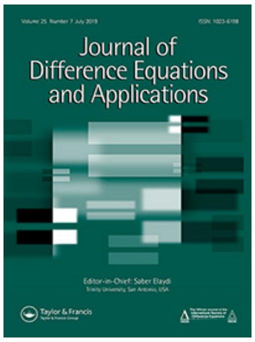

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Proof of a basic hypergeometric supercongruence modulo the fifth power of a cyclotomic polynomial

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ABSTRACT
By means of the q-Zeilberger algorithm, we prove a basic hypergeometric supercongruence modulo the fifth power of the cyclotomic polynomial $\Phi_n(q)$. This result appears to be quite unique, as in the existing literature so far no basic hypergeometric supercongruences modulo a power greater than the fourth of a cyclotomic polynomial have been proved. We also establish a couple of related results, including a parametric supercongruence.

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1. Introduction

In 1997, Van Hamme [27] conjectured that 13 Ramanujan-type series including

$$\sum_{k=0}^{\infty}(-1)^k(4k+1)\frac{(\frac{1}{2})_k^3}{k!^3}=\frac{2}{\pi}$$

admit nice p-adic analogues, such as

$$\sum_{k=0}^{\frac{p-1}{2}}(-1)^k(4k+1)\frac{(\frac{1}{2})_k^3}{k!^3}\equiv p\,(-1)^{\frac{p-1}{2}} \pmod{p^3},$$

where $(a)_n=a(a+1)\dots(a+n-1)$ denotes the Pochhammer symbol and p is an odd prime. Up to present, all of the 13 supercongruences have been confirmed. See [21,24] for historic remarks on these supercongruences. Recently, q-analogues of congruences and supercongruences have caught the interests of many authors [1–7,8–20,23,25,26,29]. In particular, the first author and Zudilin [16] devised a method, called 'creative microscoping', to prove quite a few q-supercongruences by introducing an additional parameter a. In [13], the authors of this paper proved many additional q-supercongruences by the creative microscoping method. Supercongruences modulo a higher integer power of a prime, or, in

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the q-case, of a cyclotomic polynomial, are very special and usually difficult to prove. As far as we know, until now the result

$$\sum_{k=0}^{\frac{n-1}{2}}[4k+1]\frac{(q;q^2)_k^4}{(q^2;q^2)_k^4} \equiv q^{\frac{1-n}{2}}[n]+\frac{(n^2-1)(1-q)^2}{24}q^{\frac{1-n}{2}}[n]^3 \pmod{[n]\Phi_n(q)^3}, \quad (1)$$

for an odd positive integer n, due to the first author and Wang [15], is the unique q-supercongruence modulo $[n]\Phi_n(q)^3$ in the literature that was completely proved. (Several similar conjectural q-supercongruences are stated in [13] and in [16].) The purpose of this paper is to establish an even higher q-congruence, namely modulo a fifth power of a cyclotomic polynomial. Specifically, we prove the following three theorems. (The first two together confirm a conjecture by the authors [13, Conjecture 5.4] .)

Theorem 1.1: *Let $n > 1$ be a positive odd integer. Then*

$$\sum_{k=0}^{\frac{n+1}{2}}[4k-1]\frac{(q^{-1};q^2)_k^4}{(q^2;q^2)_k^4}q^{4k} \equiv -(1+3q+q^2)[n]^4 \pmod{[n]^4\Phi_n(q)} \quad (2a)$$

and

$$\sum_{k=0}^{n-1}[4k-1]\frac{(q^{-1};q^2)_k^4}{(q^2;q^2)_k^4}q^{4k} \equiv -(1+3q+q^2)[n]^4 \pmod{[n]^4\Phi_n(q)}. \quad (2b)$$

Theorem 1.2: *Let $n > 1$ be a positive odd integer. Then*

$$\sum_{k=0}^{\frac{n+1}{2}}[4k-1]\frac{(aq^{-1};q^2)_k(q^{-1}/a;q^2)_k(q^{-1};q^2)_k^2}{(aq^2;q^2)_k(q^2/a;q^2)_k(q^2;q^2)_k^2}q^{4k} \equiv 0 \pmod{[n]^2(1-aq^n)(a-q^n)}$$

and

$$\sum_{k=0}^{n-1}[4k-1]\frac{(aq^{-1};q^2)_k(q^{-1}/a;q^2)_k(q^{-1};q^2)_k^2}{(aq^2;q^2)_k(q^2/a;q^2)_k(q^2;q^2)_k^2}q^{4k} \equiv 0 \pmod{[n]^2(1-aq^n)(a-q^n)}.$$

The $a=-1$ case of Theorem 1.2 admits an even stronger q-congruence.

Theorem 1.3: *Let $n > 1$ be a positive odd integer. Then*

$$\sum_{k=0}^{\frac{n+1}{2}}[4k-1]\frac{(q^{-2};q^4)_k^2}{(q^4;q^4)_k^2}q^{4k} \equiv -q^n(1-q+q^2)[n]_{q^2}^2 \pmod{[n]_{q^2}^2\Phi_n(q^2)} \quad (3a)$$

and

$$\sum_{k=0}^{n-1}[4k-1]\frac{(q^{-2};q^4)_k^2}{(q^4;q^4)_k^2}q^{4k} \equiv -(1-q+q^2)[n]_{q^2}^2 \pmod{[n]_{q^2}^2\Phi_n(q^2)}. \quad (3b)$$

In the above q-supercongruences and in what follows:

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$$

is the q-shifted factorial,

$$[n] = [n]_q = 1 + q + \cdots + q^{n-1}$$

is the q-number,

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}$$

is the q-binomial coefficient and $\Phi_n(q)$ is the nth cyclotomic polynomial of q. Note that the congruences in Theorem 1.1 modulo $[n]\Phi_n(q)^2$ and the congruences in Theorem 1.2 modulo $[n](1-aq^n)(a-q^n)$ have already been proved by the authors in [13, Equations (5.5) and (5.10)].

2. Proof of Theorem 1.1 by the Zeilberger algorithm

The Zeilberger algorithm [cf. 22] can be used to find that the functions

$$f(n,k) = (-1)^k \frac{(4n-1)(-\frac{1}{2})_n^3(-\frac{1}{2})_{n+k}}{(1)_n^3(1)_{n-k}(-\frac{1}{2})_k^2},$$

$$g(n,k) = (-1)^{k-1} \frac{4(-\frac{1}{2})_n^3(-\frac{1}{2})_{n+k-1}}{(1)_{n-1}^3(1)_{n-k}(-\frac{1}{2})_k^2}$$

satisfy the relation

$$(2k-3)f(n,k-1) - (2k-4)f(n,k) = g(n+1,k) - g(n,k).$$

Of course, given this relation, it is not difficult to verify by hand that it is satisfied by the above pair of doubly indexed sequences $f(n,k)$ and $g(n,k)$.

Here we use the convention $1/(1)_m = 0$ for all negative integers m. We now define the q-analogues of $f(n,k)$ and $g(n,k)$ as follows:

$$F(n,k) = (-1)^k q^{(k-2)(k-2n+1)} \frac{[4n-1](q^{-1};q^2)_n^3(q^{-1};q^2)_{n+k}}{(q^2;q^2)_n^3(q^2;q^2)_{n-k}(q^{-1};q^2)_k^2},$$

$$G(n,k) = \frac{(-1)^{k-1}q^{(k-2)(k-2n+3)}(q^{-1};q^2)_n^3(q^{-1};q^2)_{n+k-1}}{(1-q)^2(q^2;q^2)_{n-1}^3(q^2;q^2)_{n-k}(q^{-1};q^2)_k^2},$$

where we have used the convention that $1/(q^2;q^2)_m = 0$ for $m = -1,-2,\ldots$. Then the functions $F(n,k)$ and $G(n,k)$ satisfy the relation

$$[2k-3]F(n,k-1) - [2k-4]F(n,k) = G(n+1,k) - G(n,k). \tag{4}$$

Indeed, it is straightforward to obtain the following expressions:

$$\frac{F(n,k-1)}{G(n,k)} = \frac{q^{2n-4k+6}(1-q)(1-q^{4n-1})(1-q^{2k-3})^2}{(1-q^{2n-2k+2})(1-q^{2n})^3},$$
$$\frac{F(n,k)}{G(n,k)} = -\frac{q^{4-2k}(1-q)(1-q^{4n-1})(1-q^{2n+2k-3})}{(1-q^{2n})^3},$$
$$\frac{G(n+1,k)}{G(n,k)} = \frac{q^{4-2k}(1-q^{2n-1})^3(1-q^{2n+2k-3})}{(1-q^{2n})^3(1-q^{2n-2k+2})}.$$

It is easy to verify the identity

$$\frac{q^{2n-4k+6}(1-q^{4n-1})(1-q^{2k-3})^3}{(1-q^{2n-2k+2})(1-q^{2n})^3} + \frac{q^{4-2k}(1-q^{2k-4})(1-q^{4n-1})(1-q^{2n+2k-3})}{(1-q^{2n})^3}$$
$$= \frac{q^{4-2k}(1-q^{2n-1})^3(1-q^{2n+2k-3})}{(1-q^{2n})^3(1-q^{2n-2k+2})} - 1,$$

which is equivalent to (4). (Alternatively, we could have established (4) by only guessing $F(n,k)$ and invoking the q-Zeilberger algorithm [28].)

Let $m > 1$ be an odd integer. Summing (4) over n from 0 to $(m+1)/2$, we get

$$[2k-3]\sum_{n=0}^{\frac{m+1}{2}} F(n,k-1) - [2k-4]\sum_{n=0}^{\frac{m+1}{2}} F(n,k) = G\left(\frac{m+3}{2},k\right) - G(0,k)$$
$$= G\left(\frac{m+3}{2},k\right). \tag{5}$$

We readily compute

$$G\left(\frac{m+3}{2},1\right) = \frac{q^{m-1}(q^{-1};q^2)^4_{(m+3)/2}}{(1-q)^2(q^2;q^2)^4_{(m+1)/2}(1-q^{-1})^2}$$
$$= \frac{q^{m-3}[m]^4}{[m+1]^4(-q;q)^8_{(m-1)/2}} {\begin{bmatrix} m-1 \\ (m-1)/2 \end{bmatrix}}^4 \tag{6a}$$

and

$$G\left(\frac{m+3}{2},2\right) = -\frac{(q^{-1};q^2)^3_{(m+3)/2}(q^{-1};q^2)_{(m+5)/2}}{(1-q)^2(q^2;q^2)^3_{(m+1)/2}(q^2;q^2)_{(m-1)/2}(q^{-1};q^2)^2_2}$$
$$= -\frac{q^{-2}[m]^4[m+2]}{[m+1]^3(-q;q)^8_{(m-1)/2}} {\begin{bmatrix} m-1 \\ (m-1)/2 \end{bmatrix}}^4. \tag{6b}$$

Combining (5) and (6), we have

$$\begin{aligned}\sum_{n=0}^{\frac{m+1}{2}} F(n,0) &= \frac{[-2]}{[-1]}\sum_{n=0}^{\frac{m+1}{2}} F(n,1) + \frac{1}{[-1]}G\left(\frac{m+3}{2},1\right)\\ &= \frac{1+q}{q}G\left(\frac{m+3}{2},2\right) - qG\left(\frac{m+3}{2},1\right)\\ &= -\frac{(1+q)[m]^4[m+1][m+2]+q^{m+1}[m]^4}{q^3[m+1]^4(-q;q)_{(m-1)/2}^8}\begin{bmatrix} m-1\\ (m-1)/2\end{bmatrix}^4,\end{aligned}$$

i.e.

$$\sum_{n=0}^{\frac{m+1}{2}}[4n-1]\frac{(q^{-1};q^2)_n^4}{(q^2;q^2)_n^4}q^{4n} = -\frac{(1+q)[m]^4[m+1][m+2]+q^{m+1}[m]^4}{q[m+1]^4(-q;q)_{(m-1)/2}^8}\begin{bmatrix} m-1\\ (m-1)/2\end{bmatrix}^4. \tag{7}$$

By [4, Lemma 2.1] (or [3, Lemma 2.1]), we have $(-q;q)_{(m-1)/2}^2 \equiv q^{(m^2-1)/8}$ $(\text{mod } \Phi_m(q))$. Moreover, it is easy to see that

$$\begin{aligned}\begin{bmatrix} m-1\\ (m-1)/2\end{bmatrix} &= \prod_{k=1}^{(m-1)/2}\frac{1-q^{m-k}}{1-q^k}\\ &\equiv \prod_{k=1}^{(m-1)/2}\frac{1-q^{-k}}{1-q^k} = (-1)^{(m-1)/2}q^{(1-m^2)/8} \pmod{\Phi_m(q)},\end{aligned}$$

and $[m]$ is relatively prime to $(-q;q)_{(m-1)/2}$. It follows from (7) that

$$\sum_{n=0}^{\frac{m+1}{2}}[4n-1]\frac{(q^{-1};q^2)_n^4}{(q^2;q^2)_n^4}q^{4n} \equiv -((1+q)^2+q)[m]^4 \pmod{[m]^4\Phi_m(q)}.$$

Concluding, the congruence (2a) holds.

Similarly, summing (4) over n from 0 to $m-1$, we get

$$[2k-3]\sum_{n=0}^{m-1}F(n,k-1) - [2k-4]\sum_{n=0}^{m-1}F(n,k) = G(m,k),$$

and so

$$\begin{aligned}\sum_{n=0}^{m-1}[4n-1]\frac{(q^{-1};q^2)_n^4}{(q^2;q^2)_n^4}q^{4n} &= \frac{1+q}{q}G(m,2) - qG(m,1)\\ &= -\frac{(1+q)[2m-2][2m-1]+q^{2m-2}}{q(-q;q)_{m-1}^8}\begin{bmatrix} 2m-2\\ m-1\end{bmatrix}^4. \qquad (8)\end{aligned}$$

It is easy to see that

$$\frac{1}{[m]}\begin{bmatrix}2m-2\\m-1\end{bmatrix}=\frac{1}{[m-1]}\begin{bmatrix}2m-2\\m-2\end{bmatrix}\equiv(-1)^{m-2}q^{2-\binom{m-1}{2}}\pmod{\Phi_m(q)},$$

and $(-q;q)_{m-1}\equiv 1 \pmod{\Phi_m(q)}$ [4]. The proof of (2b) then follows easily from (8).

3. Proof of Theorems 1.2 and 1.3

Proof of Theorem 1.2: It is easy to see by induction on N that

$$\sum_{k=0}^{N}[4k-1]\frac{(aq^{-1};q^2)_k(q^{-1}/a;q^2)_k(q^{-1};q^2)_k^2}{(aq^2;q^2)_k(q^2/a;q^2)_k(q^2;q^2)_k^2}q^{4k}$$
$$=\frac{(aq;q^2)_N(q/a;q^2)_N((a+1)^2q^{2N+1}-a(1+q)(1+q^{4N+1}))}{q(a-q)(1-aq)(aq^2;q^2)_N(q^2/a;q^2)_N(-q;q)_N^4}\begin{bmatrix}2N\\N\end{bmatrix}^2. \qquad (9)$$

For $N=(n+1)/2$ or $N=n-1$, we see that $(aq;q^2)_N(q/a;q^2)_N$ contains the factor $(1-aq^n)(1-q^n/a)$. Moreover,

$$\frac{[(n+1)/2]}{[n]}\begin{bmatrix}n\\(n-1)/2\end{bmatrix}=\begin{bmatrix}n-1\\(n-1)/2\end{bmatrix}$$

is a polynomial in q. Since $[(n+1)/2]$ and $[n]$ are relatively prime, we conclude that $\begin{bmatrix}n\\(n-1)/2\end{bmatrix}$ is divisible by $[n]$. Therefore, $\begin{bmatrix}n+1\\(n+1)/2\end{bmatrix}=(1+q^{(n+1)/2})\begin{bmatrix}n\\(n-1)/2\end{bmatrix}$ is also divisible by $[n]$. It is also well known that $\begin{bmatrix}2n-2\\n-1\end{bmatrix}$ is divisible by $[n]$. Moreover, it is easy to see that $[n]$ is relatively prime to $1+q^m$ for any non-negative integer m. The proof then follows from (9) by taking $N=(n+1)/2$ and $N=n-1$. ■

Proof of Theorem 1.3: For $a=-1$, the identity (9) reduces to

$$\sum_{k=0}^{N}[4k-1]\frac{(q^{-2};q^4)_k^2}{(q^4;q^4)_k^2}q^{4k}=-\frac{(-q;q^2)_N^2(1+q^{4N+1})}{q(1+q)(-q^2;q^2)_N^2(-q;q)_N^4}\begin{bmatrix}2N\\N\end{bmatrix}^2$$
$$=-\frac{(1+q^{4N+1})}{q(1+q)(-q^2;q^2)_N^4}\begin{bmatrix}2N\\N\end{bmatrix}_{q^2}^2. \qquad (10)$$

Note that, in the proof of Theorem 1.2, we have proved that $\begin{bmatrix}2N\\N\end{bmatrix}_{q^2}$ is divisible by $[n]_{q^2}$ for both $N=(n+1)/2$ and $N=n-1$. Moreover, $[n]_{q^2}$ is relatively prime to $(-q^2;q^2)_m$ for $m\geqslant 0$. Hence the right-hand side of (10) is congruent to 0 modulo $[n]_{q^2}^2$ for $N=(n+1)/2$ or $N=n-1$. To further determine the right-hand side of (10) modulo $[n]_{q^2}^2\Phi_n(q^2)$, we need only to use the same congruences (with $q\mapsto q^2$) used in the proof of Theorem 1.1. ■

4. Immediate consequences

Notice that for $n=p^r$ being an odd prime power, $\Phi_{p^r}(q)=[p]_{q^{p^{r-1}}}$ holds. This observation was used in [15] to extend (1) to a supercongruence modulo $[p^r][p]^3_{q^{p^{r-1}}}$. In the same vein, we immediately deduce from Theorem 1.1 the following result:

Corollary 4.1: *Let p be an odd prime and r a positive integer. Then*

$$\sum_{k=0}^{\frac{p^r+1}{2}}[4k-1]\frac{(q^{-1};q^2)_k^4}{(q^2;q^2)_k^4}q^{4k} \equiv -(1+3q+q^2)[p^r]^4 \pmod{[p^r]^4[p]_{q^{p^{r-1}}}} \tag{11a}$$

and

$$\sum_{k=0}^{p^r-1}[4k-1]\frac{(q^{-1};q^2)_k^4}{(q^2;q^2)_k^4}q^{4k} \equiv -(1+3q+q^2)[p^r]^4 \pmod{[p^r]^4[p]_{q^{p^{r-1}}}}. \tag{11b}$$

The $q \to 1$ limiting cases of these two identities yield the following supercongruences:

Corollary 4.2: *Let p be an odd prime and r a positive integer. Then*

$$\sum_{k=0}^{\frac{p^r-1}{2}} \frac{4k+3}{16(k+1)^4\,256^k}\binom{2k}{k}^4 \equiv 1-5p^{4r} \pmod{p^{4r+1}} \tag{12a}$$

and

$$\sum_{k=0}^{p^r-2} \frac{4k+3}{16(k+1)^4\,256^k}\binom{2k}{k}^4 \equiv 1-5p^{4r} \pmod{p^{4r+1}}. \tag{12b}$$

Similarly, we deduce from Theorem 1.3 the following result:

Corollary 4.3: *Let p be an odd prime and r a positive integer. Then*

$$\sum_{k=0}^{\frac{p^r+1}{2}}[4k-1]\frac{(q^{-2};q^4)_k^2}{(q^4;q^4)_k^2}q^{4k} \equiv -q^{p^r}(1-q+q^2)[p^r]_{q^2}^2 \pmod{[p^r]_{q^2}^2[p]_{q^{2p^{r-1}}}} \tag{13a}$$

and

$$\sum_{k=0}^{p^r-1}[4k-1]\frac{(q^{-2};q^4)_k^2}{(q^4;q^4)_k^2}q^{4k} \equiv -(1-q+q^2)[p^r]_{q^2}^2 \pmod{[p^r]_{q^2}^2[p]_{q^{2p^{r-1}}}}. \tag{13b}$$

The $q \to 1$ limiting cases of these two identities yield the following supercongruences:

Corollary 4.4: *Let p be an odd prime and r a positive integer. Then*

$$\sum_{k=0}^{\frac{p^r-1}{2}} \frac{4k+3}{4(k+1)^2\,16^k}\binom{2k}{k}^2 \equiv 1-p^{2r} \pmod{p^{2r+1}} \tag{14a}$$

and

$$\sum_{k=0}^{p^r-2} \frac{4k+3}{4(k+1)^2\,16^k}\binom{2k}{k}^2 \equiv 1-p^{2r} \pmod{p^{2r+1}}. \tag{14b}$$

The supercongruences in Corollaries 4.2 and 4.4 are remarkable since they are valid for arbitrarily high prime powers. Swisher [24] had empirically observed several similar but different hypergeometric supercongruences and stated them without proof.

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