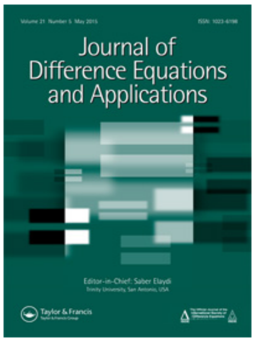

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Location and multiplicities of eigenvalues for a star graph of Stieltjes strings

Vyacheslav Pivovarchik[a,1] and Christiane Tretter[b,c]*

[a]*Institute of physics and mathematics, South Ukrainian national pedagogical university, Staroportofrankovskaya STR. 26, Odessa 65020, Ukraine;* [b]*Mathematisches institut, Universität Bern, Sidlerstr. 5, 3012 Bern, Switzerland;* [c]*Matematiska institutionen, Stockholms universitet, 10691 Stockholm, Sweden*

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The equations of motion of a star graph of Stieltjes strings with prescribed number of masses on each edge, with or without a mass at the central vertex, lead to a system of second order difference equations. At the central vertex Dirichlet or Neumann conditions are imposed while all pendant vertices are subject to Dirichlet conditions. We establish necessary and sufficient conditions on the location and multiplicities of two (finite) sequences of numbers $\{\zeta_k\}$ and $\{\lambda_k\}$ to be the corresponding Dirichlet and Neumann eigenvalues. Moreover, we derive necessary and sufficient conditions for one (finite) sequence $\{\lambda_k\}$ to be the Neumann eigenvalues of such a star graph. Here the possible multiplicities play a key role; the conditions on them are formulated by means of the notion of vector majorization. Our results include, as a special case, some earlier results for star-patterned matrix inverse problems where only multiplicities, not the location of eigenvalues, are prescribed.

Keywords: Eigenvalue; multiplicity; star graph; point mass; inverse problem; continued fraction; transversal vibration; Dirichlet boundary condition; Neumann boundary condition

MSC (2010) Classification: Primary 39A60; Secondary 05C50; 15A18; 15A29; 39A70; 70F17; 70J30

1. Introduction

Finite dimensional direct and inverse spectral problems for systems of difference equations arise in many fields of physics, such as vibrations of strings, electrical circuits synthesis etc. (see e.g. [18,9,10,5,7]). In the nice review [3], Cox, Embree, and Hokanson resurrected interest in beaded strings, i.e. massless threads supporting a finite number of point masses, by blending theory with experimental measurements (see [4] and also [11]). Such strings are also called *Stieltjes strings* since one method to solve the inverse problem, developed by Gantmakher and Krein in [8], uses Stieltjes' work on continued fractions (see [23] and the interesting reviews [25,24]). Within the last three decades, direct and inverse spectral problems on *graphs*, in particular on trees or star graphs, have attracted a lot of attention, stimulated by possible applications in quantum computing and nanoelectronics (see e.g. [6,16,2]).

In this paper we consider inverse problems for star graphs of beaded strings, which recently revealed connections to the problem of possible eigenvalue multiplicities for star-patterned matrices (see [21]). There are different settings for finding necessary and

*Corresponding author. Email: tretter@math.unibe.ch

sufficient conditions on one, or two respectively, (finite) sequences to be the Neumann, and Dirichlet respectively, spectra of a problem generated by Stieltjes string equations on a star graph. Here we consider the case where Neumann and Dirichlet conditions are imposed at the central vertex of the star, while Dirichlet boundary conditions are imposed at all pendant vertices. In [21] a complete description of the corresponding direct and inverse problems, including a constructive solution of the latter, was given in the case when the numbers of point masses on the edges were *not* part of the given data. An interesting feature of the Neumann eigenvalues proved there is that out of two neighbouring Neumann eigenvalues one must be simple.

The aim of the present paper is to study the problem where, in addition to the number of edges and the lengths of all edges, also *the number of masses on each edge is prescribed*. The latter results in restrictions on the eigenvalue multiplicities that need to be satisfied in the inverse problem. Our main results include necessary (Theorem 2.7) and sufficient (Theorem 3.3) conditions for *two* (finite) sequences $\{\pm\zeta_k\}$ and $\{\pm\lambda_k\}$, counted with multiplicities, to be the Dirichlet and Neumann spectra of a star graph of Stieltjes strings as described above, as well as necessary (Theorem 2.9) and sufficient (Theorem 3.4) conditions for *one* (finite) sequence $\{\pm\lambda_k\}$, counted with multiplicities, to be the Neumann spectrum. Examples illustrate the convenient applicability of these conditions (Section 4).

The necessary and sufficient conditions may be elegantly formulated by means of the notion of *majorization*, going back to the early work of Muirhead [20] (see also [12,19]). For example, suppose that there are q edges and n_i masses on the i-th edge enumerated such that $n_1 \geq n_2 \geq \cdots \geq n_q$ and a mass $M \geq 0$ at the central vertex. Denote by r_D the number of distinct positive values in the given sequence $\{\zeta_k\}$, by $(p_j^{\downarrow}(D))_{j=1}^{r_D}$ the vector of all their multiplicities in decreasing order, and by N_j the number of edges for which the number of masses is $\geq j$ for $j = 1, 2, \ldots, n_1$. Then the vector $(N_j)_{j=1}^{n_1}$ majorizes the vector $(p_j^{\downarrow}(D))_{j=1}^{r_D}$,

$$(N_1, N_2, \ldots, N_{n_1}) \succ (p_1^{\downarrow}(D), p_2^{\downarrow}(D), \ldots, p_{r_D}^{\downarrow}(D)),$$

which means that $n_1 \leq r_D$ and

$$\sum_{j=1}^{n_1} N_j = \sum_{j=1}^{r_D} p_j^{\downarrow}(D) \quad \text{and} \quad \sum_{j=1}^{\kappa} N_j \geq \sum_{j=1}^{\kappa} p_j^{\downarrow}(D) \quad (\kappa = 1, 2, \ldots, n_1 - 1).$$

The necessary and sufficient conditions for one (finite) sequence to be a Neumann spectrum are similar, but slightly more involved. They include the condition that *every other* element in the sequence must be simple, that, e.g. for $M > 0$, the number of *simple* elements in the sequence must be greater than or equal to the maximal number $n_1 + 1$ of masses on one string, and that the number of *distinct* elements in the sequence must be greater than or equal to $n_1 + n_2 + 1$ where n_1, n_2 are the two largest numbers of masses on a string.

Our results generalize results for star-patterned matrices, especially [13, Thm. 9], where the concept of majorization due to Muirhead was used before. While there only *multiplicities* of eigenvalues were prescribed and the existence of a tree-patterned matrix with these eigenvalue multiplicities was proved, we show the existence, and provide a method of construction, when a set of *eigenvalues with prescribed multiplicities*, together with the lengths of the q strings, is given.

Finally, we remark that the results of this paper may be of wider interest. In fact, the second order difference operators describing Stieltjes strings are special cases of the generalized second order derivatives $\frac{\mathrm{d}}{\mathrm{d}m}(\frac{\mathrm{d}}{\mathrm{d}x})$ describing *general* strings with mass distribution m (see [14] and the very recent paper [15, Sect. 8]). Hence many properties of the spectral theory of star graphs of Stieltjes strings carry over to star graphs of general strings (see [22]).

2. Direct problem: necessary conditions

Throughout this paper, we consider a plane star graph of q Stieltjes strings, $q \in \mathbb{N}$, $q \geq 2$, joined at the central vertex called the root where a mass $M \geq 0$ is placed and with all q pendant vertices fixed; here, following Gantmakher and Krein (see [8,23]), a Stieltjes string is a thread (i.e. an elastic string of zero density) bearing a finite number of point masses.

In the sequel, we label the edges of the star graph by $j = 1, 2, \ldots, q$ such that the j-th edge carries $n_j > 0$ point masses $m_k^{(j)}$ in its interior $(k = 1, 2, \ldots, n_j)$ and $n_1 \geq n_2 \geq \cdots \geq n_q$; here we do not count the possible mass at the centre and there are no masses at the pendant vertices. The masses $m_k^{(j)}$ subdivide the j-th edge into $n_j + 1$ intervals of length $l_k^{(j)}$ $(k = 0, 1, \ldots, n_j)$ where we count both, masses and intervals between them, from the exterior towards the centre; the length of the j-th edge is denoted by $l_j := \sum_{k=0}^{n_j} l_k^{(j)}$. The total number of masses on the star graph without the mass M in the centre is denoted by $n := \sum_{j=1}^{q} n_j$.

We assume that this web is stretched and study the small transverse vibrations $v_k^{(j)}(t)$ of the masses $m_k^{(j)}$ in two different cases (keeping the notation in [21]):

(N1) the mass M at the central vertex is free to move in the direction orthogonal to the equilibrium position of the strings (Neumann problem),

(D1) the mass M at the central vertex is fixed (Dirichlet problem).

Following [8, Chapt. III.1] (see also [21, Sect. 2]), separation of variables $v_k^{(j)}(t) = u_k^{(j)} \mathrm{e}^{\mathrm{i}\lambda t}$ leads to the following systems of difference equations for the displacement amplitudes $u_k^{(j)}$ $(k = 0, 1, 2, \ldots, n_j, j = 1, 2, \ldots, q)$ in the above Neumann and Dirichlet problem:

Neumann problem (N1). If the central vertex carrying the mass $M \geq 0$ can move freely, we obtain

$$-\left(\frac{u_{k+1}^{(j)} - u_k^{(j)}}{l_k^{(j)}} - \frac{u_k^{(j)} - u_{k-1}^{(j)}}{l_{k-1}^{(j)}} \right) = m_k^{(j)} \lambda^2 u_k^{(j)} \quad (k = 1, 2, \ldots, n_j,\ j = 1, 2, \ldots, q), \tag{2.1}$$

$$u_{n_1+1}^{(1)} = u_{n_2+1}^{(2)} = \cdots = u_{n_q+1}^{(q)}, \tag{2.2}$$

$$\sum_{j=1}^{q} \frac{u_{n_j+1}^{(j)} - u_{n_j}^{(j)}}{l_{n_j}^{(j)}} = M\lambda^2 u_{n_1+1}^{(1)}, \tag{2.3}$$

$$u_0^{(j)} = 0 \quad (j = 1, 2, \ldots, q). \tag{2.4}$$

Dirichlet problem (D1). If all strings are clamped at the central vertex, the problem decouples and consists of the q separate problems on the edges with Dirichlet boundary conditions at both ends,

$$-\left(\frac{u_{k+1}^{(j)}-u_k^{(j)}}{l_k^{(j)}}-\frac{u_k^{(j)}-u_{k-1}^{(j)}}{l_{k-1}^{(j)}}\right)=m_k^{(j)}\lambda^2u_k^{(j)}\quad (k=1,2,\ldots,n_j,\ j=1,2,\ldots,q), \tag{2.5}$$

$$u_{n_j+1}^{(j)}=0, \tag{2.6}$$

$$u_0^{(j)}=0 \tag{2.7}$$

for all $j=1,2,\ldots,q$.

Throughout this paper, we use the following notation for the eigenvalues of the spectral problems (N1), (D1) and their multiplicities.

Notation 2.1. *We denote by*

(1) $\Lambda_N := \begin{cases} \{\lambda_{\pm k}\}_{k=1}^{n+1} & \text{if } M>0,\\ \{\lambda_{\pm k}\}_{k=1}^{n} & \text{if } M=0,\end{cases}\quad \lambda_{-k}=-\lambda_k,\ 0<\lambda_k\le\lambda_{k'}$ *for* $0<k<k'$, *the eigenvalues of the Neumann problem* (2.1)–(2.4) *on the star graph,*

(2) $\Lambda_D := \{\zeta_{\pm k}\}_{k=1}^{n}=\bigcup_{j=1}^{q}\{\nu_{\pm\kappa}^{(j)}\}_{\kappa=1}^{n_j},\ \zeta_{-k}=-\zeta_k,\ 0<\zeta_k\le\zeta_{k'}$ *for* $0<k<k'$, *the eigenvalues of the Dirichlet problem* (D1) *on the star graph where*

(3) $\left\{\nu_{\pm\kappa}^{(j)}\right\}_{\kappa=1}^{n_j},\ \nu_{-\kappa}^{(j)}=-\nu_{\kappa}^{(j)},\ 0<\nu_{\kappa}^{(j)}<\nu_{\kappa'}^{(j)}$ *for* $0<\kappa<\kappa'$, *are the distinct eigenvalues of the Dirichlet problem* (2.5)–(2.7) *on the j-th edge for* $j=1,2,\ldots,q$,

(4) $\tilde\Lambda_N=\{\tilde\lambda_{\pm k}\}_{k=1}^{r_N},\ \tilde\lambda_{-k}=-\tilde\lambda_k,\ 0<\tilde\lambda_k<\tilde\lambda_{k'}$ *for* $0<k<k'$ *the set of distinct Neumann eigenvalues,*

(5) $\tilde\Lambda_D=\left\{\tilde\zeta_{\pm k}\right\}_{k=1}^{r_D},\ \tilde\zeta_{-k}=-\tilde\zeta_k,\ 0<\tilde\zeta_k<\tilde\zeta_{k'}$ *for* $0<k<k'$ *the set of distinct Dirichlet eigenvalues,*

(6) $(p_k(N))_{k=1}^{r_N}$ *and* $(p_k(D))_{k=1}^{r_D}$ *the vectors of multiplicities of the distinct positive Neumann and Dirichlet eigenvalues* $\tilde\lambda_k$ *and* $\tilde\zeta_k$ *for* $k>0$.

Remark 2.2.

(i) *By definition of* $p_k(N)$ *and* $p_k(D)$ *as multiplicities, clearly,*

$$\sum_{k=1}^{r_N}p_k(N)=\begin{cases} n+1 & \text{if}\quad M>0,\\ n & \text{if}\quad M=0,\end{cases}\qquad \sum_{k=1}^{r_D}p_k(D)=n. \tag{2.8}$$

(ii) *The number* r_D *of distinct positive Dirichlet eigenvalues satisfies* $r_D\ge n_1$ *since* n_1 *is the maximal number of masses on one string, labelled as the first, and* $\nu_1^{(1)}<\nu_2^{(1)}<\cdots<\nu_{n_1}^{(1)}$.

Note that since the equations in (N1) and (D1) only depend on λ^2, the Neumann and Dirichlet eigenvalues lie symmetrically to the origin and the multiplicities of $\tilde\lambda_{-k}=-\tilde\lambda_k$

and $\tilde{\lambda}_k$ as well as of $\tilde{\zeta}_{-k} = -\tilde{\zeta}_k$ and $\tilde{\zeta}_k$ coincide. Moreover, by [21, Thm. 2.5], 0 is neither a Neumann nor a Dirichlet eigenvalue.

In order to derive necessary conditions on the multiplicities of Neumann and Dirichlet eigenvalues, we need the notion of (vector) majorization which goes back to Muirhead [20] for the case of vectors of integers and was generalized to vectors of non-negative numbers by Hardy, Littlewood, and Polyá (see [12, 2.18] or [19]).

Definition 2.3. *Let $x = (x_i)_{i=1}^s$ and $y = (y_i)_{i=1}^t$ be two vectors with non-negative entries ordered decreasingly, $x_s \geq x_{s-1} \geq \cdots \geq x_1 \geq 0$, $y_t \geq y_{t-1} \geq \cdots \geq y_1 \geq 0$. If $s = t$, then x is said to* majorize *y, written as $x \succ y$, if*

$$:\Leftrightarrow \quad \sum_{i=1}^{t} x_i = \sum_{i=1}^{t} y_i, \quad \sum_{i=1}^{\tau} x_i \geq \sum_{i=1}^{\tau} y_i \quad (\tau = 1, 2, \ldots, t-1). \tag{2.9}$$

If $s \neq t$, we fill up the shorter vector with zeros, $\tilde{x} := (x_i)_{i=1}^{\max\{s,t\}}$, $\tilde{y} := (y_i)_{i=1}^{\max\{s,t\}}$ with $x_i = 0$ for $i = s+1, \ldots, \max\{s,t\}$, $y_i = 0$ for $i = t+1, \ldots, \max\{s,t\}$. Then x is said to majorize *y, $x \succ y$, if $\tilde{x}$ majorizes $\tilde{y}$, $\tilde{x} \succ \tilde{y}$.*

Remark 2.4. *If a vector $x = (x_i)_{i=1}^s$ majorizes a vector $(y_i)_{i=1}^t$, then the number of non-zero entries of x is less or equal to the number of non-zero entries of y,*

$$x \succ y \;\Rightarrow\; \#\{i \in \{1, \ldots, s\} : x_i > 0\} \leq \#\{i \in \{1, \ldots, t\} : y_i > 0\}.$$

In fact, denote the two numbers by s_0, t_0 and assume $s_0 > t_0$. Then $x \succ y$ if and only if $\tilde{x} \succ \tilde{y}$ where $\tilde{x} = (x_i)_{i=1}^{s_0}$ and $\tilde{y} = (y_i)_{i=1}^{s_0}$ with $y_i := 0$ for $i = t_0 + 1, \ldots, s_0$. *By* (2.9) *this implies*

$$\sum_{i=1}^{s_0} x_i = \sum_{i=1}^{s_0} y_i = \sum_{i=1}^{t_0} y_i \leq \sum_{i=1}^{t_0} x_i \leq \sum_{i=1}^{s_0} x_i,$$

and hence we have equality everywhere. Since all x_i are non-negative, this shows that $x_i = 0$ for $i = t_0 + 1, \ldots, s_0$, a contradiction to the assumption.

Notation 2.5. *For a vector $x = (x_i)_{i=1}^s \in \mathbb{R}^s$ we denote by $x^{\downarrow} = (x_i^{\downarrow})_{i=1}^s \in \mathbb{R}^s$ the vector with the same entries but ordered decreasingly, i.e.*

$$x_1^{\downarrow} \geq x_2^{\downarrow} \geq \cdots \geq x_s^{\downarrow}, \quad x_i^{\downarrow} = x_{\pi(i)}, \quad i = 1, 2, \ldots, s,$$

for some permutation π of $\{1, 2, \ldots, s\}$.

The following elementary lemma on the inversion of the non-increasing function $\{1, 2, \ldots, q\} \to \mathbb{N}$, $j \mapsto n_j$, will be used throughout this paper.

Lemma 2.6. *Let $q \in \mathbb{N}$, $q \geq 2$, $n_j \in \mathbb{N}$ $(j = 1, 2, \ldots, q)$ with $n_1 \geq n_2 \geq \cdots \geq n_q$, and set $n := \sum_{j=1}^{q} n_j$, $n_{q+1} := 0$. For $i = 1, 2, \ldots, n_1$ define*

$$N_i := \#\{j \in \{1, 2, \ldots, q\} : n_j \geq i\} = \max\{j \in \{1, 2, \ldots, q\} : n_j \geq i\}.$$

Then $N_1 \geq N_2 \geq \cdots \geq N_{n_1}$ and

(i) $\sum_{i=1}^{n_1} N_i = n$;
(ii) $N_i > 1$ $(i = 1, 2, \ldots, n_2)$, $N_i = 1$ $(i = n_2 + 1, n_2 + 2, \ldots, n_1)$;
(iii) $\#\{i \in \{1, 2, \ldots, n_1\} : N_i = j\} = n_j - n_{j+1}$ $(j = 1, 2, \ldots, q)$;
(iv) $n_j = \#\{i \in \{1, 2, \ldots, n_1\} : N_i \geq j\} = \max\{i \in \{1, 2, \ldots, n_1\} : N_i \geq j\}$ $(j = 1, 2, .., q)$.

Proof. Claims (i), (ii), (iii) follow from the definition of N_i, and claim (iv) from (iii). □

THEOREM 2.7. *Let Λ_N be the set of all Neumann eigenvalues $\lambda_{\pm k}$ of* (N1), $\lambda_k > 0$, $\lambda_{-k} = -\lambda_k$ $(k > 0)$, *and Λ_D the set of all Dirichlet eigenvalues $\zeta_{\pm k}$ of* (D1), $\zeta_k > 0$, $\zeta_{-k} = -\zeta_k$ $(k > 0)$ (*see Notation* 2.1 (1) *and* (2)), *both counted with multiplicities. Denote by r_D the number of distinct positive Dirichlet eigenvalues, by $p^{\downarrow}(D) = (p_i^{\downarrow}(D))_{i=1}^{r_D}$ the vector of their multiplicities in decreasing order, and by N_j the number of edges for which the number of masses is $\geq j$ $(j = 1, 2, \ldots, n_1)$, i.e. $N_i := \#\{j \in \{1, 2, \ldots, q\} : n_j \geq i\}$. Then*

(1) $$\begin{cases} 0 < \lambda_1^2 < \zeta_1^2 \leq \cdots \leq \lambda_n^2 \leq \zeta_n^2 < \lambda_{n+1}^2 & \text{if } M > 0, \\ 0 < \lambda_1^2 < \zeta_1^2 \leq \ldots \leq \lambda_n^2 \leq \zeta_n^2 & \text{if } M = 0; \end{cases}$$

(2) $\zeta_{k-1} = \lambda_k$ *if and only if* $\lambda_k = \zeta_k$;
(3) $(N_1, N_2, \ldots, N_{n_1}) \succ (p_1^{\downarrow}(D), p_2^{\downarrow}(D), \ldots, p_{r_D}^{\downarrow}(D))$.

Proof. The first two claims (1) and (2) were proved in [21, Thm. 2.5]. In order to prove (3), we first note that all entries in the vectors in (3) are non-zero and $n_1 \leq r_D$ by Remark 2.2.

The maximal multiplicity of a Dirichlet eigenvalue is equal to the number q of strings, which is in turn equal to N_1 (the number of strings carrying at least 1 mass), i.e. $p_1^{\downarrow}(D) \leq q = N_1$. If $p_1^{\downarrow}(D) = q$ is maximal, then in order to achieve the next highest multiplicity $p_2^{\downarrow}(D)$ there are only Dirichlet eigenvalues left on edges with at least 2 masses, i.e. $p_2^{\downarrow}(D) \leq N_2$ so that altogether $p_1^{\downarrow}(D) + p_2^{\downarrow}(D) \leq q + N_2 = N_1 + N_2$; in the general case $p_1^{\downarrow}(D) \leq q$, in order to achieve the multiplicity $p_2^{\downarrow}(D)$ there are only those Dirichlet eigenvalues left on edges with one mass that have not contributed to the highest multiplicity $p_1^{\downarrow}(D)$ and on edges with at least 2 masses, i.e. $p_2^{\downarrow}(D) \leq (N_1 - p_1^{\downarrow}(D)) + N_2$ and hence

$$p_1^{\downarrow}(D) + p_2^{\downarrow}(D) \leq N_1 + N_2.$$

Inductively, the same reasoning yields that

$$\sum_{i=1}^{\tau} p_i^{\downarrow}(D) \leq \sum_{i=1}^{\tau} N_i, \quad \tau = 1, 2, \ldots, n_1 - 1. \tag{2.10}$$

The total number of Dirichlet eigenvalues counted with multiplicities is n, see (2.8). Together with Lemma 2.6 (i), this implies

$$\sum_{i=1}^{r_D} p_i^{\downarrow}(D) = n = \sum_{i=1}^{n_1} N_i. \tag{2.11}$$

Now (3) follows from the inequalities (2.10) and the equality (2.11). □

The following properties of the Dirichlet and Neumann eigenvalues and their multiplicities, which follow from Theorem 2.7, will be used throughout the paper.

Note that the last item below is the well-known property that the maximal multiplicity of a Dirichlet eigenvalue and a Neumann eigenvalue is q and $q-1$, respectively (see [21, Thm. 2.5 iii)].

PROPOSITION 2.8.

(i) *If we set*

$$\begin{aligned} \kappa_N(l) &:= \#\{\lambda_k \in \tilde{\Lambda}_N \cap (0,\infty) : \lambda_k \textit{ has multiplicity } l\} \quad (l = 1,2,\ldots,q-1), \\ \kappa_D(l) &:= \#\{\zeta_k \in \tilde{\Lambda}_D \cap (0,\infty) : \zeta_k \textit{ has multiplicity } l\} \quad (l = 1,2,\ldots,q), \end{aligned} \tag{2.12}$$

and r_D denotes the number of distinct positive Dirichlet eigenvalues, then

$$\kappa_N(1) = \begin{cases} \kappa_D(2) + r_D + 1 & \textit{if } M > 0. \\ \kappa_D(2) + r_D & \textit{if } M = 0, \end{cases}$$

$$\kappa_N(l) = \kappa_D(l+1) \quad (l = 2,3,\ldots,q-1).$$

(ii) *If r_N denotes the number of distinct positive Neumann eigenvalues and $\tilde{r}_D$ the number of positive Dirichlet eigenvalues with multiplicity greater than one, then*

$$r_N = \begin{cases} r_D + \tilde{r}_D + 1 & \textit{if } \ M > 0, \\ r_D + \tilde{r}_D & \textit{if } \ M = 0, \end{cases} \tag{2.13}$$

and

$$\left.\begin{pmatrix} p_1^{\downarrow}(D) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ p_{r_D}^{\downarrow}(D) \end{pmatrix} = \begin{pmatrix} p_1^{\downarrow}(N)+1 \\ \vdots \\ p_{\tilde{r}_D}^{\downarrow}(N)+1 \\ p_{\tilde{r}_D+1}^{\downarrow}(N) \\ \vdots \\ p_{r_D}^{\downarrow}(N) \end{pmatrix} \succ \begin{pmatrix} p_1^{\downarrow}(N) \\ \vdots \\ p_{\tilde{r}_D}^{\downarrow}(N) \\ p_{\tilde{r}_D+1}^{\downarrow}(N) \\ \vdots \\ p_{r_D}^{\downarrow}(N) \\ \left.\begin{matrix} 1 \\ \vdots \\ 1 \end{matrix}\right\} \tilde{r}_D \end{pmatrix} = \begin{pmatrix} p_1^{\downarrow}(N) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ p_{r_D+\tilde{r}_D}^{\downarrow}(N) \end{pmatrix} \right\} = \begin{cases} r_N - 1, & M > 0, \\ r_N, & M = 0; \end{cases} \tag{2.14}$$

(iii) $\begin{cases} (N_1,N_2,\dots,N_{n_1}) \succ (p_1^{\downarrow}(N),p_2^{\downarrow}(N),\dots,p_{r_N-1}^{\downarrow}(N)), & M>0,\\ (N_1,N_2,\dots,N_{n_1}) \succ (p_1^{\downarrow}(N),p_2^{\downarrow}(N),\dots,p_{r_N}^{\downarrow}(N)), & M=0. \end{cases}$

(iv) $p_j(D) \le q \ (j=1,2,\dots,r_D), \quad p_j(N) \le q-1 \ (j=1,2,\dots,r_N)$.

Proof.

(i) By Theorem 2.7 (1) and (2), the only two possibilities for a simple Neumann eigenvalue λ_k with $k \in \{2,\dots,n\}$ to appear is either strictly between two Dirichlet eigenvalues or coinciding with a double Dirichlet eigenvalue, i.e.

$$\zeta_{k-1} < \lambda_k < \zeta_k \quad \text{or} \quad \lambda_{k-1} < \zeta_{k-1} = \lambda_k = \zeta_k < \lambda_{k-1};$$

in any case there appears the simple eigenvalue λ_1 and, if $M>0$, the simple eigenvalue λ_{n+1}. This shows that $\kappa_N(1) = r_D + \kappa_D(2) + 1$ if $M>0$ and $\kappa_N(1) = r_D + \kappa_D(2)$ if $M=0$.

If λ_k is an eigenvalue of multiplicity equal to $l \ge 2$, then $k \ne 1$ and, if $M>0$, also $k \ne n+1$ and there exists a $k_0 \in \{2,\dots,n-l\}$ such that $k_0 \le k$ and $\lambda_{k_0-1} < \lambda_{k_0} = \lambda_{k_0+1} = \dots = \lambda_{k_0+l-1} < \lambda_{k_0+l}$. Then Theorem 2.7 (1) and (2) show that

$$\lambda_{k_0-1} < \zeta_{k_0-1} = \lambda_{k_0} = \zeta_{k_0} = \lambda_{k_0+1} = \dots = \zeta_{k_0+l-2} = \lambda_{k_0+l-1} = \zeta_{k_0+l-1} < \lambda_{k_0+l}.$$

Hence to each Neumann eigenvalue of multiplicity $l \ge 2$ there corresponds a Dirichlet eigenvalue of multiplicity $l+1$, and the same holds vice versa.

(ii) Using the above relations for $\kappa_N(l)$ we find, e.g. for $M>0$,

$$r_N = \sum_{l=1}^{q-1} \kappa_N(l) = \kappa_D(2) + r_D + 1 + \sum_{l=2}^{q-1} \kappa_D(l+1) = r_D + 1 + \sum_{l=2}^{q} \kappa_D(l)$$
$$= r_D + \tilde{r}_D + 1;$$

for $M=0$, one only has to omit $+1$ from the second equality on. In order to prove (2.14), we note that Theorem 2.7 (2) yields that for the vectors $(p_j^{\downarrow}(D))_{j=1}^{r_D}$, $(p_j^{\downarrow}(N))_{j=1}^{r_N}$ of Dirichlet and Neumann multiplicities ordered decreasingly, we have

$$p_j^{\downarrow}(D) > 1, \quad p_j^{\downarrow}(N) = p_j^{\downarrow}(D) - 1 \qquad (j=1,2,\dots,\tilde{r}_D), \tag{2.15}$$

$$p_j^{\downarrow}(D) = 1, \quad p_j^{\downarrow}(N) = p_j^{\downarrow}(D) \qquad (j=\tilde{r}_D+1,\dots,r_D). \tag{2.16}$$

This implies the first and the last equality in (2.14) if we note (2.13). The majorization property claimed in (2.14) is obvious from Definition 2.3 since the number of components where $+1$ is added on the left hand side is equal to $\tilde{r}_D$ and coincides with the number of new components 1 added on the right hand side.

(iii) The claim is immediate from Theorem 2.7 (3) and from (2.14) since the majorization property is transitive.

(iv) It suffices to prove that $p_1^{\downarrow}(D) \le q$, $p_1^{\downarrow}(N) \le q-1$. The first claim follows from the first inequality $N_1 \ge p_1^{\downarrow}(D)$ of the majorization property (3) in Theorem 2.7

since $N_1 = q$ by definition. Together with (2.15), (2.16) and the assumption that $q \geq 2$ we obtain

$$p_1^{\downarrow}(N) = \begin{cases} p_1^{\downarrow}(D) - 1 \leq q - 1 & \text{if } \tilde{r}_D \geq 1, \\ p_1^{\downarrow}(D) = 1 \leq q - 1 & \text{if } \tilde{r}_D = 0. \end{cases} \qquad \square$$

In the following we consider the properties of the set of Neumann eigenvalues alone. Here we distinguish the case when there is a mass at the central vertex, i.e. $M > 0$, and when there is no mass there, i.e. $M = 0$.

The effect of this central mass is that it adds the simple eigenvalues $\pm\lambda_{n+1}$ and, viewed as functions of M, the Neumann eigenvalues $\pm\lambda_k(M)$ tend monotonically to the Dirichlet eigenvalues $\pm\zeta_{k-1}$ as M grows; more precisely, with $\lambda_{n+1}(0) := \infty$, we have that $\lambda_k(M) \searrow \zeta_{k-1}$ $(k = 2, 3, \ldots, n+1)$ and $\lambda_1(M) \searrow 0$ for $M \to \infty$ (see [21, Prop. 2.8]).

THEOREM 2.9. *Let Λ_N be the set of all Neumann eigenvalues $\lambda_{\pm k}$ of* (N1), *$\lambda_k > 0$, $\lambda_{-k} = -\lambda_k$ $(k > 0)$, counted with multiplicities* (*see Notation* 2.1 (1)). *Denote by r_N the number of distinct positive Neumann eigenvalues $\tilde{\lambda}_k$, by $(p_i(N))_{i=1}^{r_N}$ the vector of their multiplicities, and by $p^{\downarrow}(N) = (p_i^{\downarrow}(N))_{i=1}^{r_N}$ the corresponding vector of multiplicities in decreasing order. If $M > 0$, then*

(1) $0 < \lambda_1^2 < \lambda_2^2 \leq \lambda_3^2 \leq \cdots \leq \lambda_{n-1}^2 \leq \lambda_n^2 < \lambda_{n+1}^2$;
(2) *if $p_i(N) > 1$, then $p_{i-1}(N) = p_{i+1}(N) = 1$ $(i = 2, \ldots, r_N - 1)$;*
(3) $\{N_1 - 1, N_2 - 1, \ldots, N_{n_1} - 1\} \succ \{p_1^{\downarrow}(N), p_2^{\downarrow}(N), \ldots, p_{r_N - n_1 - 1}^{\downarrow}(N)\}$.

If $M = 0$, then

(1′) $0 < \lambda_1^2 < \lambda_2^2 \leq \lambda_3^2 \leq \cdots \leq \lambda_{n-2}^2 \leq \lambda_{n-1}^2 \leq \lambda_n^2$;
(2′) *if $p_i(N) > 1$, then $p_{i-1}(N) = p_{i+1}(N) = 1$ $(i = 2, \ldots, r_N - 1)$, and if $p_{r_N}(N) > 1$, then $p_{r_N - 1}(N) = 1$;*
(3′) $\{N_1 - 1, N_2 - 1, \ldots, N_{n_1} - 1\} \succ \{p_1^{\downarrow}(N), p_2^{\downarrow}(N), \ldots, p_{r_N - n_1}^{\downarrow}(N)\}$.

The following are necessary conditions for assumptions (1), (2), and (3) that are easy to check.

Remark 2.10. *For $M > 0$, claims* (1) *and* (2) *imply*

(4) *the number of simple Neumann eigenvalues is at least $[r_N/2] + 1$,*
(5) *the number of simple Neumann eigenvalues is at least $n_1 + 1$,*

while (3) *implies*

(6) $r_N \geq n_1 + n_2 + 1$.

For $M = 0$, claims (1′) *and* (2′) *imply*

(4′) *the number of simple Neumann eigenvalues is at least $[(r_N + 1)/2]$,*
(5′) *the number of simple Neumann eigenvalues is at least n_1,*

while (3′) *implies*

(6′) $r_N \geq n_1 + n_2$.

Note that conditions (4) *and* (5) *are not comparable and analogously for* (4′) *and* (5′).

Proof. Let $M > 0$; the proofs for $M = 0$ are analogous. Claim (4) follows immediately from (2).

In Proposition 2.8. (i) we showed that (1) and (2) imply that the number of simple Neumann eigenvalues satisfies $\kappa_N(1) \geq r_D + 1$; on the other hand, $r_D \geq n_1$ by Remark 2.2 (ii). Thus $\kappa_N(1) \geq n_1 + 1$ which is Claim (5).

Claim (6) is a consequence of the majorization property (3). Indeed, by Remark 2.4, the number of non-zero entries in majorizing vector $\{N_1 - 1, N_2 - 1, \ldots, N_{n_1} - 1\}$ is less than or equal to the number of non-zero entries in the majorized vector $\{p_1^{\downarrow}(N), p_2^{\downarrow}(N), \ldots, p^{\downarrow}_{r_N - n_1 - 1}(N)\}$. Since the number of the former is equal to n_2 by Lemma 2.6 (ii) and the number of the latter is trivially less than or equal to the number $r_N - n_1 - 1$ of entries of the majorized vector, we obtain $n_2 \leq r_N - n_1 - 1$. □

Proof of Theorem 2.9. Claims (1) and (1′) are immediate from Theorem 2.7 (1). Claims (2) and (2′) follow from Theorem 2.7 (2) (see [21, Cor. 2.6]).

For the proof of (3) we use that by Proposition 2.8 (iii) and Remark 2.10 (5), which follows by (1) and (2), we have

$$\begin{aligned}(N_1, N_2, \ldots, N_{n_1}) &\succ (p_1^{\downarrow}(N), p_2^{\downarrow}(N), \ldots\ldots\ldots\ldots\ldots\ldots\ldots, p^{\downarrow}_{r_N-1}(N)) \\ &= (p_1^{\downarrow}(N), p_2^{\downarrow}(N), \ldots, p^{\downarrow}_{r_N-n_1-1}(N), \underbrace{1, \ldots, 1}_{n_1}),\end{aligned}$$

and hence, if we subtract 1 in the n_1 components on the left and in the last n_1 components on the right and note that $N_1 \geq p_1^{\downarrow}(D) = p_1^{\downarrow}(N) + 1$ by (2.10), (2.14),

$$(N_1 - 1, N_2 - 1, \ldots, N_{n_1} - 1) \succ (p_1^{\downarrow}(N), p_2^{\downarrow}(N), \ldots, p^{\downarrow}_{r_N-n_1-1}(N), \underbrace{0, \ldots, 0}_{n_1}),$$

which proves (3). The proof of (3′) is analogous. □

3. Inverse problem

In this section we consider the problem of recovering the sequences $\left\{m_k^{(j)}\right\}_{k=1}^{n_j}$ of masses on each edge, the lengths $\left\{l_k^{(j)}\right\}_{k=0}^{n_j}$ of subintervals between them, and the central mass $M \geq 0$. Given are the Neumann and Dirichlet spectra $\{\lambda_k\}$ and $\{\zeta_k\}$ on the whole star graph, the total lengths l_j of the strings, and also the numbers of masses n_j on the j-th string.

In [17] trees with $M = 0$ at the root were considered in the case where not just the numbers n_j were given, but also the distribution of the Dirichlet eigenvalues onto the q edges. For trees that are stars with root at the central vertex, the following result is an immediate corollary of [17, Thm. 3.3].

Proposition 3.1 ([17]). *Let* $q \in \mathbb{N}$, $q \geq 2$, $\{l_j\}_{j=1}^q \subset (0, \infty)$, $n \in \mathbb{N}$. *Suppose a sequence* $\Lambda_N := \{\lambda_{\pm k}\}_{k=1}^n$ *and* q *sequences* $\Lambda_D^{(j)} = \left\{\nu_{\pm\kappa}^{(j)}\right\}_{\kappa=1}^{n_j}$ *with* $\{n_j\}_{j=1}^q \subset \mathbb{N}$, $n_1 \geq n_2 \geq \cdots \geq n_q$, *are given such that* $\lambda_k > 0$, $\lambda_{-k} = -\lambda_k$ $(k = 1, 2, \ldots, n)$, $\nu_{-\kappa}^{(j)} = -\nu_\kappa^{(j)}$, $0 < \nu_\kappa^{(j)} < \nu_{\kappa'}^{(j)}$ *for* $\kappa < \kappa'$ $(\kappa, \kappa' = 1, 2, \ldots, n_j,\ j = 1, 2, \ldots, q)$, *and* $\sum_{j=1}^q n_j = n$. *Assume further that the sets* Λ_N *and* $\Lambda_D := \{\zeta_{\pm k}\}_{k=1}^n := \cup_{j=1}^q \Lambda_D^{(j)}$ *satisfy the conditions*

(1) $0 < \lambda_1^2 \leq \zeta_1^2 \leq \cdots \leq \lambda_n^2 \leq \zeta_n^2$;
(2) $\lambda_k = \zeta_{k-1}$ *if and only if* $\lambda_k = \zeta_k$.

Then there exist a collection of masses $\left\{m_k^{(j)}\right\}_{k=1}^{n_j}$ *and lengths* $\left\{l_k^{(j)}\right\}_{k=0}^{n_j}$ $(j = 1, 2, \dots, q)$ *with* $\sum_{k=0}^{n_j} l_k^{(j)} = l_j$ *such that the corresponding spectral problems* (N1) *and* (D1) *with* $M = 0$ *have the sets* $\Lambda_N = \{\lambda_{\pm k}\}_{k=1}^{n}$ *and* $\Lambda_D = \cup_{j=1}^{q} \Lambda_D^{(j)}$ *as Neumann and Dirichlet eigenvalues.*

Proof. The claim follows from [17, Thm. 3.3] in the special case when the subtrees T_j consist only of edge number j with corresponding length $L_{i,j} = l_j$ for $j = 1, 2, \dots, q$. □

Remark 3.2. *It was shown in* [1, *Thm.* 3.2] *that if all* λ_k *are simple, then the solution of the above inverse problem is unique. We remark that uniqueness is lost as soon as one* λ_k *is not simple.*

Theorem 3.3. *Let* $q \in \mathbb{N}$, $q \geq 2$, $\{l_j\}_{j=1}^{q} \subset (0, \infty)$, $n \in \mathbb{N}$. *Suppose that sequences* $\Lambda_N := \{\lambda_{\pm k}\}_{k=1}^{n+1}$, $\Lambda_D := \{\zeta_{\pm k}\}_{k=1}^{n}$ *are given such that* λ_k, $\zeta_k > 0$, $\lambda_{-k} = -\lambda_k$, $\zeta_{-k} = -\zeta_k$ *for* $k > 0$, *and let* $\{n_j\}_{j=1}^{q} \subset \mathbb{N}$, $n_1 \geq n_2 \geq \cdots \geq n_q$, *with* $\sum_{j=1}^{q} n_j = n$. *Define* $N_i := \#\{j \in \{1, 2, \dots, q\} : n_j \geq i\}$ $(i = 1, 2, \dots, n_1)$, *denote by* r_D *the number of distinct positive elements in* Λ_D, *by* $p_k(D)$ *their multiplicities* $(k = 1, 2, \dots, r_D)$, *and let* $(p_k^{\downarrow}(D))_{k=1}^{r_D}$ *be the corresponding vector of multiplicities in decreasing order. Then the conditions*

(1) $0 < \lambda_1^2 < \zeta_1^2 \leq \cdots \leq \lambda_n^2 \leq \zeta_n^2 < \lambda_{n+1}^2$;
(2) $\zeta_{k-1} = \lambda_k$ *if and only if* $\lambda_k = \zeta_k$;
(3) $(N_1, N_2, \dots, N_{n_1}) \succ (p_1^{\downarrow}(D), p_2^{\downarrow}(D), \dots, p_{r_D}^{\downarrow}(D))$;

are necessary and sufficient such that there exist a collection of (*positive*) *masses* $\left\{m_k^{(j)}\right\}_{k=1}^{n_j}$, *a mass* $M > 0$, *and lengths* $\left\{l_k^{(j)}\right\}_{k=0}^{n_j}$ $(j = 1, 2, \dots, q)$ *with* $\sum_{k=0}^{n_j} l_k^{(j)} = l_j$ *such that the spectral problems* (N1) *and* (D1) *on the corresponding star graph have the sets* Λ_N *and* Λ_D *as Neumann and Dirichlet eigenvalues.*

If the sequence Λ_N *is of the form* $\Lambda_N = \{\lambda_{\pm k}\}_{k=1}^{n}$ *and* (1) *is replaced by*

(1′) $0 < \lambda_1^2 < \zeta_1^2 \leq \cdots \leq \lambda_n^2 \leq \zeta_n^2$;

then the above continues to hold with $M = 0$.

Proof. First we show that it is possible to divide the elements of the set $\{\zeta_k\}_{k=1}^{n}$ into q groups $\left\{\nu_\kappa^{(j)}\right\}_{\kappa=1}^{n_j}$ for $j = 1, 2, \dots, q$ such that

$$\Lambda_D = \{\zeta_{\pm k}\}_{k=1}^{n} := \bigcup_{j=1}^{q} \left\{\nu_{\pm\kappa}^{(j)}\right\}_{\kappa=1}^{n_j}, \tag{3.1}$$

with $\nu_{-\kappa}^{(j)} = -\nu_\kappa^{(j)}$ and $0 < \nu_\kappa^{(j)} < \nu_{\kappa'}^{(j)}$ for $0 < \kappa < \kappa'$. To this end, denote by $\tilde{r}_D$ the number of multiple positive elements in Λ_D.

For the choice of the first group of n_1 elements we note that all entries in the vectors in (3) are non-zero and hence the majorization property (3) implies that $n_1 \leq r_D$. Thus we can choose n_1 elements such that the first element has multiplicity $p_1^{\downarrow}(D)$ in Λ_D, the second element has multiplicity $p_2^{\downarrow}(D)$ up to the $\tilde{r}_D$-th element having multiplicity $p_{\tilde{r}_D}^{\downarrow}(D)$ in Λ_D, and $n_1 - \tilde{r}_D$ elements of multiplicity 1 from the tail of the vector

$(p_1^{\downarrow}(D), p_2^{\downarrow}(D), \ldots, p_{r_D}^{\downarrow}(D))$. We denote the n_1 selected elements, arranged in increasing order, by $\{\nu_\kappa^{(1)}\}_{\kappa=1}^{n_1}$. Again by assumption (3), we conclude that

$$\begin{aligned}(N_1 - 1, N_2 - 1, \ldots, N_{n_1} - 1)&\\ \succ (p_1^{\downarrow}(D) - 1, \ldots, p_{\tilde r_D}^{\downarrow}(D) - 1, p_{\tilde r_D+1}^{\downarrow}(D), \ldots, p_{r_D-n_1+\tilde r_D}^{\downarrow}(D))&\\ = (p_1^{\downarrow}(D) - 1, \ldots, p_{\tilde r_D}^{\downarrow}(D) - 1, 1, \ldots, 1).&\end{aligned} \tag{3.2}$$

For the choice of the second group of n_2 elements we note that the number of non-zero entries of the majorizing vector in (3.2) is n_2 by Lemma 2.6 (ii) and hence the majorization property (3.2) implies that $n_2 \le r_D - n_1 + \tilde r_D$ (see Remark 2.4). Thus we can choose n_2 elements such that the first element has multiplicity $p_1^{\downarrow}(D) - 1$ in $\Lambda_D \backslash \{\nu_k^{(1)}\}_{k=1}^{n_1}$, the second element has multiple multiplicity $p_2^{\downarrow}(D) - 1$ etc. as for the first group. We denote the n_2 selected elements, arranged in increasing order, by $\{\nu_\kappa^{(2)}\}_{\kappa=1}^{n_2}$.

Due to the majorization assumption (3), together with the property that $n_j = \#\{i \in \{1, 2, \ldots, n_1\} : N_i \ge j\}$ by Lemma 2.6 (iv) and with $\sum_{j=1}^q n_j = n$, we may continue like this to obtain q groups of elements $\{\nu_\kappa^{(j)}\}_{\kappa=1}^{n_j}$ $(j = 1, 2, \ldots, q)$ such that (3.1) holds.

In the case where $\Lambda_N = \{\lambda_{\pm k}\}_{k=1}^n$, we can now apply Proposition 3.1 to finish the proof in the case $M = 0$. In the case where $\Lambda_N = \{\lambda_{\pm k}\}_{k=1}^{n+1}$, we can use the chosen decomposition (3.1) as [21, (2.24)] in the proof of [21, Thm. 2.5] to prove the claim in the case $M > 0$. □

In the next theorem we derive sufficient conditions for one sequence of numbers to be the Neumann eigenvalues of a star graph of Stieltjes strings.

THEOREM 3.4. *Let $q \in \mathbb{N}$, $q \ge 2$, $\{l_j\}_{j=1}^q \subset (0, \infty)$, $n \in \mathbb{N}$. Suppose that a sequence $\Lambda_N := \{\lambda_{\pm k}\}_{k=1}^{n+1}$, is given such that $\lambda_k > 0$, $\lambda_{-k} = -\lambda_k$ for $k > 0$, and let $\{n_j\}_{j=1}^q \subset \mathbb{N}$, $n_1 \ge n_2 \ge \cdots \ge n_q$, with $\sum_{j=1}^q n_j = n$. Define $N_i := \#\{j \in \{1, 2, \ldots, q\} : n_j \ge i\}$ $(i = 1, 2, \ldots, n_1)$. Denote by r_N the number of distinct positive elements in the sequence $\{\lambda_k\}_{k=1}^{n+1}$, by $(p_i(N))_{i=1}^{r_N}$ the vector of their multiplicities, and by $p^{\downarrow}(N) = (p_i^{\downarrow}(N))_{i=1}^{r_N}$ the corresponding vector of multiplicities in decreasing order. Then the conditions*

1. *$0 < \lambda_1^2 < \lambda_2^2 \le \lambda_3^2 \le \cdots \le \lambda_{n-1}^2 \le \lambda_n^2 < \lambda_{n+1}^2$;*
2. *if $p_i(N) > 1$, then $p_{i-1}(N) = p_{i+1}(N) = 1$ $(i = 2, \ldots, r_N - 1)$;*
3. *$\{N_1 - 1, N_2 - 1, \ldots, N_{n_1} - 1\} \succ \{p_1^{\downarrow}(N), p_2^{\downarrow}(N), \ldots, p_{r_N-n_1-1}^{\downarrow}(N)\}$;*

are necessary and sufficient such that there exist a collection of (positive) masses $\{m_k^{(j)}\}_{k=1}^{n_j}$, a mass $M > 0$, and lengths $\{l_k^{(j)}\}_{k=0}^{n_j}$ $(j = 1, 2, \ldots, q)$ with $\sum_{k=0}^{n_j} l_k^{(j)} = l_j$ such that the spectral problem (N1) on the corresponding star graph has the set Λ_N as Neumann eigenvalues.

If the sequence Λ_N is of the form $\Lambda_N = \{\lambda_{\pm k}\}_{k=1}^n$ and the conditions (1)–(3) are replaced by the conditions (1′)–(3′) in Theorem 2.9, then the above continues to hold with central mass $M = 0$.

Remark 3.5. *For $M > 0$, necessary conditions for (1) and (2) are*

(4) *the number of simple positive elements in Λ_N is at least $[r_N/2] + 1$,*
(5) *the number of simple positive elements in Λ_N is at least $n_1 + 1$;*

while a necessary condition for (3) is

(6) $r_N \ge n_1 + n_2 + 1$;

for $M = 0$, *necessary conditions for* $(1')$ *and* $(2')$ *are* $(4')$ *and* $(5')$ *in Remark* 2.10, *while a necessary condition for* $(3')$ *is* $(6')$ *in Remark* 2.10.

Note that conditions (4) *and* (5) *are not comparable and analogously for* $(4')$ *and* $(5')$.

Proof. The necessity of (4), (5), (6) and of $(4')$, $(5')$, $(6')$, respectively, follows in the same way as in the proof of Remark 2.10. □

Proof of Theorem 3.4. In the following we will show that the assumptions above ensure the existence of a sequence $\Lambda_D := \{\zeta_{\pm k}\}_{k=1}^n$, $\zeta_k > 0$, $\zeta_{-k} := -\zeta_k$ for $k > 0$, such that the two sequences Λ_N and Λ_D satisfy the assumptions of Theorem 3.3. To this end, in the case $M = 0$, we set $\lambda_{n+1} := \infty$ for convenience.

Following condition (2), we choose $\{\zeta_k\}_{k=1}^n$, as follows: If $k \in \{1, 2, \ldots, n\}$ and $\lambda_k < \lambda_{k+1}$ are two positive elements of Λ_N of multiplicity 1, then we choose ζ_k strictly in between, i.e. $\zeta_k \in (\lambda_k, \lambda_{k+1})$, with multiplicity 1; if one of λ_k, λ_{k+1} is multiple, say $\lambda_{k+1} = \cdots = \lambda_{k+p_{j_0}(N)}$ with multiplicity $p_{j_0}(N) \geq 2$ for some $j_0 \in \{1, 2, \ldots, r_N\}$ (whence $k < n$), then we choose $\zeta_k = \zeta_{k+1} = \cdots = \zeta_{k+p_{j_0}(N)} = \lambda_{k+1}$ with multiplicity $p_{j_0}(N) + 1 \geq 3$. Note that, for this particular choice, there are no elements in $\{\zeta_{\pm k}\}_{k=1}^n$ with multiplicity 2.

If we define the numbers $\kappa_N(l)$, $\kappa_D(l)$ as in (2.12), then $\kappa_D(2) = 0$ for the above choice of $\Lambda_D = \{\zeta_{\pm k}\}_{k=1}^n$. Hence, if $\tilde{r}_N$ denotes the number of multiple positive elements in Λ_N, then

$$r_N - \tilde{r}_N = \kappa_N(1) = \begin{cases} r_D + 1 & \text{if } M > 0, \\ r_D & \text{if } M = 0; \end{cases} \tag{3.3}$$

here, for the case $M > 0$, we have to observe that the additional element λ_{n+1} is always simple due to the last strict inequality in $(1')$.

If we use claim (5) already proved and add $+1$ in the n_1 components of the majorizing vector $(N_1 - 1, N_2 - 1, \ldots, N_{n_1} - 1)$ in condition (3) and, on the right hand side, add $+1$ in the first $\tilde{r}_N$ components and $n_1 - \tilde{r}_N$ new components 1, we conclude that, for $M > 0$,

$$(N_1, \ldots, N_{n_1}) \succ (p_1^{\downarrow}(N) + 1, \ldots, p_{\tilde{r}_N}^{\downarrow}(N) + 1, p_{\tilde{r}_N+1}^{\downarrow}(N), \ldots, p_{r_N-n_1-1}^{\downarrow}(N), \overbrace{1, \ldots, 1}^{n_1 - \tilde{r}_N}); \tag{3.4}$$

note that this is possible since $r_N - \tilde{r}_N \geq n_1 + 1$, i.e. $r_N - n_1 - 1 \geq \tilde{r}_N$, by (5). For $M = 0$, according to $(5')$, the group of unchanged elements in the middle of the vector on the right hand side of (3.4) needs to be replaced by $p_{\tilde{r}_N+1}^{\downarrow}(N), \ldots, p_{\tilde{r}_N-n_1}^{\downarrow}(N)$. By (3.3), the number of components on the right hand side is equal to $r_N - n_1 - 1 + (n_1 - \tilde{r}_N) = r_N - 1 - \tilde{r}_N = r_D$ for $M > 0$ and equal to $r_N - n_1 + (n_1 - \tilde{r}_N) = r_N - \tilde{r}_N = r_D$ for $M = 0$. Moreover, by the definition of $\tilde{r}_N$ as the number of multiple positive elements of Λ_N, we have

$$p_k^{\downarrow}(N) > 1, \quad p_k^{\downarrow}(D) = p_k^{\downarrow}(N) + 1 \quad (k = 1, 2, \ldots, \tilde{r}_N),$$

$$p_k^{\downarrow}(N) = 1, \quad p_k^{\downarrow}(D) = p_k^{\downarrow}(N) = 1 \quad (k = \tilde{r}_N + 1, \ldots, r_N - n_1),$$

$$p_k^{\downarrow}(D) = 1 \qquad (k = r_N - n_1 + 1, \ldots, r_D).$$

This shows that the majorized vector on the right hand side of (3.4) is, in fact, equal to $(p_1(D), \dots, p_{r_D}(D))$ and hence

$$(N_1, \dots, N_{n_1}) \succ (p_1(D), \dots, p_{r_D}(D)).$$

Thus we have shown that the sequences Λ_N and Λ_D satisfy all assumptions of Theorem 3.3 which yields the claim. □

Remark 3.6. *Theorem* 3.4 *is related to* [13, *Thm.* 9] *about possible eigenvalue multiplicities of star-patterned matrices as follows.*

Our sets $\{\lambda_k^2\}_{k=1}^{n+1}$ *in case of* $M > 0$ *and* $\{\lambda_k^2\}_{k=1}^{n}$ *in case* $M = 0$ *are the spectrum of the generalized eigenvalue problem*

$$\mathcal{L}x = \lambda \mathcal{M} x \tag{3.5}$$

where the mass matrix $\mathcal{M}$ *is the* $(n+1)\times(n+1)$ *diagonal matrix*

$$\mathcal{M} := \operatorname{diag}\Big(M, m^{(1)}_{n_1}, m^{(1)}_{n_1-1}, \dots, m^{(1)}_1, m^{(2)}_{n_2}, m^{(2)}_{n_2-1}, \dots, m^{(2)}_1, \dots, m^{(q)}_{n_q}, m^{(q)}_{n_q-1}, \dots, m^{(q)}_1\Big)$$

and the stiffness matrix $\mathcal{L}$ *is the star-patterned* $(n+1)\times(n+1)$ *block matrix*

$$\mathcal{L} := \left\{ \begin{array}{c|c|c|c|c}
\displaystyle\sum_{j=1}^{q} \frac{1}{l^{(j)}_{n_j}} & -\frac{1}{l^{(1)}_{n_1}} \quad 0 \quad \cdots\cdots \quad 0 & -\frac{1}{l^{(2)}_{n_2}} \quad 0 \quad \cdots\cdots \quad 0 & \cdots\cdots & -\frac{1}{l^{(q)}_{n_q}} \quad 0 \quad \cdots\cdots \quad 0 \\ \hline
\begin{matrix} -\frac{1}{l^{(1)}_{n_1}} \\ 0 \\ \vdots \\ 0 \end{matrix} & \mathcal{L}_1 & 0 & \cdots\cdots & 0 \\ \hline
\begin{matrix} -\frac{1}{l^{(2)}_{n_2}} \\ 0 \\ \vdots \\ 0 \end{matrix} & 0 & \mathcal{L}_2 & & \\ \hline
\begin{matrix} \vdots \\ \vdots \end{matrix} & \begin{matrix} \vdots \\ \vdots \end{matrix} & & \ddots & \\ \hline
\begin{matrix} -\frac{1}{l^{(q)}_{n_q}} \\ 0 \\ \vdots \\ 0 \end{matrix} & 0 & & & \mathcal{L}_q
\end{array} \right\}$$

in which the blocks $\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_q$ *are the stiffness matrices of the q individual strings, i.e.* $\mathcal{L}_j$ *is the* $n_j \times n_j$ *matrix*

$$\mathcal{L}_j := \begin{pmatrix} \frac{1}{l^{(j)}_{n_j}} + \frac{1}{l^{(j)}_{n_j-1}} & -\frac{1}{l^{(j)}_{n_j-1}} & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ -\frac{1}{l^{(j)}_{n_j-1}} & \frac{1}{l^{(j)}_{n_j-2}} + \frac{1}{l^{(j)}_{n_j-1}} & -\frac{1}{l^{(j)}_{n_j-2}} & 0 & & & & 0 \\ 0 & -\frac{1}{l^{(j)}_{n_j-2}} & \frac{1}{l^{(j)}_{n_j-3}} + \frac{1}{l^{(j)}_{n_j-2}} & -\frac{1}{l^{(j)}_{n_j-3}} & & & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & & \ddots & \ddots & \ddots & 0 \\ 0 & & & & & -\frac{1}{l^{(j)}_{2}} & \frac{1}{l^{(j)}_{2}} + \frac{1}{l^{(j)}_{1}} & -\frac{1}{l^{(j)}_{1}} \\ 0 & 0 & \cdots\ \cdots & \cdots & \cdots & 0 & -\frac{1}{l^{(j)}_{1}} & \frac{1}{l^{(j)}_{1}} + \frac{1}{l^{(j)}_{0}} \end{pmatrix}$$

for $j = 1, 2, \ldots, q$. *Note that for* $M > 0$ *the eigenvalue problem* (3.5) *is equivalent to the spectral problem for the star-patterned matrix* $\mathcal{M}^{-1/2}\mathcal{L}\mathcal{M}^{-1/2}$. *For details we refer the reader to* [21*, Sect.* 4].

While Theorem 3.4 *provides necessary and sufficient conditions for a sequence* $\{\lambda_k^2\}_{k=1}^{n+1}$, *counted with multiplicities, to be the eigenvalues of a star-patterned matrix* $\mathcal{M}^{-1/2}\mathcal{L}\mathcal{M}^{-1/2}$, [13, *Thm.* 9] *provides necessary and sufficient conditions on a vector* $(p_i(N))_{i=1}^{r_N}$ *of integers to be the vector of multiplicities of eigenvalues of a star-patterned matrix. Moreover, our method allows to construct the matrices* $\mathcal{M}$ *and* $\mathcal{L}$ *explicitly if a set of lengths* $\{l_j\}_{j=1}^{q}$ *is given* (*see Example* 4.2 *below*).

The correspondence of the conditions on the multiplicities is as follows. Condition (3) *in Theorem* 3.4 *is condition* (*d*) *in* [13*, Thm.* 9]. *Condition* (6) *in Remark* 3.5 *is condition* (*a*) *in* [13*, Thm.* 9] (*that* (*a*) *is not really needed was already mentioned in* [13*, Rem. on p.* 19]). *Condition* (4) *in Remark* 3.5 *is condition* (*c*) *in* [13*, Thm.* 9]. *Condition* (*b*) *in* [13*, Thm.* 9] *is automatically satisfied since we require the number of given* λ_k^2 *counted with multiplicities to be equal to* $n+1$ (*note the different definition of* n *in* [13*, Thm.* 9]). *Condition* (5) *in Remark* 3.5 *is mentioned in* [13*, Rem. on p.* 19].

4. Examples

In this section we show how the necessary conditions in Theorem 3.3 may be conveniently used to decide whether two sequences of numbers can be the Neumann and Dirichlet spectra of a star graph of Stieltjes strings. The same applies for Theorem 3.4 with just one sequence being the Neumann spectrum.

Example 4.1.

(a) Given $q = 5$, $n_1 = 4$, $n_2 = 3$, $n_3 = 2$, $n_4 = n_5 = 1$, and hence $n = \sum_{j=1}^{5} n_j = 11$. The sequence $\{\pm\lambda_k\}_{k=1}^{12}$ given by

$$\lambda_1^2 = 1,\ \ \lambda_2^2 = 2,\ \ \lambda_3^2 = \lambda_4^2 = \lambda_5^2 = 3,\ \ \lambda_6^2 = 4,$$
$$\lambda_7^2 = 5,\ \ \lambda_8^2 = \lambda_9^2 = \lambda_{10}^2 = \lambda_{11}^2 = 6,\ \ \lambda_{12}^2 = 7,$$

satisfies conditions (1), (2) of Theorem 3.4. However, it cannot be the sequence of Neumann eigenvalues of a star graph of 5 Stieltjes strings of any lengths and any central mass $M > 0$ with the above numbers of masses on the strings. This follows since the number r_N of distinct elements in this sequence is

$$r_N = 7 < 8 = n_1 + n_2 + 1$$

and hence the necessary condition (6) in Remark 3.5 for the majorization condition (3) in Theorem 3.4 is not satisfied.

(b) Given $q = 7$, $n_1 = 5$, $n_2 = 4$, $n_3 = 3$, $n_4 = 2$, $n_5 = n_6 = n_7 = 1$. The sequence $\{\pm\lambda_k\}_{k=1}^{17}$ given by

$$\lambda_1^2 = 1,\ \ \lambda_2^2 = \lambda_3^2 = \lambda_4^2 = \lambda_5^2 = \lambda_6^2 = \lambda_7^2 = 2,\ \ \lambda_8^2 = 3,$$
$$\lambda_9^2 = \lambda_{10}^2 = \lambda_{11}^2 = \lambda_{12}^2 = 4,\ \ \lambda_{13}^2 = 5,\ \ \lambda_{14}^2 = 6,\ \ \lambda_{15}^2 = 7,\ \ \lambda_{16}^2 = 8,\ \ \lambda_{17}^2 = 9,$$

satisfies conditions (1), (2) of Theorem 3.4, and also the necessary condition (6′) in Remark 3.5 because

$$r_N = 9 = n_1 + n_2.$$

However, it cannot be the sequence of Neumann eigenvalues of a star graph of Stieltjes strings of 7 strings of any lengths and central mass $M = 0$ with the above numbers of masses on the strings. This follows since

$$N_1 = 7,\ N_2 = 4,\ N_3 = 3,\ N_4 = 2,\ N_5 = 1,$$

$$p_1^{\downarrow}(N) = 6,\ p_2^{\downarrow}(N) = 4,\ p_j^{\downarrow}(N) = 1,\ j = 3, 4, \dots, 9,$$

and hence

$$\begin{aligned}(N_1 - 1, N_2 - 1, N_3 - 1, N_4 - 1, N_5 - 1) = (6, 3, 2, 1, 0) &\not\succ (6, 4, 1, 1)\\ &= (p_1^{\downarrow}(N), p_2^{\downarrow}(N), p_3^{\downarrow}(N), p_4^{\downarrow}(N))\end{aligned}$$

because $6 + 3 \not\geq 6 + 4$. Thus the necessary majorization condition (3′) in Theorem 3.4 is not satisfied.

Example 4.2. Given $q = 3$, $n_1 = 3$, $n_2 = n_3 = 2$, $l_1 = 12$, $l_2 = l_3 = 1$, and

$$\lambda_1^2 = 3 - \sqrt{5},\ \ \zeta_1^2 = 1,\ \ \lambda_2^2 = \frac{5}{2},\ \ \zeta_2^2 = \lambda_3^2 = \zeta_3^2 = \lambda_4^2 = \zeta_4^2 = 4, \tag{4.1}$$

$$\lambda_5^2 = 3 + \sqrt{5},\ \ \zeta_5^2 = \lambda_6^2 = \zeta_6^2 = \lambda_7^2 = \zeta_7^2 = 6. \tag{4.2}$$

Then $n = 7$ and

$$N_1 = 3,\ \ N_2 = 3,\ \ N_3 = 1,\quad r_D = 3,\ \ r_N = 5.$$

The sequences $\{\pm\lambda_k\}_{k=1}^{7}$ and $\{\pm\zeta_k\}_{k=1}^{7}$ are interlacing as required in assumption (1′) of Theorem 2.9. Since every other Neumann eigenvalue is simple, also assumption (2′) of

Theorem 2.9 is satisfied. Further,

$$\sum_{i=1}^{r_N} p_i^{\downharpoonleft}(N) = 2+2+1+1+1 = 7 = n,$$

which is assumption (3′) of Theorem 2.9. Finally,

$$(N_1, N_2, N_3) = (3,3,1) \succ (3,3,1) = (p_1^{\downharpoonleft}(D), p_2^{\downharpoonleft}(D), p_3^{\downharpoonleft}(D)),$$

$$(N_1 - 1, N_2 - 1, N_3 - 1) = (2,2,0) \succ (2,2) = (p_1^{\downharpoonleft}(N), p_2^{\downharpoonleft}(N)).$$

The last relation shows that the majorization assumption (3′) of Theorem 2.9 holds.

Hence, by Theorem 3.3, there exists a star graph of Stieltjes strings with the above numbers of masses on each string and lengths of strings and with central mass $M = 0$ such that the sequences $\{\pm\lambda_k\}_{k=1}^7$, $\{\pm\zeta_k\}_{k=1}^7$ are the corresponding Dirichlet and Neumann eigenvalues.

In fact, the solution of this inverse problem can be calculated explicitly using the constructive proof of [21, Thm. 2.9] in the case $M = 0$; note that the latter is immediate from the fact that there are $n = 7$ elements in both given sequences. Sticking to the notation in [21, Sect. 2], we have

$$\begin{aligned}
\frac{\phi_{N,3}(z)}{\phi_{D,3}(z)} &= \left(\sum_{j=1}^{3} \frac{1}{l_j}\right) \frac{\prod_{k=1}^{7}\left(1-(z/\lambda_k^2)\right)}{\prod_{k=1}^{7}\left(1-\left(z/\zeta_k^2\right)\right)} \\
&= \left(\frac{1}{12}+\frac{1}{1}+\frac{1}{1}\right) \frac{(1-z/(3-\sqrt{5}))(1-z/(5/2))(1-z/(3+\sqrt{5}))}{(1-z)(1-z/4)(1-z/6)} \\
&= 5 - \frac{1}{2-2z} - \frac{1}{4-z} - \frac{1}{6-z} - \frac{2}{2-z/2} - \frac{2}{2-z/3} \\
&= 5 - \frac{1}{2-2z} - \frac{5}{4-z} - \frac{7}{6-z} \\
&= \frac{\phi_N^{(1)}(z)}{\phi_D^{(1)}(z)} + \frac{\phi_N^{(2)}(z)}{\phi_D^{(2)}(z)} + \frac{\phi_N^{(3)}(z)}{\phi_D^{(3)}(z)}
\end{aligned}$$

with

$$\begin{aligned}
\frac{\phi_N^{(1)}(z)}{\phi_D^{(1)}(z)} &= 1 - \frac{1}{2-2z} - \frac{1}{4-z} - \frac{1}{6-z}, \\
\frac{\phi_N^{(i)}(z)}{\phi_D^{(i)}(z)} &= 2 - \frac{1}{2-z/2} - \frac{1}{2-z/3}, \quad i = 2,3.
\end{aligned}$$

The following continued fraction expansions were computed with Maple and may be readily verified:

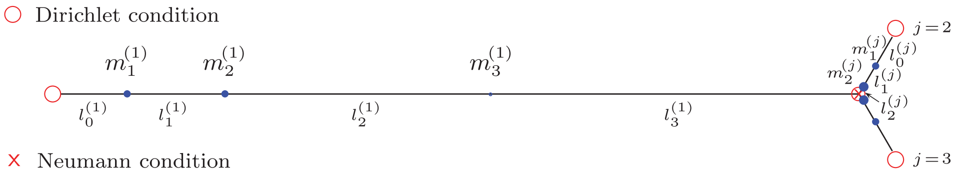

Figure 1. Star graph of Stieltjes strings solving the inverse problem in Ex. 4.2.

$$\frac{\phi_D^{(1)}(z)}{\phi_N^{(1)}(z)} = \left(1 - \frac{1}{2-2z} - \frac{1}{4-z} - \frac{1}{6-z}\right)^{-1}$$

$$= 1 + \cfrac{1}{-\frac{2}{5}z + \cfrac{1}{\frac{25}{17} + \cfrac{1}{-\frac{289}{840}z + \cfrac{1}{\frac{1176}{391} + \cfrac{1}{-\frac{529}{4200}z + \cfrac{1}{\frac{150}{23}}}}}}},$$

$$\frac{\phi_D^{(i)}(z)}{\phi_N^{(i)}(z)} = \left(2 - \frac{1}{2-z/2} - \frac{1}{2-z/3}\right)^{-1}$$

$$= \frac{1}{2} + \cfrac{1}{-\frac{4}{5}z + \cfrac{1}{\frac{25}{54} + \cfrac{1}{-\frac{243}{40}z + \frac{1}{27}}}}, \quad i = 2,3.$$

Therefore the star graph with central mass $M = 0$ consisting of 3 strings with masses and lengths of intervals between given by

$$l_0^{(1)} = 1, \quad l_1^{(1)} = \frac{25}{17}, \quad l_2^{(1)} = \frac{1176}{391}, \quad l_3^{(1)} = \frac{150}{23}, \quad m_1^{(1)} = \frac{2}{5}, \quad m_2^{(1)} = \frac{289}{840}, \quad m_3^{(1)} = \frac{529}{4200},$$

$$l_0^{(i)} = \frac{1}{2}, \quad l_1^{(i)} = \frac{25}{54}, \quad l_2^{(i)} = \frac{1}{27}, \quad m_1^{(i)} = \frac{4}{5}, \quad m_2^{(i)} = \frac{243}{40}, \quad i = 2,3,$$

(see Figure 1) has the sequences $\{\pm\lambda_k\}_{k=1}^7$ and $\{\pm\zeta_k\}_{k=1}^7$ given by (4.1) as Neumann and Dirichlet eigenvalues.

Remark 4.3. *We remark that the constructive procedure in* [21, *Thm.* 2.9] *used above for the case* $M = 0$ *also applies in the case* $M > 0$.

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Note

1. Email: v.pivovarchik@paco.net

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