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Formal functional equations and generalized Lie–Gröbner series

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ABSTRACT

Studying the translation equation $F(s+t,x) = F(s,F(t,x))$, $s,t \in \mathbb{C}$, for $F_t(x) = F(t,x) = \sum_{\nu\geq 1} c_\nu(t)x^\nu$, $t \in \mathbb{C}$, or the associated system of cocycle equations in rings of formal power series it is well known that the coefficient functions of their solutions are polynomials in additive and generalized exponential functions. Replacing these functions by indeterminates we obtain formal functional equations. Applying formal differentiation operators to these formal equations we obtain different types of formal differential equations. They can be solved in order to get explicit representations of the coefficient functions. In the present paper we consider iteration groups of type II, i.e. solutions of the translation equation of the form $F(t,x) = x+\sum_{n\geq k} c_n(t)x^n$, $t \in \mathbb{C}$, where $k \geq 2$ and $c_k: \mathbb{C} \to \mathbb{C}$ is an additive function different from 0. They correspond to formal iteration groups $G(y,x) \in (\mathbb{C}[y])[\![x]\!]$ of type II, which turn out to be the Lie–Gröbner series $LG_y(x) = \sum_{n\geq 0} \frac{1}{n!}D^n(x)y^n$. Here the Lie–Gröbner operator D is defined by $D(f(x)) = f'(x)H(x)$ for $f \in \mathbb{C}[\![x]\!]$ where H is the formal generator of G. Using this particular form of the formal iteration group G we are able to find short proofs and elegant representations of the solutions of the cocycle equations. In connection with the second cocycle equation we study the generalized Lie–Gröbner operator $\mathcal{D}(f) = \left(\sum_{j=1}^{k-1} -\kappa_j x^j\right) f(x) + f'(x)H(x)$, $f \in \mathbb{C}[\![x]\!]$, where $\kappa_1,\dots,\kappa_{k-1} \in \mathbb{C}$ are given. It yields the corresponding generalized Lie–Gröbner series $\mathcal{LG}_y(x) = \sum_{n\geq 0} \frac{1}{n!}\mathcal{D}^n(x)y^n$ which appears in the presentation of the solution of the second cocycle equation.

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1. Introduction

In [4] we have introduced the method of 'formal functional equations' to solve the translation equation (and the associated system of cocycle equations) in rings of formal power series over $\mathbb{C}$ in the case of iteration groups of type I. In [5–7] we applied this method also for the translation equation and the associated cocycle equations in rings of formal power series over $\mathbb{C}$ in the case of iteration groups of type II. Formal functional equations in connection with the translation equation were also studied by D. Gronau [10,11]. Now

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we will show that for iteration groups of type II the method of Lie–Gröbner series allows to present some elegant proofs.

Let $\mathbb{C}[\![x]\!]$ be the ring of formal power series $f(x) = \sum_{\nu\geq 0} c_\nu x^\nu$ over $\mathbb{C}$ in the indeterminate x. For a detailed introduction to formal power series we refer the reader to [1] and [16]. Together with addition $+$ and multiplication $\cdot$ the set $\mathbb{C}[\![x]\!]$ forms a commutative ring. If $f \neq 0$, then the order of f is defined as $\operatorname{ord}(f) = \min\{n \geq 0 \mid c_n \neq 0\}$. Moreover, $\operatorname{ord}(0) = \infty$. The *composition* $\circ$ of formal series is defined as follows: Let $f, g \in \mathbb{C}[\![x]\!]$, $\operatorname{ord}(g) \geq 1$, then $(f \circ g)(x)$ is $f(g(x)) = \sum_{n\geq 0} c_\nu [g(x)]^\nu$. This series converges with respect to the order topology in $\mathbb{C}[\![x]\!]$. Consider

$$\Gamma = \{f \in \mathbb{C}[\![x]\!] \mid f(x) = c_1 x + \ldots,\ c_1 \neq 0\} = \{f \in \mathbb{C}[\![x]\!] \mid \operatorname{ord}(f) = 1\}$$

then $(\Gamma, \circ)$ is the group of all invertible formal power series (with respect to $\circ$).

We consider the translation equation

$$F(s+t, x) = F\big(s, F(t,x)\big), \qquad s, t \in \mathbb{C}, \tag{T}$$

for $F_t(x) = F(t,x) = \sum_{\nu\geq 1} c_\nu(t) x^\nu \in \Gamma$, $t \in \mathbb{C}$. (Cf. the introduction of [4] for the motivation to study (T) and basic results on its solutions $(F_t)_{t\in\mathbb{C}}$.) A family $(F_t)_{t\in\mathbb{C}}$ which satisfies (T) is called iteration group, and neglecting the trivial iteration group, there are two types of such groups, namely iteration groups of type I where the coefficient c_1 is a generalized exponential function different from 1, and iteration groups of type II, where $c_1 = 1$.

It is known that for each iteration group of type II there exists an integer $k \geq 2$ such that

$$F(t,x) = x + \sum_{n\geq k} c_n(t) x^n, \qquad t \in \mathbb{C},$$

where $c_k\colon \mathbb{C} \to \mathbb{C}$ is an additive function different from 0. A family $(F_t)_{t\in\mathbb{C}}$ is an iteration group of type II, if and only if the system

$$\begin{aligned}
c_n(s+t) &= c_n(s) + c_n(s), \qquad\qquad k \leq n \leq 2k-2,\\
c_{2k-1}(s+t) &= c_{2k-1}(s) + c_{2k-1}(t) + k c_k(s) c_k(t)\\
c_{2k}(s+t) &= c_{2k}(s) + c_{2k}(t) + k c_k(s) c_{k+1}(t) + (k+1) c_{k+1}(s) c_k(t)\\
c_n(s+t) &= c_n(s) + c_n(t) + k c_k(s) c_{n-(k-1)}(t)\\
&\quad + \big(n-(k-1)\big) c_{n-(k-1)}(s) c_k(t)\\
&\quad + \tilde{P}_n\big(c_k(s), \ldots, c_{n-k}(s), c_k(t), \ldots, c_{n-k}(t)\big), \qquad n > 2k,
\end{aligned} \tag{1}$$

is satisfied for all $s, t \in \mathbb{C}$, where $\tilde{P}_n$ are universal polynomials which are linear in $c_k(s), \ldots, c_{n-k}(s)$. Comparing coefficients in $c_\nu(s+t) = c_\nu(t+s)$, $\nu \geq 2k$, we can prove that there exists a sequence of polynomials $(P_n)_{n\geq k}$ so that

$$c_n(s) = P_n(c_k(s)), \qquad \forall s \in \mathbb{C},\ n \geq k,$$

and

$$F(s,x) = x + c_k(s) x^k + \sum_{n>k} P_n(c_k(s)) x^n, \qquad s \in \mathbb{C}.$$

According to (1) these polynomials must satisfy

$$\begin{aligned}P_n(c_k(s)+c_k(t)) &= P_n(c_k(s+t)) = c_n(s+t)\\ &= P_n(c_k(s)) + P_n(c_k(t)) + kc_k(s)P_{n-(k-1)}(c_k(t))\\ &\quad + \big(n-(k-1)\big)P_{n-(k-1)}(c_k(s))c_k(t)\\ &\quad + \tilde{P}_n\big(c_k(s),\dots,P_{n-k}(c_k(s)),c_k(t),\dots,P_{n-k}(c_k(t))\big),\end{aligned} \tag{2}$$

for all $s,t\in\mathbb{C}$ and $n\geq k$, where $P_j=0$ for $j<k$ and $\tilde{P}_j=0$ for $j\leq 2k$.

Since the image of c_k contains infinitely many elements we can prove for any polynomial $Q(x,y)\in\mathbb{C}[x,y]$ that $Q(c_k(s),c_k(t))=0$ for all $s,t\in\mathbb{C}$ implies $Q=0$. From (2) we obtain by replacing $c_k(s)$ and $c_k(t)$ by independent variables y,z, that

$$\begin{aligned}P_n(y+z) = P_n(y) + P_n(z) + kyP_{n-(k-1)}(z) + \big(n-(k-1)\big)P_{n-(k-1)}(y)z\\ + \tilde{P}_n\big(y,\dots,P_{n-k}(y),z,\dots,P_{n-k}(z)\big)\end{aligned} \tag{3}$$

for all $n\geq k$.

Writing $G(y,x)=x+yx^k+\sum_{n\geq k+1}P_n(y)x^n\in(\mathbb{C}[y])[\![x]\!]$ we deduce from (3) that G satisfies the *formal translation equation of type II*

$$G(y+z,x)=G(y,G(z,x)) \tag{Tform}$$

in $(\mathbb{C}[y,z])[\![x]\!]$. We call $G(y,x)$ a *formal iteration group of type II*. It also satisfies the condition

$$G(0,x)=x. \tag{B}$$

Iteration groups of type II and formal iteration groups of this type are related in the following way.

Theorem 1: *$F(s,x)=x+c_k(s)x^k+\sum_{n>k}P_n(c_k(s))x^n$ is a solution of (T) if and only if $G(y,x)=x+yx^k+\sum_{n>k}P_n(y)x^n$ is a solution of (Tform) and (B).*

For formal series $f(x),H(x)\in\mathbb{C}[\![x]\!]$ consider the differential operator

$$D\colon\mathbb{C}[\![x]\!]\to\mathbb{C}[\![x]\!],\qquad D(f):=f'(x)H(x).$$

Iterative powers of D are defined as

$$D^n(f)=\begin{cases}f & \text{if } n=0,\\ D(D^{n-1}(f)) & \text{if } n>0.\end{cases}$$

All operators D^n are linear, thus for $f_1,f_2\in\mathbb{C}[\![x]\!]$ and $c_1,c_2\in\mathbb{C}$ we have

$$D^n\big(c_1f_1+c_2f_2\big)=c_1D^n(f_1)+c_2D^n(f_2),\qquad n\geq 0,$$

and moreover D satisfies the product rule

$$D(f_1\cdot f_2)=D(f_1)f_2+f_1D(f_2),$$

and more general

$$D^n(f_1 \cdot f_2) = \sum_{j=0}^{n} \binom{n}{j} D^j(f_1)D^{n-j}(f_2), \qquad n \geq 0.$$

The *Lie–Gröbner* series of $f = f(x) \in \mathbb{C}[\![x]\!]$ is the series

$$LG(f) := LG_y(f) := \sum_{n\geq 0} \frac{1}{n!} D^n(f(x))y^n \in \mathbb{C}[\![x,y]\!].$$

For a detailed introduction to Lie–Gröbner series see [8] or [9, chapter 1]. The operator LG is linear and multiplicative which means that

$$LG\big(c_1f_1 + c_2f_2\big) = c_1LG(f_1) + c_2LG(f_2)$$

and

$$LG\big(f_1f_2\big) = LG(f_1)LG(f_2)$$

hold true for all $f_1, f_2 \in \mathbb{C}[\![x]\!]$ and $c_1, c_2 \in \mathbb{C}$.

For $k \in \mathbb{N}$ and $f(x) = \sum_{n\geq 0} c_nx^n \in \mathbb{C}[\![x]\!]$ let $f_k(x) = \sum_{n=0}^{k} c_nx^n \in \mathbb{C}[x]$. Then

$$LG(f_k(x)) = LG\Big(\sum_{n=0}^{k} c_nx^n\Big) = \sum_{n=0}^{k} c_nLG(x)^n = f_k(LG(x)).$$

Since f is the limit of f_k with respect to the order topology we obtain the Commutation Theorem (cf. [8, Satz 7 (Vertauschungssatz)] or [9, Theorem 6 (Commutation Theorem) p. 17]).

Theorem 2: *For any power series $f(x) \in \mathbb{C}[\![x]\!]$ we have*

$$LG(f(x)) = f(LG(x)). \tag{4}$$

We note that Lie–Gröbner-series in the context of iteration groups have already been used by St. Scheinberg [22] and also by L. Reich and J. Schwaiger in [21].

2. Formal iteration groups of type II are Lie–Gröbner series

In order to determine all formal iteration groups of type II we are looking for relations between the solutions $G(y,x)$ of (Tform) and the formal generator $H(x)$ of G defined by

$$\frac{\partial}{\partial y}G(y,x)|_{y=0} = x^k + \sum_{n>k} h_nx^n = H(x).$$

Here $h_k = 1$. (Notice that in the situation of an analytic iteration group the coefficient of x^k in $H(x)$ may be different from 1.)

Differentiation of (Tform) with respect to z together with the mixed chain rule and putting $z = 0$ yields

$$\frac{\partial}{\partial y}G(y,x) = H(x)\frac{\partial}{\partial x}G(y,x). \tag{PDform}$$

In other words $\frac{\partial}{\partial y}G(y,x) = D(G(y,x))$, where $D(f(x)) := H(x)\frac{\partial}{\partial x}f(x)$ as above, $f \in \mathbb{C}[\![x]\!]$.

Since the solutions of (Tform) are elements of $(\mathbb{C}[y])[\![x]\!]$ it is possible to write them in the form

$$G(y,x) = \sum_{n\geq 0}\phi_n(x)y^n \in (\mathbb{C}[\![x]\!])[\![y]\!].$$

This allows us to rewrite (PDform) and (B) as

$$\sum_{n\geq 1} n\phi_n(x)y^{n-1} = \sum_{n\geq 0} D(\phi_n(x))y^n \tag{5}$$

$$\phi_0(x) = x. \tag{6}$$

(5) is satisfied if and only if

$$\phi_{n+1}(x) = \frac{1}{n+1}D(\phi_n(x)) \tag{5_n}$$

holds true for all $n \geq 0$.

By induction we derive from (5_n) that

$$\phi_0(x) = \frac{D^0(x)}{0!} \quad \text{and} \quad \phi_n(x) = \frac{D^n(x)}{n!}, \qquad n \geq 1.$$

Thus

$$G(y,x) = \sum_{n\geq 0}\phi_n(x)y^n = \sum_{n\geq 0}\frac{1}{n!}D^n(x)y^n = LG_y(x),$$

where

$$D\colon \mathbb{C}[\![x]\!] \to \mathbb{C}[\![x]\!], \qquad D(f(x)) := H(x)f'(x)$$

is the *Lie–Gröbner operator.*

Theorem 3:

(1) *If G is a solution of* (Tform) *and* (B), *then it is a solution of* (PDform), *whence it is the Lie–Gröbner series $LG_y(x)$ where H is the formal generator of G.*

(2) *For any generator $H(x) = x^k + \sum_{n>k} h_n x^n$, $k \geq 2$, the unique solution $G(y,x) = LG_y(x)$ of* (5) *and* (6) *is a solution of* (Tform) *and* (B).

Proof: The first assertion is proved above. Now we show that $G(y,x) = LG_y(x)$ is a formal iteration group of type II:

$$\begin{aligned}
G(y, G(z,x)) &= \sum_{n\geq 0} \frac{1}{n!} y^n D^n \Big(\sum_{\nu\geq 0} \frac{1}{\nu!} z^\nu D^\nu(x) \Big) = \sum_{n\geq 0} \sum_{\nu\geq 0} \frac{1}{n!} \frac{1}{\nu!} y^n z^\nu D^{n+\nu}(x) \\
&= \sum_{N\geq 0} \sum_{n=0}^{N} \frac{1}{N!} \frac{N!}{n!(N-n)!} y^n z^{N-n} D^N(x) \\
&= \sum_{N\geq 0} \frac{1}{N!} \Big(\sum_{n=0}^{N} \binom{N}{n} y^n z^{N-n} \Big) D^N(x) \; = \sum_{N\geq 0} \frac{1}{N!} (y+z)^N D^N(x) \\
&= G(y+z, x).
\end{aligned}$$

□

In other words the composition of two Lie–Gröbner series is again a Lie–Gröbner series.

$$LG_y(LG_z(x)) = LG_{y+z}(x) \tag{7}$$

Let $G(y,x)$ be a formal iteration group of type II with formal generator $H(x)$. Since $G(y,x) = LG_y(x)$ we get as an immediate consequence of the Commutation Theorem (Theorem 2) the Commutation Theorem for iteration groups of type II.

Theorem 4: *Let $G(y,x)$ be a formal iteration group of type II. Then for any power series $K(x)$ of order at least 1 we have*

$$G(y, K(x)) = K(G(y,x)). \tag{8}$$

Remark 5: Let H be the formal generator of the formal iteration group G of type II. Since $H(x) = D(x)$ we obtain

$$H(G(y,x)) = D(LG_y(x)) = \sum_{n\geq 0} \frac{1}{n!} D^{n+1}(x) y^n = \frac{\partial}{\partial y} LG_y(x) = \frac{\partial}{\partial y} G(y,x).$$

Moreover

$$\begin{aligned}
H(x) \frac{\partial}{\partial x} LG_y(x) = D(LG_y(x)) &= \sum_{n\geq 0} \frac{1}{n!} D^{n+1}(x) y^n = \sum_{n\geq 0} \frac{1}{n!} D^n(H(x)) y^n \\
&= LG_y(H(x)) = H(LG_y(x)).
\end{aligned}$$

Thus

$$H(x) \frac{\partial}{\partial x} G(y,x) = H(G(y,x)),$$

or equivalently

$$\frac{\partial}{\partial x} LG_y(x) = \frac{H(LG_y(x))}{H(x)}.$$

Remark 6: The general idea of formal functional equations and Lie–Gröbner series is the following: We start with a functional equation like the translation equation or the cocycle equations (introduced later). From these equations we determine formal equations by replacing independent values by independent variables. In order to solve these formal equations we derive by purely algebraic differentiation and by applying mixed chain rules some differential equations, which we are able to solve. After reordering the summands of a solution we derive a representation as a (generalized) Lie–Gröbner series. Finally we prove that this (generalized) Lie–Gröbner series is a solution of the formal equation we wanted to solve.

This idea will be applied to the first and second cocycle equations. In connection with the problem of a covariant embedding of the linear functional equation $\varphi(p(x)) = a(x)\varphi(x) + b(x)$ with respect to an iteration group $(F(t,x))_{t\in\mathbb{C}}$ (cf. [2,3]) we have to solve the two cocycle equations

$$\alpha(s+t,x) = \alpha(s,x)\alpha\big(t,F(s,x)\big), \qquad s,t\in\mathbb{C}, \tag{Co1}$$

$$\beta(s+t,x) = \beta(s,x)\alpha\big(t,F(s,x)\big) + \beta\big(t,F(s,x)\big), \qquad s,t\in\mathbb{C}, \tag{Co2}$$

under the boundary conditions

$$\alpha(0,x) = 1, \qquad \beta(0,x) = 0,$$

for

$$\alpha(s,x) = \sum_{n\geq 0}\alpha_n(s)x^n, \qquad \beta(s,x) = \sum_{n\geq 0}\beta_n(s)x^n.$$

These cocycle equations appear also in other settings, see e.g. [12,13,15,20], or [14].

3. The first cocycle equation

We study the *first cocycle equation*

$$\alpha(s+t,x) = \alpha(s,x)\alpha\big(t,F(s,x)\big), \qquad s,t\in\mathbb{C}, \tag{Co1}$$

for

$$\alpha(t,x) = \sum_{n\geq 0}\alpha_n(t)x^n, \qquad t\in\mathbb{C},$$

under the boundary condition

$$\alpha(0,x) = 1 \tag{B1}$$

where $(F_t)_{t\in\mathbb{C}}$ is an iteration group of type II. Then α_0 is a generalized exponential function and $\hat{\alpha}(t,x) := \frac{\alpha(t,x)}{\alpha_0(t)}$ is also a solution of (Co1). By substitution into the logarithmic series we obtain that $\gamma(t,x) := \log(\hat{\alpha}(t,x)) = \sum_{n\geq 1}\gamma_n(t)x^n$ is a solution of

$$\gamma(s+t,x) = \gamma(s,x) + \gamma\big(t,F(s,x)\big) \tag{Co1log}$$

satisfying $\gamma(0,x) = 0$, if and only $\hat{\alpha}(t,x)$ is a solution of (Co1) and (B1).

By comparing coefficients it is easy to prove

Lemma 7: *Let $F(t,x) = x + \sum_{n\geq k} P_n(c_k(t))x^n$ be an iteration group of type II, then each coefficient function $\gamma_n(t)$ of a solution γ of (Co1log) is a polynomial $\hat{P}_n(c_k(t))$, $t \in \mathbb{C}$. Moreover for all $s, t \in \mathbb{C}$ we have*

$$\sum_{n\geq 1} \hat{P}_n(c_k(s) + c_k(t))x^n = \sum_{n\geq 1} \hat{P}_n(c_k(s))x^n + \sum_{n\geq 1} \hat{P}_n(c_k(t)) \left[x + \sum_{r\geq k} P_r(c_k(s))x^r \right]^n .$$

Replacing $c_k(s)$ and $c_k(t)$ by independent variables y, z, we obtain the formal first cocycle equation in $(\mathbb{C}[y,z])[\![x]\!]$

$$\Gamma(y+z,x) = \Gamma(y,x) + \Gamma\big(z, G(y,x)\big) \tag{Co1form}$$

for

$$\Gamma(y,x) = \sum_{n\geq 1} \hat{P}_n(y)x^n \in (\mathbb{C}[y])[\![x]\!]$$

together with

$$\Gamma(0,x) = 0 \tag{B1$'$}$$

where $G(y,x) = x + yx^k + \sum_{n>k} P_n(y)x^n$ is a formal iteration group of type II.

As a consequence we easily obtain

Theorem 8: *Let $c_k \neq 0$ be an additive function. Then $\gamma(s,x) = \sum_{n\geq 1} \tilde{P}_n(c_k(s))x^n$ is a solution of (Co1log) satisfying $\gamma(0,x) = 0$ if and only if $\Gamma(y,x) = \sum_{n\geq 1} \tilde{P}_n(y)x^n$ is a solution of (Co1form) satisfying (B1$'$).*

Differentiation of (Co1form) with respect to z together with the mixed chain rule and putting $z = 0$ yields

$$\frac{\partial}{\partial y}\Gamma(y,x) = K(x) + H(x)\frac{\partial}{\partial x}\Gamma(y,x), \tag{Co1PD}$$

where $K(x) := \frac{\partial}{\partial y}\Gamma(y,x)|_{y=0}$ and $H(x)$ is the formal generator of the formal iteration group $G(y,x)$. We call K the *generator* of the solution Γ of (Co1form).

Thus $\frac{\partial}{\partial y}\Gamma(y,x) = K(x) + D(\Gamma(y,x))$, where $D(f(x)) := H(x)\frac{\partial}{\partial x}f(x)$.

Since the solution of (Co1form) is an element of $(\mathbb{C}[y])[\![x]\!]$ it is possible to write it in the form

$$\Gamma(y,x) = \sum_{n\geq 1} \psi_n(x)y^n \in (\mathbb{C}[\![x]\!])[\![y]\!].$$

This allows us to rewrite (Co1PD) as

$$\sum_{n\geq 1} n\psi_n(x)y^{n-1} = K(x) + \sum_{n\geq 1} D(\psi_n(x))y^n. \tag{9}$$

(9) is satisfied if and only if

$$\psi_1(x) = K(x)$$

and

$$\psi_n(x) = \frac{1}{n} H(x)\psi'_{n-1}(x) \tag{5_n}$$

holds true for all $n \geq 2$.

By induction we prove that the unique solution of (9) with ((B1′)) is

$$\Gamma(y,x) = \sum_{n\geq 1} \frac{1}{n!} D^{n-1}(K(x))y^n, \tag{10}$$

where $D\colon \mathbb{C}[\![x]\!] \to \mathbb{C}[\![x]\!], D(f(x)) := H(x)f'(x)$. This is a generalization of a Lie–Gröbner series.

If $\Phi(x) \in \mathbb{C}[\![x]\!]$ is a formal series, so that $\varphi(x) = \Phi'(x)$, then we introduce

$$\int_y^z \varphi(\xi)\,\mathrm{d}\xi := \Phi(z) - \Phi(y).$$

Given $\varphi \in \mathbb{C}[\![x]\!]$, the series $\Phi(x) = \int_0^x \varphi(\xi)\,\mathrm{d}\xi$ is the unique primitive of φ satisfying $\Phi(0) = 0$.

Since

$$\begin{aligned}\int_0^y LG_\xi(x)\,\mathrm{d}\xi &= \int_0^y \sum_{n\geq 0} \frac{1}{n!} D^n(x)\xi^n\,\mathrm{d}\xi = \sum_{n\geq 0} \frac{1}{(n+1)!} D^n(x)y^{n+1} \\ &= \sum_{n\geq 1} \frac{1}{n!} D^{n-1}(x)y^n\end{aligned}$$

we obtain

$$\Gamma(y,x) = \int_0^y LG_\xi(K(x))\,\mathrm{d}\xi.$$

Theorem 9:

(1) *If Γ is a solution of (Co1form) and (B1′) with given generator K, then it is a solution of (Co1PD). Thus it has a representation of the form (10) where $D(f) = f'H$ and H is the formal generator of the iteration group G.*

(2) *Let G be a formal iteration group of type II with formal generator H. For any series $K(x)$ of order at least 1 the unique solution $\Gamma(y,x)$ of (Co1PD) and (B1′) is a solution of (Co1form).*

Proof: The first assertion is clear. The proof of the second assertion is based on the Commutation Theorem for iteration groups of type II (Theorem 4).

Let x, y, z be distinct indeterminates.

$$\begin{aligned}\Gamma(y,x)+\Gamma(z,G(y,x)) &= \int_0^y LG_\xi(K(x))\,\mathrm{d}\xi + \int_0^z LG_\xi(K(G(y,x)))\,\mathrm{d}\xi \\ &\overset{(8)}{=} \int_0^y LG_\xi(K(x))\,\mathrm{d}\xi + \int_0^z LG_\xi(LG_y(K(x)))\,\mathrm{d}\xi \\ &\overset{(7)}{=} \int_0^y LG_\xi(K(x))\,\mathrm{d}\xi + \int_0^z LG_{\xi+y}(K(x))\,\mathrm{d}\xi \\ &= \int_0^y LG_\xi(K(x))\,\mathrm{d}\xi + \int_y^{y+z} LG_\eta(K(x))\,d\eta \\ &= \int_0^{y+z} LG_\xi(K(x))\,\mathrm{d}\xi = \Gamma(y+z,x).\end{aligned}$$

It is obvious that $\Gamma(0,x)=0$. □

Theorem 10: *For each generator $K(x)=\sum_{n\geq 1}\kappa_n x^n \in \mathbb{C}[\![x]\!]$ there exists a solution α of (Co1) and (B1) so that*

$$\alpha(s,x)=\alpha_0(s)\frac{E(G(c_k(s),x))}{E(x)}\prod_{j=1}^{k-1}\exp\left(\kappa_j\int_0^y [G(\sigma,x)]^j d\sigma\,|_{y=c_k(s)}\right),$$

where α_0 is a generalized exponential function, $E(x)=1+\ldots\in\mathbb{C}[\![x]\!]$, c_k an additive function and $G(y,x)$ a formal iteration group of type II.

Moreover, each solution of (Co1) and (B1) is of this form.

Proof: Let $K_1(x)=\sum_{j=1}^{k-1}\kappa_j x^j$ and $K_2(x)=\sum_{j\geq k}\kappa_j x^j$. According to (10) we have

$$\begin{aligned}\Gamma(y,x) &= \sum_{n\geq 1}\frac{1}{n!}D^{n-1}\big(K_1(x)+K_2(x)\big)y^n \\ &= \sum_{n\geq 1}\frac{1}{n!}\big(D^{n-1}(K_1(x))+D^{n-1}(K_2(x))\big)y^n \\ &= \sum_{n\geq 1}\frac{1}{n!}D^{n-1}(K_1(x))y^n + \sum_{n\geq 1}\frac{1}{n!}D^{n-1}(K_2(x))y^n.\end{aligned}$$

Since the order of K_2 is at least k, H is a divisor of K_2, thus there exists $\tilde{K}_2\in\mathbb{C}[\![x]\!]$ so that $K_2=H\tilde{K}_2$. Then there is a unique $\hat{K}_2$ so that $\hat{K}_2'(x)=\tilde{K}_2(x)$ and $\hat{K}_2(0)=0$. Therefore $K_2(x)=D(\hat{K}_2(x))$ and

$$\begin{aligned}\sum_{n\geq 1}\frac{1}{n!}D^{n-1}(K_2(x))y^n &= \sum_{n\geq 1}\frac{1}{n!}D^{n}(\hat{K}_2(x))y^n \\ &= LG_y(\hat{K}_2(x))-\hat{K}_2(x) \overset{(8)}{=} \hat{K}_2(G(y,x))-\hat{K}_2(x).\end{aligned}$$

Moreover

$$\sum_{n\geq 1}\frac{1}{n!}D^{n-1}(K_1(x))y^n = \int_0^y LG_\xi(K_1(x))\,\mathrm{d}\xi \overset{(4)}{=} \int_0^y K_1(LG_\xi(x))\,\mathrm{d}\xi$$
$$= \int_0^y \sum_{j=1}^{k-1}\kappa_j[LG_\xi(x)]^j\,\mathrm{d}\xi \overset{(8)}{=} \sum_{j=1}^{k-1}\kappa_j\int_0^y [G(\xi,x)]^j\,\mathrm{d}\xi.$$

Consequently,

$$\Gamma(y,x) = \sum_{j=1}^{k-1}\kappa_j\int_0^y [G(\xi,x)]^j\,\mathrm{d}\xi + \hat{K}_2(G(y,x)) - \hat{K}_2(x).$$

Let $E(x) = \exp(\hat{K}_2(x)) = 1 + \ldots$, and

$$P(y,x) = \exp\left(\sum_{j=1}^{k-1}\kappa_j\int_0^y [G(\xi,x)]^j\,\mathrm{d}\xi\right) = \prod_{j=1}^{k-1}\exp\left(\kappa_j\int_0^y [G(\xi,x)]^j\,\mathrm{d}\xi\right),$$

then

$$\exp(\Gamma(y,x)) = P(y,x)\frac{E(G(y,x))}{E(x)}$$

and

$$\alpha(s,x) = \alpha_0(s)P(c_k(s),x)\frac{E(G(c_k(s),x))}{E(x)}.$$

□

We note in passing that each of the three factors of $\alpha(s,x)$ is a solution of (Co1) and (B1), thus they are units in $\mathbb{C}[[x]]$. Moreover $P(y,x)$ and $\frac{E(G(y,x))}{E(x)}$ satisfy

$$F(y+z,x) = F(y,x)F(z,G(y,x)) \text{ together with } F(0,x) = 1 \qquad \text{(Co1form}')$$

which follows from (Co1form) and (B1′).

Let

$$\tilde{P}(y,x) = \frac{1}{P(y,x)} = \exp\left(\sum_{j=1}^{k-1}-\kappa_j\int_0^y [G(\xi,x)]^j\,\mathrm{d}\xi\right),$$

then $\tilde{P}(0,x) = 1$ and

$$\frac{\partial}{\partial y}\tilde{P}(y,x) = \tilde{P}(y,x)\frac{\partial}{\partial y}\left(\sum_{j=1}^{k-1}-\kappa_j\int_0^y [G(\xi,x)]^j\,\mathrm{d}\xi\right)$$
$$= \tilde{P}(y,x)\sum_{j=1}^{k-1}-\kappa_j[G(y,x)]^j.$$

Therefore

$$\left(\frac{\partial}{\partial y}\tilde{P}(y,x)\right)\Big|_{y=0} = \sum_{j=1}^{k-1} -\kappa_j x^j. \tag{11}$$

By construction the coefficients of x^n in $P(y,x)$ and $\tilde{P}(y,x)$ are polynomials in y.

4. The second cocycle equation

Let $(F_t)_{t\in\mathbb{C}}$ be an iteration group of type II, $F(t,x) = G(c_k(t),x)$, where $G(y,x)$ is a formal iteration group of type II, $c_k \neq 0$ an additive function, and α a solution of (Co1) and (B1), then we study the *second cocycle equation*

$$\beta(s+t,x) = \beta(s,x)\alpha\big(t,F(s,x)\big) + \beta\big(t,F(s,x)\big), \qquad s,t\in\mathbb{C}, \tag{Co2}$$

for

$$\beta(s,x) = \sum_{n\geq 0}\beta_n(s)x^n$$

under the boundary condition

$$\beta(0,x) = 0. \tag{B2}$$

Using the particular form of α given in Theorem 10 we study

$$\Delta(s,x) = \frac{\beta(s,x)}{E(x)\alpha(s,x)} = \frac{\beta(s,x)}{\alpha_0(s)E\big(G(c_k(s),x)\big)P(c_k(s),x)}, \qquad s\in\mathbb{C}. \tag{12}$$

Theorem 11: *The series β satisfies (Co2) and (B2) if and only if Δ satisfies*

$$\Delta(0,x) = 0 \tag{B2$'$}$$

and

$$\Delta(s+t,x) = \Delta(s,x) + \frac{1}{\alpha_0(s)}\tilde{P}(c_k(s),x)\Delta\big(t,G(c_k(s),x)\big), \qquad s,t\in\mathbb{C}. \tag{Co2$'$}$$

Obviously, $\Delta(s,x)$ depends on the non-trivial additive function c_k and on the generalized exponential function α_0. We write $\Delta(s,x) = \sum_{n\geq 0}\Delta_n(s)x^n$.

In [7] we distinguish four cases which cover all possible choices of $P(y,x)$.

(1) $\alpha_0 \neq 1$.
(2) $\alpha_0 = 1$ and $\kappa_1 = \cdots = \kappa_{r-1} = 0$ where either $r < k-1$ and $\kappa_r \neq 0$, or $r = k-1$ and $\kappa_{k-1} \notin \mathbb{N}_0$, then $P(U,x) = 1 + \kappa_r U x^r + \ldots$.
(3) $\alpha_0 = 1$ and $\kappa_1 = \cdots = \kappa_{k-1} = 0$, then $P(y,x) = 1$. In this situation Δ_0 can be an arbitrary additive mapping.
(4) $\alpha_0 = 1, \kappa_1 = \cdots = \kappa_{k-2} = 0$, and $\kappa_{k-1} = n_1 \in \mathbb{N}$, then $P(U,x) = 1 + n_1 U x^{k-1} + \ldots$. In this situation an additional additive function can occur in Δ_j for $j \geq n_1$.

In each case we determine a formal equation from (Co2$'$) by replacing independent values $c_k(s), c_k(t)$, $s,t\in\mathbb{C}$ by indeterminates U and V. If $\alpha_0 \neq 1$, then the independent values

$c_k(s), c_k(t), \alpha_0(s), \alpha_0(t)$, $s, t \in \mathbb{C}$ are replaced by indeterminates U, V, S and T (cf. [4, Lemma 16.6]). If the additionally occurring additive function A is not a scalar multiple of c_k, then A and c_k are linearly independent, and according to [7, Lemma 2] the values $c_k(s), c_k(t), A(s), A(t)$, $s, t \in \mathbb{C}$ can be replaced by four indeterminates U, V, σ, τ. These four formal equations are combined in

$$R(ST, U+V, \sigma+\tau, x) = R(S, U, \sigma, x) + S^{\lambda}\tilde{P}(U, x)R(T, V, \tau, G(U, x)) \qquad \text{(Co2form)}$$

which we want to solve under the boundary condition

$$R(1, 0, 0, x) = 0, \qquad \text{(B2}''\text{)}$$

where $R(S, U, \sigma, x) \in (\mathbb{C}[S, U, \sigma])[\![x]\!]$ and $\lambda \in \{0, 1\}$.

Let H be the generator of the formal iteration group G of type II of order k, and consider

$$\tilde{P}(U, x) = \exp\left(\sum_{j=1}^{k-1} -\kappa_j \int_0^U [G(\xi, x)]^j \, \mathrm{d}\xi\right)$$

for some $\kappa_1, \ldots, \kappa_{k-1} \in \mathbb{C}$. Then for any $f \in \mathbb{C}[\![x]\!]$ we have from (PDform), (B) and (11) that

$$\begin{aligned}
&\frac{\partial}{\partial U}\left(\tilde{P}(U, x)f(G(U, x))\right)\Big|_{U=0} \\
&\quad = \left(\frac{\partial}{\partial U}\tilde{P}(U, x)\right) f(G(U, x))\Big|_{U=0} + \tilde{P}(U, x)f'(G(U, x))\frac{\partial}{\partial U}G(U, x)\Big|_{U=0} \\
&\quad = \left(\sum_{j=1}^{k-1} -\kappa_j x^j\right) f(x) + f'(x)H(x)
\end{aligned}$$

since $f'(G(U, x))\frac{\partial}{\partial U}G(U, x) = f'(G(U, x))H(x)\frac{\partial}{\partial x}G(U, x) = H(x)\frac{\partial}{\partial x}f(G(U, x))$. If we consider the *generalized Lie–Gröbner operator*

$$\mathcal{D}\colon \mathbb{C}[\![x]\!] \to \mathbb{C}[\![x]\!], \qquad \mathcal{D}(f(x)) := \mathcal{D}_y(f(x)) := \Big(\sum_{j=1}^{k-1} -\kappa_j x^j\Big) f(x) + f'(x)H(x)$$

then $\mathcal{D}(f(x)) = \left(\sum_{j=1}^{k-1} -\kappa_j x^j\right) f(x) + D(f(x))$ and

$$\frac{\partial}{\partial U}\left(\tilde{P}(U, x)f(G(U, x))\right)\Big|_{U=0} = \mathcal{D}(f). \qquad (13)$$

If $\kappa_1 = \cdots = \kappa_{k-1} = 0$, then $\mathcal{D} = D$.

The next two technical lemmata are easy to prove.

Lemma 12: *For $f_1, f_2 \in \mathbb{C}[\![x]\!]$, $c_1, c_2 \in \mathbb{C}$ we have*

$$\mathcal{D}(c_1 f_1 + c_2 f_2) = c_1\mathcal{D}(f_1) + c_2\mathcal{D}(f_2),$$

and

$$\begin{aligned}\mathcal{D}(f_1f_2) &= \mathcal{D}(f_1)f_2 + f_1f_2'H = \mathcal{D}(f_2)f_1 + f_2f_1'H\\ &= \mathcal{D}(f_1)f_2 + f_1D(f_2) = \mathcal{D}(f_2)f_1 + f_2D(f_1)\\ &= \Big(\sum_{j=1}^{k-1} -\kappa_j x^j\Big) f_1(x)f_2(x) + D(f_1(x)f_2(x)).\end{aligned}$$

Lemma 13: *For $f \in \mathbb{C}[\![x]\!]$ we have*

$$\mathcal{D}\big(\tilde{P}(U,x)f(G(U,x))\big) = \frac{\partial}{\partial U}\left(\tilde{P}(U,x)f(G(U,x))\right).$$

We also mention the following rule for the generalized Lie–Gröbner operator $\mathcal{D}$:

Lemma 14: *Let $f(x) \in \mathbb{C}[\![x]\!]$. For $n \geq 0$ we have*

$$\tilde{P}(V,x)\mathcal{D}_z^n(f(z))\big|_{z=G(V,x)} = \frac{\partial^n}{\partial V^n}\left(\tilde{P}(V,x)f(G(V,x))\right).$$

Proof: By induction we prove the assertion together with the claim that the series $\mathcal{D}_z^n(f(z))|_{z=G(V,x)}$ is of the form $\hat{f}(G(V,x))$ for some $\hat{f} \in \mathbb{C}[\![x]\!]$. For $n=0$ the assertions are obvious. Assume that the assertions hold true for $n \geq 0$. Then $\mathcal{D}_z^n(f(z))|_{z=G(V,x)} = \hat{f}(G(V,x))$ for some $\hat{f} \in \mathbb{C}[\![x]\!]$ and

$$\begin{aligned}\mathcal{D}_z^{n+1}(f(z))\big|_{z=G(V,x)} &= \mathcal{D}_z(\mathcal{D}_z^n(f(z)))\big|_{z=G(V,x)} = \mathcal{D}_z(\hat{f}(z))\big|_{z=G(V,x)}\\ &= \left(\sum_{j=0}^{k-1} -\kappa_j z^j \hat{f}(z) + H(z)\hat{f}'(z)\right)\Bigg|_{z=G(V,x)}\\ &= \left(\sum_{j=0}^{k-1} -\kappa_j [G(V,x)]^j \hat{f}(G(V,x)) + H(G(V,x))\hat{f}'(G(V,x))\right).\end{aligned}$$

Multiplying this by $\tilde{P}(V,x)$ and using $H(G(V,x)) = \frac{\partial}{\partial V}G(V,x)$ we obtain

$$\begin{aligned}&\left(\frac{\partial}{\partial V}\tilde{P}(V,x)\right)\hat{f}(G(V,x)) + \tilde{P}(V,x)\frac{\partial}{\partial V}\hat{f}(G(V,x))\\ &= \frac{\partial}{\partial V}\big(\tilde{P}(V,x)\hat{f}(G(V,x))\big)\\ &= \frac{\partial}{\partial V}\left(\frac{\partial^n}{\partial V^n}\tilde{P}(V,x)f(G(V,x))\right)\\ &= \left(\frac{\partial^{n+1}}{\partial V^{n+1}}\tilde{P}(V,x)f(G(V,x))\right).\end{aligned}$$

□

Let

$$\mathcal{LG}(x) := \mathcal{LG}_y(x) := \sum_{n\geq 0} \frac{1}{n!}\mathcal{D}^n(x)y^n$$

be the *generalized Lie–Gröbner* series for the generalized Lie–Gröbner operator $\mathcal{D}$.

Now we prove a generalization of the Commutation Theorem (Theorem 2).

Theorem 15: *For $f \in \mathbb{C}[[x]]$ we have*

$$\mathcal{LG}_U(f(x)) = \tilde{P}(U,x)f(G(U,x)) = \tilde{P}(U,x)f(LG_U(x)).$$

***Proof*:** We prove that both

$$\Phi(U,x) = \mathcal{LG}_U(f(x)) \quad \text{and} \quad \Psi(U,x) = \tilde{P}(U,x)f(G(U,x))$$

satisfy

$$\frac{\partial}{\partial U}R(U,x) = \mathcal{D}(R(U,x)) \quad \text{and} \quad R(0,x) = f(x) \tag{14}$$

and (14) has a unique solution. Simple computations show that

$$\begin{aligned}\frac{\partial}{\partial U}\Phi(U,x) &= \sum_{n\geq 1}\frac{n}{n!}\mathcal{D}^n(f(x))U^{n-1} = \sum_{n\geq 0}\frac{1}{n!}\mathcal{D}^{n+1}(f(x))U^n \\ &= \mathcal{D}(\mathcal{LG}(f(x))) = \mathcal{D}(\Phi(U,x))\end{aligned}$$

and $\Phi(0,x) = f(x)$. By Lemma 13 we have

$$\mathcal{D}(\Psi(U,x)) = \mathcal{D}\big(\tilde{P}(U,x)f(G(U,x))\big) = \frac{\partial}{\partial U}\big(\tilde{P}(U,x)f(G(U,x))\big) = \frac{\partial}{\partial U}\Psi(U,x)$$

and $\Psi(0,x) = f(x)$.

Now we write $R(U,x) = \sum_{n\geq 0} R_n(x)U^n$. If it is a solution of (14), then $R_0(x) = f(x)$, and

$$\sum_{n\geq 1} nR_n(x)U^{n-1} = \mathcal{D}\left(\sum_{n\geq 0} R_n(x)U^n\right).$$

Therefore

$$nR_n(x) = \mathcal{D}(R_{n-1}(x)), \qquad n \geq 1,$$

and by induction we obtain

$$R_n(x) = \frac{1}{n!}\mathcal{D}^n(f(x)), \qquad n \geq 1,$$

and R is uniquely determined by (14). The uniqueness would also follow from (the formal part of) a uniqueness theorem for parameter dependent differential equations in the complex domain. □

Lemma 16: *For $f, f_1, f_2 \in \mathbb{C}[\![x]\!]$, $c_1, c_2 \in \mathbb{C}$ we have*

$$\mathcal{LG}(c_1 f_1 + c_2 f_2) = c_1 \mathcal{LG}(f_1) + c_2 \mathcal{LG}(f_2)$$

and

$$\mathcal{LG}_U(\mathcal{LG}_V(f)) = \mathcal{LG}_{U+V}(f).$$

***Proof*:** The first assertion is trivial. Concerning the second we apply Theorem 15 obtaining

$$\begin{aligned}\mathcal{LG}_U(\mathcal{LG}_V(f(x))) &= \tilde{P}(U,x)\mathcal{LG}_V(f(G(U,x))) \\ &= \tilde{P}(U,x)\tilde{P}(V,G(U,x))f(G(U,G(V,x))) \\ &= \tilde{P}(U+V,x)f(G(U+V,x)) = \mathcal{LG}_{U+V}(f(x))\end{aligned}$$

what follows from (Co1form′) for $\tilde{P}$ and (Tform) for G. □

An immediate consequence of Lemma 13 and Theorem 15 is

Corollary 17: *For $f \in \mathbb{C}[\![x]\!]$ we have*

$$\mathcal{D}\big(\mathcal{LG}_U(f)\big) = \frac{\partial}{\partial U}\big(\mathcal{LG}_U(f)\big).$$

In general, the generalized Lie–Gröbner operator $\mathcal{LG}$ is not multiplicative. To be more precise:

Theorem 18: *The generalized Lie–Gröbner operator $\mathcal{LG}$ is multiplicative if and only if all the coefficients of $\tilde{P}(U,x)$ vanish, i.e. $\kappa_1 = \cdots = \kappa_{k-1} = 0$.*

***Proof*:** According to Theorem 15 we have $\mathcal{LG}(x^2) = \tilde{P}(U,x)[G(U,x)]^2$ and $[\mathcal{LG}(x)]^2 = [\tilde{P}(U,x)G(U,x)]^2$. If $\mathcal{LG}$ is multiplicative, then $\tilde{P}(U,x) = 1$ what is equivalent to $\kappa_1 = \cdots = \kappa_{k-1} = 0$.

Conversely, if $\kappa_1 = \cdots = \kappa_{k-1} = 0$, then the generalized operator $\mathcal{LG}$ coincides with the Lie–Gröbner operator LG which is multiplicative. □

The next Lemma describes a relation between a series f and its image $\mathcal{D}(f)$. Its proof is straight forward.

Lemma 19: *For $f_1, f_2 \in \mathbb{C}[\![x]\!]$ we have*

$$\mathcal{D}(f_1) = f_2 \Longleftrightarrow \frac{\partial}{\partial U}\mathcal{LG}_U(f_1) = \mathcal{LG}_U(f_2) \Longleftrightarrow \mathcal{LG}_U(f_1) - f_1 = \int_0^U \mathcal{LG}_\xi(f_2)\,\mathrm{d}\xi.$$

Lemma 20: *For $f \in \mathbb{C}[\![x]\!]$ the following assertions are equivalent.*

$$1\colon \mathcal{D}(f) = 0, \qquad 2\colon \frac{\partial}{\partial U}\left(\tilde{P}(U,x)f(G(U,x))\right) = 0,\ 3\colon \frac{\partial}{\partial U}\mathcal{LG}_U(f(x)) = 0,$$
$$4\colon \mathcal{D}(\mathcal{LG}(f)) = 0,\ 5\colon \mathcal{LG}(f) - f = 0.$$

***Proof*:** The second and third assertion are equivalent by Theorem 15, the third and fourth according to Corollary 17. Due to formula (13) $\mathcal{D}(f(x)) = 0$ is equivalent to

$\frac{\partial}{\partial V}\left(\tilde{P}(V,x)f(G(V,x))\right)\big|_{V=0}=0$. Replacing x by $G(U,x)$ we obtain from (Co1form′) for $\tilde{P}$ and (Tform) for G that

$$\begin{aligned}0&=\frac{\partial}{\partial V}\Big(\tilde{P}\big(V,G(U,x)\big)f\big(G(V,G(U,x))\big)\Big)\Big|_{V=0}\\&=\frac{\partial}{\partial V}\left(\frac{\tilde{P}(U+V,x)}{\tilde{P}(U,x)}f\big(G(U+V,x)\big)\right)\Big|_{V=0}\\&=\frac{1}{\tilde{P}(U,x)}\frac{\partial}{\partial U}\left(\tilde{P}(U,x)f\big(G(U,x)\big)\right)\end{aligned}$$

and the assertion follows since $\tilde{P}(U,x)\neq 0$.

Conversely, assume that $\frac{\partial}{\partial U}\mathcal{L}\mathcal{G}_U(f(x))=0$, then

$$0=\frac{\partial}{\partial U}\sum_{n\geq 0}\frac{1}{n!}\mathcal{D}^n(f(x))U^n=\sum_{n\geq 1}\frac{n}{n!}\mathcal{D}^n(f(x))U^{n-1}=\sum_{n\geq 0}\frac{1}{n!}\mathcal{D}^{n+1}(f(x))U^n$$

and consequently $\mathcal{D}^j(f)=0$ for all $j\geq 1$. Consequently the first four assertions are equivalent.

If $\mathcal{D}(f)=0$, then $\mathcal{D}^n(f)=0$ for $n\geq 1$, thus $\mathcal{L}\mathcal{G}_U(f)-f=\sum_{n\geq 1}\frac{1}{n!}\mathcal{D}^n(f)U^n=0$. If conversely $\mathcal{L}\mathcal{G}(f)-f=0$, then necessarily $\mathcal{D}(f)=0$. □

In order to get detailed information on the coefficient functions of a solution of (Co2form) we deduce the following differential equations from the second cocycle equation. By differentiating (Co2form) with respect to S (U and σ) and setting $S=1$, $U=0$, and $\sigma=0$ we get

$$T\frac{\partial}{\partial T}R(T,V,\tau,x)=N_1(x)+\lambda R(T,V,\tau,x),\tag{Co2PD1}$$
$$\frac{\partial}{\partial V}R(T,V,\tau,x)=N_2(x)+\mathcal{D}(R(T,V,\tau,x)),\tag{Co2PD2}$$
$$\frac{\partial}{\partial \tau}R(T,V,\tau,x)=N_3(x),\tag{Co2PD3}$$

where

$$N_1(x)=\frac{\partial}{\partial S}R(S,0,0,x)|_{S=1},\qquad N_2(x)=\frac{\partial}{\partial U}R(1,U,0,x)|_{U=0},$$

and

$$N_3(x)=\frac{\partial}{\partial\sigma}R(1,0,\sigma,x)|_{\sigma=0}$$

are the three *generators* of R.

We consider a solution $R(S,U\sigma,x)$ of (Co2form) as an element of $(\mathbb{C}[S,\sigma])[\![U,x]\!]$, and write it in the form

$$R(S,U\sigma,x)=\sum_{n\geq 0}R_n(S,\sigma,x)U^n$$

with $R_n(S,\sigma,x)\in(\mathbb{C}[S,\sigma])[\![x]\!]$.

First we study the situation $\lambda=0$.

Theorem 21: *Let $\lambda = 0$.*

(1) *If R is a solution of* (Co2form) *and* (B2″)*, then R satisfies the three equations* (Co2PD1)–(Co2PD3)*, and is of the form*

$$R(S,U,\sigma,x) = \sigma N_3(x) + \sum_{n\geq 1} \frac{1}{n!}\mathcal{D}^{n-1}(N_2(x))U^n.$$

This is a generalization of a generalized Lie–Gröbner series. Moreover the generators must satisfy the conditions $N_1 = 0$ and $\mathcal{D}(N_3(x)) = 0$.

(2) *If $N_1 = 0$, $\mathcal{D}(N_3(x)) = 0$, then the system consisting of* (Co2PD1)–(Co2PD3), (B2″) *has a unique solution,*

$$R(S,U,\sigma,x) = \sigma N_3(x) + \int_0^U \mathcal{L}\mathcal{G}_\xi(N_2(x))\,\mathrm{d}\xi$$

which is the substitution of N_2 into a primitive of a generalized Lie–Gröbner series. Moreover this solution satisfies (Co2form).

Proof: If R satisfies (Co2form) and (B2″) then also the three differential equations. From (Co2PD3) we deduce $R(S,U,\sigma,x) = \sigma N_3(x)+\tilde{R}(S,U,x)$. Using this in (Co2PD1) we have $S\frac{\partial}{\partial S}\tilde{R}(S,U,x) = N_1(x)$. Since the left hand side is a multiple of S whereas the right hand side does not depend on S it follows that $N_1 = 0$. Consequently $\tilde{R}(S,U,x)$ does not depend on S, $\tilde{R}(S,U,x) = \hat{R}(U,x)$, and $R(S,U,\sigma,x) = \sigma N_3(x) + \hat{R}(U,x)$. From (Co2PD2) we get

$$\frac{\partial}{\partial U}\hat{R}(U,x) = N_2(x) + \mathcal{D}(\sigma N_3(x) + \hat{R}(U,x)).$$

If we introduce coefficient functions of $\hat{R}$, $\hat{R}(U,x) = \sum_{n\geq 0}\hat{R}_n(x)U^n$, the power series R satisfies

$$\sum_{n\geq 1} n\hat{R}_n(x)U^{n-1} = N_2(x) + \sigma\mathcal{D}(N_3(x)) + \sum_{n\geq 0}\mathcal{D}(\hat{R}_n(x))U^n.$$

From the boundary condition (B2″) it follows that $\hat{R}_0(x) = 0$, so also $\hat{R}_0'(x) = 0$. Therefore, we get

$$\hat{R}_1(x) = N_2(x) + \sigma\mathcal{D}(N_3(x)).$$

The left hand side does not depend on σ, whence $\hat{R}_1(x) = N_2(x)$ and $\mathcal{D}(N_3(x)) = 0$. Therefore $2\hat{R}_2(x) = \mathcal{D}(\hat{R}_1(x)) = \mathcal{D}(N_2(x))$, and by induction we derive $n\hat{R}_n(x) = \mathcal{D}(\hat{R}_{n-1}(x)) = \mathcal{D}(\frac{1}{(n-1)!}\mathcal{D}^{n-2}(N_2(x)))$ and consequently

$$\hat{R}_n(x) = \frac{1}{n!}\mathcal{D}^{n-1}(N_2(x)), \qquad n\geq 1.$$

Now we prove the second assertion. If $R(S,U,\sigma,x)$ is a solution of the three formal differential equations and the boundary condition, then from $\lambda = 0$, $N_1(x) = 0$ and (Co2PD1) we obtain that $R(S,U,\sigma,x)$ is of the form $\tilde{R}(U,\sigma,x)$. Moreover, (Co2PD3) implies that it is of the form $\sigma N_3(x) + \hat{R}(U,x)$, where $\hat{R}(U,x) = \sum_{n\geq 0}\hat{R}_n(U)x^n$ still must be determined. From $R(1,0,0,x) = 0$ it follows that $\hat{R}_0(x) = 0$. Comparing the coefficients of U^j in (Co2PD2) we determine by induction that $\hat{R}_n(x) = \frac{1}{n!}\mathcal{D}^{n-1}(N_2(x))$ for $n \geq 1$. Therefore, the solution $R(S, U, \sigma, x)$ is uniquely determined as

$$\begin{aligned} R(S, U, \sigma, x) &= \sigma N_3(x) + \sum_{n\geq 1} \frac{1}{n!}\mathcal{D}^{n-1}(N_2(x))U^n \\ &= \sigma N_3(x) + \int_0^U \sum_{n\geq 0} \frac{1}{n!}\mathcal{D}^n(N_2(x))\xi^n \, \mathrm{d}\xi \\ &= \sigma N_3(x) + \int_0^U \mathcal{L}\mathcal{G}_\xi(N_2(x)) \, \mathrm{d}\xi. \end{aligned}$$

Using this form in (Co2form) we obtain

$$R(ST, U+V, \sigma+\tau, x) = (\sigma+\tau)N_3(x) + \int_0^{U+V} \mathcal{L}\mathcal{G}_\xi(N_2(x)) \, \mathrm{d}\xi.$$

Computing $R(S, U, \sigma, x) + \tilde{P}(U, x)R(T, V, \tau, G(U, x))$ we obtain

$$\sigma N_3(x) + \int_0^U \mathcal{L}\mathcal{G}_\xi(N_2(x)) \, \mathrm{d}\xi + \tilde{P}(U, x)\left(\tau N_3(G(U, x)) + \int_0^V \mathcal{L}\mathcal{G}_\xi(N_2(G(U, x))) \, \mathrm{d}\xi\right).$$

According to Lemma 20 from $\mathcal{D}(N_3(x)) = 0$ it follows $\frac{\partial}{\partial U}(\tilde{P}(U, x)N_3(G(U, x))) = 0$, thus $\tilde{P}(U, x)N_3(G(U, x))$ does not depend on U and

$$\tilde{P}(U, x)N_3(G(U, x)) = \tilde{P}(0, x)N_3(G(0, x)) = N_3(x).$$

Consequently, $\sigma N_3(x) + \tilde{P}(U, x)\tau N_3(G(U, x)) = (\sigma+\tau)N_3(x)$.

Sine $\tilde{P}(U, x)$ satisfies (Co1form′), G satisfies (Tform) and according to Theorem 15 we derive

$$\begin{aligned} \tilde{P}(U, x)\int_0^V \mathcal{L}\mathcal{G}_\xi(N_2(G(U, x))) \, \mathrm{d}\xi &= \int_0^V \tilde{P}(U, x)\tilde{P}(\xi, G(U, x))N_2(G(\xi, G(U, x))) \, \mathrm{d}\xi \\ &= \int_0^V \tilde{P}(U+\xi, x)N_2(G(U+\xi, x)) \, \mathrm{d}\xi \\ &= \int_U^{U+V} \tilde{P}(\eta, x)N_2(G(\eta, x)) \, \mathrm{d}\eta \\ &= \int_U^{U+V} \tilde{P}(\eta, x)N_2(LG_\eta(x)) \, \mathrm{d}\eta \\ &= \int_U^{U+V} \mathcal{L}\mathcal{G}_\eta(N_2(x)) \, \mathrm{d}\eta \end{aligned}$$

and

$$\begin{aligned}&\int_0^U \mathcal{L}\mathcal{G}_\xi(N_2(x))\,\mathrm{d}\xi + \tilde{P}(U,x)\int_0^V \mathcal{L}\mathcal{G}_\xi(N_2(G(U,x)))\,\mathrm{d}\xi\\ &= \int_0^U \mathcal{L}\mathcal{G}_\xi(N_2(x))\,\mathrm{d}\xi + \int_U^{U+V} \mathcal{L}\mathcal{G}_\xi(N_2(x))\,\mathrm{d}\xi\\ &= \int_0^{U+V} \mathcal{L}\mathcal{G}_\xi(N_2(x))\,\mathrm{d}\xi\end{aligned}$$

what finishes the proof. □

For $\lambda = 1$ we obtain the following

Theorem 22: *Let $\lambda = 1$.*

(1) *If R is a solution of (Co2form) and (B2″), then R satisfies the three Equations (Co2PD1)–(Co2PD3), and is of the form*

$$R(S,U,\sigma,x) = S\sum_{n\geq 0}\frac{1}{n!}\mathcal{D}^n(N_1(x))U^n - N_1(x) = S\mathcal{L}\mathcal{G}_U(N_1(x)) - N_1(x).$$

Moreover the generators must satisfy $\mathcal{D}(N_1(x)) = N_2(x)$ and $N_3 = 0$.

(2) *If $N_3 = 0$, $\mathcal{D}(N_1(x)) = N_2(x)$, then the system consisting of (Co2PD1)–(Co2PD3), (B2″) has a unique solution of the form given above. Moreover this solution satisfies (Co2form).*

***Proof*:** From (Co2PD3) we deduce $R(S,U,\sigma,x) = \sigma N_3(x) + \tilde{R}(S,U,x)$. Using this in (Co2PD1) we have $S\frac{\partial}{\partial S}\tilde{R}(S,U,x) = N_1(x) + \sigma N_3(x) + \tilde{R}(S,U,x)$, or equivalently $S\frac{\partial}{\partial S}\tilde{R}(S,U,x) - N_1(x) - \tilde{R}(S,U,x) = \sigma N_3(x)$. Since the right hand side is a multiple of σ whereas the left hand side does not depend on σ it follows that $N_3 = 0$. Consequently $S\frac{\partial}{\partial S}\tilde{R}(S,U,x) = N_1(x) + \tilde{R}(S,U,x)$. Writing $\tilde{R}(S,U,x)$ as $\sum_{n\geq 0}\tilde{R}_n(U,x)S^n$ necessarily $R_0(U,x) = -N_1(x)$ and $N_1(x) + \tilde{R}(S,U,x) = \sum_{n\geq 1}\tilde{R}_n(U,x)S^n$. From the equation above we deduce $\sum_{n\geq 1}(n-1)\tilde{R}_n(U,x)S^n = 0$ which means that $\tilde{R}_n(U,x) = 0$ for $n \geq 2$. Consequently $R(S,U,\sigma,x) = -N_1(x) + S\tilde{R}_1(U,x)$ where still $\tilde{R}_1(U,x)$ must be determined. Therefore we now represent it as $\sum_{n\geq 0}\hat{R}_n(x)U^n$.

From (Co2PD2) we get

$$S\frac{\partial}{\partial U}\tilde{R}_1(U,x) = N_2(x) + \mathcal{D}(-N_1(x) + S\tilde{R}_1(U,x)).$$

This means

$$S\sum_{n\geq 1}n\hat{R}_n(x)U^{n-1} = N_2(x) - \mathcal{D}(N_1(x)) + S\sum_{n\geq 0}\mathcal{D}(\hat{R}_n(x))U^n.$$

Thus $N_2(x) = \mathcal{D}(N_1(x))$ since these terms do not depend on S. From the boundary condition (B2″) it follows that $\hat{R}_0(x) = N_1(x)$. Moreover, $n\hat{R}_n(x) = \mathcal{D}(\hat{R}_{n-1}(x))$ and by induction we derive

$$\hat{R}_n(x) = \frac{1}{n!}\mathcal{D}^n(N_1(x)) = \frac{1}{n!}\mathcal{D}^{n-1}(N_2(x)), \qquad n \geq 1.$$

Now we prove the second assertion. If $R(S,U,\sigma,x)$ is a solution of the three formal differential equations and the boundary condition, then from $N_3 = 0$ and (Co2PD3) we get $R(S,U,\sigma,x) = \tilde{R}(S,U,x)$. Due to $\lambda = 1$ and (Co2PD1) it follows that $\tilde{R}(S,U,x) = -N_1(x) + S\sum_{n\geq 0}\hat{R}_n(x)U^n$, where $\hat{R}_n(x)$, $n \geq 0$, still must be determined. From $R(1,0,0,x) = 0$ we get $\hat{R}_0(x) = N_1(x)$ and (Co2PD2) becomes

$$S\sum_{n\geq 1} n\hat{R}_n(x)U^{n-1} = N_2(x) + \mathcal{D}(-N_1(x)) + S\sum_{n\geq 0}\mathcal{D}(\hat{R}_n(x))U^n.$$

Comparing the coefficients of S^0 we derive that $N_2(x) = \mathcal{D}(N_1(x))$. Moreover, by induction we obtain that $\hat{R}_n(x) = \frac{1}{n!}\mathcal{D}^n(N_1(x))$ for $n \geq 0$. Therefore, the solution $R(S,U,\sigma,x)$ is uniquely determined as

$$R(S,U,\sigma,x) = S\sum_{n\geq 0}\frac{1}{n!}\mathcal{D}^n(N_1(x))U^n - N_1(x) = S\mathcal{L}\mathcal{G}_U(N_1(x)) - N_1(x).$$

Since $\tilde{P}(U,x)$ satisfies (Co1form′), G satisfies (Tform) and according to Theorem 15 we derive

$$\begin{aligned}
&R(S,U,\sigma,x) + S\tilde{P}(U,x)R(T,V,\tau,G(U,x)) \\
&\quad = S\tilde{P}(U,x)N_1(G(U,x)) - N_1(x) + S\tilde{P}(U,x)\big(T\tilde{P}(V,G(U,x))N_1(G(V,G(U,x))) \\
&\qquad - N_1(G(U,x))\big) \\
&\quad = -N_1(x) + ST\tilde{P}(U,x)\tilde{P}(V,G(U,x))N_1(G(U+V,x)) \\
&\quad = ST\tilde{P}(U+V,x)N_1(G(U+V,x)) - N_1(x) \\
&\quad = ST\mathcal{L}\mathcal{G}_{U+V}(N_1(x)) - N_1(x) \\
&\quad = R(ST,U+V,\sigma+\tau,x),
\end{aligned}$$

whence R satisfies (Co2form). □

In a similar way by differentiation of (Co2form) with respect to T (V and τ) and substituting $T = 1$, $V = 0$, and $\tau = 0$ we obtain another system of differential equations, namely

$$S\frac{\partial}{\partial S}R(S,U,\sigma,x) = S^{\lambda}\mathcal{L}\mathcal{G}(N_1(x)), \tag{Co2D1}$$

$$\frac{\partial}{\partial U}R(S,U,\sigma,x) = S^{\lambda}\mathcal{L}\mathcal{G}(N_2(x)), \tag{Co2D2}$$

$$\frac{\partial}{\partial \sigma}R(S,U,\sigma,x) = S^{\lambda}\mathcal{L}\mathcal{G}(N_3(x)), \tag{Co2D3}$$

where

$$N_1(x) = \frac{\partial}{\partial S}R(S,0,0,x)|_{S=1}, \qquad N_2(x) = \frac{\partial}{\partial U}R(1,U,0,x)|_{U=0},$$

and

$$N_3(x) = \frac{\partial}{\partial \sigma} R(1, 0, \sigma, x)|_{\sigma=0}$$

are the three generators of R.

Working with a method different from the application of Lie–Gröbner series we proved the following two theorems in [7] describing the solutions of this system together with the boundary condition (B2″). It is possible to apply the method of Lie–Gröbner series also for this problem. Comparing these proofs we see that *especially in the situation* $\lambda = 0$ *the method of Lie–Gröbner series allows more elegant and simpler proofs.*

Theorem 23: *Let* $\lambda = 0$.

(1) *If R is a solution of* (Co2form) *and* (B2″)*, then R satisfies the three Equations* (Co2D1)–(Co2D3)*, and it has a representation as a generalized Lie–Gröbner series of the form*

$$R(S, U, \sigma, x) = \sigma N_3(x) + \int_0^U \mathcal{LG}_\xi(N_2(x))\, \mathrm{d}\xi.$$

Moreover the generators must satisfy the conditions $N_1 = 0$ *and* $\mathcal{D}(N_3(x)) = 0$.

(2) *If* $N_1 = 0$, $\mathcal{D}(N_3(x)) = 0$*, then the system consisting of* (Co2D1)–(Co2D3), (B2″) *has a unique solution,*

$$R(S, U, \sigma, x) = \sigma N_3(x) + \int_0^U \mathcal{LG}_\xi(N_2(x))\, \mathrm{d}\xi$$

which is a generalized Lie–Gröbner series as above. Moreover this solution satisfies (Co2form).

Theorem 24: *Let* $\lambda = 1$.

(1) *If R is a solution of* (Co2form) *and* (B2″)*, then R satisfies the three equations* (Co2D1)–(Co2D3)*, and is of the form*

$$R(S, U, \sigma, x) = S \sum_{n \geq 0} \frac{1}{n!} \mathcal{D}^n(N_1(x))U^n - N_1(x) = S\mathcal{LG}_U(N_1(x)) - N_1(x).$$

Moreover the generators must satisfy $\mathcal{D}(N_1(x)) = N_2(x)$ *and* $N_3 = 0$.

(2) *If* $N_3 = 0$, $\mathcal{D}(N_1(x)) = N_2(x)$*, then the system consisting of* (Co2D1)–(Co2D3), (B2″) *has a unique solution, of the form given above. Moreover this solution satisfies* (Co2form).

In conclusion both systems of differential equations have the same solution.

The necessary conditions on the generators N_j of R are expressed as conditions on $\mathcal{D}(N_j)$. Now we will analyze them more thoroughly. The consequences of $\mathcal{D}(f) = 0$ for $f \in \mathbb{C}[\![x]\!]$ depend on the particular form of $\tilde{P}(U, x)$.

Theorem 25: *Let* $H(x) = x^k + \ldots$ *be a formal generator of a formal iteration group of type II, where* $k \geq 2$*, and* $\kappa_1, \ldots, \kappa_{k-1} \in \mathbb{C}$*. Assume that* $\mathcal{D}(f) = \left(\sum_{j=1}^{k-1} -\kappa_j x^j\right) f(x) + f'(x)H(x) = 0$ *for some* $f \in \mathbb{C}[\![x]\!]$.

(1) *If* $\kappa_1 = \cdots = \kappa_{k-1} = 0$*, then* $f(x) \in \mathbb{C}$ *is constant.*

(2) *If* $\kappa_1 = \dots = \kappa_{r-1} = 0$ *where either* $r < k-1$ *and* $\kappa_r \neq 0$, *or* $r = k-1$ *and* $\kappa_{k-1} \notin \mathbb{N}_0$, *then* $\tilde{P}(U,x) = 1 - \kappa_r U x^r + \dots$ *and* $f = 0$.
(3) *If* $\kappa_1 = \dots = \kappa_{k-2} = 0$ *and* $\kappa_{k-1} = n_1 \in \mathbb{N}$, *then* $\tilde{P}(U,x) = 1 - n_1 U x^{k-1} + \dots$ *and* $f(x) = f_{n_1} x^{n_1} + \sum_{n>n_1} \Psi_n(f_{n_1}) x^n$, *where* $f_{n_1} \in \mathbb{C}$ *can be arbitrarily chosen and the coefficients* $\Psi_n(f_{n_1})$ *are uniquely determined polynomials in* f_{n_1} *for* $n > n_1$.

Proof: In the first case $\mathcal{D}(f) = f'H = D(f) = 0$, consequently f is constant.

In the second and third case assume that $f(x) = \sum_{n\geq 0} f_n x^n$, then $\mathcal{D}(f(x)) = \sum_{n\geq 0} f_n \mathcal{D}(x^n)$. Moreover for $n \geq 0$ we have

$$\mathcal{D}(x^n) = \sum_{j=r}^{k-1} (-\kappa_j) x^{j+n} + n x^{n-1} H(x) = \begin{cases} -\kappa_r x^{r+n} + \dots & \text{if } r < k-1, \\ (-\kappa_{k-1} + n) x^{k-1+n} + \dots & \text{if } r = k-1, \end{cases}$$

thus $\operatorname{ord}(\mathcal{D}(x^n)) = r + n$ and consequently $f = 0$ is the unique solution of $\mathcal{D}(f) = 0$.

In the third case $\kappa_{k-1} = n_1 \in \mathbb{N}$, thus $\operatorname{ord}(\mathcal{D}(x^n)) = k - 1 + n$ only for $n \neq n_1$, and $\operatorname{ord}(\mathcal{D}(x^{n_1})) > k - 1 + n_1$. Comparing coefficients of x^n in $\mathcal{D}(f) = 0$ we obtain that $f_n = 0$ for $n < n_1$, the coefficient f_{n_1} is not determined by this equation, actually it can be chosen arbitrarily in $\mathbb{C}$, and the coefficients f_n, $n > n_1$, are uniquely determined depending on $f_{n_1} \in \mathbb{C}$. □

Consider $f_1, f_2 \in \mathbb{C}[\![x]\!]$, then

$$\mathcal{D}(f_1) = \mathcal{D}(f_2) \Longleftrightarrow \mathcal{D}(f_1 - f_2) = 0.$$

Now for given $N \in \mathbb{C}[\![x]\!]$, $N \neq 0$, we want to solve the inhomogeneous equation $\mathcal{D}(f) = N$. It is enough to find a particular solution, since by adding all solutions of the homogeneous equation (cf. Theorem 25) we obtain all solutions of the inhomogeneous equation.

Theorem 26: *Consider some* $N \in \mathbb{C}[\![x]\!]$. *Let* $H(x) = x^k + \dots$ *be a formal generator of a formal iteration group of type II, where* $k \geq 2$, *and for* $\kappa_1, \dots, \kappa_{k-1} \in \mathbb{C}$ *let* $\mathcal{D}(f) = \left(\sum_{j=1}^{k-1} -\kappa_j x^j\right) f(x) + f'(x) H(x)$ *for* $f \in \mathbb{C}[\![x]\!]$.

(1) *If* $\kappa_1 = \dots = \kappa_{k-1} = 0$, *then* $\mathcal{D}(f) = N$ *has a solution if and only if* $\operatorname{ord}(N) \geq k$.
(2) *If* $\kappa_1 = \dots = \kappa_{r-1} = 0$ *where either* $r < k-1$ *and* $\kappa_r \neq 0$, *or* $r = k-1$ *and* $\kappa_{k-1} \notin \mathbb{N}_0$, *then* $\mathcal{D}(f) = N$ *has a solution if and only if* $\operatorname{ord}(N) \geq r$.
(3) *Assume* $\kappa_1 = \dots = \kappa_{k-2} = 0$ *and* $\kappa_{k-1} = n_1 \in \mathbb{N}$. *If* $\operatorname{ord}(N) < k-1$, *then there is no solution of* $\mathcal{D}(f) = N$. *Assume that* $\operatorname{ord}(N) \geq k-1$. *Then there exist uniquely determined coefficients* $f_0, \dots, f_{n_1-1}$ *so that* $\operatorname{ord}\left(N(x) - \sum_{n=0}^{j} f_n \mathcal{D}(x^n)\right) \geq k + j$, $0 \leq j \leq n_1 - 1$. *If, moreover,* $\operatorname{ord}\left(N(x) - \sum_{n=0}^{n_1-1} f_n \mathcal{D}(x^n)\right) \geq k + n_1$, *which means that the coefficients* $f_0, \dots, f_{n_1-1}$ *satisfy a certain polynomial relation which yields the coefficient of* x^{k-1+n_1} *of* N, *then* f_{n_1} *can be chosen arbitrarily, say* $f_{n_1} = 0$. *All further coefficients* f_n, $n > n_1$, *are then uniquely determined (depending on* f_{n_1}*).*

Proof: In the first case $\operatorname{ord}(\mathcal{D}(f)) = \operatorname{ord}(f'H) \geq \operatorname{ord}(H) = k$. If $\operatorname{ord}(N) \geq k$, then $N/H \in \mathbb{C}[\![x]\!]$ and $f(x) = \int_0^x N(\xi)/H(\xi)\, \mathrm{d}\xi$ satisfies $\mathcal{D}(f) = N$.

According to the proof of Theorem 25 in the second case we have $\operatorname{ord}(\mathcal{D}(f)) \geq r$. If $\operatorname{ord}(N) \geq r$, then f is uniquely determined by $\mathcal{D}(f) = N$.

In the third case we obtain from the proof of Theorem 25 that $\operatorname{ord}(\mathcal{D}(f)) \geq k-1$. If $\operatorname{ord}(N) \geq k-1$, then comparison of coefficients (or an application of the Theory of Briot–Bouquet equations, cf.[19, Section 5.2] [17, Section 11.1], [18, Section 12.6]) yields the assertion. □

Finally we want to give the explicit form of the solutions β of (Co2). According to (12) we have

$$\beta(s,x) = E(x)\alpha(s,x)\Delta(s,x) = \alpha_0(s)P(c_k(s),x)E(G(c_k(s),x))\Delta(s,x),$$

where $E(x) = 1 + \ldots \in \mathbb{C}[\![x]\!]$, α_0 is a generalized exponential function, $c_k \neq 0$ is an additive function, $G(y,x)$ is a formal iteration group of type II, $\kappa_1, \ldots, k_{k-1} \in \mathbb{C}$, $P(y,x) = \exp\left(\sum_{j=1}^{k-1} \kappa_j \int_0^y [G(\xi,x)]^j \,\mathrm{d}\xi\right)$, and $\Delta(s,x) = R(\alpha_0(s)^{-1}, c_k(s), A(s), x)$, where R is a solution of (Co2form), and A is another additive function so that c_k and A are linearly independent.

- The situation $\alpha_0 \neq 1$ corresponds to $\lambda = 1$, thus by Theorem 22 we have

$$\Delta(s,x) = R(\alpha_0(s)^{-1}, c_k(s), A(s), x) = \alpha_0(s)^{-1}\mathcal{L}\mathcal{G}_U(N_1(x))|_{U=c_k(s)} - N_1(x)$$

and from Theorem 15 follows

$$\beta(s,x) = \alpha_0(s)P(c_k(s),x)E(G(c_k(s),x))\left(\alpha_0(s)^{-1}\frac{N_1(G(c_k(s),x))}{P(c_k(s),x)} - N_1(x)\right).$$

- The situation $\alpha_0 = 1$ corresponds to $\lambda = 0$, thus by Theorem 21 we have

$$\Delta(s,x) = R(1, c_k(s), A(s), x) = A(s)N_3(x) + \int_0^U \mathcal{L}\mathcal{G}_\xi(N_2(x))\,\mathrm{d}\xi\Big|_{U=c_k(s)}$$

where $\mathcal{D}(N_3) = 0$.
If $\kappa_1 = \cdots = \kappa_{r-1} = 0$ where either $r < k-1$ and $\kappa_r \neq 0$, or $r = k-1$ and $\kappa_{k-1} \notin \mathbb{N}_0$, then by Theorem 25 we have $N_3 = 0$. Writing $N_2(x)$ as $\sum_{j=0}^{r-1} n_j x^j + \tilde{N}_2(x)$ where $\operatorname{ord}(\tilde{N}_2) \geq r$, by Theorem 26 there exists a unique $\hat{N}_2 \in \mathbb{C}[\![x]\!]$ so that $\mathcal{D}(\hat{N}_2) = \tilde{N}_2$, and we derive

$$\begin{aligned}\int_0^U \mathcal{L}\mathcal{G}_\xi(N_2(x))\,\mathrm{d}\xi &= \int_0^U \sum_{j=0}^{r-1} n_j \mathcal{L}\mathcal{G}_\xi(x^j)\,\mathrm{d}\xi + \int_0^U \sum_{n\geq 0} \frac{1}{n!}\mathcal{D}^n(\mathcal{D}(\hat{N}_2(x)))\xi^n\,\mathrm{d}\xi \\ &= \int_0^U \sum_{j=0}^{r-1} n_j \frac{[G(\xi,x)]^j}{P(\xi,x)}\,\mathrm{d}\xi + \mathcal{L}\mathcal{G}_U(\hat{N}_2(x)) - \hat{N}_2(x).\end{aligned}$$

In conclusion, again using Theorem 15, we deduce

$$\beta(s,x) = P(c_k(s),x)E(G(c_k(s),x))\left(\frac{\hat{N}_2(G(c_k(s),x))}{P(c_k(s),x)} - \hat{N}_2(x) + Q(s,x)\right),$$

where

$$Q(s,x) = \int_0^U \sum_{j=0}^{r-1} n_j \frac{[G(\xi,x)]^j}{P(\xi,x)} \, \mathrm{d}\xi \big|_{U=c_k(s)}.$$

- If $\kappa_1 = \cdots = \kappa_{k-1} = 0$, then by Theorem 25 we have $N_3 = c \in \mathbb{C}$. Similarly as in the previous case we obtain

$$\beta(s,x) = P(c_k(s),x)E(G(c_k(s),x)) \left(cA(s) + \frac{\hat{N}_2(G(c_k(s),x))}{P(c_k(s),x)} - \hat{N}_2(x) + Q(s,x) \right),$$

where

$$Q(s,x) = \int_0^U \sum_{j=0}^{k-1} n_j \frac{[G(\xi,x)]^j}{P(\xi,x)} \, \mathrm{d}\xi \big|_{U=c_k(s)}.$$

- If $\kappa_1 = \cdots = \kappa_{k-2} = 0$ and $\kappa_{k-1} = n_1 \in \mathbb{N}$, then by Theorem 25 we have $N_3(x) = cx^{n_1} + \sum_{n>n_1} \Psi_n(c)x^n$ with $c \in \mathbb{C}$ and uniquely determined coefficients $\Psi_n(c), n > n_1$. Writing $N_2(x)$ as $\sum_{j=0}^{k-2} n_j x^j + \tilde{N}_2(x)$ where $\operatorname{ord}(\tilde{N}_2) \geq k-1$, by Theorem 26 there exist a series $\hat{N}_2 \in \mathbb{C}[\![x]\!]$ and a constant $b \in \mathbb{C}$ so that $\mathcal{D}(\hat{N}_2) + bx^{k+n_1-1} = \tilde{N}_2$, and we conclude that

$$\begin{aligned} \beta(s,x) &= P(c_k(s),x)E(G(c_k(s),x)) \\ &\quad \times \left(A(s)N_3(x) + \frac{\hat{N}_2(G(c_k(s),x))}{P(c_k(s),x)} - \hat{N}_2(x) + Q(s,x) \right), \end{aligned}$$

where

$$Q(s,x) = \int_0^U \sum_{j=0}^{k-2} \left(n_j \frac{[G(\xi,x)]^j}{P(\xi,x)} + b\frac{[G(\xi,x)]^{k+n_1-1}}{P(\xi,x)} \right) \mathrm{d}\xi \big|_{U=c_k(s)}.$$

These results generalize the representations of β given in Theorems 16, 17 and 18 of [3] or the second and third item of Theorem 2.8 of [2].

Combining the two systems of differential equations for the formal second cocycle R we obtain the following system of three formal *Aczél–Jabotinsky equations*

$$N_1(x) + \lambda R(S,U,\sigma,x) = S^\lambda \mathcal{L}\mathcal{G}(N_1(x)), \tag{Co2AJ1}$$

$$N_2(x) + \mathcal{D}(R(S,U,\sigma,x)) = S^\lambda \mathcal{L}\mathcal{G}(N_2(x)), \tag{Co2AJ2}$$

$$N_3(x) = S^\lambda \mathcal{L}\mathcal{G}(N_3(x)), \tag{Co2AJ3}$$

where

$$N_1(x) = \frac{\partial}{\partial S} R(S,0,0,x)|_{S=1}, \qquad N_2(x) = \frac{\partial}{\partial U} R(1,U,0,x)|_{U=0},$$

and

$$N_3(x) = \frac{\partial}{\partial \sigma} R(1,0,\sigma,x)|_{\sigma=0}$$

are the three generators of R. Only (Co2AJ2) is a differential equation for R.

Theorem 27: *Let $\lambda = 1$.*

(1) *If R is a solution of (Co2form) and (B2″), then R satisfies the three equations (Co2AJ1)–(Co2AJ3), and is of the form*

$$R(S,U,\sigma,x) = S\sum_{n\geq 0}\frac{1}{n!}\mathcal{D}^n(N_1(x))U^n - N_1(x) = S\mathcal{L}\mathcal{G}_U(N_1(x)) - N_1(x).$$

Moreover the generators must satisfy $\mathcal{D}(N_1(x)) = N_2(x)$ and $N_3 = 0$.

(2) *If $N_3 = 0$, $\mathcal{D}(N_1(x)) = N_2(x)$, then the system consisting of (Co2AJ1)–(Co2AJ3) has a unique solution of the form given above. Moreover this solution satisfies (Co2form) and (B2″).*

***Proof*:** Let R be a solution of (Co2form) and (B2″), then R satisfies the three equations. From (Co2AJ1) we immediately obtain the form of R as $R(S,U,\sigma,x) = S\mathcal{L}\mathcal{G}(N_1(x)) - N_1(x)$. According to (Co2AJ3) the series N_3 must vanish since the left hand side does not depend on S, whereas the right hand side depends on S. Finally (Co2AJ2) can be written as $N_2(x) + S\mathcal{L}\mathcal{G}(\mathcal{D}(N_1(x))) - \mathcal{D}(N_1(x)) = S\mathcal{L}\mathcal{G}(N_2(x))$ what implies $N_2(x) = \mathcal{D}(N_1(x))$. This also guarantees that $\mathcal{L}\mathcal{G}(\mathcal{D}(N_1(x))) = \mathcal{L}\mathcal{G}(N_2(x))$.

If conversely R satisfies the three equations where $N_3 = 0$ and $\mathcal{D}(N_1(x)) = N_2(x)$, then R is of the form $R(S,U,\sigma,x) = S\mathcal{L}\mathcal{G}_U(N_1(x)) - N_1(x)$ and according to Theorem 22 it is a solution of (Co2form) and (B2″). □

In the situation $\lambda = 0$ we do not obtain the same solutions as for the two other systems of formal equations. Now (Co2AJ1) and (Co2AJ3) read as $N_j(x) = \mathcal{L}\mathcal{G}(N_j(x))$, $j \in \{1,3\}$, which means $\mathcal{L}\mathcal{G}(N_j(x)) - N_j(x) = 0$. Thus $\mathcal{D}(N_j) = 0$ according to Lemma 20. From (Co2AJ2) we deduce

$$\mathcal{D}(R(S,U,\sigma,x)) = \mathcal{L}\mathcal{G}(N_2(x)) - N_2(x) = \mathcal{D}\left(\int_0^U \mathcal{L}\mathcal{G}_\xi(N_2(x))\,\mathrm{d}\xi\right)$$

thus by Theorem 25

$$R(S,U,\sigma,x) = \int_0^U \mathcal{L}\mathcal{G}_\xi(N_2(x))\,\mathrm{d}\xi + \tilde{R}(S,U,\sigma,x)$$

where $\mathcal{D}(\tilde{R}(S,U,\sigma,x)) = 0$. Again by Theorem 25 if all the coefficients κ_j, $1 \leq j \leq k-1$, of $\tilde{P}(U,x)$ are equal to zero, then $\tilde{R}(S,U,\sigma,x)$ can be any element of $\mathbb{C}[S,U,\sigma]$ since it must be constant with respect to x. Taking still into account that $N_1(x) = c_1 \in \mathbb{C}$, $N_2(x)$, $N_3(x) = c_3 \in \mathbb{C}$ are the generators of R the polynomial $\tilde{R}$ still has to satisfy $\frac{\partial}{\partial S}\tilde{R}(S,0,0,x)|_{S=1} = c_1$, $\frac{\partial}{\partial \sigma}\tilde{R}(1,0,\sigma,x)|_{\sigma=0} = c_3$ and $\frac{\partial}{\partial U}\tilde{R}(1,U,0,x)|_{U=0} = 0$. For example for any positive integer j

$$R(S,U,\sigma,x) = -\frac{c_1}{j} + \frac{c_1}{j}S^j + c_3\sigma + S\sigma U + \int_0^U \mathcal{L}\mathcal{G}_\xi(N_2(x))\,\mathrm{d}\xi$$

is a solution of the three formal Aczél–Jabotinsky equations, the boundary condition (B2″), and the three conditions imposed by the generators of R but not of (Co2form). Also in the situation $\kappa_1 = \cdots = \kappa_{k-2} = 0$ and $\kappa_{k-1} = n_1 \in \mathbb{N}$ there exist solutions of the three formal Aczél–Jabotinsky equations which are not solutions of (Co2form).

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