Journal of Difference Equations and Applications

ISSN: 1023-6198 (Print) 1563-5120 (Online) Journal homepage: https://www.tandfonline.com/loi/gdea20

Stability of dynamical structures under perturbation of the generating function

J. Smítal & T.H. Steele

To cite this article: J. Smítal & T.H. Steele (2009) Stability of dynamical structures under perturbation of the generating function, Journal of Difference Equations and Applications, 15:1, 77-86, DOI: 10.1080/10236190802563761

To link to this article: https://doi.org/10.1080/10236190802563761

Journal of Difference Equations and Applications
Vol. 15, No. 1, January 2009, 77–86

Stability of dynamical structures under perturbation of the generating function

J. Smítal[a]* and T.H. Steele[b1]

[a]*Mathematical Institute, Silesian University, Opava, Czech Republic;* [b]*Department of Science and Mathematics, American University of Rome, Roma, Italy*

(*Received 13 October 2008; final version received 14 October 2008*)

We consider the set valued functions C, NW and $\mathcal{L}$ taking f in $\mathcal{C}(I, I)$ to its centre $C(f)$, its set of nonwandering points $NW(f)$ and its collection of ω-limit sets $\mathcal{L}(f) = \{\omega(x, f) : x \in I\}$, and consider how these sets are affected by pertubations of f. Our main results characterize those functions g in $\mathcal{C}(I, I)$ at which C, NW and $\mathcal{L}$ are continuous. In particular, we show that either of the maps C and NW is continuous at g if and only if one of the following conditions is satisfied: (i) The map ω which takes a function f to its set $\omega(f)$ of ω-limit points is continuous at g; (ii) the periodic orbits of g which are p-stable, i.e. stable with respect to small perturbations of g, are dense in the set $CR(g)$ of chain recurrent points of g; (iii) $CR(g) = \omega(g)$ and the p-stable periodic orbits of g are dense in the set of periodic points of g.

Keywords: centre of a system; chain recurrent points; Hausdorff metric; nonwandering points; omega-limit points; stability under perturbation

2000 Mathematics Subject Classification: Primary 37B20; 37E05; 26A18; 54H20

1. Introduction and main results

We shall be concerned with the class $\mathcal{C}(I, I)$ of continuous self-maps of the unit interval $I = [0,1]$. For f in $\mathcal{C}(I,I)$ and any integer $n \geq 0, f^n$ denotes the nth iterate of f. Let $P(f)$ denote the set of periodic points of f. For each $x \in I$, we denote by $\omega(x,f)$ the ω-limit set of f generated by x, i.e. the set of limit points of the sequence $\{f^k(x)\}_{k\geq 0}$. Let $\omega(f) = \cup_{x\in I}\omega(x,f)$ and $\mathcal{L}(f) = \{\omega(x,f) : x \in I\}$.

If $x \in P(f)$ has period n, and if any neighbourhood of x contains points u, v such that $f^n(u) < u$ and $f^n(v) > v$ (i.e. if $f^n(y) - y$ is unisigned in no neighbourhood of x) then x is an *essential periodic point* which cannot be removed by small perturbations of f. We let $S_0(f)$ represent the essential periodic points of f. We also say that a periodic orbit A of f of period $n \geq 1$ is a *p-stable periodic orbit* if for any $\varepsilon > 0$ there is a $\delta > 0$ such that any $g \in \mathcal{C}(I,I)$ with $\|f - g\| < \delta$ has a periodic orbit B of period n satisfying $\rho_H(A,B) < \varepsilon$, where ρ_H denotes the Hausdorff metric. We denote by $S(f)$ the union of p-stable periodic orbits of f. It follows that

$$\bigcup_{j\geq 0} f^j(S_0(f)) \subseteq S(f), \tag{1}$$

so that a periodic orbit of an essential periodic point is p-stable (for the argument see Section 2). We also denote by $(\mathcal{K},\rho_H)$ the class of nonempty closed sets $K \subseteq I = [0,1]$ endowed with the

*Corresponding author. Email: jaroslav.smital@math.slu.cz

ISSN 1023-6198 print/ISSN 1563-5120 online

DOI: 10.1080/10236190802563761
http://www.informaworld.com

Hausdorff metric ρ_{H}, and $(\mathcal{K}^*, \rho_{\mathrm{H}}^*)$ consists of the nonempty closed subsets of $\mathcal{K}$. Other terms, definitions and previously known results are provided in Section 2.

Bruckner [5] posed several questions regarding the iterative stability of a continuous function f of the interval with respect to small perturbations. In particular, how are the set $\omega(f)$ of ω-limit points and the collection $\mathcal{L}(f)$ of ω-limit sets of f affected by slight changes in that function? As one sees from examples in Refs. [5,12], in general, both these sets may be affected dramatically by arbitrarily small perturbations. Some results concerning the continuity structure of the maps $\omega: f \mapsto \omega(f)$ and $\mathcal{L}: f \mapsto \mathcal{L}(f)$ were obtained in Refs. [13,14]; the following is proved in Ref. [13].

Proposition 1.1. *The map* $\omega: (\mathcal{C}(I,I), \|\cdot\|) \to (\mathcal{K}, \rho_{\mathrm{H}})$ *is continuous at* g *if and only if the p-stable periodic orbits of* g *are dense in the set of chain recurrent points of* g.

The next result is from Steele [14]; note that, by Blokh et al. [3], the set $\mathcal{L}(f)$ is a compact subset of $(\mathcal{K}, \rho_{\mathrm{H}})$ so that the map $\mathcal{L}$ is correctly defined.

Proposition 1.2. *The map* $\mathcal{L}: (\mathcal{C}(I,I), \|\cdot\|) \to (\mathcal{K}^*, \rho_{\mathrm{H}}^*)$ *is upper semi continuous at* g *if and only if* $\mathcal{L}(g)$ *contains all sets* $L \in \mathcal{K}$ *with* $g(L) = L$ *such that, for every proper closed subset* F *of* L, $F \cap \overline{g(L \setminus F)} \neq \emptyset$.

In this paper we build upon these results. In particular, in Section 3 we show that the condition from Proposition 1.1 characterizes the points of continuity of the maps C and NW taking f in $\mathcal{C}(I,I)$ to its centre $C(f)$, and its set of nonwandering points $NW(f)$, respectively.

Recall that by the classical Birkhoff definition, the *centre* $C(f)$ of f is the nonwandering set of the nonwandering set of the nonwandering set ..., continued by transfinite induction until one gets nothing smaller. However, for any map f in $\mathcal{C}(I,I)$, Sharkovsky [8] proved that $C(f) = \overline{P(f)}$, i.e. it is the closure of the set $P(f)$ of periodic points. See also Ref. [1]. We say that $x \in I$ is a *nonwandering point* of f and write $x \in NW(f)$ if, for any neighbourhood U of x there is an $n > 0$ so that $f^n(U) \cap U \neq \emptyset$.

Now, let $\varepsilon > 0$ be given, and take x and y to be points in I. An ε-*chain* from x to y with respect to a function f is a finite set of points $\{x_0, x_1, \ldots, x_n\}$ in I with $x = x_0$, $y = x_n$, and $|f(x_{k-1}) - x_k| < \varepsilon$ for $k = 1,2,\ldots,n$. We call x a *chain recurrent point* of f if there is an ε-chain from x to itself for any $\varepsilon > 0$, and write $x \in CR(f)$. It is well-known [1] that

$$f \in \mathcal{C}(I,I) \Rightarrow S(f) \subseteq P(f) \subseteq \overline{P(f)} = C(f) \subseteq \omega(f) \subseteq NW(f) \subseteq CR(f) \tag{2}$$

and $C(f), \omega(f), NW(f)$ and $CR(f)$ are closed sets in I.

Our main result is the following extension of Proposition 1.1:

Theorem 1.3. Let $\varphi: (\mathcal{C}(I,I), \|\cdot\|) \to (\mathcal{K}, \rho_{\mathrm{H}})$ *be any of the maps* $f \mapsto \omega(f)$, $f \mapsto C(f)$, $f \mapsto NW(f)$. *Then the following conditions are equivalent:*

(i) φ *is continuous at* $g \in \mathcal{C}(I,I)$;
(ii) *the set of points in p-stable periodic orbits of* g *is dense in* $CR(g)$;
(iii) $\omega(g) = CR(g)$ *and the set of points in p-stable periodic orbits of* g *is dense in the set of periodic points of* g.

Whereas Steele [14] contains a characterization of the points of continuity of the map $\mathcal{L}$ restricted to the set of functions $f \in \mathcal{C}(I,I)$ with zero topological entropy, in Section 4 we characterize the points of continuity of $\mathcal{L}$ without any restrictions on the elements of $\mathcal{C}(I,I)$. Our main result there is the following theorem which solves a problem from Ref. [5].

THEOREM 1.4. *The map* $\mathcal{L} : (\mathcal{C}(I,I), \|\cdot\|) \to (\mathcal{K}^*, \rho_H^*)$ *is continuous at* $g \in \mathcal{C}(I,I)$ *if and only if all of the following holds:*

(i) *the periodic points of g are dense in* $CR(g)$,
(ii) *all the periodic points of g belong to p-stable periodic orbits, and*
(iii) *if* $L \in \mathcal{K}$, $g(L) = L$ *and, for any proper closed subset F of L,* $F \cap \overline{g(L \setminus F)} \neq \emptyset$, *then* $L \in \mathcal{L}(g)$.

The following problem is related to Theorem 1.3. We conjecture that the answer is positive.

Problem 1.5. Let CR denote the map taking any $f \in \mathcal{C}(I,I)$ to the set $CR(f)$ of chain recurrent points of f. Is it true that CR is continuous at a $g \in \mathcal{C}(I,I)$ if and only if ω is continuous at g?

The next Section 2 presents much of the terminology and many of the essential facts used in what follows.

2. Preliminaries

Here we introduce terminology, notation and basic results. If no specific reference is given, the result can be found in Ref. [1] (or in most standard books on low-dimensional dynamics). We already defined p-stable periodic orbits. To prove (1), suppose $x_0 \in S_0(f)$ with period n, and $\omega(x_0,f) = \{x_0,x_1,\ldots,x_{n-1}\}$ is the periodic orbit generated by x_0. Since $x_0 \in S_0(f)$, whenever g is sufficiently close to f, g must have a periodic point y_0 close to x_0 such that $g^n(y_0) = y_0$. Because f and g are both continuous and g is uniformly close to f, it follows that $y_i = g^i(y_0)$ must be close to $x_i = f^i(x_0)$, for any i.

In connection with (1) it would be interesting to know other relations between essential periodic points and p-stable periodic orbits (even though this is not really necessary for the paper). In particular, does every p-stable periodic orbit contain at least one essential periodic point? Is any point in a p-stable periodic orbit essential? It seems that the answer to both questions is positive but there is no regular proof.

In addition to the usual, Euclidean metric on I, we will be working in three metric spaces. Within $\mathcal{C}(I,I)$ we will use the uniform metric $\|\cdot\|$. Our second metric space $(\mathcal{K},\rho_H)$ is the space of all nonempty closed sets $\mathcal{K}$ in I endowed with the Hausdorff metric ρ_H given by $\rho_H(E,F) = \inf\{\delta > 0; E \subseteq B_\delta(F), F \subseteq B_\delta(E)\}$, where $B_\delta(F)$ is the open δ-neighbourhood of F. This space is compact [6] and, for any $f \in \mathcal{C}(I,I)$, $\mathcal{L}(f)$ is its compact subspace [3]. Our final metric space $(\mathcal{K}^*, \rho_H^*)$ consists of the nonempty closed subsets of $\mathcal{K}$. Thus, $K \in \mathcal{K}^*$ if K is a nonempty family of nonempty closed sets in I such that K is closed in $\mathcal{K}$ with respect to ρ_H. We endow $\mathcal{K}^*$ with the metric ρ_H^* so that K_1 and K_2 are close with respect to ρ_H^* if each member of K_1 is close to some member of K_2 with respect to ρ_H, and vice versa. This metric space is also compact [5].

A set valued function $\varphi : (\mathcal{C}(I,I),\|\cdot\|) \to (\mathcal{K},\rho_H)$ is *upper semicontinuous* at $g \in \mathcal{C}(I,I)$ if, for any $\varepsilon > 0$, there exists $\delta > 0$ so that $\varphi(f) \subseteq B_\varepsilon(\varphi(g))$ whenever $\|f - g\| < \delta$. Similarly, φ is *lower semicontinuous* at g if for any $\varepsilon > 0$ there exists $\delta > 0$ so that $\varphi(g) \subseteq B_\varepsilon(\varphi(f))$ whenever $\|f - g\| < \delta$. The following result is well-known (see, e.g., Proposition V.38 in Ref. [1]) and will be used in what follows.

Proposition 2.1. *The map* $CR : (\mathcal{C}(I,I), \|\cdot\|) \to (\mathcal{K}, \rho_H)$ *is upper semicontinuous.*

It is well-known that any ω-limit set of a map $f \in \mathcal{C}(I,I)$ is contained in a *maximal* ω*-limit set* (maximal with respect to inclusion). This has been proved by Sharkovsky [9] and further developed by Blokh [2]. Maximal ω-limit sets can be of three possible types: cycles (which are the only finite ω-limit sets), basic sets, and solenoids. If an infinite maximal ω-limit set $\tilde{\omega}$ contains a periodic point then it is a *basic set*. Otherwise $\tilde{\omega}$ is a *solenoidal set*. For any maximal solenoidal ω-limit set $\tilde{\omega}$ of a map $f \in \mathcal{C}(I,I)$ there is an *associated system of periodic intervals* $\{J_k\}_{k=1}^{\infty}$ which is a nested family of compact periodic intervals such that, for any k, J_k has period n_k (so that n_{k+1} is a multiple of n_k), $n_k < n_{k+1}$, and

$$\tilde{\omega} \subseteq \bigcap_{k=1}^{\infty} \bigcup_{i=0}^{n_k-1} f^i(J_k) \subset CR(f). \tag{3}$$

Here an interval $J \subseteq I$ is *periodic with period* $k \geq 1$ if $f^k(J) = J$ and if $J, f(J), \ldots, f^{k-1}(J)$ are pairwise disjoint. If (3) is satisfied then, for any $x \in \tilde{\omega}$, there is a sequence $\{i(k,x)\}_{k\geq 0}$ of positive integers such that $x \in \cap_{k=1}^{\infty} f^{i(k,x)}(J_k) =: M_x$, and M_x is a singleton or a compact interval. If M_x is a singleton for any $x \in \tilde{\omega}$ then $\tilde{\omega}$ is a minimal set; recall that A is a minimal set for f if $\omega(x,f) = A$, for any $x \in A$. Such a set is perfect, i.e. nonempty, closed, without isolated points. On the other hand, if a solenoidal ω-limit set is not minimal then it contains countably many isolated points [2]. If $b \in \tilde{\omega}$ is such a point then M_b is a nondegenerate wandering interval, i.e. $f^j(M_b) \cap M_b = \emptyset$. It follows that the interior of M_b is disjoint from the nonwandering set of f. Thus, in view of (3) we have the following result which is implicitly contained in Ref. [2] and will be used in what follows:

Lemma 2.2. *If a maximal solenoidal* ω*-limit set* $\tilde{\omega}$ *of* $f \in \mathcal{C}(I,I)$ *contains an isolated point then* $NW(f) \neq CR(f)$.

We also need the following result.

Lemma 2.3. *Let* $f \in \mathcal{C}(I,I)$ *and let* W *be a minimal set contained in a solenoidal* ω*-limit set of* f. *Then* $W \subset \overline{P(f)}$. *Moreover, if* $P(f)$ *has empty interior then* $W \subset \overline{S_0(f)}$.

Proof. The first statement is well-known (see, e.g., Proposition IV.15 in Ref. [1]). To prove the second one, let W be a minimal solenoidal set with associated system $\{J_k\}_{k\geq 0}$ of compact periodic intervals such that J_k has period n_k, and $n_k \to \infty$. Let $a \in W$, and let U be a neighbourhood of a. By (3), for any $x \in W$ there is a sequence $\{i(k,x)\}_{k\geq 0}$ with $0 \leq i(k,x) < n_k$ such that $x \in \cap_{k=1}^{\infty} f^{i(k,x)}(J_k) =: M_x$. Since W is a perfect set, and since M_x can be a nondegenerate interval only for countably many x, there is a $y \in W \cap U$ and a $k > 0$ such that $y \in f^{i(k,y)}(J_k) \subset U$. Then $L := f^{i(k,y)}(J_k)$ is a compact periodic interval of period n_k which contains $r := n_{k+1}/n_k \geq 2$ of periodic intervals of f^{n_k} of period r hence, a periodic orbit $p_1 < p_2 < \cdots < p_r$ of f^{n_k} of period r (this actually proves the first statement). Let A be the set of fixed points of f^{n_k} contained in the interval (p_1, p_r). Since $A \subset L$, any $z \in A$ is a periodic point of f of period n_k, and if $P(f)$ has empty interior, then A is nowhere dense and closed. Therefore $f^{n_k}(p_1) > p_1$ and $f^{n_k}(p_r) < p_r$ imply the existence of a $p \in A$ such that any its neighbourhood contains points u, v satisfying $u < p < v$, $f^{n_k}(u) > u$ and $f^{n_k}(v) < v$. Thus, $p \in S_0(f) \cap U$ and consequently, $a \in \overline{S_0(f)}$. □

Concerning basic sets, we recall a few important properties (see, e.g., Ref. [2]): Any basic set $\tilde{\omega}$ of an $f \in \mathcal{C}(I,I)$ is perfect, and the periodic points of f are dense in it. There is a minimal compact interval J and an integer $k \geq 1$ such that $J, f(J), \dots, f^{k-1}(J)$ are non-overlapping intervals which cover $\tilde{\omega}$. The sets $\tilde{\omega} \cap f^j(J) =: \tilde{\omega}_j$ form the *decomposition* of $\tilde{\omega}$ into a maximal system of periodic portions which may have common endpoints. If $k = 1$ then $\tilde{\omega}$ is *indecomposable*. Obviously, any $\tilde{\omega}_j$ is an indecomposable basic set for f^k. The map on a basic set is transitive but, in what follows, we need more (cf. [2] or Lemma 3.3 in Ref. [11]). Assume $\tilde{\omega}$ is an indecomposable basic set for an $f \in \mathcal{C}(I,I)$, and $J = [a, b]$ is the minimal invariant interval containing $\tilde{\omega}$. Then, for any compact intervals $U, V \subset (a,b)$,

$$U \cap \tilde{\omega} \,\text{is uncountable} \Rightarrow \operatorname{Int} f^n(U) \supset V, \text{ for any sufficiently large } n. \tag{4}$$

We use basic sets in proofs of Lemmas 3.6 and 3.7.

3. Mapping f to $C(f)$ and $NW(f)$

The main result of this section is the following theorem.

THEOREM 3.1. *Each of the maps $f \mapsto C(f)$ and $f \mapsto NW(f)$ is continuous at $g \in \mathcal{C}(I,I)$ if and only if $\overline{S(g)} = CR(g)$.*

Proof. Characterization of those g at which C is continuous follows by Lemmas 3.2, 3.5 and 3.12, and by (1). The result concerning NW follows by Lemmas 3.2, 3.4, 3.10 and 3.11, and by (1). □

LEMMA 3.2. *Let $g \in \mathcal{C}(I,I)$ for which $\overline{S(g)} = CR(g)$. Then the maps taking f in $\mathcal{C}(I,I)$ to $C(f), \omega(f), NW(f)$ and $CR(f)$ in $(\mathcal{K}, \rho_H)$ are all continuous at g.*

Proof. By Proposition 2.1, CR is upper semicontinuous at g so that if $\varepsilon > 0$ then there exists $\delta_1 > 0$ for which $CR(f) \subseteq B_\varepsilon(CR(g))$ whenever $\|f - g\| < \delta_1$. It is easy to see that the map $f \mapsto \overline{S(f)}$ is lower semicontinuous. In particular, there exists $\delta_2 > 0$ so that $\overline{S(g)} \subseteq B_\varepsilon(\overline{S(f)})$ should $\|f - g\| < \delta_2$. If $\delta = \min\{\delta_1, \delta_2\}$ and $\|f - g\| < \delta$, then $\overline{S(g)} \subseteq B_\varepsilon\big(\overline{S(f)}\big) \subseteq B_\varepsilon(\omega(f)) \subseteq B_\varepsilon(NW(f)) \subseteq B_\varepsilon(CR(f)) \subseteq B_{2\varepsilon}(CR(g)) = B_{2\varepsilon}\big(\overline{S(g)}\big)$. It follows that all of our maps are continuous at g. □

LEMMA 3.3. *Let $f \in \mathcal{C}(I,I)$. Then for any $\varepsilon > 0$ there exists g in $\mathcal{C}(I,I)$ so that $\|f - g\| < \varepsilon$ and $CR(f) \subseteq B_\varepsilon\big(\overline{S(g)}\big)$.*

Proof. Since $CR(f)$ is compact, for any $\varepsilon > 0$ there exists an ε-net of $CR(f)$ composed of chain recurrent points. Fix $\varepsilon > 0$, and take $\{x_1, x_2, \dots, x_n\} \subseteq CR(f)$ to be an ε-net of $CR(f)$. Now, from Lemma V.49 in Ref. [1], we may perturb f to a new function g so that $\|f - g\| < \varepsilon$ and $\{x_1, x_2, \dots, x_n\} \subseteq S(g)$. This gives $CR(f) \subseteq B_\varepsilon(\{x_1, x_2, \dots, x_n\}) \subseteq B_\varepsilon(\overline{S(g)})$. □

LEMMA 3.4. *If the map $f \mapsto NW(f)$ is upper semicontinuous at g then $NW(g) = CR(g)$.*

Proof. Suppose that $NW(g) \subsetneq CR(g)$. Then there exist $f_n \to g$ so that $\overline{S(f_n)} \to CR(g)$; this follows by Lemma 3.3 and Proposition 2.1. Since $\overline{S(f_n)} \subseteq NW(f_n) \subseteq CR(f_n)$ for each n, we have

$\lim_{n\to\infty}\overline{S(f_n)} = \lim_{n\to\infty} NW(f_n) = \lim_{n\to\infty} CR(f_n) = CR(g)$. In particular, NW is not upper semicontinuous at g. □

Since $\overline{S(f)} \subseteq C(f)$, the proof of Lemma 3.4 remains valid when NW is everywhere replaced with C. Thus, we have the following.

LEMMA 3.5. *If the map $f \mapsto C(f)$ is upper semicontinuous at g then $C(g) = CR(g)$.*

We now turn our attention to the lower semicontinuity of the maps NW and C.

LEMMA 3.6. *If $P(f)$ has empty interior then any basic set is a subset of $\overline{S_0(f)}$.*

Proof. Suppose W is a basic set with $W_0 \cup W_1 \cup \cdots \cup W_{m-1}$ the decomposition of W into its maximal system of periodic portions. Let U be a neighbourhood of a point in W, and assume, contrary to what we wish to show that $U \cap S_0(f) = \emptyset$. There exists $V \subseteq U$ a compact interval such that $V \cap W$ is uncountable, and none of the points $\max W_i$ and $\min W_i$, $0 \le i < m$, belongs to V. Such a V exists since W is perfect. Then, by (4), $\mathrm{Int} f^k(V) \supseteq V$, for some $k > 0$. It follows that there exist x and y in V such that $f^k(x) > x$ and $f^k(y) < y$. Since f^k is continuous there is a point z_0 between x and y such that $f^k(z_0) = z_0$. Without loss of generality we may assume that $f^k(w) \ge w$ in a neighbourhood of z_0. If $\mathrm{Fix}(\varphi)$ denotes the set of fixed points of φ, and $F = \{z \in \mathrm{Fix}(f^k) \cap [x,y]; f^k(w) - w \ge 0 \text{ on some neighbourhood of } z\}$ then $F \ne \emptyset$. Assume $y > z_0$ and let $z_1 = \sup F$. (If $y < z_0$ the argument is similar, with $z_1 = \inf F$.) Since $\mathrm{Fix}(f^k)$ is closed, $z_1 \in \mathrm{Fix}(f^k)$ and obviously, $f^k(w) \ge w$, for w in a left neighbourhood of z_1. Since $\mathrm{Int}(P(f)) = \emptyset$, any left neighbourhood of z_1 contains a point w with $f^k(w) > w$. If any right neighbourhood of z_1 contains a point w with $f^k(w) < w$ then $z_1 \in S_0(f)$ and we are done. Otherwise, since $P(f)$ has empty interior, there is a point $w_1 \in (z_1,y)$ with $f^k(w_1) > w_1$ and since $f^k(y) < y$ there is the minimal point $z_2 \in (w_1,y)$ with $f^k(z_2) = z_2$. Then, by the minimality of z_2, $f^k(w) > w$ if $z_1 < w < z_2$. On the other hand, $z_2 \notin F$ hence, any right neighbourhood of z_2 contains a point w with $f^k(w) < w$. Thus, $z_2 \in S_0(f)$. This proves $W \subseteq \overline{S_0(f)}$. □

LEMMA 3.7. *If the map $f \mapsto NW(f)$ is continuous at g then any infinite maximal ω-limit set of g either is a basic set, or a minimal solenoidal set. In particular, $\overline{P(g)} = \omega(g)$.*

Proof. By Lemmas 3.4 and 2.2, any infinite solenoidal ω-limit set of g is minimal. Hence, the maximal ω-limit sets of g are cycles, basic sets or minimal solenoidal sets. Since periodic points are dense in any basic set, $\overline{P(g)} = \omega(g)$ follows by Lemma 2.3. □

LEMMA 3.8. *If $P(g)$ has nonempty interior then neither $f \mapsto C(f)$ nor $f \mapsto NW(f)$ is lower semicontinuous at g.*

Proof. If $P(g)$ has a nonempty interior then it contains a compact interval L. Denote by P_n the set of periodic points of g of period n. By the Baire category theorem there is an $m \ge 1$ such that $P_m \cap L$ contains a compact interval $V = [a,b]$. Then, for any $\varepsilon > 0$ there is a $\varphi \in \mathcal{C}(I,I)$ which is the identity outside of V, $\varphi(x) > x$ for $a < x < b$, $\varphi(V) = V$, and $\|g \circ \varphi - g\| < \varepsilon$. Then V is a periodic interval of $f = g \circ \varphi$ such that $NW(f) \cap (a,b) = \emptyset$ and hence, $C(f) \cap (a,b) = \emptyset$. □

The next result is known, see, e.g., Lemma IV.9 in Ref. [1].

LEMMA 3.9. *Let $f \in \mathcal{C}(I,I)$, let J be an interval and j, k positive integers. If $x, f^j(x), y, f^k(y) \in J$ and $J \cap P(f) = \emptyset$, then $x < f^j(x)$ implies $y < f^k(y)$.*

LEMMA 3.10. *If the map $f \mapsto NW(f)$ is continuous at g then $\overline{S_0(g)} = \omega(g)$.*

Proof. Let NW be continuous at g. Then, by Lemma 3.7, $\overline{P(g)} = \omega(g)$. So assume $\overline{S_0(g)} \subsetneq \overline{P(g)}$. Then there is a compact interval U such that $U \cap \overline{S_0(g)} = \emptyset$ but $U \cap P(g) \neq \emptyset$. By Lemma 3.8 we may assume that $P(g)$ has empty interior. Then, by Lemmas 2.3, 3.6 and 3.7, one can take U so that $U \cap \omega(g) = U \cap P(g) =: P \neq \emptyset$. By (2), P is a closed set. To prove the discontinuity of NW at g it suffices to show that there are $p \in P$, and $\eta > \delta > 0$ such that, for any $\varepsilon > 0$ there is a map $f \in \mathcal{C}(I,I)$ with $f(x) = g(x)$ for $x \notin (p - \eta, p + \eta)$, $\|f - g\| < \varepsilon$, and $(p - \delta, p + \delta) \cap NW(f) = \emptyset$.

Assume first that P contains an isolated point p of period $m \geq 1$, and let $V = (p - \eta, p + \eta) \subset U$ be such that $V \cap P = \{p\}$. Without loss of generality assume $g^m(x) > x$ for $x \in V$, $x \neq p$, and $g^j(V) \cap V = \emptyset$ whenever $0 < j < m$. Then, for any $y \in V$, $y \neq p$, and any $j \geq 1$, $g^{mj}(y) > y$ since otherwise there would be a $k > 1$ such that $g^{mk}(y) < y$, and a $z > p$, sufficiently close to p, with $g^{mk}(z) > z$. This would imply existence of a periodic point $\neq p$ of g in V. Thus, by Lemma 3.9,

$$g^{mj}(x) > x, \text{ and } g^k(x) \in V \text{ implies } g^k(x) > x, \quad \text{whenever } x \in V, x \neq p, \text{ and } j, k > 0. \tag{5}$$

Let $\delta > 0$ be such that, for $W = (p - \delta, p + \delta)$, $g^{2m}(W) \subseteq V$. Let $\varphi \in \mathcal{C}(I,I)$ be the identity outside of V, $\varphi(x) > x$ for $x \in V$. Given $\varepsilon > 0$, φ can be chosen such that

$$\|g \circ \varphi - g\| < \varepsilon, (g \circ \varphi)^j(V) \cap V = \emptyset \text{ if } 0 < j < m, \text{ and } (g \circ \varphi)^m(W) \subset V. \tag{6}$$

Denote $f = g \circ \varphi$. Thus, the graph of f^m is obtained by shifting any point of the graph of g^m over V horizontally to the left. Take a point $a \in W$. By (6) there is a tiny neighbourhood $G \subset W$ of a such that $f^m(G) = g^m(\varphi(G)) > \varphi(G) > G$ but $f^j(G) \cap V = \emptyset$ for $0 < j < m$. (Here $A > B$ for sets means $x > y$, for any $x \in A$, $y \in B$.) Let $z \in f^m(G)$. The point z may enter V again. If so let $k_1, k_2, \ldots$ be the corresponding moments. Then it follows from the above that $f^{k_1}(z) = g^{k_1}(\varphi(z)) > z$ and hence, $\varphi(z) \notin G$. By induction we get that $f^{k_j}(z)$ is a growing sequence, hence in fact $f^k(y) \notin G$ for any $y \in G$, $k \in \mathbb{N}$. This proves that $a \notin NW(f)$ as desired.

If P contains no isolated points denote by P_n the set of points in P of period n. Since P is closed some P_m has nonempty interior (with respect to the relative topology on P), by the Baire category theorem. Hence there is a compact interval $V \subset U$ such that $V \cap P = V \cap P_m$ is a perfect set. We may assume that the endpoints of V are in P_m. Since $P(g)$ has empty interior, $V \cap P_m$ is a Cantor-type set. Without loss of generality we may assume that $g^m(x) \geq x$ for $x \in V$, since otherwise there would be an essential periodic point in P_m. Then the argument is similar as before. □

LEMMA 3.11. *If the map $f \mapsto NW(f)$ is continuous at g then $\omega(g) = NW(g)$.*

Proof. Let $NW_l(g)$ be the set of points x in $NW(g)$ such that for any left open neighbourhood U of x there is an $n > 0$ such that $U \cap g^n(U) \neq \emptyset$, and let $NW_r(g)$ be defined similarly with right neighbourhood U. It is well-known [10] that

$$\omega(g) = NW_l(g) \cup NW_r(g), \tag{7}$$

and the set $NW(g)\backslash\omega(g)$, if nonempty, consists of isolated points (cf. also [1]).

Now to prove the lemma assume $\omega(g) \neq NW(g)$, and let $a \in NW(g)\backslash\omega(g)$. Then there is an $\varepsilon > 0$ such that $(a - \varepsilon, a + \varepsilon) \cap NW(g) = \{a\}$. Moreover, since a is nonwandering, for any $\delta > 0$ there are infinitely many k such that $g^k((a - \delta, a + \delta)) \cap (a - \delta, a + \delta) \neq \emptyset$. Since $a \notin \omega(g)$, (7) implies that either $g^k((a - \delta, a)) \cap (a, a + \delta) \neq \emptyset$, for infinitely many k, or $g^k((a, a + \delta)) \cap (a - \delta, a) \neq \emptyset$, for infinitely many k. By symmetry, we may assume the first case and hence, by (7), there is a $\delta_0 > 0$ such that $g^k((a, a + \delta_0)) \cap (a - \delta_0, a) = \emptyset$, for any $k \geq 0$. Let $f_n \to g$ such that $f_n(x) = g(x)$ for $x \notin (a - \delta_n, a)$, $\delta_n < \delta_0$, $\delta_n \to 0$, and let $f_n(x) = g(a)$ for $x \in (a - \delta_n/2, a]$. Then $NW(f_n) \cap (a - \varepsilon, a + \varepsilon) = \emptyset$ and consequently, $\rho_{\mathrm{H}}(NW(f_n), NW(g)) \geq \varepsilon$. □

Lemma 3.12. *If the map $f \mapsto C(f)$ is lower semicontinuous at g then $\overline{S(g)} = C(g)$.*

Proof. Let $g \in \mathcal{C}(I,I)$ for which $\overline{S(g)} \subsetneq C(g)$. By (2), $C(g) = \overline{P(g)}$, and by Lemma 3.8 we may assume that $P(g)$ has empty interior. By Lemmas 2.3 and 3.6 there is U an open interval such that $U \cap \overline{S(g)} = \emptyset$, but that $U \cap \omega(g) = U \cap P(g)$ is nonempty and comprised of periodic points. We may now proceed as in the proof of Lemma 3.10 to show that, for a $p \in U \cap P(g)$ there is a neighbourhood $V \subseteq U$ of p such that for any $\delta > 0$, there is some $f \in \mathcal{C}(I,I)$ for which $\|f - g\| < \delta$, yet $V \cap P(f) = \emptyset$. □

Proof of Theorem 1.3. It follows by Proposition 1.1 and Theorem 3.1. □

4. Mapping f to $\mathcal{L}(f)$

This section is devoted to the proof of Theorem 1.4.

Lemma 4.1. *If $f \mapsto \mathcal{L}(f)$ is lower semicontinuous at $g \in \mathcal{C}(I,I)$ then $P(g) = S(g)$.*

Proof. Let $x \in P(g)\backslash S(g)$, say of period n. Then, by (1), there is an open interval N_i containing $x_i = g^i(x)$ so that $g^n(x_i) - x_i$ is unisigned on N_i, for $0 \leq i < n$. We may assume that $N_i \cap N_j = \emptyset$ whenever $i \neq j$. Let $\varepsilon > 0$ so that $B_\varepsilon(x_i) \subset N_i$ and $\overline{g(B_\varepsilon(x_i))} \subseteq N_{i+1}$, for any i, letting $N_n = N_0$. Then there is a $\delta > 0$ such that, for any i, $\overline{f(B_\varepsilon(x_i))} \subset N_{i+1}$, whenever $\|f - g\| < \delta$. Then there is an $f \in C(I,I)$ so that N_0 contains no periodic point of f of period n, and $\|f - g\| < \delta$. Indeed, if, e.g., $g^n(x) \geq x$, for $x \in N_0$, it suffices to take $f(x) = g(x) + \delta/2$ for $x \in B_\varepsilon(x_{n-1})$, $f(x) = g(x)$ for $x \notin N_{n-1}$, and extend f properly onto the whole interval I.

We show that $\rho_{\mathrm{H}}(\omega(x,g), \omega(y,f)) > \varepsilon$ for all y in I. Suppose, to the contrary, that there exists $y^* \in I$ so that $\rho_{\mathrm{H}}(\omega(x,g), \omega^*) < \varepsilon$, where $\omega^* = \omega(y^*, f)$. Then $\omega^* \subseteq B_\varepsilon(\omega(x,g)) \subseteq \cup_{i=0}^{n-1} N_i$, and by choosing f as we did we know that $f(\omega^* \cap N_i) = \omega^* \cap N_{i+1}$ for any i. Thus, $f^n(\omega^* \cap N_0) = \omega^* \cap N_0$ so that the convex closure of $(\omega^* \cap N_0)$ contains a periodic point of f of period n. This, however, contradicts our earlier choice of f. □

Proposition 4.2. *Let $g \in \mathcal{C}(I,I)$, and let $\overline{P(g)} = CR(g)$. Then any ω-limit set of f is in the Hausdorff closure of the periodic orbits, and any solenoidal ω-limit set is minimal, i.e. without isolated points.*

Proof. The first part is proved in Ref. [7], Theorem 4.6; see also Ref. [4]. So it suffices to show that no solenoidal ω-limit set W has isolated points. This follows by Lemma 2.2, since $\overline{P(f)} \subseteq NW(f)$, by (2). □

Lemma 4.3. *Let $g \in \mathcal{C}(I,I)$ such that* (i) $\overline{S(g)} = CR(g)$ *and* (ii) $P(g) = S(g)$. *Then $f \mapsto \mathcal{L}(f)$ is lower semicontinuous at g.*

Proof. Let $\omega \in \mathcal{L}(g)$. Then, by (i) and Proposition 4.2, ω is approximable by periodic orbits. By (ii), these orbits are p-stable whence $f \mapsto \mathcal{L}(f)$ is lower semicontinuous at g. □

Proof of Theorem 1.4. Suppose the map $\mathcal{L}$ is continuous at g. It follows immediately from the definition of the map ω that it, too, must be continuous there. This implies (i), by Proposition 1.1. Condition (ii) follows by Lemma 4.1, and condition (iii) by Proposition 1.2. Conversely, (i) and (ii) imply that $\mathcal{L}$ is lower semicontinuous, by Lemma 4.3, and (iii) implies the upper semicontinuity of $\mathcal{L}$, by Proposition 1.2. □

Acknowledgements

The main part of the paper was written during the second author's visit at the Silesian University. The research was supported, in part, by projects GA 201/06/0318 from the Czech Science Foundation, and MSM 4781305904 from the Czech Ministry of Education. The authors also wish to thank the referee for a careful reading of the manuscript and for the many useful comments and suggestions that markedly improve the paper.

Note

1. Email: t.steele@aur.edu

References

[1] L. Block and W. Coppel, *Dynamics in One Dimension*, Lecture Notes in Mathematics, vol. 1513, Springer, Berlin, 1991.
[2] A.M. Blokh, *The 'Spectral' Decomposition for One-Dimensional Maps*, Dynamics reported, 1–59, Dynam. Report. Expositions Dynam. Systems (N. S.), vol. 4, Springer, Berlin, 1995.
[3] A.M. Blokh, A.M. Bruckner, P.D. Humke, and J. Smítal, *The space of ω-limit sets of a continuous map of the interval*, Trans. Amer. Math. Soc. 348 (1996), pp. 1357–1372.
[4] A.M. Blokh, *Density of periodic orbits in ω-limit sets with the Hausdorff metric*, Real Anal. Exchange 24 (1998/99), pp. 503–521.
[5] A.M. Bruckner, *Stability in the family of ω-limit sets of continuous self maps of the interval*, Real Anal. Exchange 22 (1996/97), pp. 52–57.
[6] A.M. Bruckner, J.B. Bruckner, and B.S. Thomson, *Real Analysis*, Prentice-Hall, Upper Saddle River, NJ, 1997.
[7] E. D'Aniello and T.H. Steele, *Approximating ω-limit sets by periodic orbits*, Aequationes Math. 75 (2008), pp. 93–102.
[8] A.N. Šarkovskii, *Non-wandering points and the centre of a continuous mapping of the line into itself*, Dokl. Akad. Nauk Ukrain. SSR Ser. A (7) (1964), pp. 865–868 (Ukrainian).
[9] A.N. Šarkovskii, *The partially ordered system of attracting sets*, Dokl. Akad. Nauk SSSR 170(6) (1966), pp. 1384–1386.
[10] A.N. Šarkovskii, *About one Birkhoff's theorem*, Dokl. Akad. Nauk Ukrain. SSR Ser. A (5) (1967), pp. 429–432 (Ukrainian).

[11] B. Schweizer and J. Smítal, *Measures of chaos and a spectral decomposition of dynamical systems on the interval*, Trans. Amer. Math. Soc. 344 (1994), pp. 737–854.
[12] T.H. Steele, *Iterative stability in the class of continuous functions*, Real Anal. Exchange 24 (1998/99), pp. 765–780.
[13] T.H. Steele, *Continuity structure of* $f \to \cup_{x\in I}\omega(x,f)$ *and* $f \to \{\omega(x,f) : x \in I\}$, Real Anal. Exchange 25 (1999/2000), pp. 421–428.
[14] T.H. Steele, *The persistence of ω-limit sets under perturbation of the generating function*, Real Anal. Exchange 26 (2000/01), pp. 963–974.