathbb{L}^2_{\mathbb{K}}$

$$\begin{cases} dp^1(t) &= -\nabla_m H(t)dt + q^1(t)dB(t), t \in [0,T], \\ p^1(T) &= \nabla_m h(T). \end{cases} \tag{3.5}$$

The forward SDE $\lambda^0 \in \mathcal{S}^2$

$$\begin{cases} d\lambda^0(t) &= \partial_y H(t)dt + \partial_z H(t)dB(t), t \in [0,T], \\ \lambda^0(0) &= \partial_y \phi(0), \end{cases} \tag{3.6}$$

and $\lambda^1 \in \mathcal{S}^2_{\mathbb{K}}$

$$\begin{cases} d\lambda^1(t) &= \nabla_n H(t)dt, t \in [0,T], \\ \lambda^1(0) &= \nabla_n \phi(0). \end{cases} \tag{3.7}$$

Remark 3.3. The real-valued linear system of FBSDE (3.4) and (3.6) have a unique solution by Proposition 3.1 since the coefficients satisfy condition (A3). However, Equation (3.5) is equivalent to the degenerate BSDE

$$p^1(t) = \nabla_m h(T) + \int_t^T \{\nabla_m f(s) + p^0(s)\nabla_m b(s) + q^0(s)\nabla_m \sigma(s)\}ds \\ - \int_t^T q^1(t)dB(t).$$

We take conditional expectation to obtain

$$p^1(t) = E\left[\nabla_m h(T) + \int_t^T \{\nabla_m f(s) + p^0(s)\nabla_m b(s) + q^0(s)\nabla_m \sigma(s)\}ds | \mathcal{F}_t\right].$$

Similarly, a solution for (3.7) is given by

$$\lambda^1(t) = \nabla_n \phi(0) + \int_0^t \{\nabla_n f(s) + \lambda^0(s)\nabla_n g(s)\}ds.$$

Before stating and proving sufficient and necessary conditions of optimality, we need the following result, which is Lemma 2.3 in Agram and Øksendal [7].

Lemma 3.4. *Suppose that X(t) is an Itô process of the form*

$$\begin{cases} dX(t) = \theta(t)dt + \gamma(t)dB(t), \quad t \in [0,T], \\ X(0) = x_0 \in \mathbb{R}, \end{cases}$$

where θ, γ are adapted processes.

Then the map $M(t) : [0,T] \to \mathcal{M}_0$ is absolutely continuous.

It follows that $t \mapsto M(t)$ is differentiable for t-*a.e.* We will in the following use the notation

$$M'(t) = \frac{d}{dt} M(t).$$

In fact, it is proven in [7] that if $M(t) \in \mathcal{M}^k$ then $M'(t) \in \mathcal{M}^{k-4}; k \in \mathbb{Z}$.

3.1. Sufficient optimality conditions

We state and prove a type of a verification theorem.

Theorem 3.5. *Suppose that $\hat{u} \in \mathcal{U}_{\mathbb{G}}$ with corresponding solutions $\hat{X}(t), (\hat{Y}(t), \hat{Z}(t))$, $(p^0(t), q^0(t)), (p^1(t), q^1(t)), \lambda^0(t), \lambda^1(t)$ to* Equations (3.1), (3.2), *(3.4), (3.5), (3.6) and (3.7), respectively. Suppose that*

- $x, m \mapsto h(k, m)$,
- $y, n \mapsto \phi(y, n), x \mapsto \psi(x)$,
- $x, y, z, m, n, u \mapsto H(\cdot, x, y, z, m, n, u)$,

are concave functions P-a.s for each $t \in [0,T]$. Moreover,

$$E\big[\hat{H}(t)|\mathcal{G}_t\big] = \max_{u \in U} E\big[\check{H}(t)|\mathcal{G}_t\big],$$

P-a.s for all $t \in [0,T]$. Then $\hat{u}$ is an optimal control.

Proof. We show that $J(u) - J(\hat{u}) \leq 0$, for an arbitrary u and a fixed optimal $\hat{u} \in \mathcal{U}_{\mathbb{G}}$.

We introduce first the following notation $\forall \Phi \in \{\sigma, f, H, h, g, \phi, M, N, M', N'\}$ and $\forall t$,

$$\delta\Phi(t) = \check{\Phi}(t) - \hat{\Phi}(t),$$

and

$$\delta M'(t) = \delta\left(\frac{d}{dt} M(t)\right) = \frac{d}{dt}(\delta M(t)\Big).$$

From the definition of the Hamiltonian (3.3), we have

$$\begin{aligned} \delta f(t) = \delta H(t) - \delta b(t) p^0(t) - \delta\sigma(t) q^0(t) \\ - \langle p^1(t), \delta M'(t)\rangle - \langle \lambda^1(t), \delta N'(t)\rangle, \end{aligned}$$

and

$$\begin{aligned} J(u) - J(\hat{u}) = E\bigg[\int_0^T \{\delta H(t) - \delta b(t) p^0(t) - \delta\sigma(t) q^0(t) - \langle p^1(t), \delta M'(t)\rangle \\ - \langle \lambda^1(t), \delta N'(t)\rangle\} dt + \delta h(T) + \delta\Phi(0)\bigg]. \end{aligned} \tag{3.8}$$

We use the concavity of h and ϕ as well as the boundary values of Equations (3.4), (3.5), (3.6) and (3.7)

$$\begin{aligned}\delta h(T)+\delta\phi(0) &\leq \partial_x h(T)\delta X(T)+\langle\nabla_m h(T),\delta M(T)\rangle\\ &\quad+\partial_x\phi(0)\delta Y(0)+\langle\nabla_n\phi(0),\delta N(0)\rangle\\ &= p^0(T)\delta X(T)-\lambda^0(T)\delta X(T)+\langle p^1(T),\delta M(T)\rangle\\ &\quad+\lambda^0(0)\delta Y(0)+\langle\lambda^1(0),\delta N(0)\rangle.\end{aligned} \tag{3.9}$$

Applying Itô formula to $p^0(t)\delta X(t), \langle p^1(t),\delta M(t)\rangle, \lambda^0(t)\delta Y(t)$ and $\langle\lambda^1(t),\delta N(t)\rangle$, yields the following duality relations:

$$\begin{aligned}E\big[p^0(T)\delta X(T)\big]-E\big[\lambda^0(T)\partial_x\psi(T)\big] &= E\left[\int_0^T p^0(t)\delta b(t)dt-\int_0^T \delta X(t)\partial_x H(t)dt\right]\\ &\quad+E\left[\int_0^T q^0(t)\delta\sigma(t)dt\right]-E\big[\lambda^0(T)\partial_x\psi(T)\big],\end{aligned} \tag{3.10}$$

$$E\big[\langle p^1(T),\delta M(T)\rangle\big]=E\left[\int_0^T\langle p^1(t),\delta M'(t)\rangle dt-\int_0^T\langle\nabla_m\hat{H}(t),\delta M(t)\rangle dt\right], \tag{3.11}$$

$$\begin{aligned}E\big[\lambda^0(T)\delta Y(T)\big]-E\big[\lambda^0(0)\delta Y(0)\big] &= -E\left[\int_0^T\lambda^0(t)\delta g(t)dt\right]+E\left[\int_0^T\delta Y(t)\partial_y\hat{H}(t)dt\right]\\ &\quad+E\left[\int_0^T\delta Z(t)\partial_z\hat{H}(t)dt\right].\end{aligned} \tag{3.12}$$

Concavity of ψ gives

$$\begin{aligned}E\big[\lambda^0(0)\delta Y(0)\big] &= E\big[\lambda^0(T)\delta\psi(T)\big]+E\left[\int_0^T\lambda^0(t)\delta g(t)dt\right]-E\left[\int_0^T\delta Y(t)\partial_y\hat{H}(t)dt\right]\\ &\quad-E\left[\int_0^T\delta Z(t)\partial_z\hat{H}(t)dt\right]\\ &\leq E\big[\lambda^0(T)\partial_x\psi(T)\delta X(T)\big]+E\left[\int_0^T\lambda^0(t)\delta g(t)dt\right]\\ &\quad-E\left[\int_0^T\delta Y(t)\partial_y\hat{H}(t)dt\right]-E\left[\int_0^T\delta Z(t)\partial_z\hat{H}(t)dt\right],\\ E\big[\lambda^1(T)\delta N(T)\big]-E\big[\langle\lambda^1(0),\delta N(0)\rangle\big] &= E\left[\int_0^T\langle\lambda^1(t),\delta N'(t)\rangle+\langle\nabla_n\hat{H}(t),\delta N(t)\rangle dt\right].\end{aligned} \tag{3.13}$$

By the concavity of H, we obtain

$$\begin{aligned}\delta H(t) &\leq \partial_x\hat{H}(t)\delta X(t)+\partial_y\hat{H}(t)\delta Y(t)+\partial_z\hat{H}(t)\delta Z(t)\\ &\quad+\langle\nabla_m\hat{H}(t),\delta M(t)\rangle+\langle\nabla_n\hat{H}(t),\delta N(t)\rangle+\partial_u\hat{H}(t)\delta u(t).\end{aligned} \tag{3.14}$$

Finally, by substituting the derived duality relations (3.10), (3.11), (3.12) and (3.13) in (3.8) and using the estimates (3.9), (3.14), we obtain

$$J(u) - J(\hat{u}) \leq E\left[\int_0^T \partial_u \hat{H}(t)\delta u(t)\right].$$

Using the tower property and the fact that $u(t)$ is $\mathbb{G}$-adapted the desired result follows

$$J(u) - J(\hat{u}) \leq E\left[\int_0^T E\left[\partial_u \hat{H}(t)|\mathcal{G}_t\right]\delta u(t)dt\right] \leq 0,$$

and thus, $\hat{u}$ is optimal. □

3.2. Necessary optimality conditions

Given an arbitrary but fixed control $u \in \mathcal{U}_{\mathbb{G}}$, we define

$$u^\rho := \hat{u} + \rho u, \rho \in [0, 1]. \tag{3.15}$$

Note that, the convexity of U and $\mathcal{U}_{\mathbb{G}}$ guarantees that $u^\rho \in \mathcal{U}_{\mathbb{G}}, \rho \in [0, 1]$. We denote by $X^\rho := X^{u^\rho}$ and by $\hat{X} := X\hat{u}$, the solution processes corresponding to u^ρ and $\hat{u}$, respectively.

For each $t_0 \in [0, T]$ and all bounded $\mathcal{G}_{t_0}$-measurable random variables α, the process

$$u(t) \quad = \quad \alpha \mathbf{1}_{(t_0, T]}(t),$$

belongs to $\mathcal{U}_{\mathbb{G}}$.

In general, if $K^{\hat{u}}(t)$ is a process depending on $\hat{u}$, we define the operator D on K by

$$DK\hat{u}(t) := D^u K^{\hat{u}}(t) = \frac{d}{d\rho} K^{\hat{u}+\rho u}(t)|_{\rho=0}, \tag{3.16}$$

whenever the derivative exists.

Define the following derivative processes

$$\begin{aligned}
DX(t) &:= \frac{d}{d\rho} X^{\hat{u}+\rho u}(t)|_{\rho=0} = \mathcal{X}(t), \\
DY(t) &:= \frac{d}{d\rho} Y^{\hat{u}+\rho u}(t)|_{\rho=0} = \mathcal{Y}(t), \\
DZ(t) &:= \frac{d}{d\rho} Z^{\hat{u}+\rho u}(t)|_{\rho=0} = \mathcal{Z}(t), \\
DN(t) &:= \frac{d}{d\rho} N^{\hat{u}+\rho u}(t)|_{\rho=0}, \\
DM(t) &:= \frac{d}{d\rho} M^{\hat{u}+\rho u}(t)|_{\rho=0}, \\
DN'(t) &:= \frac{d}{d\rho}\frac{d}{dt} M^{\hat{u}+\rho u}(t)|_{\rho=0}, \\
DM'(t) &:= \frac{d}{d\rho}\frac{d}{dt} M^{\hat{u}+\rho u}(t)|_{\rho=0},
\end{aligned}$$

such that

$$\begin{cases} d\mathcal{X}(t) & = \{\partial_x b(t)\mathcal{X}(t) + \langle \nabla_m b(t), DM(t)\rangle + \partial_u b(t)u(t)\}dt \\ & +\{\partial_x \sigma(t)\mathcal{X}(t) + \langle \nabla_m \sigma(t), DM(t)\rangle + \partial_u \sigma(t)u(t)\}dB(t), t \in [0,T], \\ \mathcal{X}(0) & = 0, \end{cases} \tag{3.17}$$

and

$$\begin{cases} d\mathcal{Y}(t) & = -\{\partial_x g(t)\mathcal{X}(t) + \partial_y g(t)\mathcal{Y}(t) + \partial_z g(t)\mathcal{Z}(t) + \langle \nabla_m g(t), DM(t)\rangle \\ & +\langle \nabla_n g(t), DN(t)\rangle + \partial_u g(t)u(t)\}dt + \mathcal{Z}(t)dB(t), t \in [0,T], \\ \mathcal{Y}(T) & = \partial_x \psi(T)\mathcal{X}(T). \end{cases} \tag{3.18}$$

Remark 3.6. Equations (3.17), (3.18) are linear FBSDE with bounded coefficients, then by Proposition 3.1 they have a unique solution.

Theorem 3.7. *Let $\hat{u} \in \mathcal{U}_{\mathbb{G}}$ be the optimal control and $\mathcal{X}(t), (\mathcal{Y}(t), \mathcal{Z}(t)), (p^0(t), q^0(t)), (p^1(t), q^1(t)), \lambda^0(t), \lambda^1(t)$ be the corresponding solutions to the* Equations (3.17), (3.18), (3.4), (3.5), (3.6), (3.7), *respectively. Then, the following statements are equivalent*

(i) $\frac{d}{d\rho} J(\hat{u} + \rho u)|_{\rho=0} = 0$ *for all bounded* $\beta \in \mathcal{U}_{\mathbb{G}}$.

(ii) $E[\frac{\partial}{\partial u}\hat{H}(t)|\mathcal{G}_t] = 0$ *for all* $t \in [0,T]$.

Proof We first prove Theorem 3.7 by assuming (i) and aiming to show (ii)

$$\begin{aligned} 0 &= \frac{d}{d\rho} J(u+\rho u)|_{\rho=0} \\ &= E\Big[\int_0^T \frac{d}{d\rho} f^\rho(t)|_{\rho=0} dt + p^0(T)\mathcal{X}(T) - \lambda^0(T)\partial_x \psi(T)\mathcal{X}(T) + \langle p^1(T), DM(T)\rangle \\ &+ \lambda^0(0)\mathcal{Y}(0) + \langle \lambda^1(0), DN(0)\rangle\Big] \end{aligned}$$

{we substitute $f(t)$ from Equation (3.3)}

$$\begin{aligned} &= E\Big[\int_0^T \frac{d}{d\rho}\{H^\rho(t) - p^0(t)b^\rho(t) - q^0(t)\sigma^\rho(t) - \lambda^0(t)g^\rho(t) - \langle p^1(t), M^{\rho'}(t)\rangle \\ &\quad + \langle \lambda^1(t), N^{\rho'}(t)\rangle\}|_{\rho=0} dt + p^0(T)\mathcal{X}(T) - \lambda^0(T)\mathcal{X}(T) + \langle p^1(T), DM(T)\rangle \\ &\quad + \lambda^0(0)\mathcal{Y}(0) + \langle \lambda^1(0), DN(0)\rangle\Big], \end{aligned}$$

by using the chain rule, we obtain

$$\frac{d}{d\rho}H^{\rho}(t)|_{\rho=0} = \partial_x H(t)\mathcal{X}(t) + \partial_y H(t)\mathcal{Y}(t) + \partial_z H(t)\mathcal{Z}(t) + \langle \nabla_m H(t), DM(t)\rangle$$
$$+ \langle \nabla_n H(t), DN(t)\rangle + \partial_u H(t)u(t),$$
$$\frac{d}{d\rho}p^0(t)b^{\rho}(t)|_{\rho=0} = p^0(t)\partial_x b(t)\mathcal{X}(t) + p^0(t)\langle \nabla_m b(t), DM(t)\rangle + p^0(t)\partial_u b(t)u(t),$$
$$\frac{d}{d\rho}q^0(t)\sigma^{\rho}(t)|_{\rho=0} = q^0(t)\partial_x \sigma(t)\mathcal{X}(t) + q^0(t)\langle \nabla_m \sigma(t), DM(t)\rangle + q^0(t)\partial_u \sigma(t)u(t),$$
$$\frac{d}{d\rho}\lambda^0(t)g^{\rho}(t)|_{\rho=0} = \lambda^0(t)\partial_x g(t)\mathcal{X}(t) + \lambda^0(t)\partial_y g(t)\mathcal{Y}(t) + \lambda^0(t)\partial_z g(t)\mathcal{Z}(t)$$
$$+ \lambda^0(t)\langle \nabla_m g(t), DM(t)\rangle + \lambda^0(t)\langle \nabla_n g(t), DN(t)\rangle$$
$$+ \lambda^0(t)\partial_u g(t)u(t),$$
$$\frac{d}{d\rho}\langle p^1(t), M^{\rho'}(t)\rangle|_{\rho=0} = \langle p^1(t), DM'(t)\rangle,$$

and

$$\frac{d}{d\rho}\langle \lambda^1(t), N^{\rho'}(t)\rangle|_{\rho=0} = \langle \lambda^1(t), DN'(t)\rangle.$$

We apply Itô formula to $p^0(t)\mathcal{X}(t), \langle p^1(t), DM(t)\rangle, \lambda^0(t)\mathcal{Y}(t)$ and $\langle \lambda^1(t), DN(t)\rangle$ then we take the expectation, we obtain the following important duality relations:

$$E[p^0(T)\mathcal{X}(T)] = E\bigg[\int_0^T \{p^0(t)\partial_x b(t)\mathcal{X}(t) + p^0(t)\langle \nabla_m b(t), DM(t)\rangle$$
$$+ p^0(t)\partial_u b(t)u(t) - \partial_x H(t)\mathcal{X}(t) + q^0(t)\partial_x \sigma(t)\mathcal{X}(t)$$
$$+ q^0(t)\langle \nabla_m \sigma(t), DM(t)\rangle + q^0(t)\partial_u \sigma(t)u(t)\}dt\bigg],$$
$$E[\langle p^1(T), DM(T)\rangle] = E\bigg[\int_0^T \langle p^1(t), DM'(t)\rangle - \langle \nabla_m H(t), DM(t)\rangle dt],$$
$$E[\lambda^0(T)\mathcal{Y}(T)] - E[\lambda^0(0)\mathcal{Y}(0)] = E\bigg[\int_0^T \{-\lambda^0(t)\partial_x g(t)\mathcal{X}(t)$$
$$- \lambda^0(t)\partial_y g(t)\mathcal{Y}(t) - \lambda^0(t)\partial_z g(t)\mathcal{Z}(t)$$
$$- \lambda^0(t)\langle \nabla_m g(t), DM(t)\rangle$$
$$- \lambda^0(t)\langle \nabla_n g(t), DN(t)\rangle - \lambda^0(t)\partial_u g(t)u(t)$$
$$+ \partial_y H(t)\mathcal{Y} + \partial_z H(t)\mathcal{Z}(t)\}dt\bigg],$$
$$E[\langle \lambda^1(T), DN(T)\rangle] - E[\langle \lambda^1(0), DN(0)\rangle] = E\bigg[\int_0^T \{\langle \lambda^1(t), DN'(t)\rangle$$
$$+ \langle \nabla_n H(t), DN(t)\rangle\}dt\bigg].$$

By substituting the derived duality relations and the partial derivatives of $f(t)$ the desired result follows. This proof can be reversed to prove $(ii) \Rightarrow (i)$. We omit the details. □

4. Mean-field risk minimization

4.1. Mean-field dynamic risk measure

In this section, we are interested in a particular class of MF-BSDE of the following form

$$\begin{cases} dY(t) &= -\mathbf{f}(t, Y(t), E[Y(t)], Z(t))dt + Z(t)dB(t), t \in [0, T], \\ Y(T) &= \xi, \end{cases} \tag{4.1}$$

where

$$\mathbf{f}(t, Y(t), E[Y(t)], Z(t)) := -r(t)Y(t) - r'(t)E[Y(t)] + F(t, Z(t)).$$

We assume that the generator $(y, \bar{y}, z) \to \mathbf{f}(t, Y(t), E[Y(t)], Z(t)) : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is $\mathbb{F}$-adapted, uniformly Lipschitz and concave, and the terminal condition $\xi \in L^2(\Omega, \mathcal{F}_T)$.

Definition 4.1. *Define $\mathcal{E}_t : (T; \xi) \to \mathcal{E}_t(T; \xi)$ by*

$$\mathcal{E}_t(T; \xi) \quad = \quad -Y_t(T; \xi), \mathrm{t} \in [0, \mathrm{T}],$$

*where $Y_t(T; \xi)$ is a component of the solution of the MF-BSDE (4.1) with terminal horizon T, terminal condition ξ and driver **f**. Then $\mathcal{E}_t(T; \xi)$ is a dynamic risk measure induced by the MF-BSDE (4.1).*

We may remark that the driver $\mathbf{f}$ depends linearly on Y and its expected value $E[Y]$, and nonlinearly on Z. This is interpreted as a market with interest rates $(r(t), r'(t))$. We can reformulated this as a problem with a driver independent of Y and $E[Y]$ by discounting the financial position ξ. We assume that the instantaneous interest rates $r(t)$ and $r'(t)$ are deterministic. We denote by $\mathcal{E}_\cdot$, the corresponding discounted risk measure.

Define the discounted process

$$Y^r(t) := e^{-\int_0^t (r(s)+r'(s))ds} Y(t).$$

Then Y^r with driver

$$F^r(\cdot, t, Z(t)) := e^{-\int_0^t (r(s)+r'(s))ds} F(\cdot, t, e^{-\int_0^t (r(s)+r'(s))ds} Z(t)),$$

and the terminal value $\xi^r := e^{-\int_0^t (r(s)+r'(s))ds} \xi$ is a part of the solution of the associated BSDE. We obtain also a discounted risk measure accordingly

$$\mathcal{E}_0(\xi, T) = \mathcal{E}_0^r(e^{-\int_0^t (r(s)+r'(s))ds} \xi, T).$$

This discounted risk measure is translation-invariant because F^r does not depend on Y, we have for $\xi \in L^2(\Omega, \mathcal{F}_T)$ and $a \in \mathbb{R}$,

$$\begin{aligned} \mathcal{E}_0(\xi + a e^{\int_0^t (r(s)+r'(s))ds}, T) &= \mathcal{E}_0^r(e^{-\int_0^t (r(s)+r'(s))ds} \xi + a, T) \\ &= \mathcal{E}_0^r(e^{-\int_0^t (r(s)+r'(s))ds} \xi, T) - a \\ &= \mathcal{E}_0(\xi, T) - a. \end{aligned}$$

Similarly, we can get for each $t \in [0, T]$, that

$$\mathcal{E}(\xi, T) = \mathcal{E}^r\left(e^{-\int_0^t (r(s)+r'(s))ds}\xi, T\right)$$

is translation-invariant.

4.2. Optimal portfolio with mean-field risk minimization

Consider a financial market with two investment possibilities:

(i) Safe, or risk-free asset with unit price

$$S_0(t) = 1, t \in [0, T].$$

(ii) Risky asset with unit price

$$dS_1(t) = S_1(t)[b_0(t)dt + \sigma_0(t)dB(t)], t \in [0, T].$$

Let $\pi(t)$ be a self-financing portfolio invested in the risky asset at time t. We want to minimize the risk $\varphi(X^\pi(T))$ of the terminal value of the wealth process $X^\pi(t)$ corresponding to a portfolio π which satisfies the linear SDE

$$\begin{cases} dX^\pi(t) &= \pi(t)X^\pi(t)[b_0(t)dt + \sigma_0(t)dB(t)], \quad t \in [0, T], \\ X^\pi(0) &= x_0, \end{cases} \tag{4.2}$$

such that

$$\varphi(X^\pi(T)) = -Y^\pi(0)$$

where $Y^\pi(t)$ satisfies a MF-BSDE

$$\begin{cases} -dY^\pi(t) &= [-r_0(t)E[Y^\pi(t)] + F(Z(t))]dt - Z(t)dB(t), \quad t \in [0, T], \\ Y^\pi(T) &= X^\pi(T). \end{cases} \tag{4.3}$$

Here we assume that $b_0(t)$, $\sigma_0(t)$, $r_0(t)$ are given deterministic functions and $F : \mathbb{R} \to \mathbb{R}$ is some given concave function. We want to find $\hat{\pi} \in \mathcal{U}_G$ such that

$$\inf_{\pi \in \mathcal{U}_G} (-Y^\pi(0)) = -Y\hat{\pi}(0).$$

Define the Hamiltonian H that corresponds to our problem by

$$H(t, x, z, \bar{y}, \pi, p^0, q^0, \lambda^0, \lambda^1) = p^0 b_0 \pi x + q^0 \sigma_0 \pi x + \lambda^0(r_0 \bar{y} + F(z)) + \langle \lambda^1, \bar{y} \rangle.$$

The couple (p^0, q^0) solution of the following BSDE

$$\begin{cases} dp^0(t) &= -\left[p^0(t)b_0(t)\pi(t) + q^0(t)\sigma_0(t)\pi(t)\right]dt + q^0(t)dB(t), t \in [0, T], \\ p^0(T) &= \lambda^0(T), \end{cases}$$

and (p^1, q^1) satisfies

$$\begin{cases} dp^1(t) &= q^1(t)dB(t), t \in [0, T], \\ p^1(T) &= 0. \end{cases}$$

The equation for λ^0 is given by the forward SDE

$$\begin{cases} d\lambda^0(t) &= \partial_z F(Z(t))\lambda^0(t)dB(t), t \in [0,T], \\ \lambda^0(0) &= 1, \end{cases} \tag{4.4}$$

and λ^1 satisfies

$$\begin{cases} d\lambda^1(t) &= -r_0(t)\lambda^0(t)dt, t \in [0,T], \\ \lambda^1(0) &= 0. \end{cases}$$

The first order necessary optimality condition gives

$$\hat{p}^0(t)b_0(t)\hat{X}(t) + \hat{q}^0(t)\sigma_0(t)\hat{X}(t) = 0,$$

where we denoted by $\hat{X}(t) = X^{\hat{\pi}}(t)$ and so on. Since $\hat{X}(t) > 0$ for all t P-a.s., we obtain

$$\hat{p}^0(t)b_0(t) + \hat{q}^0(t)\sigma_0(t) = 0, \tag{4.5}$$

which implies

$$\begin{cases} d\hat{p}^0(t) &= \hat{q}^0(t)dB(t) = -\dfrac{b_0(t)}{\sigma_0(t)}\hat{p}^0(t)dB(t), t \in [0,T], \\ \hat{p}^0(T) &= \hat{\lambda}^0(T), \end{cases}$$

this together with Equation (4.4), yields

$$\hat{p}^0(t) = \hat{\lambda}^0(t), \ \ \hat{q}^0(t) = \partial_z F(\hat{Z}(t))\hat{\lambda}^0(t).$$

From (4.5), we get

$$\partial_z F(\hat{Z}(t)) = -\frac{b_0(t)}{\sigma_0(t)}.$$

For example, if we choose

$$F(z) = -\frac{1}{2}z^2. \tag{4.6}$$

That is

$$\hat{Z}(t) = \frac{b_0(t)}{\sigma_0(t)}.$$

Substituting the expression of $\hat{Z}(t)$ above into the MF-BSDE (4.3), we obtain

$$\begin{cases} d\hat{Y}(t) &= -\left[-r_0(t)E\left[\hat{Y}(t)\right] - \dfrac{1}{2}\left(\dfrac{b_0(t)}{\sigma_0(t)}\right)^2\right]dt - \dfrac{b_0(t)}{\sigma_0(t)}dB(t), \ \ t \in [0,T], \\ \hat{Y}(T) &= \hat{X}(T). \end{cases} \tag{4.7}$$

Consequently

$$-dE\left[\hat{Y}(t)\right] = \left[-r_0(t)E\left[\hat{Y}(t)\right] - \frac{1}{2}\left(\frac{b_0(t)}{\sigma_0(t)}\right)^2\right]dt,$$

thus

$$E\left[\hat{Y}(t)\right] = \exp\left(-\int_0^t r_0(s)ds\right)\left[\hat{Y}(0) + \frac{1}{2}\int_0^t \frac{b_0^2(s)}{\sigma_0^2(s)}\exp\left(\int_0^s r_0(\alpha)d\alpha\right)ds\right]. \tag{4.8}$$

Define $\Gamma(t)$ to be the solution of the linear SDE

$$\begin{cases} d\Gamma(t) &= -\frac{b_0(t)}{\sigma_0(t)}\Gamma(t)dB(t), \ \ t \in [0, T], \\ \Gamma(0) &= 1, \end{cases}$$

or explicitly

$$\Gamma(t) = \exp\left(-\int_0^t \frac{b_0(s)}{\sigma_0(s)}dB(s) - \frac{1}{2}\int_0^t \left(\frac{b_0(s)}{\sigma_0(s)}\right)^2 ds\right), \ \ t \in [0, T]. \tag{4.9}$$

By the Girsanov theorem of change of measures, we know that there exists an equivalent local martingale measure $Q \ll P$, such that

$$dQ = \Gamma(T)dP \text{ on } \mathcal{F}_T,$$

with $\Gamma(T) = \frac{dQ}{dP}$ is the Radon–Nikodym derivative of Q with respect to P on $\mathcal{F}_T$.

Substituting (4.8), (4.9) into (4.7) we have

$$\begin{aligned} \hat{X}(T) = \hat{Y}(T) = \hat{Y}(0) + \exp\left(-\int_0^t r_0(s)ds\right)\left[\hat{Y}(0) + \frac{1}{2}\int_0^t \frac{b_0^2(s)}{\sigma_0^2(s)}\exp\left(\int_0^s r_0(\alpha)d\alpha\right)\right] \\ + \frac{1}{2}\int_0^T \left(\frac{b_0(s)}{\sigma_0(s)}\right)^2 ds + \int_0^T \frac{b_0(s)}{\sigma_0(s)}dB(s) \\ = \hat{Y}(0) + \exp\left(-\int_0^t r_0(s)ds\right)\left[\hat{Y}(0) + \frac{1}{2}\int_0^t \frac{b_0^2(s)}{\sigma_0^2(s)}\exp\left(\int_0^s r_0(\alpha)d\alpha\right)ds\right] - \ln\Gamma(t). \end{aligned}$$

Taking the expectation but now with respect to the new measure Q, we get

$$\begin{aligned} -\hat{Y}(0) &= -x_0 - \exp\left(-\int_0^t r_0(s)ds\right)\left[\hat{Y}(0) + \frac{1}{2}\int_0^t \frac{b_0^2(s)}{\sigma_0^2(s)}\exp\left(\int_0^s r_0(\alpha)d\alpha\right)ds\right] - E_Q[\ln\Gamma(T)] \\ &= \frac{1}{1 - \exp\left(-\int_0^t r_0(s)ds\right)}\left\{-x_0 - \exp\left(-\int_0^t r_0(s)ds\right)\left[\frac{1}{2}\int_0^t \frac{b_0^2(s)}{\sigma_0^2(s)}\exp\left(\int_0^s r_0(\alpha)d\alpha\right)ds\right]\right. \\ &\quad \left. - E[\Gamma(T)\ln\Gamma(T)]\right\}, \end{aligned} \tag{4.10}$$

where $E[\Gamma(T)\ln\Gamma(T)]$ is the entropy of Q with respect to P.

Since we obtained the optimal value of $\hat{Y}(0)$, we can get the corresponding optimal terminal wealth $\hat{X}(T)$.

Summarizing, we have the following conclusion:

Theorem 4.2. *Suppose that (4.6) holds. Then the minimal risk of our problem is given by (4.10).*

Funding

This research was carried out with support of the Norwegian Research Council, within the research project Challenges in Stochastic Control, Information and Applications (STOCONINF), project number 250768/F20; and U.S. Air Force Office of Scientific Research under grant number FA9550-17-1-0259.

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