THE CAPACITY REGION OF THE GAUSSIAN Z-INTERFERENCE CHANNEL WITH GAUSSIAN INPUT AND WEAK INTERFERENCE

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ABSTRACT

We consider a wireless communication scenario with two transmit-receive pairs where each of the transmitters has a message for its corresponding receiver and only one of the receivers face interference from the undesired transmitter. In our research, we focused on devising optimal ways to manage this undesired interference and characterize the best communication rates for both transmit-receive pairs. Currently, this problem of interference is dealt with by restricting the two communications in different frequency or time bands. We explore the possibility of achieving better rates by allowing them to operate in the same band. Such channels were identified about 4 decades ago, but the maximum rate of communication when the transmitters have a power constraint is still unknown. In this work, we characterize the best rates for this channel under a reasonable practical constraint of using Gaussian signals at both the transmitters.

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CHAPTER 1. OVERVIEW

We consider the scenario where two users try to communicate with another two users, but they face interference from each other. If only one of the two channels face interference from the other channel, then we call such a channel a Z-Interference Channel.

Before we describe the problem, we will first go through the basics and address questions like what is information. Chapter 1 introduces the reader to information theory and related concepts of entropy and mutual information. Proper definitions for all related entropy terms, mutual information and the capacity of the channel are provided. It also discusses the concept of Rate-Distortion theory, which will be used later in our work.

Then, we introduce the Interference channel and provide the motivations for investigating into such channels in Chapter 2. We familiarize ourselves with the current state of knowledge by reviewing various ongoing research projects in this field. We define our notations and lay the background for our work on Z-Interference Channel.

Next, we define the Z-Interference channel in details in Chapter 3. We are interested in the weak interference regime and we discuss the best known achievable region for such channels.

Now that we have defined the Z-Interference channel and seen achievable regions, we try to derive an upper bound for such channels. We analyze the derived upper bound and find that the bound is actually loose. Based on the intuitions gained from the derivation, we derive another bound for the Z-Interference channel and with a specific example show that this outer bound matches with the best known achievable scheme. Thus, we show the optimality of the known achievable schemes, which was previously unknown.

Finally, we conclude this work by discussing related problems to the Z-Interference channels. The Appendix shows the proofs that we skipped in our main work.

CHAPTER 2. INTRODUCTION

Information theory answers two fundamental questions in communication theory: What is the ultimate data compression (answer: entropy), and what is the ultimate transmission rate of communication (answer: channel capacity). Information theory has relations to physics (statistical mechanics), mathematics (probability theory), electrical engineering (communication theory), and computer science (algorithmic complexity). But. here we talk about it in the aspect of communication theory.

In the early 1940s it was thought to be impossible to send information at a positive rate with negligible probability of error. Shannon surprised the communication theory community by proving that the probability of error could be made nearly zero for all communication rates below channel capacity. The capacity can be computed simply from the noise characteristics of the channel. Shannon further argued that random processes such as music and speech have an irreducible complexity below which the signal cannot be compressed. This he named the entropy, in deference to the parallel use of this word in thermodynamics, and argued that if the entropy of the source is less than the capacity of the channel, asymptotically error-free communication can be achieved.

Before we go into formal definitions, we first need to understand what information is. When a person hears something and learns something new, then he gained some information. However, if it did not convey anything new, then he did not gain any information. This suggests that information conveys something that was not known. Therefore, stating the obvious does not count as giving information. If any outcome is deterministic, then it does not contain information. On the other hand, consider the toss of a fair coin, the outcome could not be known with certainty before the coin is tossed. As a result, knowing the outcome of the coin toss reveals some information about the random experiment of tossing the coin. Hence, information is always accompanied by some amount of uncertainty to the event of interest.

The best way to model uncertainty is through Random Variables (RV). Information content of a RV is related to the uncertainty present in the RV. A continuous random variable is described by its probability distribution function (PDF) and a discrete random variable is described by its probability mass function (PDF) . Now let us see how to characterize the uncertainty in a RV from its PDF/PMF by a quantity called *entropy*.

We will now define the required terms formally and take a detailed look at Shannon's theory. The definitions and notations of the information-theoretic terms those we state here are taken by [1]. All the logarithms are taken with respect to base 2.

2.1. Entropy

Let X be a discrete random variable which can take values from the set $\mathcal{X}$. Let the PMF of X be denoted by $p(x) = Pr\{X = x\},\ x \in \mathcal{X}$.

Definition 1. *The entropy $H(X)$ of a discrete random variable X is defined by*

$$H(X) = -\sum_{x \in \mathcal{X}} p(x) \log p(x) \tag{2.1}$$

If the logarithm is to the base 2, then the unit of entropy is *bits*. If the base of the logarithm is e, then entropy is measured in *nats*.

Let us go back to the example of tossing a coin. If we model this by a random variable X then X will be a binary random variable taking values from $\{heads,\ tails\}$. If we represent the probability of getting heads by p, i.e $Pr(X = heads) = p$, then $Pr(X = tails) = 1 - p$.

The entropy of X is given by,

$$H(X) = -p \log p - (1-p)\log(1-p) \tag{2.2}$$

Now, let us look at the case when $p = 1$. Then the entropy is,

$$H(X) = -1 \log 1 - (1-1)\log(1-1) = 0$$

where we have used the fact that $0 \log 0 = 0$, which is justified by the fact that $\lim_{x\to 0} x \log(x) = 0$bits.

So, we see that when $p = 1$, i.e. when the event is deterministic, the entropy is zero. Thus, we see how entropy represents the randomness present in a RV. $p = 1$ or $p = 0$ implies zero randomness and for such a X, we have $h(X) = 0$.

It is easy to show that the expression in (2.2) is maximized when $p = \frac{1}{2}$. Now, intuitively as well, the RV has maximum randomness when both outcomes are equally likely. Thus, we see that entropy is maximized when the randomness of RV is maximum. This maximum value is given by

$$H(X) = -\frac{1}{2}\log\frac{1}{2} - (1-\frac{1}{2})\log(1-\frac{1}{2}) = 1\text{bits}$$

So, entropy is a measure of randomness of a RV. We have seen it for the case of discrete RV. Now, let us generalize it to the case of continuous RV - the *differential entropy*.

Definition 2. *The differential entropy $h(X)$ of a continuous random variable X with PDF $f_X(x)$ is defined as,*

$$h(X) = -\int_S f_X(x) \log f_X(x) dx \tag{2.3}$$

where S is the support set of the random variable X.

Now, let us turn our attention to the randomness in a pair of random variables. We can extend the concept of entropy to *joint entropy* and *conditional entropy.* Later, we shall see that joint entropy can be expanded using chain rule and can be written as a sum of conditional entropy terms. The relationship between joint entropy and conditional entropy will play an important role later on. Here, we define them in terms of discrete random variables, but the extension to continuous random variables follows naturally.

Definition 3. *The joint entropy $H(X, Y)$ of a pair of discrete random variables (X, Y) with a joint distribution $p(x, y)$ is defined as,*

$$H(X, Y) = -\sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x, y) \log p(x, y) \tag{2.4}$$

where X and Y take values from $\mathcal{X}$ and $\mathcal{Y}$ respectively.

Definition 4. *If X and Y are two random variables with joint distribution $p(x, y)$, then the conditional entropy of Y given the random variable X, denoted by $H(X, Y)$, is defined as,*

$$H(Y|X) = -\sum_{x \in \mathcal{X}} p(x) H(Y|X = x) = \sum_{x \in \mathcal{X}} p(x) \sum_{y \in \mathcal{Y}} p(y|x) \log p(y|x) \tag{2.5}$$

where X and Y take values from $\mathcal{X}$ and $\mathcal{Y}$ respectively.

The relation between joint entropy and conditional entropies is given by the Chain rule of entropy.

Chain Rule of Entropy: Let $X_1, X_2, ..., X_n)$ be drawn according to $p(x_1, ..., x_n)$. Then,

$$H(X_1, ..., X_n) = \sum_{i=1}^{n} H(X_i|X_{i-1}, ..., X_n) \tag{2.6}$$

For two random variables X and Y, we can relate entropy, conditional entropy and joint entropy by using chain rule of entropy:

$$H(X, Y) = H(X) + H(Y|X) = H(Y) + H(X|Y) \tag{2.7}$$

This equation can be interpreted as the fact the entropy of a pair of variables is the sum of entropy of one RV and the conditional entropy of the other, given the first one.

Now, we define the information content in one random variable regarding another random variable though *mutual information*. We will soon see that mutual information between two random variables is the maximum possible rate of information transfer without any error.

2.2. Mutual Information

Mutual information between two random variables is the amount of information one variable carries about the other. It is the reduction in the uncertainty of one random variable due to the knowledge of the other. Again, we define it first with respect to discrete random variables and the extension to continuous random variables follows naturally.

Definition 5. *If X and Y are two random variables with joint distribution $p(x, y)$ and marginal distribution $p(x)$ and $p(y)$ respectively, then the mutual information*

$I(X;Y)$ *is given by,*

$$I(X;Y) = -\sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p(x,y)\log\frac{p(x,y)}{p(x)p(y)} \tag{2.8}$$

Note that the definition is symmetric with respect to X and Y, i.e. $I(X;Y) = I(Y;X)$. Thus, X has as much information about Y as much Y has about X.

Now let us see how mutual information is related to entropy.

2.3. Relationship between Entropy and Mutual Information

We can write mutual information $I(X;Y)$ as,

$$I(X:Y) = H(X) - H(X|Y) = H(Y) - H(Y|X) \tag{2.9}$$

This equations shows that mutual information can be interpreted as the reduction in the uncertainty of one random variable due to the knowledge of the other random variable.

It follows the definition of mutual information that,

$$I(X;Y) \leq H(X),\ I(X;Y) \leq H(Y),\ \text{and } I(X;Y) \geq 0$$

Also, note that,

$$I(X;X) = H(X) - H(X|X) = H(X) \tag{2.10}$$

This shows that the entropy of X is the amount of information that X carries about itself. For this reason, entropy is sometimes called *self-information.*

Mutual information between the input X and the output of the channel Y is actually the rate of information transfer on a channel. To see this, consider the situation when X is exactly conveyed to the output without any distortion, i.e. $Y =$

X. Then,

$$I(X;Y) = H(Y) - H(Y|X) = H(X) - H(X|X) = H(X)$$

Now, $H(X)$ is the amount of information in X and when $Y = X$ all of the information is obtained at the output. Mutual information gives us exactly this quantity.

If Mutual information does indeed give us the rate of information transfer on a channel, then it leads us to think that the maximum possible mutual information will actually be the maximum rate of information transfer on that channel, or the *capacity* of the channel. So, let us first define capacity of the channel.

2.4. Channel Capacity

Definition 6. *The information channel capacity of a channel is defined as,*

$$C = \max_{p(x)} I(X^n; Y^n) \tag{2.11}$$

where the maximum is taken over all possible input distributions $p(x)$*.*

Shannon, in his celebrated paper, showed that it was possible to send information reliably, i.e. $Pr(error) = 0$, if the rate of transfer was less than or equal to the capacity. If the rate of communication is higher than the capacity of the channel, then there would be an non-zero probability of error. For a detailed proof, see [1], ch 7.

2.5. Rate Distortion Theory

Now we consider the problem of representing a random variable using fewer bits than is required to represent the random variable completely. Since, we are using fewer bits, the constructed random variable will be a little distorted (or quantized) version of the original random variable. The problem is how to efficiently use the given bits to come as close as the original version.

We first consider the problem of representing a single sample from the source. Let the random variable be represented be X and let the representation of X be denoted as $\hat{X}(X)$. If we are given R bits to represent X, the function $\hat{X}$ can take on 2^R values. The problem is to find the optimum set of values for $\hat{X}$ (called the reproduction points or code points) and the regions that are associated with each value $\hat{X}$. But in order to measure how good our representation is, we need to define some way to measure how close our approximation is, which leads us to distortion function (or distortion measure).

Definition 7. *A distortion function or distortion measure is a mapping $d : X \times \hat{X} \to R^+$ from the set of source alphabet-reproduction alphabet pairs into the set of nonnegative real numbers. The distortion $d(X, \hat{X})$ is a measure of the cost of representing the symbol X by the symbol $\hat{X}$.*

Examples of common distortion functions are:

Hamming distortion The Hamming distortion is given by,

$$d(x, \hat{x}) = \begin{cases} 0 \text{ if } x = \hat{x} \\ 1 \text{ if } x \neq \hat{x} \end{cases} \tag{2.12}$$

which results in a probability of error distortion, since $\mathbb{E}[d(X, \hat{X})] = Pr(X \neq \hat{X})$

Squared-error distortion The squared-error distortion is given by,

$$d(x, \hat{x}) = (x - \hat{x})^2 \tag{2.13}$$

This is the most popular distortion measure used for continuous alphabets because of its simplicity and its relationship to least-squares prediction.

So far, we have defined the distortion measure on a symbol-by-symbol basis. We now extend the definition to sequences by using the following definition:

Definition 8. *The distortion between sequences x^n and $\hat{x}^n$ is defined by*

$$d(x^n, \hat{x}^n) = \frac{1}{n} \sum_{i=1}^{n} d(x_i, \hat{x}_i) \tag{2.14}$$

So the distortion for a sequence is the average of the per symbol distortion of the elements of the sequence.

Definition 9. *A $(2^{nR}, n)$-rate distortion code consists of an encoding function,*

$$f_n : \mathcal{X}^n \to \{1, 2, ..., 2^{nR}\} \tag{2.15}$$

and a decoding(reproducing) function,

$$g_n : \{1, 2, ..., 2^{nR}\} \to \mathcal{X}^n \tag{2.16}$$

The distortion associated with the $(2^{nR}, n)$ code is defined as,

$$D = \mathbb{E}[d(X^n, g_n(f_n(X^n)))] \tag{2.17}$$

where the expectation is with respect to the probability distribution on X.

The set of n-tuples $g_n(1), g_n(2), ..., g_n(2^{nR})$ are denoted by $\hat{X}^n(1), ..., \hat{X}^n(2^{nR})$.

Definition 10. *A rate distortion pair (R, D) is said to be achievable if there exists a sequence of $(2^{nR}, n)$-rate distortion codes (f_n, g_n) with $\lim_{n\to\infty} \mathbb{E}[d(X^n, g_n(f_n(X^n)))] \leq D$.*

Definition 11. *The rate distortion region for a source is the closure of the set of achievable rate distortion pairs (R, D).*

Now, for a Gaussian source with $\mathcal{N}(0, \sigma^2)$, the rate-distortion function with squared-error distortion is given by (see [1], theorem 10.3.2),

$$R(D) = \begin{cases} \frac{1}{2}\log\left(\frac{\sigma^2}{D}\right), & 0 \le D \le \sigma^2 \\ 0, & D > \sigma^2 \end{cases} \tag{2.18}$$

When $D \le \sigma^2$, then we have $R(D) = \log\left(\frac{\sigma^2}{D}\right)$ and we can achieve this rate distortion pair if we choose a joint distribution,

$$X = \hat{X} + Z,\ \hat{X} \sim \mathcal{N}(0, \sigma^2 - D),\ Z \sim \mathcal{N}(0, D) \tag{2.19}$$

where $\hat{X}$ and Z are independent. It is easy to that $I(X; \hat{X}) = \log\left(\frac{\sigma^2}{D}\right)$ and thus the bound in (2.18) is achieved. If $D > \sigma^2$, then we can choose $\hat{X} = 0$ with probability 1, achieving $R(D) = 0$. Therefore, we can have the quantized version of a Gaussian random variable as another Gaussian random variable with variance less than that of the original random variable. We will use this important fact later in our analysis. Figure (1) illustrates this.

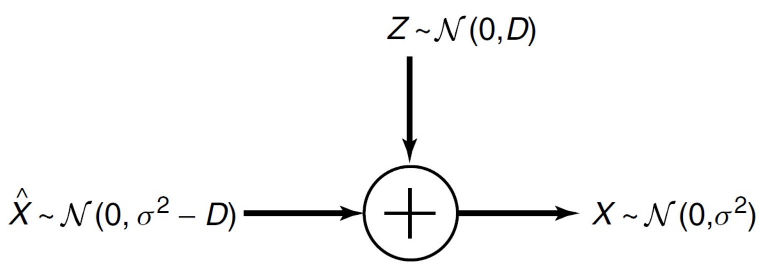

Figure 1. Quantization of Gaussian Random Variable

With this background in mind, we now define the Interference Channels in Chapter 2.

CHAPTER 3. INTERFERENCE CHANNEL

In this chapter, we define the Interference Channel and provide the motivations to study such channels. We discuss the previous works in this field and provide the current state of knowledge about Interference Channels.

The situation often occurs where several sender-receiver pairs share a common communication channel so that transmission of information from one sender to its corresponding receiver interferes with communications between the other senders and their receivers. In radio communications, for example, since the electromagnetic spectrum is a limited resource, frequency bands are often simultaneously used by several radio links that are not completely isolated. A communication channel that is shared in this manner is called an *interference channel*.

A more general concept, the *interference network*, is defined as a communication network with M senders, or input terminals, respectively, with alphabets $\mathcal{X}_1, \mathcal{X}_2, ..., \mathcal{X}_M$; N receivers, or output terminals, with alphabets $\mathcal{Y}_1, \mathcal{Y}_2, ..., \mathcal{Y}_M$, respectively; and a collection of conditional probability measures on the set of output signals, given the input signals.

An interference channel is an M-to-M network (i.e., an interference network with the same number M of senders and receivers) where a one-to-one correspondence exists between senders and receivers such that each sender communicates information only to its corresponding receiver. Thus, an interference channel has M principal links (between its corresponding terminals) and $M(M-1)$ interference links.

There are several motivations to study such kind of channels. Usually, when we don't want any interference, we would use multiplexing of some kind, such as frequency division multiplexing where we give separate bandwidth to the different users so that they don't interfere with each other. However, there may arises scenarios where such complete bandwidth separation might not be possible or too expensive to implement. For example, consider the situation where we have 10 kHz of bandwidth available for communication, but we have multiple links of voice communication. Since speech requires 10 kHz of bandwidth, there is no choice other than having the transmitter-receiver pairs to face interference from each other. In such a scenario, in order to maximize our communication rates, we need to know the capacity of the interference channel.

Another motivation to study the interference channel is the fact that we can increase our rate of communication if, instead of allowing users to transmit in their own frequency range we make them transmit in the same frequency band (thereby increasing the frequency band for each transmitter-receiver pair) and have interference with other. This can be seen from the point of view that whatever can be achieved by the users, while using their own frequency, can also be achieved from interference channel point of view - by simple frequency division multiplexing to avoid interference. Thus, the achievable rate of communication in the case of interference channel is a super-set of the rate of communication that is achievable when we remove the interference by frequency division multiplexing. This motivates us to characterize the capacity region of the interference channels in the hope that we can strictly improve our rate of communication by allowing users to interfere. Etkin et al [2] already showed that this is indeed true and we can actually strictly improve our rate of communication by allowing users to interfere. Therefore, our study of interference channels will lead to development of better rate of communications.

An interference channel has M principal links (between corresponding terminals) and $M(M-1)$ interference links. So, the capacity region of the interference channel is M-dimensional.

The capacity region $\mathcal{C}$ of an interference channel is the closure of the set of rate vectors $\mathbf{R} = (R_1, R_2, ..., R_M)$ for which jointly reliable communications are possible over the M principal links, with independent information sources at the input terminals.

Before we continue, we will first define the notations used here. We denote $n-$letter random variables by capital letters, as in $\boldsymbol{X}$. Also, we introduce the notation $\tilde{a} = \frac{1}{a}$. All logarithms are taken with respect to base 2. I represents the unitary matrix of dimension n. $\mathcal{N}(p, q)$ represents a random variable with Gaussian distribution with mean p and variance q.

The general interference channel with M users is complicated and we consider a simpler case when $M = 2$. This kind of channel is called the two-user interference channel. When the added noise at the receivers is Gaussian, then the channel is called Gaussian two-user interference channel.

For the two-user interference channel, a rate pair (R_1, R_2) is said to be achievable if there exists a sequence of $(\lceil 2^{nR_1} \rceil, \lceil 2^{nR_2} \rceil, n)$ codes, such that $P_{e,1}^{(n)} \to 0$ and $P_{e,2}^{(n)} \to 0$ as $n \to \infty$, where $P_{e,i}^{(n)}$ represents the n-letter probability of error of transmission for user i, and $i \in \{1, 2\}$. The rates are expressed in terms of bits per channel use.

A two-user Gaussian Interference Channel in *standard form* is defined as,

$$\begin{aligned} \boldsymbol{Y_1} &= \boldsymbol{X_1} + \sqrt{a}\boldsymbol{X_2} + \boldsymbol{Z_1},\ a \in \mathbb{R}^+ \\ \boldsymbol{Y_2} &= \sqrt{b}\boldsymbol{X_1} + \boldsymbol{X_2} + \boldsymbol{Z_2},\ b \in \mathbb{R}^+ \end{aligned} \tag{3.1}$$

$$\text{where } \boldsymbol{Z_u} \sim \mathcal{N}(0, I),\ \operatorname{tr}\big(\mathbb{E}[|\boldsymbol{X_u}|^2]\big) \le P_u,\ u = \{1, 2\}$$

As figure 2 illustrates, a is a dimensionless number that determines the interference strength. It may seem that we lose generality by considering only those channels with transmission coefficients and unit noise power as shown in (3.1), but as Carleial showed [3], [4], we can always apply a scaling transformation to a Gaussian interference channel with arbitrary transmission coefficients and noise powers and reduce it to an equivalent channel in this restricted class. A Gaussian interference

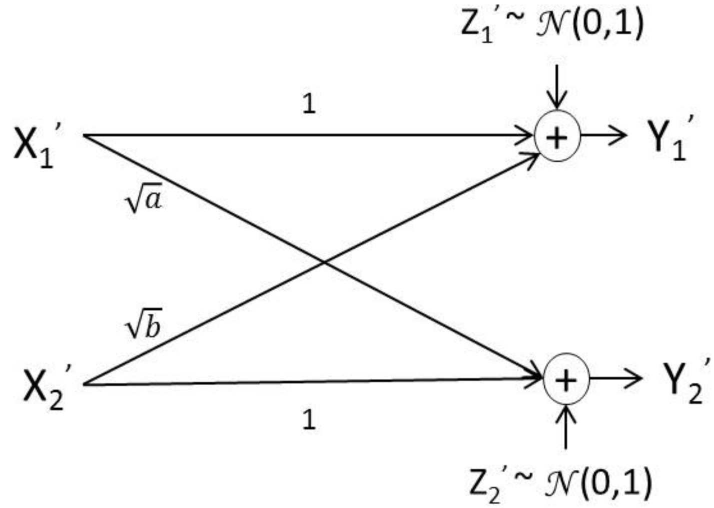

Figure 2. Gaussian Interference Channel

channel in standard form is completely specified by its interference coefficients a and b and by its available powers P_1 and P_2.

We say that the Gaussian Interference Channel has *strong interference* if $a \geq 1$ and $b \geq 1$, *mixed interference* if $a \geq 1$ and $b < 1$, or $a < 1$ and $b \geq 1$, and *weak interference* if $a < 1$ and $b < 1$. If $a = 0$ or $b = 0$, then the channel is referred to a *Z-Interference Channel*. If $ab = 1$, the channel is referred to a *degraded* Interference Channel.

Since in this paper we will mainly deal with the Z-Interference Channel, let us see a practical example of this channel in daily life and motivate ourselves to study this channel. Consider wireless mobile communication, where a mobile user communicates to a cellphone tower. Near the boundary of a tower's allotted area (called cell), a mobile user will start feeling the effect of the other nearest tower as well (figure 3). However, another user close to the other tower communicates with that tower and receivers almost no interference from the first tower (since the towers are far apart). Thus only one user receives interference from the other communication link and we have a Z-Interference Channel.

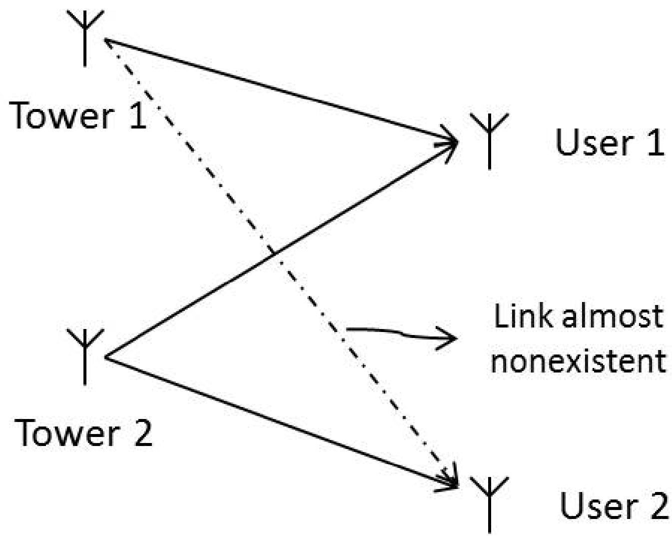

Figure 3. Z-Interference Channel example

This model is usually studied in its discrete-time form. Results obtained for the discrete-time version can be readily extended to its continuous time, band-limited counterpart in much the same way as we can obtain the capacity of continuous-time, band-limited Gaussian Shannon channels from the capacity of discrete-time Gaussian Shannon channels [5].

3.1. Literature Review

The study of this kind of channel was initiated by C. E. Shannon [6] , and furthered by R. Ahlswede [7], who gave fundamental inner and outer bounds to the capacity region.

In 1975 Carleial [3] demonstrated the striking fact that very strong interference is same as no interference. He showed that if $a \geq 1 + P_2$ and $b \geq 1 + P_1$, then the capacity region of the Gaussian interference channel is the full rectangular region described by

$$\begin{aligned} 0 \leq R_1 \leq C_1 \triangleq \frac{1}{2}\log(1+P_1) \\ 0 \leq R_2 \leq C_2 \triangleq \frac{1}{2}\log(1+P_2) \end{aligned} \tag{3.2}$$

The reason behind this counter-intuitive fact is that the interfering signals are so strong in this case that the receivers may decode them reliably even if they consider their intended signals as noise. Having decoded the interfering signals, the receivers may clean their channels by subtracting out the interference.

Later Carleial [4] established a considerably improved achievable rate region for the memoryless interference channel by applying the superposition coding technique of T. M. Cover [8] which was originally devised to study the capacity region of the broadcast channel. On the other hand, H. Sato [9] obtained various inner and outer bounds by transforming the problem to one for the associated multiple-access or broadcast channel. By 1981, Han and Kobayashi [10] and Sato [11] found the capacity region for the strong interference case, where $a \geq 1$ and $b \geq 1$. They showed that for interference parameters in this range both receivers would be able to reliably decode both messages, regardless of the particular coding technique being used. Thus the capacity region can be defined as the intersection of the capacity regions of the two multiple-access channels embedded in the interference channel. This region is the subset of rate pairs (R_1, R_2) of the rectangle given by (3.2) for which,

$$R_1 + R_2 \leq \min\left\{\frac{1}{2}\log(1 + aP_1 + P_2), \frac{1}{2}\log(1 + P_1 + bP_2)\right\} \tag{3.3}$$

In 1981, Han and Kobayashi [10] also came up with their achievable scheme, on which even the current achievable schemes are based. The key ideas behind the Han Kobayashi scheme are: rate-splitting, superposition coding and jointly decoding. Each user spits its message into two parts $W_u = (W_{u0}, W_{uu}), u \in 1, 2$, where W_{u0} $-$ the common message $-$ is to be decoded at both receivers, while W_{uu} $-$ the private message $-$ is to be decoded at intended receiver only. At the encoder side, the common and the private messages are encoded by superposition. At the decoder side, the two common messages and the intended private message are jointly decoded. However, they did not specify the optimal power split between common and private messages and we generally express the achievable rate as a union over all possible power division between common and private parts. The computation of the full Han-Kobayashi achievable rate region for a general discrete, memoryless Interference Channel is, in general, prohibitively complex, because of the huge number of degrees of freedom which are involved in the computation of its sub-regions.

By 1985, the problem of specifying a computable expression of the capacity region for the general interference channel was still open, although it had been solved for some very special cases (Carleial [3], Benzel [12]).

In 1985, Costa [13] considered the two-user Gaussian Interference Channel under weak interference and established the optimality of two extreme points in the achievable region of the general Gaussian interference channel. He proved that the class of degraded Gaussian interference channels is equivalent to the class of Z (one-sided) interference channels.

Kramer [14] derived two outer bounds on the capacity region of the two-user Gaussian interference channel. The idea of the first bound was to let a genie give each receiver just enough information to decode both messages. He showed that the outer bound derived from the genie-aided decoding strategy, unified and improved the best known outer bounds of Sato and Carleial. The second bound followed directly from existing results of Costa and Sato and possessed certain optimality properties for weak interference.

Igal Sason [15] came up with an achievable rate region for this channel, which can be achieved by time/frequency division multiplexing (TDM/ FDM). It also included the rate region which is obtained by time sharing between the two rate pairs where one of the transmitters sends its data reliably at the maximal possible rate (i.e., the maximum rate it can achieve in the absence of interference), and the other transmitter decreases its data rate to the point where both receivers can reliably decode its message. They showed that their suggested rate region is easily calculable, though it is a special case of the celebrated achievable rate region of Han and Kobayashi whose calculation is, in general, prohibitively complex. In addition, they obtained the sum-capacities of the degraded and one-sided Gaussian Interference Channels.

By 2008, the capacity of the two-user Gaussian interference channel had been open for 30 years. The understanding on this problem was been limited. The best known achievable region was due to Han and Kobayashi, but its characterization was very complicated. It was also not known how tight the existing outer bounds were.

Chong et al [16] derived a simplified description of the HanKobayashi rate region for the general interference channel. They established that their recently discovered ChongMotaniGarg rate region is a new representation of the HanKobayashi region. Then Etkin, Tse, and Wang [2] showed that the existing outer bounds can in fact be arbitrarily loose in some parameter ranges. By deriving new outer bounds, they showed that a very simple and explicit HanKobayashi type scheme can achieve to within a single bit per second per hertz (bit/s/Hz) of the capacity for all values of the channel parameters. Furthermore, they showed that the scheme was asymptotically optimal at certain high signal-to-noise ratio (SNR) regimes.

In 2010, Shang, Chen, and Kramer [17] considered the vector, or multipleinput multipleoutput, Gaussian interference channels and established the capacity regions for very strong interference and aligned strong interference. Furthermore, the sum-rate capacities were established for Z interference, noisy interference, and mixed (aligned weak/intermediate and aligned strong) interference. These results generalize known results for scalar Gaussian interference channels.

People began to suspect that the best achievable scheme known, the Han-Kobayashi scheme, was indeed the capacity region of the Z Interference Channels.

But in 2011, Costa [18] proposed an efficient scheme to transmit information over a Z Gaussian interference channel. The scheme used the concept of water filling to provide optimal power sharing among orthogonal dimensions. In the proposed solution, the notion of noisebergs (noise icebergs) arises, where noise power floats above signal power in a water filling representation of the problem, providing an improved allocation of power and degrees of freedom. The solution is best characterized by a graphical representation. We will discuss his coding scheme more in the next section.

On the other hand, M. Vaezi and H. Vincent Poor [19] showed with an explicit counterexample that the restriction to Gaussian distributions in the limiting expression for the capacity region of memoryless Gaussian interference channel falls short of achieving capacity, in general. This underlies the fact that there is something wrong with just mathematically trying to derive a upper bound for the Z-Interference channel. A similar result was shown by Verdu [20] - the restriction to Gaussian inputs in the limiting expression for the capacity regions of memoryless Gaussian interference and multiple-access channels falls short of achieving capacity even if the inputs are allowed to be dependent and non-stationary. This result of Vaezi and Poor just shows the complications of finding the outer bound of the Z Interference Channel.

In early 2016, M. Vaezi and H. Vincent Poor [21] now showed that with Gaussian codebooks, timesharing can strictly improve the HK achievable region. They showed that time-sharing with power allocation over two dimensions is enough to achieve the border of the HK inner bound, for these channels.

We summarize the current knowledge of the Z-Interference channel in the figure 4. It is clear from the foregoing discussion that, capacity region of the ZIC is known only for the scenario when the interfering link's channel gain is larger or equal to one, i.e., $a \geq 1$, which is typically referred to as the *strong interference regime*. It is well known that the optimal input distribution for *strong interference* channel is Gaussian.

In contrast, the capacity region of the ZIC with *weak* interference is not known till date. In the light of the optimality of Gaussian input for the *strong interference channel*, its only reasonable to think that Gaussian distribution might turn out to be optimal for *weak interference channel* as well. However, characterization of the capacity region of ZIC, restricted to Gaussian inputs has been a challenging problem as well. In this thesis, we solve the later problem completely, i.e., assuming that the input is Gaussian, we characterize the capacity region of the ZIC with weak interference.

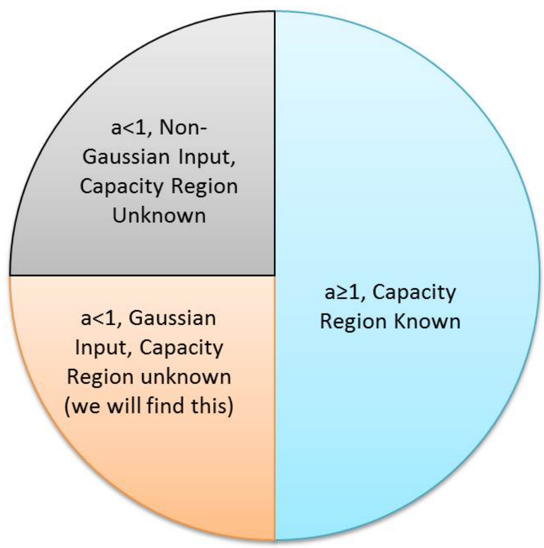

Figure 4. Current knowledge on Z-Interference Channel

In the next chapter we show how to model the Z-Interference Channel and we provide the best known achievable scheme [18].

CHAPTER 4. THE Z-INTERFERENCE CHANNEL

In this chapter we define the Z-Interference Channel in its standard form and convert it into a form that is more convenient for us. Then we describe the achieveability scheme of Costa [18] and take a deeper look at the time-sharing present in the scheme and notion of noisebergs.

The Z-Interference Channel in its standard form (see figure 5) is given by,

$$\boldsymbol{Y_1'} = \boldsymbol{X_1'} + \sqrt{a}\boldsymbol{X_2'} + \boldsymbol{Z_1'}, \tag{4.1a}$$

$$\boldsymbol{Y_2'} = \boldsymbol{X_2'} + \boldsymbol{Z_2'}, \tag{4.1b}$$

$$\text{where, } \operatorname{tr}\big(\mathbb{E}(\boldsymbol{X_1'^2})\big) \leq nP_1',\ \boldsymbol{Z_1'} \sim \mathcal{N}(0, I)$$

$$\operatorname{tr}\big(\mathbb{E}(\boldsymbol{X_2'^2})\big) \leq nP_2',\ \boldsymbol{Z_2'} \sim \mathcal{N}(0, I),\ a \in \mathbb{R}^+$$

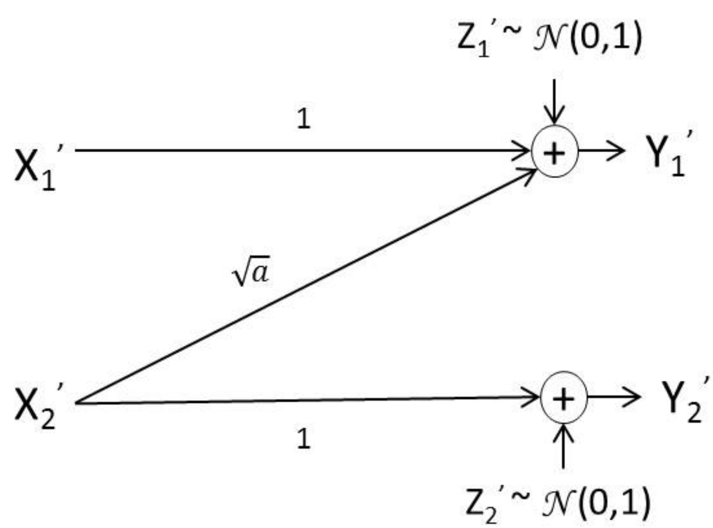

Figure 5. Z-Interference Channel in the standard form

We consider the problem under the weak interference regime, i.e. $a \in (0, 1)$. Furthermore, we restrict the input to be Gaussian distribution only.

First, we convert this channel into an equivalent channel (as shown in figure 6) by dividing both sides of (4.1a) by $\sqrt{a}$. We have,

$$\boldsymbol{Y_1} = \frac{\boldsymbol{Y_1'}}{\sqrt{a}} = \boldsymbol{X_1} + \boldsymbol{X_2} + \boldsymbol{Z_1} + \boldsymbol{Z_2} \tag{4.2a}$$

$$\boldsymbol{Y_2} = \boldsymbol{Y_2'} = \boldsymbol{X_2} + \boldsymbol{Z_2} \tag{4.2b}$$

$$\operatorname{tr}\Big(\mathbb{E}(\boldsymbol{X_1^2})\Big) \leq \frac{nP_1'}{a} = nP_1,\ \boldsymbol{Z_1} \sim \mathcal{N}\Big(0, (\tilde{a}-1)I\Big)$$

$$\operatorname{tr}\Big(\mathbb{E}(\boldsymbol{X_2^2})\Big) \leq nP_2' = nP_2,\ \boldsymbol{Z_2} \sim \mathcal{N}(0, I),\ a \in \mathbb{R}^+$$

where $\boldsymbol{X_1} = \frac{\boldsymbol{X_1'}}{\sqrt{a}}$ and $\boldsymbol{X_2} = \boldsymbol{X_2'}$.

Here have used the fact that the sum of two independent Gaussian random variables is another Gaussian random variable with variance the sum of the variances of the Gaussian random variables, i.e. if $\boldsymbol{A} \sim \mathcal{N}(O, \sigma_1)$ and $\boldsymbol{B} \sim \mathcal{N}(O, \sigma_2)$ with $C = A + B$, then $\boldsymbol{C} \sim \mathcal{N}(O, \sigma_1 + \sigma_2)$. Using this we split the noise of variance $\tilde{a}I$ into two independent noise of variances I and $(\tilde{a}-1)I$. Note that this split is possible because $a \leq 1$

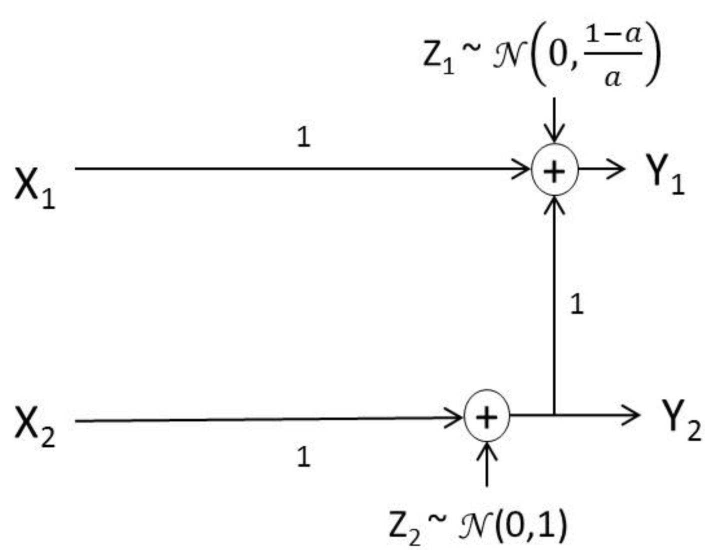

Figure 6. The degraded Z-Interference Channel

4.1. Achievable Rate

Before we derive any outer bound to the Z-Interference channel, let us take a closer look at the Han-Kobayashi based achievable scheme of Costa ([18]).

The scheme divided the channel use into two different bands, with different powers of $\boldsymbol{X_1}$ and $\boldsymbol{X_2}$ in different bands. Then, the concept of water filling was used to provide optimal power sharing among orthogonal dimensions. It turns out that the notion of noisebergs (noise icebergs) arises, where noise power floats above signal power in a water filling representation of the power distribution.

In band 1, let m_1 be the fraction of time we transmit according to band 1 scheme, $m_1 q_1$ be the power of $\boldsymbol{X_1}$, and $m_1 P_{2a}$ be the power of $\boldsymbol{X_2}$. In band 2, let m_2 be the fraction of time we transmit according to band 1 scheme, and $m_2 P_{2b}$ be the power of $\boldsymbol{X_2}$. Note that $\boldsymbol{X_1}$ has no power in band 2. When the height of band 2 is close to that of band 1, then there is no overflow of power into band 1. This situation is described by the non-overflow region and is illustrated in figure 7. At receiver one, in band 1 if we treat $m_1 P_{2a}$ as noise, then we can achieve

$$R_1 = \frac{1}{2}\log\Big(1 + \frac{q_1}{\tilde{a} + P_{2a}}\Big) \text{ and } R_2 = \frac{1}{2}\log\Big(1 + P_{2a}\Big) \tag{4.3}$$

In band 2, we can achieve:

$$R_1 = 0 \text{ and } R_2 = \frac{1}{2}\log\Big(1 + P_{2b}\Big) \tag{4.4}$$

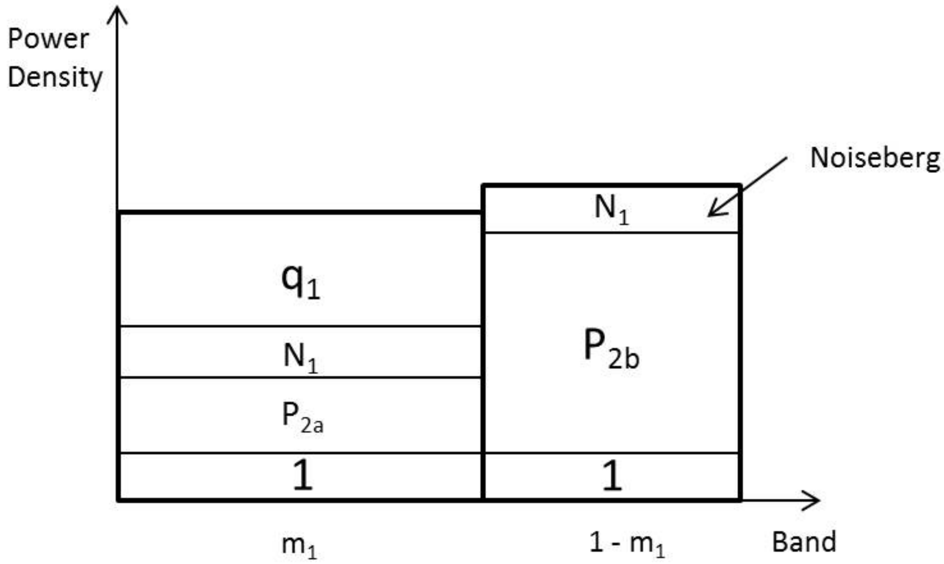

Figure 7. Non-overflow region

Since band 1 is used for m_1 fraction of time and band 2 is used for m_2 fraction of time, we have $m_1 + m_2 = 1$ and therefore $m_2 = 1 - m_1$. Thus, the achievable rate is,

$$R_1 = \frac{m_1}{2} \log \left(1 + \frac{q_1}{\tilde{a} + P_{2a}}\right) \text{ and } R_2 = \frac{m_1}{2} \log \left(1 + P_{2a}\right) + \frac{1 - m_1}{2} \log \left(1 + P_{2b}\right) \tag{4.5}$$

However, when the height of band 2 is much higher than that of band 1, then, due to water filling, excess power from band 2 overflows into band 1, giving rise to the picture in figure 8. At receiver one, in band 1 if we treat $m_1 P_{2a}$ as noise, then we can achieve

$$R_1 = \frac{1}{2} \log \left(1 + \frac{q_1}{\tilde{a} + P_{2a}}\right) \text{ and } R_2 = \frac{1}{2} \log \left(1 + P_{2a}\right) + \frac{1}{2} \log \left(1 + \frac{P_{2c}}{\tilde{a} + q_1 + P_{2a}}\right) \tag{4.6}$$

In band 2, we can achieve

$$R_1 = 0 \text{ and } R_2 = \frac{1}{2}\log\Big(1 + P_{2b}\Big) \tag{4.7}$$

Overall, the achievable rate is,

$$\begin{aligned} R_1 =& \frac{m_1}{2}\log\Big(1 + \frac{q_1}{\tilde{a} + P_{2a}}\Big) \text{ and} \\ R_2 =& \frac{m_1}{2}\log\Big(1 + P_{2a}\Big) + \frac{m_1}{2}\log\Big(1 + \frac{P_{2c}}{\tilde{a} + q_1 + P_{2a}}\Big) + \frac{1-m_1}{2}\log\Big(1 + P_{2b}\Big) \end{aligned} \tag{4.8}$$

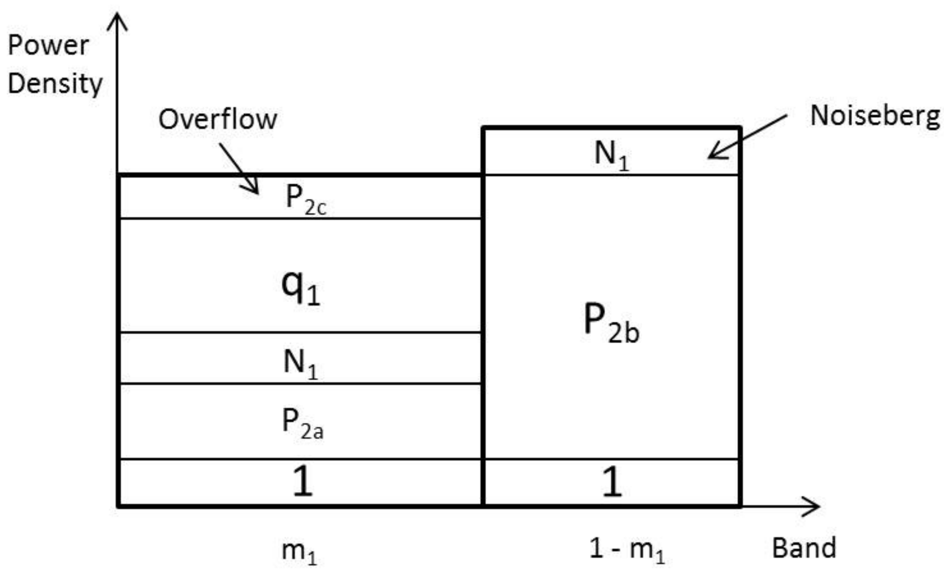

Figure 8. Overflow region

Varying m_1, q_1, P_{2a}, P_{2b} and P_{2c}, under the constraints,

$$\begin{aligned} 0 < m_1 \leq 1;\ m_1 q_1 =& P_1;\ m_1(P_{2a} + P_{2c}) + (1 - m_1)P_{2b} = P_2;\ \text{and} \\ P_{2c} =& \max\{0, P_{2b} - P_{2a} - \tilde{a} - q_1\} \end{aligned} \tag{4.9}$$

we get a set of possible achievable rates. Taking the union over these rates gives us an achievable boundary, as shown in figure 9.

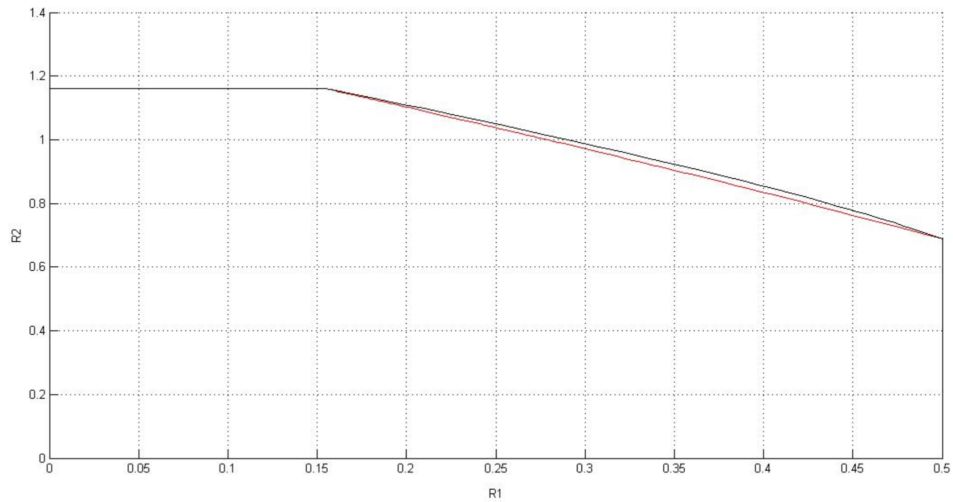

Figure 9. The Han-Kobayashi region with band splitting (top) as compared to the naive or single band Han-Kobayashi region (bottom)

Now that we have seen the best achievable scheme, we will now try to derive an upper bound for the Z-Interference Channels in the next chapter.

CHAPTER 5. UPPER BOUND FOR THE Z-INTERFERENCE CHANNEL

In this chapter, we first derive an outer bound to the Z-Interference channel. We analyze the derived outer bound and show that the bound is actually loose. Based on the intuitions gained from the derivation, we derive another bound for the Z-Interference channel and with a specific example show that this outer bound matches with the best known achievable scheme. The tightness of the upper bound shows the optimality of the known achievable scheme, which was previously unknown.

Theorem 1. *If R_1 and R_2 are achievable rates in the Z Interference Channel, then*

$$R_1 \leq \frac{1}{2}\log(1+P_1) \tag{5.1a}$$

$$R_2 \leq \frac{1}{2}\log(1+P_2) \tag{5.1b}$$

$$R_1 + \omega R_2 \leq \max_{\lambda_i, q_i, m_i} \sum_{i=1}^{4} m_i \left[\frac{(1-\omega)}{2}\log\left(\tilde{a} + q_i + \lambda_i\right) + \frac{\omega}{2}\log\left(1+\lambda_i\right) - \frac{1}{2}\log\left(\tilde{a}+\lambda_i\right)\right] + \frac{\omega}{2}\log\left(\tilde{a}+P_1+P_2\right) \tag{5.1c}$$

$$\textit{subject to } \sum_{i=1}^{4} m_i q_i \leq P_1 \tag{5.1d}$$

$$\sum_{i=1}^{4} m_i \lambda_i \leq P_2 \tag{5.1e}$$

$$\sum_{i=1}^{4} m_i = 1;\ \lambda_1 = q_4 = 0 \tag{5.1f}$$

$$\tilde{a} + q_i + \lambda_i = \tilde{a} + q_j + \lambda_j,\ \ i,j \in \{1,2,3\} \tag{5.1g}$$

$$\tilde{a} + q_i + \lambda_i \leq \tilde{a} + \lambda_4,\ \ i \in \{1,2,3\} \tag{5.1h}$$

Proof. Let us define $\boldsymbol{U}$ as a quantized version of $\boldsymbol{X_2}$ (i.e. $\boldsymbol{U} = \tilde{\boldsymbol{X}}_{\boldsymbol{2}}$), such that

$$I(\boldsymbol{U};\boldsymbol{X_2}) \leq I(\boldsymbol{U};\boldsymbol{Y_1}) \tag{5.2}$$

Note that intuitively $\boldsymbol{U}$ represents the portion of $\boldsymbol{X_2}$ which is decodable at receiver one.

Let the set of all such possible $\boldsymbol{U}$ be $\mathbb{U}$, i.e. $\mathbb{U} = \{\boldsymbol{U} : \boldsymbol{U}$ is a quantized version of $\boldsymbol{X_2}\}$.

For a given $\boldsymbol{X_2}$, note that all quantized versions of $\boldsymbol{X_2}$ (i.e. all $\boldsymbol{U} \in \mathbb{U}$) does not satisfy this property. We will only consider those which satisfy the constraint. Let $\mathcal{U}$ be the set of all $\boldsymbol{U}$ which satisfy the constraint, i.e.

$$\mathcal{U}(\boldsymbol{X_2}) = \{\boldsymbol{U} : I(\boldsymbol{U}; \boldsymbol{X_2}) \leq I(\boldsymbol{U}; \mathbf{Y_1})\} \tag{5.3}$$

Also, since $\mathbf{Y_2} = \boldsymbol{X_2} + \boldsymbol{Z_2}$, and we have a Markov chain from $\boldsymbol{U} \rightarrow \boldsymbol{X_2} \rightarrow \mathbf{Y_2}$. we can write (see [1]),

$$I(\boldsymbol{U}; \mathbf{Y_2}) \leq I(\boldsymbol{U}; \boldsymbol{X_2}) \tag{5.4}$$

Combining (5.2) and (5.4), we get,

$$I(\boldsymbol{U}; \mathbf{Y_2}) \leq I(\boldsymbol{U}; \mathbf{Y_1}) \tag{5.5}$$

Now, for a given $\boldsymbol{U} \in \mathcal{U}(\boldsymbol{X_2})$, we have,

$$nR_1 \leq I(\boldsymbol{X_1}; \mathbf{Y_1}) \leq I(\boldsymbol{X_1}, \boldsymbol{U}; \mathbf{Y_1}) \leq I(\boldsymbol{X_1}; \boldsymbol{U}) + I(\boldsymbol{X_1}; \mathbf{Y_1}|\boldsymbol{U}) \tag{5.6a}$$

$$\Rightarrow nR_1 \leq I(\boldsymbol{X_1}; \mathbf{Y_1}|\boldsymbol{U}) \tag{5.6b}$$

(5.6a) follows from the Chain rule for Mutual Information (see [1]), and (5.6b) follows from the fact $I(\boldsymbol{X_1}; \boldsymbol{U}) = 0$ (since $\boldsymbol{U}$ is independent of $\boldsymbol{X_1}$).

Similarly, by Chain rule for Mutual Information,

$$nR_2 \leq I(\boldsymbol{X_2}; \mathbf{Y_2}) \leq I(\boldsymbol{X_2}, \boldsymbol{U}; \mathbf{Y_2}) \leq I(\boldsymbol{U}; \mathbf{Y_2}) + I(\boldsymbol{X_2}; \mathbf{Y_2}|\boldsymbol{U}) \tag{5.7}$$

Combining (5.6b) and (5.7), we get for any ω,

$$
\begin{aligned}
n(R_1 + \omega R_2) &\leq \max_{\boldsymbol{X_1}, \boldsymbol{X_2}} I(\boldsymbol{X_1}; \boldsymbol{Y_1} | \boldsymbol{U}) + \omega \Big(I(\boldsymbol{X_2}; \boldsymbol{Y_2} | \boldsymbol{U}) + I(\boldsymbol{U}; \boldsymbol{Y_2}) \Big) && \text{(5.8a)} \\
&\leq I(\boldsymbol{X_1}; \boldsymbol{Y_1} | \boldsymbol{U}) + \omega I(\boldsymbol{X_2}; \boldsymbol{Y_2} | \boldsymbol{U}) + \omega I(\boldsymbol{U}; \boldsymbol{Y_1}) && \text{(5.8b)} \\
&\leq (1-\omega) h(\boldsymbol{Y_1} | \boldsymbol{U}) + \omega h(\boldsymbol{X_2} + \boldsymbol{Z_2} | \boldsymbol{U}) + \omega h(\boldsymbol{Y_1}) - h(\boldsymbol{Y_1} | \boldsymbol{U}, \boldsymbol{X_1}) && \text{(5.8c)} \\
&\leq (1-\omega) h(\boldsymbol{Y_1} | \boldsymbol{U}) + \omega h(\boldsymbol{X_2} + \boldsymbol{Z_2} | \boldsymbol{U}) - h(\boldsymbol{Y_1} | \boldsymbol{U}, \boldsymbol{X_1}) \\
&\qquad + \frac{n\omega}{2} \log \Big(\tilde{a} + P_1 + P_2 \Big) && \text{(5.8d)}
\end{aligned}
$$

where (5.8c) follows from (5.5) and (5.8d) follows from the fact that, $h(\boldsymbol{Y_1}) \leq \frac{n}{2} \log \Big(\tilde{a} + P_1 + P_2 \Big)$.

Now, we have the following lemma,

Lemma 1.

$$
\begin{aligned}
h(\boldsymbol{X_2} + \boldsymbol{Z_2} | \boldsymbol{U}) &= \frac{1}{2} \log |K_{\boldsymbol{X_2}|\boldsymbol{U}} + I| && \text{(5.9a)} \\
h(\boldsymbol{Y_1} | \boldsymbol{U}, \boldsymbol{X_1}) &= \frac{1}{2} \log |K_{\boldsymbol{X_2}|\boldsymbol{U}} + \tilde{a} I| && \text{(5.9b)} \\
h(\boldsymbol{Y_1} | \boldsymbol{U}) &= \frac{1}{2} \log \Big| K_{\boldsymbol{X_1}} + K_{\boldsymbol{X_2}|\boldsymbol{U}} + \tilde{a} I \Big| && \text{(5.9c)}
\end{aligned}
$$

Proof. See appendix A.1 □

Using (5.9a), (5.9b) and (5.9c) we can write (5.8d) as,

$$
\begin{aligned}
n(R_1 + \omega R_2) \leq \frac{(1-\omega)}{2} \log \Big| \tilde{a} I + K_{\boldsymbol{X_2}|\boldsymbol{U}} + K_{\boldsymbol{X_1}} \Big| + \frac{\omega}{2} \log \Big| I + K_{\boldsymbol{X_2}|\boldsymbol{U}} \Big| \\
- \frac{1}{2} \log \Big| \tilde{a} I + K_{\boldsymbol{X_2}|\boldsymbol{U}} \Big| + \frac{n\omega}{2} \log \Big(\tilde{a} + P_1 + P_2 \Big) \qquad \text{(5.10)}
\end{aligned}
$$

Since (5.10) holds for all distributions of $\boldsymbol{X_1}, \boldsymbol{X_2}$ and all $\boldsymbol{U} \in \mathcal{U}(\boldsymbol{X_2})$, we can write,

$$n(R_1 + \omega R_2) \leq \max_{\substack{\boldsymbol{X_1}, \boldsymbol{X_2} \\ \boldsymbol{U} \in \mathcal{U}(\boldsymbol{X_2})}} \frac{(1-\omega)}{2} \log \left| \tilde{a} I + K_{\boldsymbol{X_2}|\boldsymbol{U}} + K_{\boldsymbol{X_1}} \right| + \frac{\omega}{2} \log \left| I + K_{\boldsymbol{X_2}|\boldsymbol{U}} \right|$$
$$- \frac{1}{2} \log \left| \tilde{a} I + K_{\boldsymbol{X_2}|\boldsymbol{U}} \right| + \frac{n\omega}{2} \log \left(\tilde{a} + P_1 + P_2 \right) \quad (5.11a)$$
$$\text{subject to } \operatorname{tr}(K_{\boldsymbol{X_1}}) \leq nP_1 \quad (5.11b)$$
$$\operatorname{tr}(K_{\boldsymbol{X_2}}) \leq nP_2 \quad (5.11c)$$

Now we drop the restriction $\boldsymbol{U} \in \mathcal{U}(\boldsymbol{X_2})$ and allow $\boldsymbol{U} \in \mathbb{U}$. Relaxing the constraints can only increase our upper bound. For any $\boldsymbol{U}$, we have $\operatorname{tr}(K_{\boldsymbol{X_2}|\boldsymbol{U}}) \leq nP_2$ and therefore we can write our maximization problem as,

$$n(R_1 + \omega R_2) \leq \max_{\substack{\boldsymbol{X_1}, \boldsymbol{X_2} \\ \boldsymbol{U} \in \mathbb{U}}} \frac{(1-\omega)}{2} \log \left| \tilde{a} I + K_{\boldsymbol{X_2}|\boldsymbol{U}} + K_{\boldsymbol{X_1}} \right| + \frac{\omega}{2} \log \left| I + K_{\boldsymbol{X_2}|\boldsymbol{U}} \right|$$
$$- \frac{1}{2} \log \left| \tilde{a} I + K_{\boldsymbol{X_2}|\boldsymbol{U}} \right| + \frac{n\omega}{2} \log \left(\tilde{a} + P_1 + P_2 \right) \quad (5.12a)$$
$$\text{subject to } \operatorname{tr}(K_{\boldsymbol{X_1}}) \leq nP_1 \quad (5.12b)$$
$$\operatorname{tr}(K_{\boldsymbol{X_2}|\boldsymbol{U}}) \leq nP_2 \quad (5.12c)$$

Since $K_{\boldsymbol{X_2}|\boldsymbol{U}}$ is a positive semi-definite matrix, we can decompose $K_{\boldsymbol{X_2}|\boldsymbol{U}}$ as $V \Lambda V^{\dagger}$, where Λ is a diagonal matrix with diagonal elements λ_i and V is a unitary matrix.

We have,

$$\log|I + K_{\boldsymbol{X_2}|\boldsymbol{U}}| = \log|1+\Lambda| = \sum_i \log(1+\lambda_i) \tag{5.13a}$$

$$\log\left|\tilde{a}I + K_{\boldsymbol{X_2}|\boldsymbol{U}}\right| = \log\left|\tilde{a}+\Lambda\right| = \sum_i \log\left(\tilde{a}+\lambda_i\right) \tag{5.13b}$$

$$\log\left|\tilde{a}I + K_{\boldsymbol{X_2}|\boldsymbol{U}} + K_{\boldsymbol{X_1}}\right| = \log\left|\tilde{a}I + V\Lambda V^\dagger + K_{\boldsymbol{X_1}}\right| \tag{5.13c}$$

$$= \log\left|(V^\dagger[K_{\boldsymbol{X_1}} + V\Lambda V^\dagger + \tilde{a}I]V)\right| \tag{5.13d}$$

$$= \log\left|(V^\dagger K_{\boldsymbol{X_1}} V + \Lambda + \tilde{a}I)\right| \tag{5.13e}$$

where (5.13a) and (5.13b) follows from the fact that det(I+AB) = det(I+BA) and (5.13d) follows from the fact that det(AB) = det(BA).

Since V is a unitary matrix, if we define $\tilde{K}_{\boldsymbol{X_1}} = V^\dagger K_{\boldsymbol{X_1}} V$, then $\tilde{K}_{\boldsymbol{X_1}}$ and $K_{\boldsymbol{X_1}}$ will have the same trace constraint. If q_i are the diagonal elements of $\tilde{K}_{\boldsymbol{X_1}}$, then we have $\sum_i q_i \leq nP_1$. By Hadamund's inequality, we can write,

$$\log\left|\tilde{a}I + K_{\boldsymbol{X_2}|\boldsymbol{U}} + K_{\boldsymbol{X_1}}\right| = \log\left|(V^\dagger K_{\boldsymbol{X_1}} V + \Lambda + \tilde{a}I)\right| \leq \sum_i \log\left(\tilde{a} + q_i + \lambda_i\right) \tag{5.14}$$

With (5.13a), (5.13b) and (5.14), we can write (5.12) as,

$$n(R_1 + \omega R_2) \leq \max_{q_i,\lambda_i} \sum_{i=1}^{n} \left[\frac{(1-\omega)}{2}\log\left(\tilde{a}+q_i+\lambda_i\right) + \frac{\omega}{2}\log\left(1+\lambda_i\right) - \frac{1}{2}\log\left(\tilde{a}+\lambda_i\right)\right] + \frac{n\omega}{2}\log\left(\tilde{a}+P_1+P_2\right) \tag{5.15a}$$

$$\text{subject to } \sum_{i=1}^{n} q_i \leq nP_1 \tag{5.15b}$$

$$\sum_{i=1}^{n} \lambda_i \leq nP_2 \tag{5.15c}$$

To solve this new optimization problem, we use Lagrange Multipliers, and the study the Lagrangian,

$$L = \sum_i \left[\frac{(1-\omega)}{2} \log\Big(\tilde{a} + q_i + \lambda_i\Big) + \frac{\omega}{2} \log\Big(1 + \lambda_i\Big) - \frac{1}{2} \log\Big(\tilde{a} + \lambda_i\Big) \right]$$
$$+ \frac{n\omega}{2} \log\Big(\tilde{a} + P_1 + P_2\Big) + \alpha(nP_2 - \sum_i \lambda_i) + \beta(nP_1 - \sum_i q_i) + \sum_i \alpha_i \lambda_i + \sum_i \beta_i q_i \tag{5.16}$$

Because of the inequality constraints, we need to apply the Karush-Kuhn-Tucker (KKT) conditions. The KKT stationary and complementary slackness constraints are given by,

$$\frac{(1-w)/2}{\tilde{a} + q_i + \lambda_i} - \beta - \beta_i = 0 \tag{5.17a}$$
$$\frac{(1-w)/2}{\tilde{a} + q_i + \lambda_i} + \frac{w/2}{1 + \lambda_i} - \frac{1/2}{\tilde{a} + \lambda_i} - \alpha + \alpha_i = 0 \tag{5.17b}$$
$$\alpha(nP_2 - \sum_i \lambda_i) = 0 \tag{5.17c}$$
$$\beta(nP_1 - \sum_i q_i) = 0 \tag{5.17d}$$
$$\alpha_i \lambda_i = 0 \ \forall\ i \tag{5.17e}$$
$$\beta_i q_i = 0 \ \forall\ i \tag{5.17f}$$

Now, we have the following lemma,

Lemma 2. *The solution to the optimization problem (*5.15*), can be written as,*

$$R_1 + \omega R_2 \leq \max_{\lambda_i, q_i, m_i} \sum_{i=1}^{4} m_i \Bigg[\frac{(1-\omega)}{2} \log\Big(\tilde{a} + q_i + \lambda_i\Big) + \frac{\omega}{2} \log\Big(1 + \lambda_i\Big) - \frac{1}{2} \log\Big(\tilde{a} + \lambda_i\Big)\Bigg] + \frac{\omega}{2} \log\Big(\tilde{a} + P_1 + P_2\Big) \tag{5.18a}$$

$$\textit{subject to} \sum_{i=1}^{4} m_i q_i \leq P_1 \tag{5.18b}$$

$$\sum_{i=1}^{4} m_i \lambda_i \leq P_2 \tag{5.18c}$$

$$\sum_{i=1}^{4} m_i = 1; \;\; \lambda_1 = q_4 = 0 \tag{5.18d}$$

$$\tilde{a} + q_i + \lambda_i = \tilde{a} + q_j + \lambda_j, \;\; i, j \in \{1, 2, 3\} \tag{5.18e}$$

$$\tilde{a} + q_i + \lambda_i \leq \tilde{a} + \lambda_4, \;\; i \in \{1, 2, 3\} \tag{5.18f}$$

Proof. See appendix A.2 □

Thus, in (5.15), we only need to consider four different values of q_i and λ_i satisfying the constraints given in the lemma above. This is summarized in the figure 10.

This completes the proof of the theorem 1. □

From the outer bound given in the form of (5.1), it is not clear whether this outer bound is tight or not, i.e. whether we can achieve this rate or not.

We numerically evaluate the outer bound and compare it to the achievable region of Costa, as described earlier, for $P_1 = 2, P_2 = 2, a = 0.5$ and $P_1 = 2, P_2 = 4, a = 0.8$. They are shown in figures 11 and 12 respectively. The difference between upper bounds and achievable rates show that our outer bound is not tight.

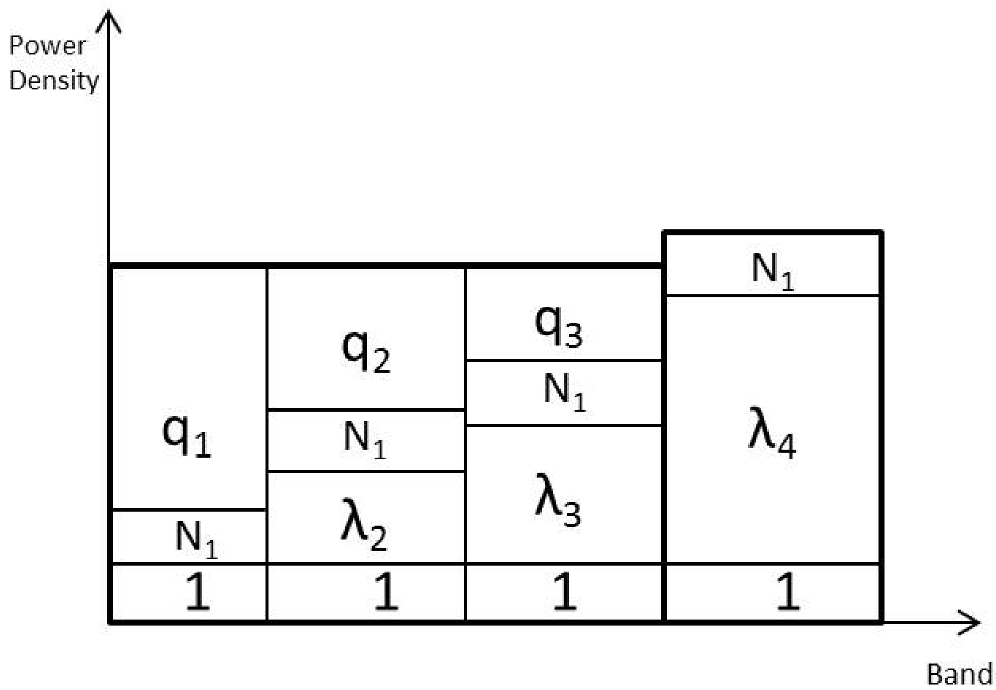

Figure 10. The four band partition: The height of the first three bands are the same and are less than or equal to the height of the last band.

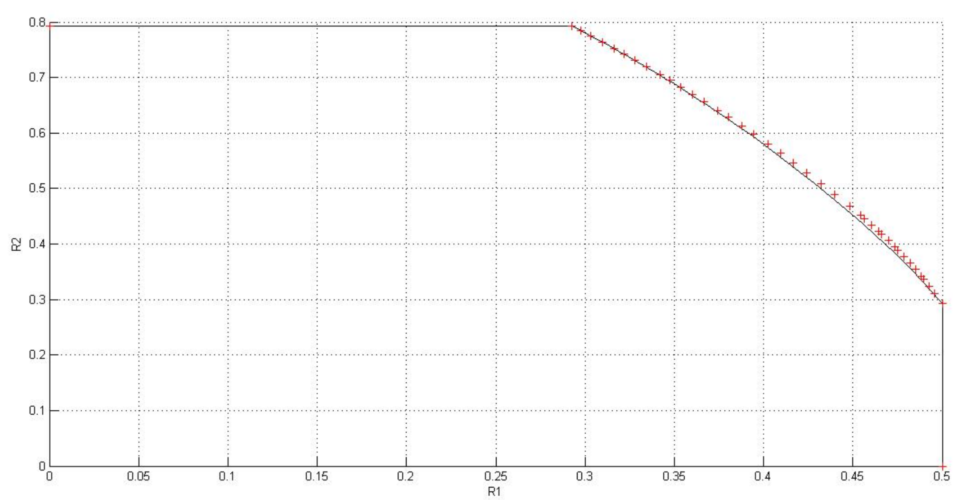

Figure 11. The achievable rate (black line) as compared to outer bound (red cross) for $P_1 = 2, P_2 = 2, a = 0.5$.

Now, in equation 5.8d, we maximized $h(\boldsymbol{Y_1})$ separately from the rest of the terms in the equation. Since simultaneous maximization always leads to a maximum

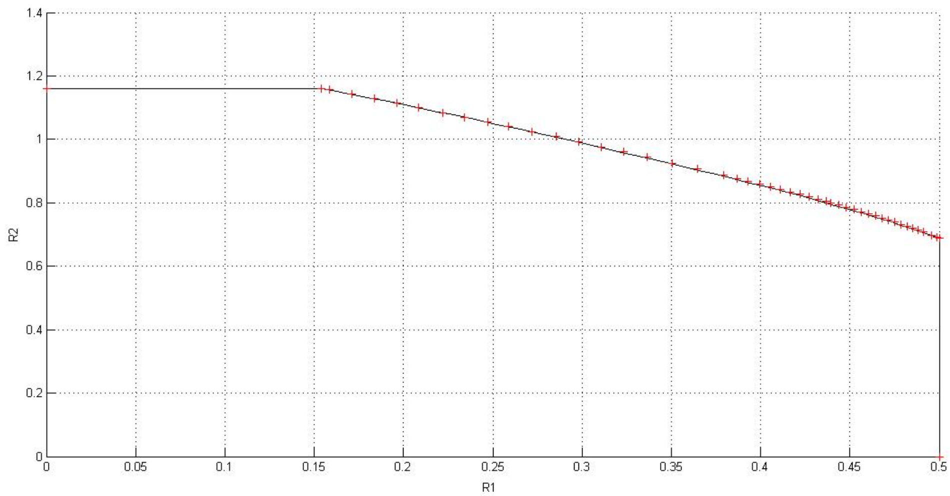

Figure 12. The achievable rate (black line) as compared to outer bound (red cross) for $P_1 = 2, P_2 = 4, a = 0.8$.

value which is less than or equal to individual maximization value, perhaps we can improve this upper bound by simultaneously maximizing all the terms in equation (5.8c).

This motivates us to find the outer bound again, keeping $h(\mathbf{Y_1}) = \frac{n}{2}\log\left|\tilde{a}I + K_{\mathbf{X_1}} + K_{\mathbf{X_2}}\right|$. This is done in Theorem 2, which we now describe.

Theorem 2. *If R_1 and R_2 are achievable rates in the Z Interference Channel, then*

$$R_1 \leq \frac{1}{2}\log(1+P_1) \tag{5.19a}$$

$$R_2 \leq \frac{1}{2}\log(1+P_2) \tag{5.19b}$$

$$n(R_1+\omega R_2) \leq \max_{\lambda_i,q_i,k_i,m_i} \sum_{i=1}^{4} m_i \Bigg[\frac{(1-\omega)}{2}\log\Big(\tilde{a}+q_i+\lambda_i\Big) + \frac{\omega}{2}\log\Big(1+\lambda_i\Big) - \frac{1}{2}\log\Big(\tilde{a}+\lambda_i\Big) + \frac{\omega}{2}\log\Big(\tilde{a}+q_i+\lambda_i+k_i\Big)\Bigg] \tag{5.19c}$$

$$\textit{subject to } \sum_{i=1}^{4} m_i q_i \leq nP_1 \tag{5.19d}$$

$$\sum_{i=1}^{4} m_i(\lambda_i+k_i) \leq nP_2 \tag{5.19e}$$

$$\sum_{i=1}^{4} m_i = 1;\ \lambda_1 = q_4 = k_4 = 0 \tag{5.19f}$$

$$k_1 = k_2 = k_3 \tag{5.19g}$$

$$\tilde{a}+q_i+\lambda_i+k_i = \tilde{a}+q_j+\lambda_j+k_j,\ \ i,j\in\{1,2,3\} \tag{5.19h}$$

$$\tilde{a}+q_i+a\lambda_i+k_i \leq \tilde{a}+\lambda_4,\ \ i\in\{1,2,3\} \tag{5.19i}$$

Proof. We define $\boldsymbol{U}$ as before. But this time we treat $h(\boldsymbol{Y_1})$ differently. In (5.8c), instead of using $h(\boldsymbol{Y_1}) \leq \frac{n}{2}\log\Big(\tilde{a}+P_1+P_2\Big)$, we keep $h(\boldsymbol{Y_1}) = \frac{n}{2}\log\Big|\tilde{a}I+K_{\boldsymbol{X_1}}+K_{\boldsymbol{X_2}}\Big|$.

Therefore, our maximization problem is,

$$n(R_1+\omega R_2) \leq \max_{\substack{\boldsymbol{X_1},\boldsymbol{X_2}\\ \boldsymbol{U}\in\mathcal{U}(\boldsymbol{X_2})}} \frac{(1-\omega)}{2}\log\Big|\tilde{a}I+K_{\boldsymbol{X_2}|\boldsymbol{U}}+K_{\boldsymbol{X_1}}\Big| + \frac{\omega}{2}\log\Big|I+K_{\boldsymbol{X_2}|\boldsymbol{U}}\Big| - \frac{1}{2}\log\Big|\tilde{a}I+K_{\boldsymbol{X_2}|\boldsymbol{U}}\Big| + \frac{\omega}{2}\log\Big|\tilde{a}I+K_{\boldsymbol{X_1}}+K_{\boldsymbol{X_2}}\Big| \tag{5.20a}$$

$$\text{subject to } \operatorname{tr}(K_{\boldsymbol{X_1}}) \leq nP_1 \tag{5.20b}$$

$$\operatorname{tr}(K_{\boldsymbol{X_2}}) \leq nP_2 \tag{5.20c}$$

Now, as defined in (5.3), $\boldsymbol{U}$ is a quantized version of $\boldsymbol{X_2}$. Note again that for any $\boldsymbol{X_2}$, not all $\boldsymbol{U} \in \mathbb{U}$ is allowed. For example, if $\boldsymbol{X_2} \sim \mathcal{N}(0, P_2 I_n)$ with $\frac{n}{2}\log(1+P_2)$ information for receiver two, then $\boldsymbol{U}$ cannot be $\boldsymbol{X_2}$ since the channel from transmitter two to receiver is degraded and cannot carry that much information. Thus, for any given $\boldsymbol{X_2}$, we can choose $\boldsymbol{U}$ from a constant to some quantized version of $\boldsymbol{X_2}$ and we see that the minimum value of $\mathrm{tr}(K_{\boldsymbol{X_2}|\boldsymbol{U}})$ may not always be 0. So, let $n\rho \leq \mathrm{tr}(K_{\boldsymbol{X_2}|\boldsymbol{U}})$. Also, for any $\boldsymbol{U}$, we have $\mathrm{tr}(K_{\boldsymbol{X_2}|\boldsymbol{U}}) \leq \mathrm{tr}(K_{\boldsymbol{X_2}}) \leq nP_2$. Therefore, $n\rho \leq \mathrm{tr}(K_{\boldsymbol{X_2}|\boldsymbol{U}}) \leq nP_2$.

The natural question that arises is what are the possible values of ρ? If we choose $\boldsymbol{X_2} = 0$, then we see that ρ can be zero. Thus, $0 \leq \rho \leq P_2$. Therefore, we can write our maximization problem as,

$$
\begin{aligned}
n(R_1 + \omega R_2) \leq \max_{\substack{\boldsymbol{X_1}, \boldsymbol{X_2} \\ \rho \in [0, P_2]}} \frac{(1-\omega)}{2} \log \left| \tilde{a} I + K_{\boldsymbol{X_2}|\boldsymbol{U}} + K_{\boldsymbol{X_1}} \right| + \frac{\omega}{2} \log \left| I + K_{\boldsymbol{X_2}|\boldsymbol{U}} \right| \\
- \frac{1}{2} \log \left| \tilde{a} I + K_{\boldsymbol{X_2}|\boldsymbol{U}} \right| + \frac{\omega}{2} \log \left| \tilde{a} I + K_{\boldsymbol{X_1}} + K_{\boldsymbol{X_2}} \right| \qquad (5.21a) \\
\text{subject to } \ \mathrm{tr}(K_{\boldsymbol{X_1}}) \leq nP_1 \qquad (5.21b) \\
n\rho \leq \mathrm{tr}(K_{\boldsymbol{X_2}|\boldsymbol{U}}) \leq nP_2 \qquad (5.21c)
\end{aligned}
$$

Since $K_{\boldsymbol{X_2}|\boldsymbol{U}}$ is a positive semi-definite matrix, we can decompose $K_{\boldsymbol{X_2}|\boldsymbol{U}}$ as $V\Lambda V^\dagger$, where Λ is a diagonal matrix with diagonal elements λ_i and V is a unitary matrix. Then (5.13) holds.

Now, we can write $K_{\boldsymbol{X_2}} = K_{\boldsymbol{X_2|U}} + K_{\boldsymbol{X_2U}} K_{\boldsymbol{U}}^{-1} K_{\boldsymbol{UX_2}}$ and therefore we have,

$$\log \left| \tilde{a}I + K_{\boldsymbol{X_1}} + K_{\boldsymbol{X_2}} \right| = \log \left| V^\dagger \Big(\tilde{a}I + K_{\boldsymbol{X_1}} + [K_{\boldsymbol{X_2|U}} + K_{\boldsymbol{X_2U}} K_{\boldsymbol{U}}^{-1} K_{\boldsymbol{UX_2}}] \Big) V \right| \tag{5.22a}$$

$$= \log \left| \tilde{a}I + V^\dagger K_{\boldsymbol{X_1}} V + V^\dagger K_{\boldsymbol{X_2|U}} V + \tilde{K} \right| \tag{5.22b}$$

$$= \log \left| \tilde{a}I + \tilde{K_{\boldsymbol{X_1}}} + \Lambda + \tilde{K} \right| \tag{5.22c}$$

where $\tilde{K} = V^\dagger K_{\boldsymbol{X_2U}} K_{\boldsymbol{U}}^{-1} K_{\boldsymbol{UX_2}} V$ and (5.22a) follows from the fact that det(AB) = det(BA). Now if the diagonal elements of $\tilde{K}$ are k_i, then by Hadamund's inequality we can write,

$$\log \left| \tilde{a}I + K_{\boldsymbol{X_1}} + K_{\boldsymbol{X_2}} \right| = \log \left| \tilde{a}I + \tilde{K_{\boldsymbol{X_1}}} + \Lambda + \tilde{K} \right| \leq \sum_i \log \Big(\tilde{a} + q_i + \lambda_i + k_i \Big) \tag{5.23}$$

With (5.13a), (5.13b), (5.14) and (5.23), we can write (5.21) as,

$$n(R_1 + \omega R_2) \leq \max_{\substack{q_i,\lambda_i,k_i \\ \rho \in [0,P_2]}} \sum_{i=1}^{n} \Bigg[\frac{(1-\omega)}{2} \log \Big(\tilde{a} + q_i + \lambda_i \Big) + \frac{\omega}{2} \log \Big(1 + \lambda_i \Big) - \frac{1}{2} \log \Big(\tilde{a} + \lambda_i \Big) + \frac{\omega}{2} \log \Big(\tilde{a} + q_i + \lambda_i + k_i \Big) \Bigg] \tag{5.24a}$$

$$\text{subject to } \sum_{i=1}^{n} q_i \leq nP_1 \tag{5.24b}$$

$$n\rho \leq \sum_{i=1}^{n} \lambda_i \tag{5.24c}$$

$$\sum_{i=1}^{n} (\lambda_i + k_i) \leq nP_2 \tag{5.24d}$$

To solve this new optimization problem, we use Lagrange Multipliers, and the study the Lagrangian, (something might be missing here)

$$
\begin{aligned}
L = \sum_{i=1}^{n} \Bigg[& \frac{(1-\omega)}{2} \log\Big(\tilde{a} + q_i + \lambda_i\Big) + \frac{\omega}{2} \log\Big(1 + \lambda_i\Big) - \frac{1}{2} \log\Big(\tilde{a} + \lambda_i\Big) \\
& + \frac{\omega}{2} \log\Big(\tilde{a} + q_i + \lambda_i + k_i\Big)\Bigg] + \alpha\Big(\sum_i \lambda_i - n\rho\Big) + \beta\Big(nP_1 - \sum_i q_i\Big) \\
& + \gamma\Big(nP_2 - \sum_i \lambda_i - \sum_i k_i\Big) + \sum_i \alpha_i\lambda_i + \sum_i \beta_i q_i + \sum_i \sigma_i k_i
\end{aligned} \tag{5.25}
$$

Because of the inequality constraints, we need to apply the Karush-Kuhn-Tucker (KKT) conditions. The KKT stationary and complementary slackness constraints are given by,

$$\frac{(1-w)/2}{\tilde{a} + q_i + \lambda_i} + \frac{w/2}{\tilde{a} + q_i + \lambda_i + k_i} - \beta - \beta_i = 0 \tag{5.26a}$$

$$\frac{(1-w)/2}{\tilde{a} + q_i + \lambda_i} + \frac{w/2}{1 + \lambda_i} - \frac{1/2}{\tilde{a} + \lambda_i} + \frac{w/2}{\tilde{a} + q_i + \lambda_i + k_i} + \alpha - \gamma + \alpha_i = 0 \tag{5.26b}$$

$$\frac{w/2}{\tilde{a} + q_i + \lambda_i + k_i} - \gamma + \sigma_i = 0 \tag{5.26c}$$

$$\alpha(nP_2 - \sum_i \lambda_i) = 0 \tag{5.26d}$$

$$\beta(nP_1 - \sum_i q_i) = 0 \tag{5.26e}$$

$$\alpha_i \lambda_i = 0 \ \forall \ i \tag{5.26f}$$

$$\beta_i q_i = 0 \ \forall \ i \tag{5.26g}$$

$$\sigma_i k_i = 0 \ \forall \ i \tag{5.26h}$$

Now, we have the following lemma,

Lemma 3. *The solution to the optimization problem (5.24), can be written as,*

$$n(R_1 + \omega R_2) \leq \max_{\lambda_i, q_i, k_i, m_i} \sum_{i=1}^{4} m_i \left[\frac{(1-\omega)}{2} \log\left(\tilde{a} + q_i + \lambda_i\right) + \frac{\omega}{2} \log\left(1 + \lambda_i\right) \right.$$
$$\left. - \frac{1}{2} \log\left(\tilde{a} + \lambda_i\right) + \frac{\omega}{2} \log\left(\tilde{a} + q_i + \lambda_i + k_i\right) \right] \quad (5.27a)$$
$$\textit{subject to} \sum_{i=1}^{4} m_i q_i \leq nP_1 \quad (5.27b)$$
$$\sum_{i=1}^{4} m_i(\lambda_i + k_i) \leq nP_2 \quad (5.27c)$$
$$\sum_{i=1}^{4} m_i = 1;\ \lambda_1 = q_4 = k_4 = 0 \quad (5.27d)$$
$$k_1 = k_2 = k_3 \quad (5.27e)$$
$$\tilde{a} + q_i + \lambda_i + k_i = \tilde{a} + q_j + \lambda_j + k_j,\ \ i, j \in \{1, 2, 3\} \quad (5.27f)$$
$$\tilde{a} + q_i + a\lambda_i + k_i \leq \tilde{a} + \lambda_4,\ \ i \in \{1, 2, 3\} \quad (5.27g)$$

Proof. See appendix A.3 □

Thus, in (5.24), we only need to consider four different values of q_i, λ_i, and k_i satisfying the constraints given in the lemma above. Figure 13 summarizes these facts.

This completes the proof of the theorem 2. □

Now let us take a look at how close the known achievable schemes are. First, note that any point on the boundary of the upper bound must lie on one of the lines defined in (5.19).

Now consider the Han-Kobayashi achievability scheme. With λ_i power of X_2 used for private information, k_i power of X_2 used for common information, and q_i

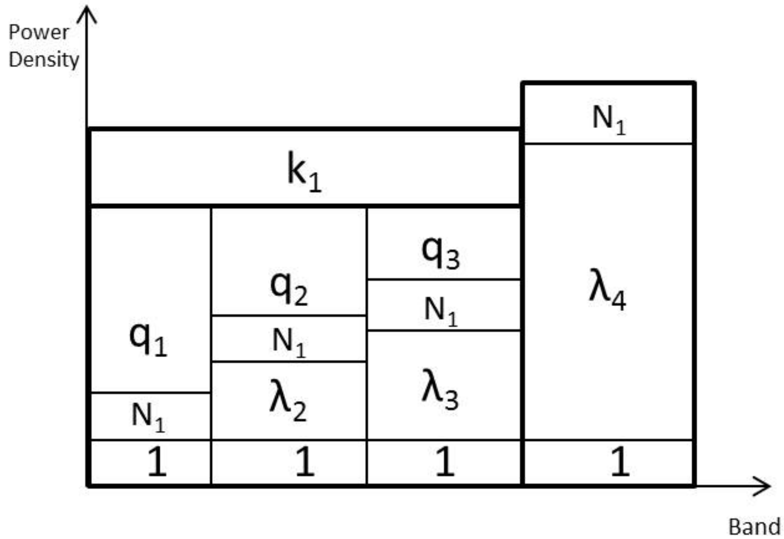

Figure 13. The four band partition: The height of the first three bands with k_i are the same and are less than or equal to the height of the last band.

power of X_1, we can achieve:

$$R_1 = \frac{1}{2}\log\left(1+\frac{q_i}{\tilde{a}+\lambda_i}\right),$$
$$R_2 = \frac{1}{2}\log\left(1+\lambda_i\right)+\frac{1}{2}\log\left(1+\frac{k_i}{\tilde{a}+\lambda_i+q_i}\right) \tag{5.28}$$

With time-sharing, we can achieve:

$$R_1 = \sum_{i=1}^{4}\frac{m_i}{2}\log\left(1+\frac{q_i}{\tilde{a}+\lambda_i}\right),$$
$$R_2 = \sum_{i=1}^{4} m_i\left[\frac{1}{2}\log\left(1+\lambda_i\right)+\frac{1}{2}\log\left(1+\frac{k_i}{\tilde{a}+\lambda_i+q_i}\right)\right] \tag{5.29}$$

And we see that for this achievability scheme, we have,

$$R_1 + \omega R_2 = \sum_{i=1}^{4} m_i \Bigg[\frac{(1-\omega)}{2} \log\Big(\tilde{a} + q_i + \lambda_i\Big) + \frac{\omega}{2} \log\Big(1 + \lambda_i\Big) \\ - \frac{1}{2} \log\Big(\tilde{a} + \lambda_i\Big) + \frac{\omega}{2} \log\Big(\tilde{a} + q_i + \lambda_i + k_i\Big) \Bigg] \qquad (5.30)$$

which is a point on the outer bound.

Thus, any point on the outer bound must be a solution of some equation of (5.19), for some ω, and that point can be achieved with Han-Kobayashi scheme as described above. Therefore, our outer bound matches with the achievable scheme.

We numerically evaluate the upper bound and compare it to the achievable scheme for $P_1 = 2, P_2 = 2, a = 0.5$ and $P_1 = 2, P_2 = 4, a = 0.8$. The results are shown in figure 14 and figure 15 respectively.

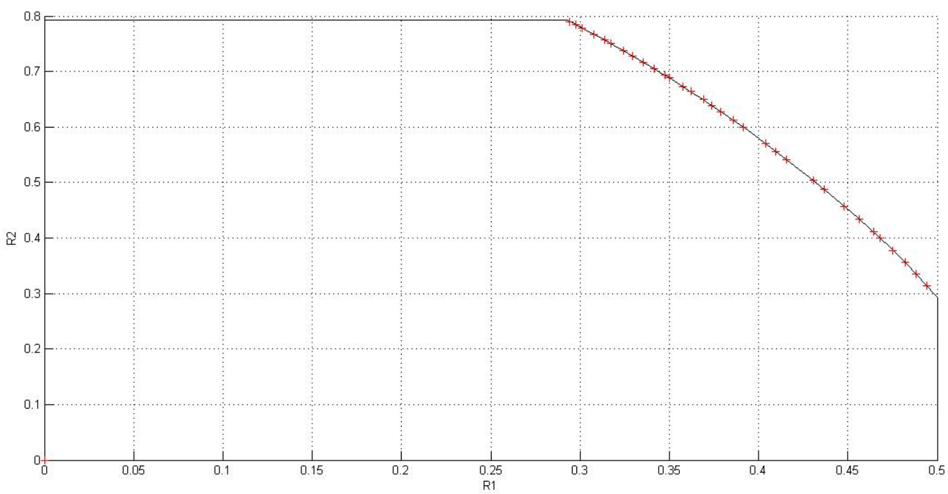

Figure 14. The achievable rate (black line) as compared to the new outer bound (red cross) for $P_1 = 2, P_2 = 2, a = 0.5$.

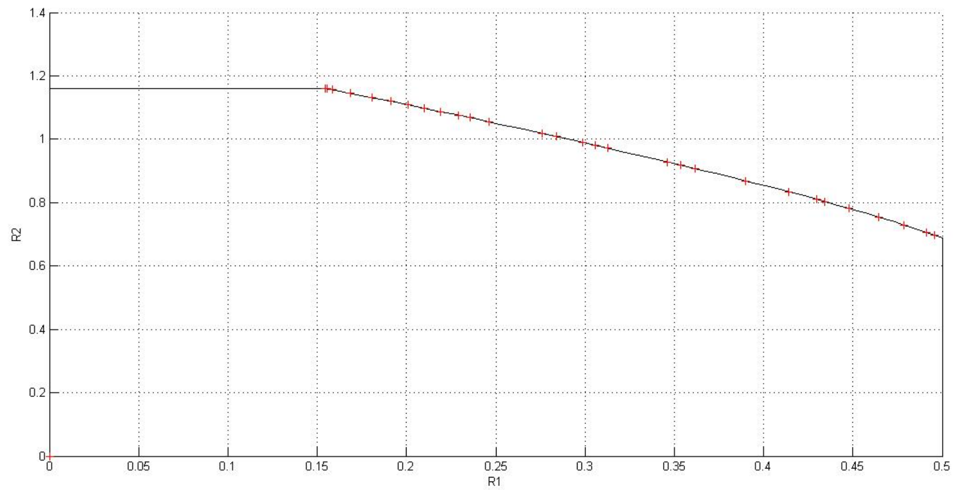

Figure 15. The achievable rate (black line) as compared to the new outer bound (red cross) for $P_1 = 2, P_2 = 4, a = 0.8$.

In the next chapter, we discuss why we were able to obtain such a tight upper bound. The concept of common information (defined in $\mathbf{U}$) deserves a closer look.

CHAPTER 6. DISCUSSION

In this section, we discuss our intuitions for our definition of $\boldsymbol{U}$. We show that it is necessary to take special care of common information while trying to derive an upper bound of Z-Interference channels. We provide the reasons for why Vaezi and Poor [19] got an upper bound which is lower than the known inner bound.

Consider a single user Gaussian point to point channel. Such channels are well studied and the capacity region is known. Let the input and output of the channel be represented by $\boldsymbol{X_1}$ and $\boldsymbol{Y_1}$ respectively. If the input average power constraint is P_1 and the noise has unit variance, then the capacity of the channel is given by $C = \frac{1}{2}\log(1 + P_1)$ bits per channel use. Now, suppose in a particular communication scenario, the transmitter is sending information at full capacity rate; it is transmitting a Gaussian signal with power P_1 (i.e. $\boldsymbol{X_1} \sim \mathcal{N}(0, P_2 I)$), and the information content of the transmitter is $\frac{1}{2}\log(1 + P_1)$ bits per channel use, which is being decoded at the receiver (receiver 1). Now suppose that this message of transmitted is also getting received at another receiver (receiver 2), but this link is stronger than the original link (see figure 16).

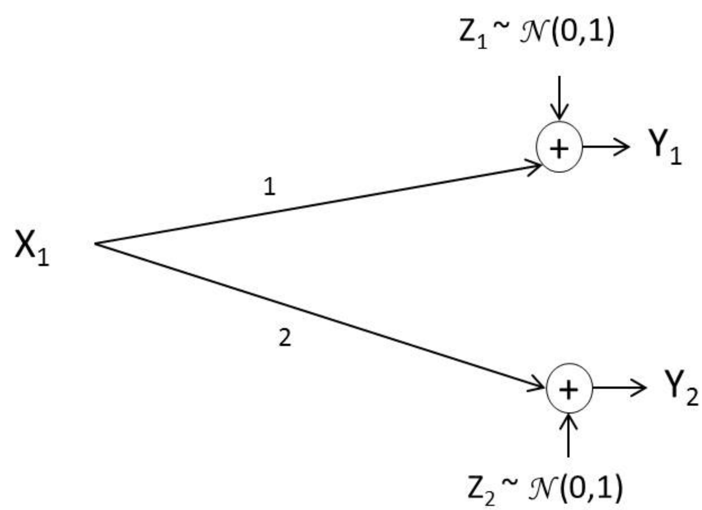

Figure 16. A simple case demonstrating common and private information

In this scenario, if the strength of the second link is two, then the outputs of the channel can be written as $\boldsymbol{Y_1} = \boldsymbol{X_1}+\boldsymbol{Z_1}$ and $\boldsymbol{Y_2} = 2\boldsymbol{X_1}+\boldsymbol{Z_2}$, where $\boldsymbol{Z_1}, \boldsymbol{Z_2} \sim \mathcal{N}(0, I)$. Information theory tells us that receiver 2, receives $I(\boldsymbol{X_1}; \boldsymbol{Y_2})$ amount of information. Clearly, $I(\boldsymbol{X_1}; \boldsymbol{Y_2}) > I(\boldsymbol{X_1}; \boldsymbol{Y_1})$, since $2\boldsymbol{X_1} + \boldsymbol{Z_2}$ contains more information about $\boldsymbol{X_1}$ than $\boldsymbol{X_1} + \boldsymbol{Z_1}$ (more signal compared to noise). Thus, receiver 2 is able to receive $\frac{1}{2}\log(1 + 2P_1)$ bits of information per channel use. However, the transmitter has only $\frac{1}{2}\log(1 + P_1)$ bits of information per channel use. Therefore, receiver 2 cannot gain any more information than what is contained in $\boldsymbol{X_1}$. Therefore, $I(\boldsymbol{X_1}; \boldsymbol{Y_2}) = I(\boldsymbol{X_1}; \boldsymbol{Y_1}) = \frac{1}{2}\log(1 + P_1)$. However, if $\boldsymbol{X_2}$ is a Gaussian distribution with any given variance, then mathematically $I(\boldsymbol{X_1}; \boldsymbol{Y_2}) > I(\boldsymbol{X_1}; \boldsymbol{Y_1})$. But, because of the nature our communication scheme, they turn out to be equal.

This conundrum between mathematical value of mutual information and operational meaning of information is explored in the context of Z-Interference Channels in the following theorem:

Theorem 3. *For the Z-Interference Channel given in equation* 4.2*, we have,*

$$h(\boldsymbol{X_2} + \boldsymbol{Z_2}) - h(\boldsymbol{X_2} + \boldsymbol{Z_1} + \boldsymbol{Z_2}) + h(\boldsymbol{Z_1} + \boldsymbol{Z_2}) - h(\boldsymbol{Z_2}) = h(\boldsymbol{W_2}|\boldsymbol{Y_1}, \boldsymbol{X_1}) \tag{6.1}$$

where $\boldsymbol{W_2}$ *is the information content of* $\boldsymbol{X_2}$ *for receiver two only (to be decoded at* $\boldsymbol{Y_2}$*).*

Proof. For any $\boldsymbol{W_2}$, we have,

$$\begin{aligned} h(\boldsymbol{W_2}|\boldsymbol{Y_1}, \boldsymbol{X_1}) &= h(\boldsymbol{W_2}|\boldsymbol{Y_2}, \boldsymbol{X_1}) + h(\boldsymbol{X_2}|\boldsymbol{W_2}, \boldsymbol{Y_2}, \boldsymbol{X_1}) \\ &= h(\boldsymbol{X_2}|\boldsymbol{Y_2}, \boldsymbol{X_1}) + h(\boldsymbol{W_2}|\boldsymbol{X_2}, \boldsymbol{Y_2}, \boldsymbol{X_1}) \end{aligned} \tag{6.2}$$

So,

$$h(\boldsymbol{W_2}|\boldsymbol{X_2}, \boldsymbol{Y_2}, \boldsymbol{X_1}) = h(\boldsymbol{W_2}|\boldsymbol{Y_2}) + h(\boldsymbol{X_2}|\boldsymbol{W_2}, \boldsymbol{Y_2}) - h(\boldsymbol{X_2}|\boldsymbol{Y_2}) \tag{6.3}$$

Similarly, we can write,

$$h(\boldsymbol{W_2}|\boldsymbol{X_2}, \boldsymbol{Y_1}, \boldsymbol{X_1}) = h(\boldsymbol{W_2}|\boldsymbol{Y_1}, \boldsymbol{X_1}) + h(\boldsymbol{X_2}|\boldsymbol{W_2}, \boldsymbol{Y_1}, \boldsymbol{X_1}) - h(\boldsymbol{X_2}|\boldsymbol{Y_1}, \boldsymbol{X_1}) \tag{6.4a}$$

$$\Rightarrow h(\boldsymbol{W_2}|\boldsymbol{X_2}, \boldsymbol{X_2}+\boldsymbol{Z_1}+\boldsymbol{Z_2}) = h(\boldsymbol{W_2}|\boldsymbol{X_2}+\boldsymbol{Z_1}+\boldsymbol{Z_2}) + h(\boldsymbol{X_2}|\boldsymbol{W_2}, \boldsymbol{X_2}+\boldsymbol{Z_1}+\boldsymbol{Z_2}) - h(\boldsymbol{X_2}|\boldsymbol{X_2}+\boldsymbol{Z_1}+\boldsymbol{Z_2}) \tag{6.4b}$$

Since $\boldsymbol{W_2}$ is contained in $\boldsymbol{X_2}$, we have $h(\boldsymbol{W_2}|\boldsymbol{X_2}, \boldsymbol{X_2}+\boldsymbol{Z_1}+\boldsymbol{Z_2}) = 0$ and $h(\boldsymbol{W_2}|\boldsymbol{X_2}, \boldsymbol{Y_2}, \boldsymbol{X_1}) = 0$. We can write,

$$0 = h(\boldsymbol{W_2}|\boldsymbol{X_2}+\boldsymbol{Z_2}) + h(\boldsymbol{X_2}|\boldsymbol{W_2}, \boldsymbol{X_2}+\boldsymbol{Z_2}) - h(\boldsymbol{X_2}|\boldsymbol{X_2}+\boldsymbol{Z_2}) \tag{6.5}$$

$$\Rightarrow h(\boldsymbol{W_2}|\boldsymbol{X_2}+\boldsymbol{Z_1}+\boldsymbol{Z_2}) = h(\boldsymbol{X_2}|\boldsymbol{X_2}+\boldsymbol{Z_1}+\boldsymbol{Z_2}) - h(\boldsymbol{X_2}|\boldsymbol{W_2}, \boldsymbol{X_2}+\boldsymbol{Z_1}+\boldsymbol{Z_2}) \tag{6.6}$$

Adding (6.5) and (6.6), we get,

$$h(\boldsymbol{W_2}|\boldsymbol{X_2}+\boldsymbol{Z_1}+\boldsymbol{Z_2}) = [h(\boldsymbol{X_2}|\boldsymbol{X_2}+\boldsymbol{Z_1}+\boldsymbol{Z_2}) - h(\boldsymbol{X_2}|\boldsymbol{X_2}+\boldsymbol{Z_2})] + h(\boldsymbol{W_2}|\boldsymbol{X_2}+\boldsymbol{Z_2}) + [h(\boldsymbol{X_2}|\boldsymbol{W_2}, \boldsymbol{X_2}+\boldsymbol{Z_2}) - h(\boldsymbol{X_2}|\boldsymbol{W_2}, \boldsymbol{X_2}+\boldsymbol{Z_1}+\boldsymbol{Z_2})] \tag{6.7}$$

Now,

$$\begin{aligned} h(\boldsymbol{X_2}|\boldsymbol{W_2}, \boldsymbol{X_2}+\boldsymbol{Z_1}+\boldsymbol{Z_2}) &\geq h(\boldsymbol{X_2}|\boldsymbol{W_2}, \boldsymbol{X_2}+\boldsymbol{Z_1}+\boldsymbol{Z_2}, \boldsymbol{Z_1}) \\ &= h(\boldsymbol{X_2}|\boldsymbol{W_2}, \boldsymbol{X_2}+\boldsymbol{Z_2}) \end{aligned}$$

$$\Rightarrow h(\boldsymbol{X_2}|\boldsymbol{W_2}, \boldsymbol{X_2}+\boldsymbol{Z_2}) - h(\boldsymbol{X_2}|\boldsymbol{W_2}, \boldsymbol{X_2}+\boldsymbol{Z_1}+\boldsymbol{Z_2}) \leq 0 \tag{6.8}$$

Using (6.8), we can write (6.7) as,

$$h(\boldsymbol{W_2}|\boldsymbol{X_2}+\boldsymbol{Z_1}+\boldsymbol{Z_2}) \leq h(\boldsymbol{X_2}|\boldsymbol{X_2}+\boldsymbol{Z_1}+\boldsymbol{Z_2}) - h(\boldsymbol{X_2}|\mathbf{Y_2}) + h(\boldsymbol{W_2}|\boldsymbol{X_2}+\boldsymbol{Z_2}) \tag{6.9a}$$

$$\leq h(\boldsymbol{X_2}|\boldsymbol{X_2}+\boldsymbol{Z_1}+\boldsymbol{Z_2}) - h(\boldsymbol{X_2}|\mathbf{Y_2}) \tag{6.9b}$$

$$\leq [h(\boldsymbol{X_2}) - h(\boldsymbol{X_2}+\boldsymbol{Z_1}+\boldsymbol{Z_2}) + h(\boldsymbol{X_2}+\boldsymbol{Z_1}+\boldsymbol{Z_2}|\boldsymbol{X_2})] \tag{6.9c}$$

$$- [h(\boldsymbol{X_2}) - h(\boldsymbol{X_2}+\boldsymbol{Z_2}) + h(\boldsymbol{X_2}+\boldsymbol{Z_2}|\boldsymbol{X_2})] \tag{6.9d}$$

$$= h(\boldsymbol{X_2}+\boldsymbol{Z_2}) - h(\boldsymbol{X_2}+\boldsymbol{Z_1}+\boldsymbol{Z_2}) + h(\boldsymbol{Z_1}+\boldsymbol{Z_2}) - h(\boldsymbol{Z_2}) \tag{6.9e}$$

Thus, we have the following:

$$h(\boldsymbol{W_2}|\boldsymbol{X_2}+\boldsymbol{Z_1}+\boldsymbol{Z_2}) \leq h(\boldsymbol{X_2}+\boldsymbol{Z_2}) - h(\boldsymbol{X_2}+\boldsymbol{Z_1}+\boldsymbol{Z_2}) + h(\boldsymbol{Z_1}+\boldsymbol{Z_2}) - h(\boldsymbol{Z_2}) \tag{6.10}$$

Now, assume that

$$h(\boldsymbol{X_2}+\boldsymbol{Z_2}) - h(\boldsymbol{X_2}+\boldsymbol{Z_1}+\boldsymbol{Z_2}) + h(\boldsymbol{Z_1}+\boldsymbol{Z_2}) - h(\boldsymbol{Z_2}) > h(\boldsymbol{W_2}|\boldsymbol{X_2}+\boldsymbol{Z_1}+\boldsymbol{Z_2}) \tag{6.11}$$

Then, we have,

$$h(\boldsymbol{X_2} + \boldsymbol{Z_2}) - h(\boldsymbol{Z_2}) - h(\boldsymbol{X_2} + \boldsymbol{Z_1} + \boldsymbol{Z_2}) + h(\boldsymbol{Z_1} + \boldsymbol{Z_2}) > h(\boldsymbol{W_2}|\boldsymbol{X_2} + \boldsymbol{Z_1} + \boldsymbol{Z_2}) \tag{6.12a}$$

$$\Rightarrow I(\boldsymbol{X_2}; \boldsymbol{Y_2}) - I(\boldsymbol{X_2}; \boldsymbol{Y_1}|\boldsymbol{X_1}) > h(\boldsymbol{W_2}|\boldsymbol{X_2} + \boldsymbol{Z_1} + \boldsymbol{Z_2}) \tag{6.12b}$$

$$\Rightarrow I(\boldsymbol{X_2}; \boldsymbol{Y_2}) > h(\boldsymbol{W_2}|\boldsymbol{X_2} + \boldsymbol{Z_1} + \boldsymbol{Z_2}) + I(\boldsymbol{X_2}; \boldsymbol{Y_1}|\boldsymbol{X_1}) \tag{6.12c}$$

Since $\boldsymbol{W_2}$ is a part of $\boldsymbol{X_2}$, we can write

$$I(\boldsymbol{X_2}; \boldsymbol{Y_1}|\boldsymbol{X_1}) = I(\boldsymbol{X_2}, \boldsymbol{W_2}; \boldsymbol{Y_1}|\boldsymbol{X_1}) \tag{6.13a}$$

$$= I(\boldsymbol{W_2}; \boldsymbol{Y_1}|\boldsymbol{X_1}) + I(\boldsymbol{X_2}; \boldsymbol{Y_1}|\boldsymbol{W_2}, \boldsymbol{X_1}) \tag{6.13b}$$

In Z-Interference Channel, no information is sent from transmitter 2 to receiver 1. Therefore, $I(\boldsymbol{X_2}; \boldsymbol{Y_1}|\boldsymbol{W_2}, \boldsymbol{X_1}) = 0$. With that in mind, we can use (6.13b) in (6.12c) and get,

$$I(\boldsymbol{X_2}; \boldsymbol{Y_2}) > h(\boldsymbol{W_2}|\boldsymbol{X_2} + \boldsymbol{Z_1} + \boldsymbol{Z_2}) + I(\boldsymbol{W_2}; \boldsymbol{Y_1}|\boldsymbol{X_1}) \tag{6.14a}$$

$$I(\boldsymbol{X_2}; \boldsymbol{Y_2}) > h(\boldsymbol{W_2}) \tag{6.14b}$$

Equation (6.14b) states that R_2 is greater than $h(\boldsymbol{W_2})$, which is not possible. Thus our initial assumption of (6.11) was wrong and we have,

$$h(\boldsymbol{X_2}+\boldsymbol{Z_2})-h(\boldsymbol{X_2}+\boldsymbol{Z_1}+\boldsymbol{Z_2})+h(\boldsymbol{Z_1}+\boldsymbol{Z_2})-h(\boldsymbol{Z_2}) \leq h(\boldsymbol{W_2}|\boldsymbol{X_2}+\boldsymbol{Z_1}+\boldsymbol{Z_2}) \tag{6.15}$$

From (6.10) and (6.15), we have

$$h(\boldsymbol{X_2} + \boldsymbol{Z_2}) - h(\boldsymbol{X_2} + \boldsymbol{Z_1} + \boldsymbol{Z_2}) + h(\boldsymbol{Z_1} + \boldsymbol{Z_2}) - h(\boldsymbol{Z_2}) = h(\boldsymbol{W_2}|\boldsymbol{Y_1}, \boldsymbol{X_1}) \quad (6.16)$$

which proves our original statement. □

Theorem 3 says that the information decodable at receiver two (i.e. at $\boldsymbol{Y_2}$) is dependent on the message content of $\boldsymbol{X_2}$ and not just the distribution of $\boldsymbol{X_2}$. For example, if $\boldsymbol{W_2} = 0$ (i.e. no specific information for receive two; in other words, no "private information"), then the information of $\boldsymbol{X_2}$ decodable at receiver two is same as the decodable amount at receiver one :

$$h(\boldsymbol{X_2} + \boldsymbol{Z_2}) - h(\boldsymbol{Z_2}) = h(\boldsymbol{X_2} + \boldsymbol{Z_1} + \boldsymbol{Z_2}) - h(\boldsymbol{Z_1} + \boldsymbol{Z_2}) \quad (6.17)$$

$$\Rightarrow I(\boldsymbol{X_2}; \boldsymbol{X_2} + \boldsymbol{Z_2}) = I(\boldsymbol{X_2}; \boldsymbol{X_2} + \boldsymbol{Z_1} + \boldsymbol{Z_2}) \quad (6.18)$$

However, mathematically this is not possible. For any Gaussian distribution $\boldsymbol{X_2}$, we have, $I(\boldsymbol{X_2}; \boldsymbol{X_2} + \boldsymbol{Z_2}) > I(\boldsymbol{X_2}; \boldsymbol{X_2} + \boldsymbol{Z_1} + \boldsymbol{Z_2})$. This is exactly the same case as was described earlier in figure 16. This characteristic of $h(\boldsymbol{X_2}+\boldsymbol{Z_1}+\boldsymbol{Z_2})-h(\boldsymbol{Z_1}+\boldsymbol{Z_2})$ is shown in figure 17, where we plot $h(\boldsymbol{X_2} + \boldsymbol{Z_1} + \boldsymbol{Z_2}) - h(\boldsymbol{Z_1} + \boldsymbol{Z_2})$ vs power of the Gaussian random variable $\boldsymbol{X_2}$ for a given $\boldsymbol{W_2}$.

So, it turns out that just specifying the distribution of the random variable $\boldsymbol{X_2}$ is not enough; we need to specify how much information is "common" and how much is "private". For the Z-Interference Channel, if we do not use $\boldsymbol{W_2}$ to describe our information and rather use $\boldsymbol{X_2}$ as our information content, then we are implicitly assuming that $\boldsymbol{X_2} = \boldsymbol{W_2}$ and the entire power of $\boldsymbol{X_2}$ is used as private information. This is exactly what was done in Vaezi and Poor's outer bound [19]. Now, to achieve the capacity for R_1, it needs a clear channel and therefore the private information content of $\boldsymbol{X_2}$ should be zero. If we do not allow common information, then the

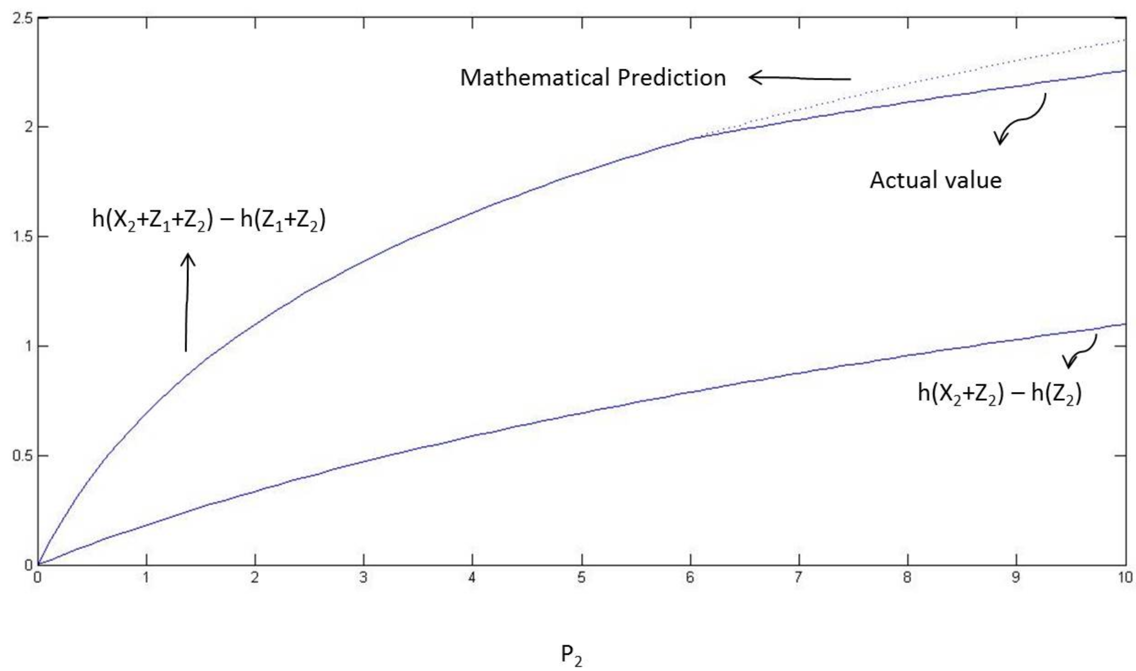

Figure 17. Nature of common information: The mutual information is no longer given by the mathematical expression

entire power of $\boldsymbol{X_2}$ is zero at this point (since we have no common information and no private information), and thus $R_2 = 0$. However, if we allow common information (as we did), then we can achieve the point $\left(\frac{1}{2}\log(1+P_1), \frac{1}{2}\log\left(1+\frac{P_2}{1+P_1}\right)\right)$, where the entire information of $\boldsymbol{X_2}$ is common information when R_1 achieves its capacity. This explains why Vaezi and Poor's outer bound turned out to be lower than the inner bound, but we were able to derive an outer bound which matches with the inner bound. For the reader's reference, we show the upper bound of Vaezi and Poor in figure 18.

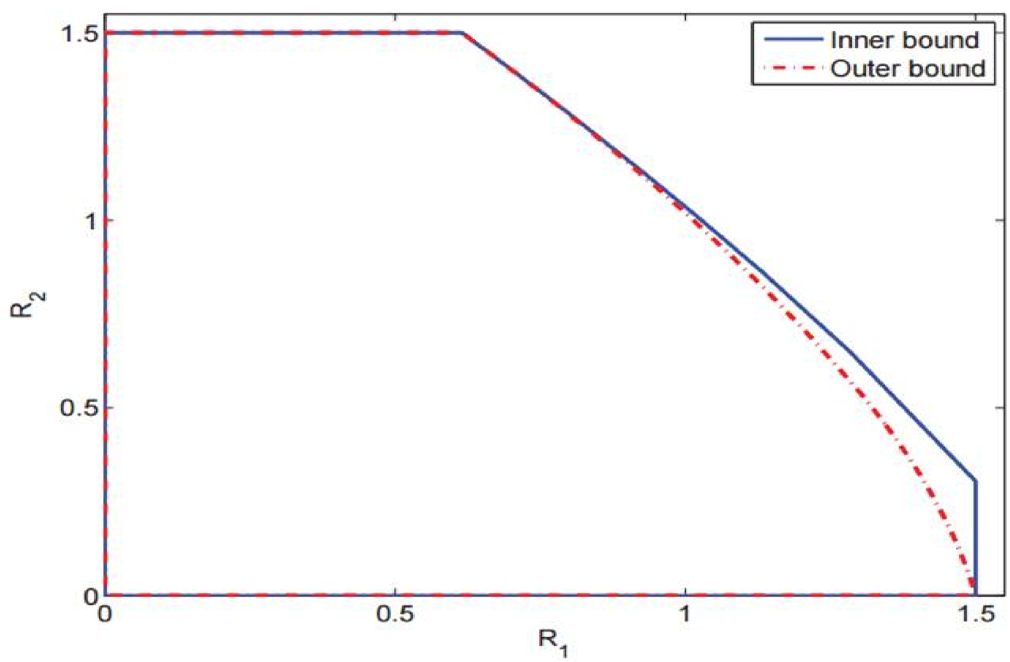

Figure 18. Vaezi and Poor's outer bound

The next chapter concludes our work and summarizes our results. We also discuss related problems where this work can be applied.

CHAPTER 7. CONCLUSION

We have considered the Z-Interference Channel, which is a simplified case of the general two user Interference Channel in the fact that one of the interference links is zero. For such a channel, we restricted the input to Gaussian signal and under that constraint characterized the capacity region under weak interference. Interference channel was identified more than 40 years ago, but the exact characterization of the capacity region is still not known to date. Many progress has been made, but it remained unclear whether the best known achievable scheme was indeed the capacity region or not. Even for the one-sided interference channels, the capacity was not known. In this work about the Z-Interference Channel, we assume that some part of interfering message can be decoded and the rest of it has to be treated as noise. Then considering all possible combinations of decodable and undecodable interfering messages at the receiver, we derive an upper bound. Finally, we show that this upper bound is actually achievable by Han-Kobayashi scheme with time sharing. Therefore, we prove that best known achievable region was indeed the capacity region.

This work sheds light on the problem of common information lurking in the Z-Interference channels. The intuition gained from this work can be applied to the problems such as the Z-Channel. The Z-Channel has the same channel model as the Z-Interference Channel, but the interference link is seen as another communication link. So the Z-Channel has three set of independent messages to be transmitted and its capacity region is therefore three dimensional. We denote the three rates by R_1, R_2, and R_{21}. When $R_{21} = 0$, we get the Z-Interference Channel (as shown by [22]). There has been many work on this channel and the capacity region is partially characterized (see [23]). Till now, the boundary on the $R_1 - R_2$ plane was not characterized. Our result just characterizes this bound on $R_1 - R_2$ plane. For the reader's reference, we show the known Z-Channel capacity region in figure 19 (taken from [23]).

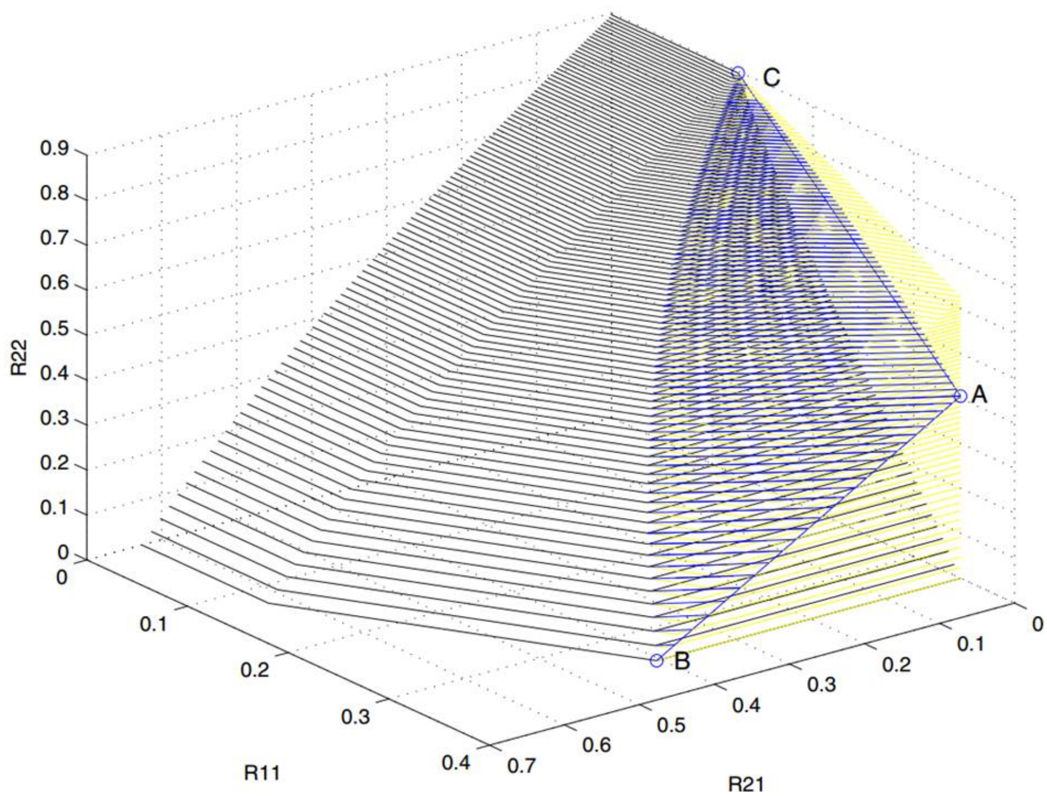

Figure 19. Known Z-Channel Capacity Region: Points B and C are known to be optimal; point A is achievable, but its optimality is unknown

Our work will also give intuitions about finding the solution to the general Z-Interference Channel (i.e. without Gaussian signaling constraint). Furthermore, since the Z-Interference Channel is a special case of the two user Interference Channel, our work can be used as a starting point for characterizing the capacity of the two user Interference Channel.

Ultimately, this work will help in the future research towards exact characterization of capacity of interference networks.

REFERENCES

[1] T. M. Cover and J. A. Thomas, *Elements of Information Theory.* John Wiley and Sons Inc, 1991.

[2] R. Etkin, D. Tse, and H. Wang, “Gaussian interference channel capacity to within one bit,” *IEEE Trans. on Inform. Th.*, vol. 54, pp. 5534–5562, Dec, 2008.

[3] A. B. Carleial, “A case where interference does not reduce the capacity,” *IEEE Trans. on Inform. Th.*, vol. 21, pp. 569–570, Sep, 1975.

[4] ——, “Interference channels,” *IEEE Trans. on Inform. Th.*, vol. 24, pp. 60–70, Jan, 1978.

[5] R. G. Gallager, *Information Theory and Reliable Communications.* New York: Wiley, 1968.

[6] C. E. Shannon, “Two-way communication channel,” in *Proc. of 4th Berkeley symp. Mathematical statistics and probability, Berkeley, CA*, vol. 1, Mar, 1961, pp. 611–644.

[7] R. Ahlswede, “The capacity region of a channel with two senders and two receivers,” *Annals Probabil.*, vol. 2, pp. 805–814, Sep, 1974.

[8] T. M. Cover, “An achievable rate region for the broadcasting channel,” *IEEE Trans. on Inform. Th.*, vol. IT-21, pp. 399–404, July 1975.

[9] H. Sato, “Two-user communication channels,” *IEEE Trans. on Inform. Th.*, vol. 23, pp. 295–304, May, 1977.

[10] T. S. Han and K. Kobayashi, “A new achievable region for the interference channel,” *IEEE Trans. on Inform. Th.*, vol. 27, pp. 49–60, Jan, 1981.

[11] H. Sato, "The capacity of Gaussian interference channel under strong interference," *IEEE Trans. on Inform. Th.*, vol. 27, pp. 786–788, Nov, 1981.

[12] R. Benzel, "The capacity region of a class of discrete additive degraded interference channels," *IEEE Trans. on Inform. Th.*, vol. 25, pp. 228–231, Mar, 1979.

[13] M. H. M. Costa, "On the gaussian interference channel," *IEEE Trans. on Inform. Th.*, vol. 31, pp. 607–615, Sep, 1985.

[14] G. Kramer, "Outer bounds on the capacity of Gaussian interference channels," *IEEE Trans. on Inform. Th.*, vol. 50, pp. 581–586, Feb, 2004.

[15] I. Sason, "On achievable rate regions for the gaussian interference channel," *IEEE Trans. on Inform. Th.*, vol. 50, pp. 1345–1356, Jun, 2004.

[16] H. Chong, M. Motani, H. Garg, and H. E. Gamal, "On the Han-Kobayashi region for the interference channel," *IEEE Trans. on Inform. Th.*, vol. 54, pp. 3188–3195, Jul, 2008.

[17] X. Shang, B. Chen, G. Kramer, and H. V. Poor, "Capacity regions and sum-rate capacities of vector Gaussian interference channels," *IEEE Trans. on Inform. Th.*, vol. 56, no. 10, pp. 5030 – 5044, Oct, 2010.

[18] M. H. M. Costa, "Noisebergs in z gaussian interference channels," *Information Theory and Applications Workshop (ITA)*, pp. 1–6, 2011.

[19] M. Vaezi and H. V. Poor, "On limiting expressions for the capacity region of gaussian interference channels," in *49th Asilomar Conference on Signals, Systems and Computers*, 2015.

[20] R. Cheng and S. Verdu, "On limiting characterizations of memoryless multiuser capacity regions," *IEEE Trans. on Inform. Th.*, vol. 39, pp. 609–612, Mar, 1993.

[21] M. Vaezi and H. V. poor, "Simplified han-kobayashi region for one-sided and mixed gaussian interference channels," *IEEE International Conference on Communications (ICC)*, May, 2016.

[22] H. Chong, M. Motani, H. Garg, and H. E. Gamal, "Capacity theorems for the z channel," *IEEE Trans. on Inform. Th.*, vol. 53, pp. 1348–1365, Mar, 2007.

[23] N. Liu and S. Ulukus, "On the capacity region of the gaussian z-channel," *Global Telecommunications Conference, GLOBECOM, IEEE.*, vol. 1, pp. 415–419, 2004.

[24] V. S. Annapureddy and V. V. Veeravalli, "Sum capacity of the Gaussian interference channels in the low interference regime," *IEEE Trans. on Inform. Th.*, vol. 55, pp. 3032 – 3050, Jul. 2009.

APPENDIX

A.1. Proof of Lemma 1

We can use [24], Lemma 6, and write

$$h(\boldsymbol{X_2} + \boldsymbol{Z_2}|\boldsymbol{U}) = \frac{1}{2}\log|K_{\boldsymbol{X_2}|\boldsymbol{U}} + I| \tag{A.1}$$

where $K_{\boldsymbol{X_2}|\boldsymbol{U}}$ is the covariance of $\boldsymbol{X_2}|\boldsymbol{U}$. [We need $\boldsymbol{X_2}$ and $\boldsymbol{U}$ to be zero-mean Gaussian random vector sequences independent of Z].

Similarly, we have

$$h(\boldsymbol{Y_1}|\boldsymbol{U}, \boldsymbol{X_1}) = h(\boldsymbol{X_2} + \boldsymbol{Z_1} + \boldsymbol{Z_2}|\boldsymbol{U}) = \frac{1}{2}\log|K_{\boldsymbol{X_2}|\boldsymbol{U}} + \tilde{a}I| \tag{A.2}$$

On the other hand,

$$h(\boldsymbol{Y_1}|\boldsymbol{U}) = h(\boldsymbol{X_1} + \boldsymbol{X_2} + \boldsymbol{Z_1} + \boldsymbol{Z_2}|\boldsymbol{U}) = \frac{1}{2}\log\left|\mathrm{cov}(\boldsymbol{X_1} + \boldsymbol{X_2} + \boldsymbol{Z_1} + \boldsymbol{Z_2}|\boldsymbol{U})\right| \tag{A.3}$$

Now, define $K_{\boldsymbol{U}}$ to be covariance of $\boldsymbol{U}$, $K_{\boldsymbol{X_1}}$ to be covariance of $\boldsymbol{X_1}$, $K = E[\boldsymbol{X_2}\boldsymbol{U}]$ and $K^* = E[\boldsymbol{U}\boldsymbol{X_2}]$. Now consider,

$$\left|\mathrm{cov}\begin{bmatrix}\boldsymbol{X_2} \\ \boldsymbol{U}\end{bmatrix}\right| = \left|\begin{bmatrix}K_{\boldsymbol{X_2}} & K \\ K^* & K_{\boldsymbol{U}}\end{bmatrix}\right| \tag{A.4a}$$

$$= \left|K_{\boldsymbol{U}}\right|\left|K_{\boldsymbol{X_2}} - KK_{\boldsymbol{U}}^{-1}K^*\right| \tag{A.4b}$$

where (A.4b) follows from Schur complement.

So,

$$h(\boldsymbol{X_2}, \boldsymbol{U}) = h\Big(\begin{bmatrix} \boldsymbol{X_2} \\ \boldsymbol{U} \end{bmatrix} \Big) = h(\boldsymbol{U}) + h(\boldsymbol{X_2}|\boldsymbol{U}) \tag{A.5a}$$

$$\Rightarrow \frac{1}{2}\log\left|2\pi e.\text{cov}\begin{bmatrix} \boldsymbol{X_2} \\ \boldsymbol{U} \end{bmatrix}\right| = \frac{1}{2}\log\Big|2\pi e.K_{\boldsymbol{U}}\Big| + \frac{1}{2}\log\Big|2\pi e.K_{\boldsymbol{X_2}|\boldsymbol{U}}\Big| \tag{A.5b}$$

$$\Rightarrow \frac{1}{2}\log\Big|K_{\boldsymbol{U}}\Big| + \frac{1}{2}\log\Big|K_{\boldsymbol{X_2}} - KK_{\boldsymbol{U}}^{-1}K^*\Big| = \frac{1}{2}\log\Big|K_{\boldsymbol{U}}\Big| + \frac{1}{2}\log\Big|K_{\boldsymbol{X_2}|\boldsymbol{U}}\Big| \tag{A.5c}$$

$$\Rightarrow K_{\boldsymbol{X_2}|\boldsymbol{U}} = K_{\boldsymbol{X_2}} - KK_{\boldsymbol{U}}^{-1}K^* \tag{A.5d}$$

where (A.5c) follows from (A.4b).

Similarly, we have

$$\left|\text{cov}\begin{bmatrix} \boldsymbol{X_1} + X_2 + \boldsymbol{Z_1} + \boldsymbol{Z_2} \\ \boldsymbol{U} \end{bmatrix}\right| = \left|\begin{bmatrix} K_{\boldsymbol{X_1}} + K_{\boldsymbol{X_2}} + \tilde{a}I & K \\ K^* & K_{\boldsymbol{U}} \end{bmatrix}\right| \tag{A.6a}$$

$$= \Big|K_{\boldsymbol{U}}\Big| \Big|K_{\boldsymbol{X_1}} + K_{\boldsymbol{X_2}} + \tilde{a}I - KK_{\boldsymbol{U}}^{-1}K^*\Big| \tag{A.6b}$$

$$= \Big|K_{\boldsymbol{U}}\Big| \Big|K_{\boldsymbol{X_1}} + \tilde{a}I - K_{\boldsymbol{X_2}|\boldsymbol{U}}\Big| \tag{A.6c}$$

Therefore,

$$h(\boldsymbol{X_1} + \boldsymbol{X_2} + \boldsymbol{Z_1} + \boldsymbol{Z_2}, \boldsymbol{U}) = h\left(\begin{bmatrix} \boldsymbol{X_1} + \boldsymbol{X_2} + \boldsymbol{Z_1} + \boldsymbol{Z_2} \\ \boldsymbol{U} \end{bmatrix} \right)$$

$$= h(\boldsymbol{U}) + h(\boldsymbol{X_1} + \boldsymbol{X_2} + \boldsymbol{Z_1} + \boldsymbol{Z_2}|\boldsymbol{U}) \tag{A.7a}$$

$$\Rightarrow \frac{1}{2}\log\left|2\pi e.\text{cov}\begin{bmatrix}\boldsymbol{X_1}+\boldsymbol{X_2}+\boldsymbol{Z_1}+\boldsymbol{Z_2}\\ \boldsymbol{U}\end{bmatrix}\right| = \frac{1}{2}\log\left|2\pi e.K_{\boldsymbol{U}}\right| + \frac{1}{2}\log\left|2\pi e.\text{cov}(\boldsymbol{X_1}+\boldsymbol{X_2}+\boldsymbol{Z_1}+\boldsymbol{Z_2}|\boldsymbol{U})\right| \tag{A.7b}$$

$$\Rightarrow \frac{1}{2}\log\left|K_{\boldsymbol{U}}\right| + \frac{1}{2}\log\left|K_{\boldsymbol{X_1}} + K_{\boldsymbol{X_2}|\boldsymbol{U}} + \tilde{a}I\right| = \frac{1}{2}\log\left|K_{\boldsymbol{U}}\right| + \frac{1}{2}\log\left|\text{cov}(\boldsymbol{X_1}+\boldsymbol{X_2}+\boldsymbol{Z_1}+\boldsymbol{Z_2}|\boldsymbol{U})\right| \tag{A.7c}$$

$$\Rightarrow \text{cov}(\boldsymbol{X_1}+\boldsymbol{X_2}+\boldsymbol{Z_1}+\boldsymbol{Z_2}|\boldsymbol{U}) = K_{\boldsymbol{X_1}} + K_{\boldsymbol{X_2}|\boldsymbol{U}} + \tilde{a}I \tag{A.7d}$$

Therefore, from (5.9c), we have.

$$h(\boldsymbol{Y_1}|\boldsymbol{U}) = \frac{1}{2}\log\left|\text{cov}(\boldsymbol{X_1}+\boldsymbol{X_2}+\boldsymbol{Z_1}+\boldsymbol{Z_2}|\boldsymbol{U})\right| = \frac{1}{2}\log\left|K_{\boldsymbol{X_1}} + K_{\boldsymbol{X_2}|\boldsymbol{U}} + \tilde{a}I\right| \tag{A.8}$$

A.2. Proof of Lemma 2

Here, for a given P_1 and P_2, we will solve the maximization problem (5.15) numerically after we simplify the problem by observing the nature of the equations.

Combining (5.17a) and (5.17f), we get

$$q_i\Big(\beta - \frac{(1-\omega)/2}{\tilde{a} + q_i + \lambda_i}\Big) = 0 \ \forall \ i \tag{A.9}$$

Without loss of generality, assume $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$. Say m is the largest integer such that $\beta < \frac{(1-\omega)/2}{\tilde{a}+\lambda_i} \ \forall i \leq m$.

Then,

$$q_i = \begin{cases} \frac{1-\omega}{2\beta} - (\tilde{a} + \lambda_i) & \forall \ i \leq m \\ 0 & \forall \ i > m \end{cases} \tag{A.10}$$

Note that this essentially divided our q_i and λ_i into two bands - one where $q_i \neq 0$ and the other with $q_i = 0$. We shall call the case $q_i \neq 0$ as band 1 and $q_i = 0$ as band 2. Let $m/n = \lambda$. Thus, for λ fraction we have band 1 and $1 - \lambda$ fraction we have band 2.

Now, for any $l < m$, we have

$$\begin{aligned} \beta - \frac{(1-\omega)/2}{\tilde{a} + \lambda_l} < 0 \\ \text{and } \beta - \frac{(1-\omega)/2}{\tilde{a} + \lambda_l + q_l} = 0 \end{aligned} \tag{A.11}$$

For any $j > m$, we have

$$\beta - \frac{(1-\omega)/2}{\tilde{a} + \lambda_j} \geq 0 \tag{A.12}$$

From (A.11) and (A.12), we have for any $l < m$ and $j > m$,

$$\tilde{a} + q_l + \lambda_l \leq \tilde{a} + \lambda_j \tag{A.13}$$

This shows that the height of Band 1 is less than or equal to the height of Band 2.

Combining (5.17b) and (5.17e), we get

$$\lambda_i\Big(\alpha - \frac{(1-\omega)/2}{\tilde{a} + q_i + \lambda_i} + \frac{1/2}{\tilde{a} + \lambda_i} - \frac{\omega/2}{1 + \lambda_i}\Big) = 0 \tag{A.14}$$

Now when $q_i = 0$, we have,

$$\lambda_i\Big(\alpha + \frac{\omega/2}{\tilde{a} + \lambda_i} - \frac{\omega/2}{1 + \lambda_i}\Big) = 0 \tag{A.15a}$$

Assuming that λ_i is not zero (since it would be sub-optimal for both q_i and λ_i to be zero simultaneously), we have

$$(\tilde{a} + \lambda_i)(1 + \lambda_i) = \frac{\omega(\tilde{a} - 1)}{2\alpha} \tag{A.15b}$$

$$\Rightarrow a\lambda_i^2 + (1 + a)\lambda_i + 1 - \frac{\omega(1 - a)}{2\alpha} = 0 \tag{A.15c}$$

This shows that λ_i can have two values, but one of them is negative (since coefficient of λ_i is positive). Since $\lambda_i \geq 0$, we see that when $q_i = 0$, λ_i can take only one value. Thus, band 2 consists of some non-zero $\lambda_i = \lambda_2$ and $q_i = 0$.

When q_i is non-zero, i.e. $i \leq m$, we have,

$$\lambda_i\Big(\alpha - \beta + \frac{1/2}{\tilde{a} + \lambda_i} - \frac{\omega/2}{1 + \lambda_i}\Big) = 0 \tag{A.16}$$

which shows that there can be three different values of λ_i within band 1.

Thus, we have divided the input power into 4 bands with different values of q_i and λ_i with $i = 0, 1, 2, 3$, and $\lambda_0 = 0$, $q_3 = 0$.

A.3. Proof of Lemma 3

Combining (5.26c) and (5.26h) we get,

$$k_i\Big(\gamma - \frac{w/2}{\tilde{a} + q_i + \lambda_i + k_i}\Big) = 0 \;\forall\; i \tag{A.17}$$

Without loss of generality, assume $q_1 + \lambda_1 \leq q_2 + \lambda_2 \leq ... \leq q_n + \lambda_n$. Say m is the largest integer such that $\gamma - \frac{w/2}{\tilde{a}+q_i+\lambda_i} < 0 \;\forall i \leq m$ and $\gamma - \frac{w/2}{\tilde{a}+q_i+\lambda_i} \geq 0 \;\forall i > m$. Then, we have,

$$k_i = \begin{cases} \frac{\omega}{2\gamma} - (\tilde{a} + q_i + \lambda_i) & \forall\; i \leq m \\ 0 & \forall\; i > m \end{cases} \tag{A.18}$$

Note that this again divides the input power into two bands - one where k_i is zero and one where it is not zero. We will call the band where $k_i \neq 0$ as band 1 and the band where $k_i = 0$ as band 2.

Now, for any $l < m$, we have

$$\begin{aligned} \gamma - &\frac{w/2}{\tilde{a} + q_l + \lambda_l} < 0 \\ \Rightarrow \gamma - &\frac{w/2}{\tilde{a} + q_l + \lambda_l + k_l} = 0 \end{aligned} \tag{A.19}$$

For any $j > m$, we have

$$\gamma - \frac{w/2}{\tilde{a} + q_j + \lambda_j} \geq 0 \tag{A.20}$$

From (A.19) and (A.20), we have for any $l < m$ and $j > m$,

$$\tilde{a} + q_l + \lambda_l + k_l \leq \tilde{a} + q_j + \lambda_j \tag{A.21}$$

Next, we will show that in band 1, $q_i \neq 0$. Suppose $q_i = 0$ for some $i \leq m$. Then the contribution towards the objective function for this particular i is:

$$\left[\frac{(1-\omega)}{2}\log(\tilde{a}+\lambda_i) + \frac{\omega}{2}\log(1+\lambda_i) - \frac{1}{2}\log(\tilde{a}+\lambda_i) + \frac{\omega}{2}\log(\tilde{a}+\lambda_i+k_i)\right]$$
$$= \omega\left[\frac{1}{2}\log(1+\lambda_i) + \frac{1}{2}\log(\tilde{a}+\lambda_i+k_i) - \frac{1}{2}\log(\tilde{a}+\lambda_i)\right]$$

It is easy to show that this expression can be maximized with $k_i' = 0$ and $\lambda_i' = \lambda_i + k_i$, i.e. having all the power in λ_i instead of k_i. However, by assumption $k_i \neq 0$. Therefore, it is sub-optimal to have $q_i = 0$ for $i \leq m$.

Now, combining (5.26a) and (5.26g) we get,

$$q_i\left(\beta - \frac{(1-w)/2}{\tilde{a}+q_i+\lambda_i} - \frac{w/2}{\tilde{a}+q_i+\lambda_i+k_i}\right) = 0 \ \forall \ i \tag{A.22}$$

Again consider $l \leq m$ and $j > m$, so that we have $k_j = 0$, $k_l \neq 0$ and $q_l \neq 0$. Since $q_l \neq 0$, by (A.22) we have,

$$\left(\beta - \frac{(1-w)/2}{\tilde{a}+q_l+\lambda_l} - \frac{w/2}{\tilde{a}+q_l+\lambda_l+k_l}\right) = 0 \tag{A.23}$$

Now, since $l \leq m$, from the ordering of $q_i + \lambda_i$ we have $q_l + \lambda_l \leq q_j + \lambda_j$. Using this fact and (A.21), we have

$$\left(\beta - \frac{(1-w)/2}{\tilde{a}+q_j+\lambda_j} - \frac{w/2}{\tilde{a}+q_j+\lambda_j}\right) > 0 \tag{A.24}$$

From equations (A.22) and (A.24), we have $q_j = 0$. Thus, whenever $k_i = 0$, we have $q_i = 0$. This simplifies our numeric analysis since in Band 2, we only have λ_i.

Combining (5.26b) and (5.26f) we get,

$$\lambda_i \left(\frac{(1-w)/2}{\tilde{a} + q_i + \lambda_i} + \frac{w/2}{1+\lambda_i} - \frac{1/2}{\tilde{a} + \lambda_i} + \frac{w/2}{\tilde{a} + q_i + \lambda_i + k_i} + \alpha - \gamma \right) = 0 \tag{A.25}$$

Now, in Band 1, we have $\tilde{a} + q_i + \lambda_i + k_i = \frac{\omega}{2\gamma} =$ constant. Then, equation (A.23) shows that $\tilde{a} + q_i + \lambda_i =$ constant. This means that in the Band 1, k_i has only one value.

Using the fact that $\tilde{a} + q_i + \lambda_i + k_i$ and $\tilde{a} + q_i + \lambda_i$ are constant in (A.25), we again get an equation similar to (A.16) of part 1, which tells us that λ_i can have three values in Band 1.