Nonlinear Observer Design using Metric Based Potentials

by

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Abstract

This thesis addresses observer design for nonlinear dynamical systems which can be approximated with dissipative Hamiltonian realizations. The design methods builds upon earlier developments that allow the approximate dissipative potential to be extracted using a homotopy operator. This potential is obtained by decomposition of the observer error associated one-form using the homotopy operator which generates the potential. A time-varying differential metric equation dependent on the Hessian of the potential and the measured output function is proposed, which is used to design a state observer. The stability of both the observer and metric equation are assessed using Lyapunov theory. A time-invariant metric is then proposed making use of the Hessian of the potential on a metric-state based observer. Using several process simulations, the approach is shown to provide an effective design alternative for nonlinear observer design.

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Chapter 1

Introduction

1.1 Motivation

The design of feedback controllers for dynamical systems requires that the controlled variables be either measured or estimated. Industrial applications commonly avoid this problem by linearizing the system about a point and using the well established linear control and observer design methodology. The observer design solution established by Luenberger states that for any linear system satisfying the necessary observability conditions an observer can be designed [31]. However, for general nonlinear systems, a nonlinear approach is required. Unfortunately, no general observer design method currently exists. The most general approach for the design of an observer using a Luenberger-like approach is the technique proposed by Kazantzis and Kravaris [26].

Existing methods of nonlinear observer design are either computationally expensive or restrictive in terms of requirements. For example, the Kravaris-Kazantzis extension of the Luenberger observer can be computationally expensive for large scale systems. Simple techniques such as the extended Kalman filter use linear approoximations of the nonlinear dynamics. As a result, it may not be convergent if the initial linearization error is too large. Most other nonlinear design methods lack generality. Therefore, the objective of this thesis aims to develop an alternative observer design approach that can be applied to more general classes of nonlinear systems.

A vector field describing the dynamics of a nonlinear system can always be expressed as the sum of a gradient of a potential function and an antisymmetric component that involves the gradient of multiple Hamiltonian functions. Although such decompositions are not unique, the Hamiltonian potentials that are generated can be used effectively in the design of controllers and observers. A subset of these systems, position based mechanical systems, are described by the Lagrangian equations, whose symmetry can be exploited to design an observer. However, non-mechanistic systems are driven by chemical or electric potential fields. These dissipative systems can be described by the Hamiltonian equations and the 'force' for action is described by the negative of the potential.[1] Assuming our system model is accurate and the system is observable, a potential function associated with the system can be obtained and used to design an observer.

1.2 Thesis Organization

The remainder of this thesis is organized into three chapters.

Chapter 2: First, the observer design problem is specified. An overview of selected nonlinear observer design methods is presented as well as their potential drawbacks. The necessary and sufficient conditions for observability are presented. The development and properties of Hamiltonian systems for control systems is shown, and other

[1] Systems are assumed physically based, ie. energy is conserved at best.

methods of physically based metric observers are presented. The second half of the chapter covers the necessary mathematical background for the development of our observer design method. Lyapunov stability theory as the main tool for the stability analysis of the observer error dynamics. This thesis proposes a homotopy operator approach for the computation of suitable potentials. This operator is reviewed along with some basic mathematical background on exterior calculus.

Chapter 3: The method of potential decomposition on an arbitrary metric using the homotopy operator is developed. Using the mathematics presented in Chapter 2, the potential-based observer design method for systems using a time-varying metric is proposed. A Lyapunov-based approach is used to prove the convergence of the proposed observer. Two simulation examples, the Van der Pol oscillator and the Chen attractor, are used to demonstrate the effectiveness of the observer.

Chapter 4: A technique for the design of a time-invariant metric is presented. The approach makes use of the Hessian of the potential to design an effective observer. The proposed metric is assumed to depend only on the observer states. As a result, the metric expression can be used directly in the observer dynamics. A Lyapunov analysis is used to establish the asymptotic stability of the observer error dynamics. Two simulation examples are then presented: a stable focus system and the two-dimensional Lotka-Volterra predator-prey system.

Chapter 5: The final chapter provides the conclusions of the thesis. A brief statement of future research is provided.

Chapter 2

Literature Review

In this thesis we consider autonomous nonlinear systems of the form:

$$\dot{x} = f(x), \tag{2.1}$$

$$y = h(x), \tag{2.2}$$

where $x \in \mathbb{R}^n$ is the vector of state variables, and $y \in \mathbb{R}^m$ are the output variables. $f(x)$ is a vector-valued C^k function of the state variables $x \in \mathbb{R}^n$, and $h(x)$ is also C^k with $k \geq 2$. The system is assumed to be stable in the sense of Lyapunov.

2.1 Observer Design and Stability

The design of an observer seeks a nonlinear dynamical system, $\dot{\hat{x}} = F(\hat{x}, h(x))$, where $h(x)$ is the observed system's output, such that the observer error dynamics:

$$\dot{e} = \dot{x} - \dot{\hat{x}}, \tag{2.3}$$

has an asymptotically stable equilibrium at the origin $e = 0$. In this chapter, a definition of stability will be provided. An overview of the mathematics used in this thesis is presented. Current nonlinear observer design methods are reviewed.

2.1.1 Nonlinear Observers

Developments in nonlinear observer design can be broadly broken down into the following areas: linearization methods, high-gain observers, differential geometric techniques, optimization-based receding horizon, and particle filters. The primary linearization method makes use of the Luenberger method applied to a local linearization of the nonlinear system. A local nonlinear Luenberger approximation was proposed by Kazantzis and Kravaris in [26]. Assuming that the eigenvalues of the Jacobian of $f(x)$ at x_0 are non-zero, the observability of the nonlinear system (2.21) it is shown that there exists a state-space transformation $z = T(x)$ such that the nonlinear system can be transformed to the linear system:

$$\dot{z} = Az + by, \tag{2.4}$$

where A is Hurwitz, and $\{A, b\}$ is controllable. This observer can be written in original coordinates as follows:

$$\dot{\hat{x}} = f(\hat{x}) + \left[\frac{\partial T}{\partial \hat{x}}(\hat{x})\right]^{-1} b(y - h(\hat{x})). \tag{2.5}$$

This approach requires solving a nonlinear partial differential equation (PDE),

$$\frac{\partial T(x)}{\partial x} = AT(x) + bh(x),\ T(0) = 0. \tag{2.6}$$

Such transformations are therefore extremely difficult to obtain in practice. Kazantzis and Kravaris [26] propose a Taylor series approach to approximate the solution of these PDEs for a class of analytic nonlinear systems. Andrieu and Praly [7] build on this method using a series of sufficient conditions that allows the computation of an approximate solution for the resulting nonlinear PDE. The approximate solution is given by:

$$T(x) = \int_{-\infty}^{0} exp(-As)B(h(\widetilde{X}(x,s)))ds, \tag{2.7}$$

where B is some exponentially decreasing function and $\widetilde{X}$ is the solution to the modified system:

$$\dot{x} = \widetilde{f}(x) = \chi(x)f(x), \tag{2.8}$$

where $\chi(x) = 1$ if x is in some region about the equilibrium and zero outside the region.

High-gain observers are another widely applied class of design methods. A single output high-gain observer was proposed by Gauthier et al. [18]. The approach considers an observer of the form:

$$\dot{\hat{x}} = f(\hat{x}) + H(h(x) - h(\hat{x})), \tag{2.9}$$

where $H \in \mathbb{R}^n$ is chosen such that for $\dot{e} = \dot{x} - \dot{\hat{x}}$ has a Jacobian which is Hurwitz. Then H is chosen to minimize the error between:

$$\dot{\eta} = A_m\eta + NL(\eta), \tag{2.10}$$

and,

$$\dot{\hat{\eta}} = A_m\hat{\eta} + \hat{N}L(\hat{\eta}) + H(y - E_1^T\hat{\eta}), \tag{2.11}$$

where A_m is the linearization of the state function at x, $y = E_1^T\eta$, and NL is the nonlinear terms dependent on a series of transforms [27]. It is then shown that for a single output system, one can choose an observer gain that depends on the Lipschitz constant of the system's nonlinear dynamics[1] and the output function [38][4].

Differential geometric observers use state-space transformation of a nonlinear system to achieve linear error dynamics. Initial work by Bestle and Zeitz [9], as well as Krener and Isidori [29], showed that nonlinear systems satisfying certain conditions can be transformed by state-space diffeomorphisms and output injection to a system that admits linear observer dynamics. In Alvarez and López [5], a compromise between robustness and speed of convergence for geometric state estimation is established.

The celebrated Kalman filter can be applied to the nonlinear problem through a successive linearization procedure [25]. The filter works by estimating the upcoming output measurement, comparing the prediction to the measurement, and based on the measurement error recursively updating all previous state estimates to minimize the least-squares error between the output predictions and measurements. For a nonlinear continuous-time system, the deterministic formulation of the extended Kalman filter is discussed in Chapter 4.

The unscented Kalman filter was developed as an improvment upon the extended Kalman filter and lead to the development of the general class of particle filters [22]. A particle filter uses a set of prior measurements to construct a probability density function that predicts the expected state variable values [8]. Numerous approaches

[1] Where if the system seperable into a linear, Ax, and nonlinear part, Ψ, $\dot{x} = Ax + \Psi(x)$.

have been proposed in the literature depending on the type of system and statistical distribution.

State estimation using the receding horizon method predicts the current state variable using various optimization techniques to minimize the error between prior state measurements and the estimated state trajectory. The method was first proposed in [25] to design optimization based observers for both continuous and discrete time to determine a current state trajectory that minimizes the error of the past measurements [30]. Discrete-time formulation is generally easier to apply and the preferred methodology in practice. Nevertheless, continuous-time formulations have also been developed [35].

2.1.2 Observability

This section provides a brief overview of the nonlinear observability conditions and follows the work of Andrieu, et. al. [6]. From their work, the necessary conditions for the asymptotic observability of an autonomous nonlinear system of the form (3.1) are reviewed. Conditions necessary for exponential observability and tunability are also given. Finally, a sufficient condition for local observability is presented.

In [6], the dynamical system is considered on a smooth Riemannian manifold, M:

$$f : M \mapsto TM,\ h : M \mapsto \mathbb{R}^p, \tag{2.12}$$

then letting $A \subset M$ be an open subset containing both the initial conditions, x_0, and the solution at the maximum time, $X(x(t_f), t_f)$. The observer is denoted by the dynamical system:

$$\dot{\hat{\xi}} = \psi(\hat{\xi}, y),\ \hat{x} = \tau(\hat{\xi}, y), \tag{2.13}$$

where τ and ψ are some smooth functions. The trajectories of the nonlinear system solution of the system, and the observer, $\hat{\xi}$, are denoted by $(X(x,t), \hat{\Xi}(\hat{\xi}, x, t))$, respectively. The state estimates are given by:

$$\hat{X} = \tau(\hat{\Xi}(\hat{\xi}, x, t), h(X(x,t))). \tag{2.14}$$

An asymptotic observer is an observer of the form (2.13) whose solutions, defined for $t \in [0, \infty)$, are such that:

$$\lim_{t\to\infty} d_g(X(x,t), \hat{X}(x, \hat{\xi}, t)) = 0. \tag{2.15}$$

Andrieu, et. al [6] provide necessary conditions for observability. If A is an open subset of M where $x \in A \subset M$ containing the trajectory of x for all time. Then $\forall x_1, x_2 \in A$ with the same outputs:

$$h(X(x_1,t)) = h(X(x_2,t)),\ \forall t \geq 0, \tag{2.16}$$

the corresponding state trajectories $X(x_1,t), X(x_2,t)$ converge to the same trajectory asymptotically. That is,

$$lim_{t\to+\infty} d_g(X(x_1,t), X(x_2,t)) = 0, \tag{2.17}$$

where d_g is the metric distance on the given Riemannian manifold. This is a nonlinear detectability condition.

The final requirement for an asymptotic observer is the invariance and attractivity of the zero error set between the given autonomous system and its observer. Assuming there exists a compact forward invariant [2] set $C = C_x \times C_{\hat{\xi}} \subset A \times \mathbb{R}^m$, and there exists $C_2 \subseteq C_x$ where $x \in C_2 \mapsto \tau^*(x) \subset C_{\hat{\xi}}$ such that:

$$\varepsilon = \{(x, \hat{\xi}) \in C_2 \times C_{\hat{\xi}} : \hat{\xi} \in \tau^*(x)\}. \tag{2.18}$$

Explicitly this requires:

- $\forall (x, \hat{\xi}) \in \varepsilon : \tau(\hat{\xi}, h(x)) = x$,
- ε is forward invariant, and
- ε is attractive in C:

  $\lim_{t \to +\infty} d_{g,m}((X(x,t), \hat{\Xi}(x, \hat{\xi}, t)), \varepsilon) = 0$,

where $d_{g,m}$ is the metric on $M \times \mathbb{R}^m$ and $d_{g,m}((x, \hat{\xi}), \varepsilon) = \min_{(x_0, \hat{\xi}_0) \in \varepsilon} d_{g,m}((x, \hat{\xi}), (x_0, \hat{\xi}_0))$. The more widely used definition of observability is given locally using the Lie derivatives of the state and the output functions [19]. The notation uses the Lie derivative defined as:

$$L_f(h_l) = \sum_{k=1}^{n} \frac{\partial h_l}{\partial x_k} f_k, \tag{2.19}$$

$$L_f^{i+1}(h_l) = L_f^i(h_l). \tag{2.20}$$

Given that the system's state trajectories are distinguishable, the system is said to

[2] (Forward Invariance) A set $A \subset X$ is forward invariant if $f(A) \subset A$, where $f : M \mapsto M$ is a continuous mapping [12].

be locally observable if the matrix:

$$O(x_0) = \begin{bmatrix} \frac{\partial L_f^0(h_1)}{\partial x}(x_0) \\ \vdots \\ \frac{\partial L_f^0(h_p)}{\partial x}(x_0) \\ \vdots \\ \frac{\partial L_f^{n-1}(h_1)}{\partial x}(x_0) \\ \vdots \\ \frac{\partial L_f^{n-1}(h_p)}{\partial x}(x_0) \end{bmatrix}, \tag{2.21}$$

has rank $= n$ in a neighbourhood of x_0. This condition is associated with the lineari-ation of the system. In general, local observability requires that the codistribution $O = [dh, dL_f h, ..., d_f^{n-1} h]$ spans $\mathbb{R}^n$. If these conditions are satisfied then f is locally observable in some neighbourhood of x_0.

2.1.3 Hamiltonian Systems

In this thesis, we consider a class of systems that can be expressed using a potential field. By the method presented in section (3.2), a potential can be obtained and used for stabilization or observer construction. Work on Hamiltonian representations of non-mechanical systems has led to the development of control and stabilization methods [34]. Some of the recent developments on the subject are briefly reviewed. Work on the control and description dissipative Hamiltonian systems was proposed by Maschke et al. [32]. They examined the use of a system's dissipation to develop a control function that stabilizes the system. A dissipative Hamiltonian dynamical system is represented by:

$$\dot{x} = f(x) + g(x)u = T(x)\frac{\partial H}{\partial x} + g(x)u, \quad (2.22)$$

where $H(x)$ is a Hamiltonian function, $T(x) \in \mathbb{R}^n \times \mathbb{R}^n$ that defines the geometry to the state space called the structure matrix[3]. If such a representation exists then (2.22) is a dissipative Hamiltonian system. Systems of this type form the basis for the approach proposed in this thesis.

The stabilization and control of Hamiltonian systems has been studied by Ortega et. al. [34]. In particular, it was shown that if any system of the form $\dot{x} = f(x) \in \mathbb{C}^1$ has an asymptotically stable equilibrium point then there exists matrix valued functions $J(x) = -J^T(x)$, $R(x) = R^T(x) \geq 0$, and a positive definite $H(x)$ such that:

$$f(x) = [J(x) - R(x)]\frac{\partial H}{\partial x}(x). \quad (2.23)$$

2.1.4 Physically Based Metric Observers

Metric based observers map a dynamical system into a given metric space where an observer can be effectively designed, then transformed back into the system's original coordinates.

The most well developed metric based nonlinear observer is the Lagrangian observer. A Lagrangian system is one that can be described by the Lagrange equation on an n-dimensional manifold, M:

$$\mathcal{L}(q, \dot{q}) = \frac{1}{2}g_{ij}(q)\dot{q}^i\dot{q}^j - U(q), \quad (2.24)$$

[3] $T(x) = J(x, u)$ - $R(x)$, where $J(x, u) = -J^T(x, u)$ and $R(x) = R^T(x) \geq 0$.

where $g_{ij}(q)$ are the elements of the intertia matrix, $q \in M$ is the coordinates denoted by q^i for $i = 1..n$, and $U(q)$ is a potential energy function. Typically, both position or velocity defined mechanical systems are expressible using this method. The Lagrangian and Euler-Lagrange equations are intrinsic[4] and define a useable symmetry that can be exploited to design an observer [3].

Aghannan and Rouchon propose a method of state estimation of the position and velocity of a mechanical Lagrangian system with position measurements using a projection onto a Riemannian manifold [3]. The proposed metric, G, is then defined by the kinetic energy, where, for a given coordinate system's velocity vectors, it takes the form:

$$KE = \frac{1}{2}\dot{x}^T G\dot{x}. \tag{2.25}$$

Using the metric curvature and distance, a convergent observer can be designed. Bonnabel builds on this method and proposes the use of a Jacobi metric for an independant estimation of the velocity components (when the position is measured) for a conservative system [10]. Another example of an energy-based symmetry exploiting observer was developed by Bonnabel et. al. for quantum systems [11].

Recently, work by Sanfelice and Praly generalizes the use of the metric projection properties for observer design [37]. They show that one can design an observer if the system satisfies the following properties. Using the notation in [37], a Riemannian metric, $P(x)$ is defined that satisfies the following expression:

$$v^T \mathcal{L}_f P(x) v \leq 0 \; \forall (x,v) \in \mathbb{R}^n \times \mathbb{R}^n \; and \; \frac{\partial h}{\partial x}(x)v = 0, \tag{2.26}$$

[4]Meaning that the relations are invariant with respect to choice of coordinates.

where v is an arbitrary vector in $\mathbb{R}^n$ and $P(x)$ is the metric expressed in the original coordinates [5]. As long as the output set $\mathcal{H}(x) = \{x : h(x) = y\}$ is monotonic then there exists a minimal geodesic for all points in the system's domain on the Riemannian manifold. Then the resulting metric observer is be defined as:

$$\dot{\hat{x}} = f(\hat{x}) - k_E(\hat{x})P(\hat{x})^{-1}\frac{\partial h}{\partial x}(\hat{x})^T\frac{\partial \delta}{\partial y_1}(h(\hat{x}), y), \tag{2.27}$$

where $\delta : \mathbb{R}^m \times \mathbb{R}^m \longmapsto \mathbb{R}^+$ and $k_E(\hat{x})$ is a gain function greater than the minimal geodesic function throughout the entire range. The difficulty wih this method is that the metric must be implicitly defined using a set of inequality relations. The computation of a suitable metric remains challenging.[6]

2.1.5 Previous Work

This thesis develops on developments in the area of Hamiltonian realization [24], Hamiltonian decomposition [21], and potential-based feedback control [23]. The initial work on Hamiltonian realization [24] using exterior calculus generalizes on work by Cheng, et. al [14] to find a dissipative realization on a given metric space for a given Hamiltonian system. It is shown that given a system expressed in the form:

$$\dot{x} = T(x)\nabla H, \tag{2.28}$$

[5]Where the coordinate on the manifold is given by $\bar{x} = \psi(x)$, then if $\bar{P}(\bar{x})$ is the metric on the manifold, $P(x) = \frac{\partial \psi}{\partial x}(x)^T\bar{P}(\bar{x})\frac{\partial \psi}{\partial x}(x)$.

[6]Sanfelice and Praly promise a follow up paper that defines the construction of such a Riemmannian metric satisfying these conditions but it is not yet published.

there exists a dissipative potential function.[7] The expression of a potential driven system can be derived using De Rham's Homotopy operator [20]. For a system with this property, we build on this work to use the potential field in the design of an observer.

2.2 Lyapunov Theory

In this section, a brief introduction to Lyapunov theory is provided. Lyapunov theory remains the only suitable method currently for the analysis of stability of nonlinear systems. This section follows the definitions and notations provided in [28].

The standard Lyapunov stability results presented are focused on nonlinear systems of the form:

$$\dot{x} = f(x). \tag{2.29}$$

Stability definitions are first presented.

Definition 1. *(Stability): The equilibrium point $x = 0$ is*

- *stable if, for each $\epsilon > 0$, there is a $\delta(\epsilon) > 0$ such that*

  $||x(0)|| < \delta \implies ||x(t)|| < \epsilon, \forall\ t \geq 0,$

- *unstable if it is not stable,*

- *asymptotically stable if it is stable and δ can be chosen such that*

  $||x(0)|| < \delta \implies \lim_{t\to\infty} x(0) = 0.$

Is can be shown that this definition of stability is consistent with the existence of a suitable Lyapunov function. As a result, this stability is often referred to as Lyapunov

[7] See the section on the potential decomposition.

stability. A Lyapunov function, $V(x)$, is a continuous function on some domain $D \subset \mathbb{R}^n$, $V{:}D \longmapsto \mathbb{R}$.

Definition 2. *(Lyapunov Stability): Let $x = 0$ be an equilibrium point and $D \subset \mathbb{R}^n$ be a domain containing the equilibrium, $x = 0$. Let $V{:}D \longmapsto \mathbb{R}$ be continuously differentiable such that:*

$$V(0) = 0 \text{ and } V(x) > 0 \text{ in } D - \{0\},$$

$$\dot{V}(x) \leq 0 \text{ in } D,$$

- *then, $x = 0$ is stable. Additionally, if:*

$$\dot{V}(x) < 0 \text{ in } D - \{0\},$$

- *then $x = 0$ is asymptotically stable.*

Thus, by the above definition, the existence of a Lyapunov function is sufficient, but not necessary, for stability of a dynamical system. For physical systems Lyapunov functions can be constructed from the conservation laws.

2.3 Exterior Calculus

The mathematics of exterior calculus is used to extract the dissipative and conservative parts of the energy-like systems. An overview of the necessary operators and properties used in the analysis are covered here. Edelen [16] gives an extensive covering of exterior calculus [16] and most advanced vector calculus texts cover differential forms and their properties [15].

2.3.1 Differential Forms

The tangent space, $T_p(\mathbb{R}^b)$, at a point, $p \in \mathbb{R}^n$, is isomorphic to $\mathbb{R}^n$, and its bundle is represented by:

$$T(\mathbb{R}^n) = \bigcup_{p \in \mathbb{R}^n} T_p(\mathbb{R}^n). \tag{2.30}$$

This mean that if X is a smooth vector field, $X \in \Gamma^\infty(\mathbb{R}^n)$, then we define the mapping:

$$X : \mathbb{R}^n \longmapsto T(\mathbb{R}^n), \tag{2.31}$$

where the mapping assigns to every point in $p \in \mathbb{R}^n$ a tangent vector $X|_p \in T(\mathbb{R}^n)$. Then, the cotangent (dual) space as the space of linear functionals on $T(\mathbb{R}^n)$:

$$T^*(\mathbb{R}^n) = \{\omega|_p : T_p\mathbb{R}^n \longmapsto \mathbb{R}\}, \tag{2.32}$$

where each $\omega|_x$ is linear. The space of linear functions on $\mathbb{R}^n$:

$$f : \mathbb{R}^n \longmapsto \mathbb{R}^n, \tag{2.33}$$

is defined as the space of zero-forms, denoted $\Lambda^0(\mathbb{R}^n)$. We define the covector field, $T^*(X)$, as the space of differential forms of degree one, $\Lambda^1(\mathbb{R}^n)$, with elements of this space called one-forms, with the representation:

$$\omega = \sum \omega_i(x) dx_i, \tag{2.34}$$

where $\{dx_i\}$ is the natural basis of the cotangent space.

The space of smooth vectors is denoted by $\Gamma^\infty(\mathbb{R}^n)$, and its standard basis by $\partial_i =$

$\frac{\partial}{\partial x_i}$. A smooth vector field can be represented as:

$$X(x) = \sum_{i=1}^{n} v_i(x)\partial_i \in \Gamma(\mathbb{R}^n), \tag{2.35}$$

where $v_i(x)$ are smooth functions at point x. As before, the space of one-forms is denoted by $\Lambda^1(\mathbb{R}^n)$, and its standard basis by dx_i. A differentiable one-form is then defined as:

$$\omega(x) = \sum_{i=1}^{n} \omega_i(x)dx_i \in \Lambda^1(\mathbb{R}^n), \tag{2.36}$$

where the coefficient $\omega(x_1, ..., x_n)$, $i = 1, ..., n$ are smooth functions. The one-form maps a subset of the manifold to the tangent bundle.

2.3.2 Operators

The required exterior calculus operators will be covered as well as their properties necessary to extract the potential function of a dynamical system. These operators are: the wedge (exterior) product, the exterior derivative, the Hodge star operator, the inner product , and the homotopy operator[8].

Using the definitions of the previous section, the wedge product defines an anti-symmetric algebra on two differential forms of degree $k \geq 1$, $\wedge : \Lambda^k \times \Lambda^l \longmapsto \Lambda^{k+l}$, with the following distributive and anti-commutative properties:

1. $\alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma$,

2. $\alpha \wedge \beta = (-1)^{deg(\alpha)deg(\beta)}\beta \wedge \alpha$,

[8]Also known as De Rham's homotopy operator

where α, β, and $\gamma \in \Lambda(\mathbb{R}^n)$. A differential form of degree $k \leq n$ can be written as:

$$\omega(x) = \sum_{i_1<i_2<...<i_k} \omega_{i_1 i_2 ... i_3}(x) dx_{i_1} \wedge dx_{i_2} \wedge ... \wedge dx_{i_k}, \tag{2.37}$$

where, $\omega(x) \in \Lambda^k(\mathbb{R}^n)$, and $\omega_{i_1 i_2 ... i_k}$ defines $\binom{n}{k}$ functions in the domain of $\Lambda^k(\mathbb{R}^n)$. The exterior derivative operates in a similar manner to the familiar derivative operator and the same on zero-forms. It is a unique operator on $\Lambda(\mathbb{R}^n)$ with the following properties:

1. $d : \Lambda^k(\mathbb{R}^n) \longmapsto \Lambda^{k+1}(\mathbb{R}^n)$,
2. $d(a\alpha + b\beta) = ad\alpha + bd\beta$,
3. $d(\alpha \wedge \beta) = (d\alpha) \wedge (d\beta)$,
4. $df = \sum_i^n (\partial f / \partial x_i) dx_i, \ f \in \Lambda^0(\mathbb{R}^n)$,
5. $d \circ d\alpha = 0$.

where $f(x) \in \Lambda^0(\mathbb{R}^n)$. Following the fourth property of the exterior derivative, all differential forms that are exact, $\alpha = d\beta$, are also closed, $d \circ d\alpha = 0$.[9] The generalization of the exterior derivative for higher order differential forms is as follows. For a smooth differential form:

$$\omega = \sum_{i=1}^{n} \sum_{j=i+1}^{n} \cdots \sum_{z=y+1}^{n} \omega_{i,j,...,z} dx_i \wedge dx_j \wedge \cdots \wedge dx_z, \ \omega_{i,j,...,z} \in \Lambda^0(\mathbb{R}^n), \tag{2.38}$$

[9] The converse, all closed forms are exact, is not always true. However on some spaces it is – The Poincare Lemma.

its exterior derivative is given by:

$$d\omega = \sum_{i=1}^{n} \left(\frac{\partial \omega}{\partial x_i} \right) \wedge dx_i. \tag{2.39}$$

The Hodge star operator or Hodge dual, $\star$, is a linear operator defined a metric space, G, equiped with an inner product, such that it satisfies:

$$\omega \wedge \star\omega = \sqrt{det(G)}\mu,\ \forall \omega \in \Lambda(\mathbb{R}^n), \tag{2.40}$$

where $\mu = \sqrt{det(G)}dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$ is the volume form. The Hodge star maps a differential form of degree $p \leq n$ to its dual of degree $n - p$:

$$\star : \Lambda^p \longmapsto \Lambda^{n-p}, \tag{2.41}$$

where $\omega \in \Lambda^p(\mathbb{R}^n)$. As a basic example, take on a 2-dimensional manifold that has a metric tensor $G = dx_1 \otimes dx_1 + dx_2 \otimes dx_2$, the Hodge star of dx_1 yields dx_2 (ie. $\star dx_1 = dx_2$ and $\star dx_2 = -dx_1$). An explicit definition for the generalized Hodge star is defined using Einstein summation [33]:

$$\star\omega = \frac{\sqrt{G}}{k!(n-k)!} \omega_{i_1 i_2 \ldots i_k} \epsilon^{i_1 \ldots i_k j_{k+1} \ldots j_{n-k}} dx_{j_{k+1}} \wedge \ldots \wedge dx_{j_n}, \tag{2.42}$$

with $\omega \in \Lambda^k(\mathbb{R}^n)$ $k \leq n$, where G is the metric tensor, $\epsilon^{i_1 \ldots i_k j_1 \ldots j_{n-k}}$ is the Levi-Civita symbol, $\omega_{i_1 \ldots i_k}$ is the product of the zero-forms making up the k-form.

Similar to the exterior derivative the interior product can be generalized to exterior calculus. It acts on a differentiable vector and a differential form of degree k over $\mathbb{R}^n$:

$$\lrcorner : \Gamma^\infty(\mathbb{R}^n) \times \Lambda^k(\mathbb{R}^n) \longmapsto \Lambda^{k-1}(\mathbb{R}^n), \tag{2.43}$$

with the following properties:

1. $\mathscr{V} \lrcorner f = 0$,
2. $\mathscr{V} \lrcorner \omega = \omega(D)$,
3. $\mathscr{V} \lrcorner(\alpha + \beta) = D \lrcorner \alpha + D \lrcorner \beta$,
4. $\mathscr{V} \lrcorner(\alpha \wedge \beta) = (D \lrcorner) \wedge \beta + (-1)^{deg(\alpha)} \alpha \wedge (D \lrcorner \beta)$,

where $\omega \in \Lambda^1(\mathbb{R}^n)$, $\forall \mathscr{V} \in \Gamma^\infty(\mathbb{R}^n)$, $\forall \alpha,\, \beta \in \Lambda^k(\mathbb{R}^n)$ with $k = 1, 2, ..., n$ and $f \in \Lambda^0(\mathbb{R}^n)$.

Homotopy Operator

The final and most central operator to this analysis is the De Rham's homotopy operator. The homotopy operator operates on a star-shaped region, S. The star-shaped region is an open region in the n-dimensional space of $\mathbb{R}^n$. A region is defined with respect to a point, p_0, that is contained in S (in this thesis p_0 will normally be assumed to be 0). If $S \subset U$ where U is a neighbourhood of p_0, then coordinate functions assign coordinates $(x_1^0,\, x_2^0,\, \dots\, ,\, x_n^0)$ to the point p_0. Then S is defined such that for any point p in S the coordinate function of U assigns coordinates to p such that the set of points with coordinates $(x_i^0 + \lambda(x_i - x_i^0)) \in S \;\forall\; \lambda \in [0,\, 1]$.[10] This vector field is defined as:

$$\mathfrak{X}(x_i) = (x_i - x_i^0)\partial_i = \left(\frac{\partial}{\partial\lambda}(x_i^0 + \lambda(x_i - x_i^0))\right)\partial_i. \tag{2.44}$$

[10] In other words, all points in S are connected to p_0 linearly.

The homotopy operator, $\mathbb{H}$, is defined on S by requiring that $\mathbb{H}$ be a linear operator on $\Lambda(S) \subset \Lambda(\mathbb{R}^n)$ that satisfies:

$$\omega = d(\mathbb{H}\omega) + \mathbb{H}d\omega, \tag{2.45}$$

where $\omega \in \Lambda^k(S)$. Let ω be a k-form with S centered around x^*. For these coordinates the operator is defined as:

$$(\mathbb{H}\omega)(x) = \int_0^1 \mathfrak{X} \lrcorner \omega(x_i^* + \lambda(x_i - x_i^*))\lambda^{k-1} d\lambda, \tag{2.46}$$

with $\mathfrak{X}$ being the associated vector field on S, $\lambda \in [0, 1]$, and $k = \deg(\omega)$. $\mathbb{H}$ is well defined on S, since both $\mathfrak{X}$ and $\omega(x,\lambda)$ are well defined on S [16]. The properties of $\mathbb{H}$ are as follows:

1. $\mathbb{H} : \Lambda^k(S) \longmapsto \Lambda^{k-1}(S)\ \forall k \geq 1\ \&\ \Lambda^0(S) \longmapsto 0$,

2. $d\mathbb{H} + \mathbb{H}d = \mathbb{I}\ \forall k \geq 1$,

3. $\mathbb{H}df(x) = f(x) - f(x^0)\ \forall k = 0$,

4. $\mathbb{H}\mathbb{H}\omega(x),\ \mathbb{H}\omega(x^0) = 0$,

5. $\mathfrak{X}\lrcorner\mathbb{H} = 0,\ \mathbb{H}\mathfrak{X}\lrcorner = 0$.

By identity (2.45), it follows that $d(\mathbb{H}\omega)$ is a closed form. By the Poincaré lemma every closed form on S is exact, so $d(\mathbb{H}\omega)$ is also exact [17]. The exact part of ω is denoted ω_e and the anti-exact part is $\omega_a = \mathbb{H}d\omega$. Thus, the homotopy operator can be used to separate a one-form into an exact and anti-exact form.

Chapter 3

Potential Based Metric Observer

3.1 Introduction

In this chapter, a method is developed for the design of a metric-based nonlinear observer for a system with a potential expression. The systems considered are autonomous nonlinear systems with a measured[1] output of the form:

$$\dot{x} = f(x),\ y = h(x) = Cx, \tag{3.1}$$

where $x \in \mathbb{R}^n$ is the vector of state variables, $y \in \mathbb{R}^m$ is the output vector, and $f(x) \in \mathbb{C}^k$ where $k \geq 2$.

The decomposition method is presented and used to obtain the error system's potential function, $P(e, \hat{x})$. The observer equation and time-varying metric differential equation are proposed to design a suitable observer. The stability of the observer and metric equation are then assessed using Lyapunov stability. Finally, non-trivial illustrative simulations are shown.

[1] Where at least one or more states are not measured.

3.2 Potential Decomposition

With the techniques developed in the previous section, we now seek to extract the system's potential function, as defined in Section 2.1.3. The potential of the error dynamics is desired in an attempt to highlight the properties of the error dynamics.[2] The error vector field is defined over $\mathbb{R}^{2n}$ as:

$$F_e = (f(x) - f(\hat{x}))\frac{\partial}{\partial x} - (f(x) - f(\hat{x}))\frac{\partial}{\partial \hat{x}}, \tag{3.2}$$

where $\hat{x}$ is the observer state, and $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial \hat{x}}$ denote the vector field coordinates.[3] In this thesis, we explore the use of two types of metric spaces: time-varying and time invariant. For both we define the metric tensor, G, that defines the distance between both state and observer points as:

$$G = \sum_{i=1}^{n}\sum_{j=i}^{n}(a_{ij}dx_i \otimes dx_j + a_{ij}d\hat{x}_i \otimes d\hat{x}_j), \tag{3.3}$$

where the time-varying metric coefficients, a_{ij}, are functions of time and state, and for time-invariant metrics a_{ij} is a function of state only. The canonical metric tensor is written as:

$$G_{can} = dx_1 \otimes dx_1 + ... + dx_n \otimes dx_n + d\hat{x}_1 \otimes d\hat{x}_1 + ... + d\hat{x}_n \otimes d\hat{x}_n, \tag{3.4}$$

with the associated volume form:

$$\mu = dx_1 \wedge ... \wedge dx_n \wedge d\hat{x}_1 \wedge ... \wedge d\hat{x}_n. \tag{3.5}$$

[2] As long as the system is stable.

[3] $\frac{\partial}{\partial x_i} \cdot dx_j = 1$ if $i = j$, and zero otherwise.

To get the error associated with the differential form, we take the interior product of the volume form, μ, along the vector field to obtain the $(2n-1)$ form:

$$j = F_e \lrcorner \mu = (-1)^{n-1} \sum_{i=1}^{n} ((f_i(x) - f_i(\hat{x}))\Lambda_i - (f_i(x) - f_i(\hat{x}))\hat{\Lambda}_i), \tag{3.6}$$

where

$$\Lambda_i = dx_1 \wedge ... \wedge dx_{i-1} \wedge dx_{i+1} \wedge ... \wedge dx_n \wedge d\hat{x}_1 \wedge ... \wedge d\hat{x}_n,$$

$$\hat{\Lambda}_i = dx_1 \wedge ... \wedge dx_n \wedge d\hat{x}_1 \wedge ... \wedge d\hat{x}_{i-1} \wedge d\hat{x}_{i+1} \wedge ... \wedge d\hat{x}_n.$$

The associated one-form, ω, is then computed using the Hodge star operator:

$$\omega = \star j. \tag{3.7}$$

The metric induced one-form can then be decomposed using the homotopy operator into the sum of three differential forms:

$$\omega = \omega_e + \omega_a + \gamma, \tag{3.8}$$

where ω_e is the exact part of ω, ω_a, the anti-exact part and γ, the harmonic part. Using the homotopy operator, the one-form, ω, can be decomposed as:

$$\omega = d\mathbb{H}\omega + \mathbb{H}d\omega, \tag{3.9}$$

where $d\mathbb{H}\omega$ is the exact or dissipative part and $\mathbb{H}d\omega$ is the anti-exact or conservative part of the covector field. For a given system with a potential expression, we can rewrite the exact part in gradient-like form:

$$\omega_e = d\mathbb{H}\omega = dP = \nabla_e P de, \tag{3.10}$$

where the function $P(x, \hat{x}) = P(e + \hat{x}, \hat{x})$ is the potential function of the system over the domain of the star-shaped region, S. Additionally, the anti-exact part of ω on the $\mathfrak{X}$ over S is:

$$\mathfrak{X} \lrcorner \mathbb{H} d\omega = 0, \tag{3.11}$$

by the fourth property of the Homotopy operator. The homotopy operator yields the potential associated with the canonical metric in error coordinates as:

$$P(e, \hat{x}) = \mathbb{H}\omega = \int_0^1 e\frac{\partial}{\partial e} \lrcorner (f(\hat{x} + \lambda e) - f(\hat{x}))d\lambda. \tag{3.12}$$

In the following, this potential will be used for the design of suitable observers.

3.3 Observer Design

The homotopy operator is used to construct an observer for the system shown in (3.1) that seeks to reduce the estimation error $e = x - \hat{x}$ with time. From the construction of the potential function given in (3.12), we note that $P(0, \hat{x}) = 0$ and $\nabla_e P(0, \hat{x}) = 0$. It is then noted using property 5 of the homotopy operator that the error dynamics follow a gradient:

$$e\frac{\partial}{\partial e} \lrcorner \omega = e^T \frac{\partial P(\hat{x} + e, \hat{x})^T}{\partial e} = e^T \Theta(\hat{x} + e, \hat{x})e, \tag{3.13}$$

where $\Theta(e,\hat{x})$, the Hessian matrix of the potential, is defined by:

$$\Theta(e,\hat{x}) = \int_0^1 \frac{\partial^2 P}{\partial x \partial x^T}(\hat{x}, \hat{x}+\lambda e)d\lambda. \tag{3.14}$$

With the output function (3.1), the following design equations are proposed. The metric equation is given by:

$$\dot{M} = -2\Theta(\hat{x}) - \beta M - MQM + 2C^T RC \ , \ M(0) = \alpha, \tag{3.15}$$

where M is a symmetric positive definite matrix of the time-varying metric coefficients, a_{ij}, α is a matrix of the initial values of the metric coefficients, and Q and β are symmetric positive matrices of scalars that define a unique and symmetric solution for M. The initial value matrix, α, must be symmetric, positive definite and in the solution space of β and Q. The proposed observer is given by the dynamical system:

$$\dot{\hat{x}} = f(\hat{x}) + M(t)^{-1}C^T R(h(x) - h(\hat{x})) = F(e,\hat{x}). \tag{3.16}$$

In the following section, the convergence of the metric based observer is proven.

3.4 Main Result

In this section, we establish the asymptotic stability of the error system, $\dot{e} = \dot{x}$ - $\dot{\hat{x}}$. First, we show that the Hessian $\Theta(e,\hat{x})$ is Lipschitz. Following that, a Lyapunov function is proposed and used to establish the observer stability.

Lemma 1. *If $f \in Lip(S)$, where $S \subset \mathbb{R}^n$, then the Hessian matrix $\Theta(x)$ is Lipschitz on S.*

Proof: First, assume that the system has a potential function, $P(e+\hat{x},\hat{x})$, and that $f(x)$ is at least Lipschitz continuous for all points in the domain of interest. Then since $\dot{e} = \frac{\partial P}{\partial e}(e+\hat{x},\hat{x}) + U(e,\hat{x})^4$, so since $f(x)$ is Lipschitz continuous then $\dot{e}$, thus $\frac{\partial P}{\partial e}(e+\hat{x},\hat{x})$ is as well. This implies that, by the equality given in (3.13), we get the Lipschitz relation:

$$|e^T \nabla_e P(x,\hat{x}) - e^T \nabla_e P(x=\hat{x},\hat{x})| \leq L||e||^2,$$

$$\Longrightarrow \ |e^T(\Theta(x,\hat{x}) - \Theta(x=\hat{x},\hat{x}))e| \leq L||e||^2.$$

Thus, Θ is Lipschitz on the star-shaped domain in some area around $e=0$. This proves the lemma. ■

Then using the Lipschitz property of Θ the local stability of the error dynamics are assessed.

Theorem 1. *Assume that there exists a β and Q such that the equation (3.15) admits a unique positive definite solution $M(t)$. Furthermore, assume that the β and Q are such that $\frac{1}{2}(\beta\lambda + \lambda^2\lambda_Q) > L$, where λ is the minimum eigenvalue of M and λ_Q that of Q. Then for this choice of β and Q the observer (3.16) is such that the origin is an asymptotically stable equilibrium of the error dynamics:*

$$\dot{e} = f(\hat{x}+e) - F(e,\hat{x}). \tag{3.17}$$

Proof: We pose the Lyapunov function for the error dynamics, $V(e,M) = \frac{1}{2}e^T Me$.

[4] Where $e^T U(e,\hat{x}) = 0$.

Then taking the derivative we get,

$$\dot{V} = e^T M \dot{e} + \frac{1}{2} e^T \dot{M} e. \tag{3.18}$$

The error dynamics are given by expanding (3.17):

$$\dot{e} = f(x) - f(\hat{x}) - M^{-1} C^T R C e.$$

Expanding $\dot{V}$, we obtain:

$$\dot{V} = e^T M(f(x) - f(\hat{x})) - e^T C^T R C e + \frac{1}{2} e^T(-2\Theta(\hat{x}) - \beta M - MQM + C^T R C)e.$$

By the equality (3.13), we obtain $e^T M(f(x) - f(\hat{x})) = e^T \Theta(e, \hat{x} + e)e$. Thus, $\dot{V}$ can be simplified to:

$$\dot{V} = e^T(\Theta(e, \hat{x} + e) - \Theta(\hat{x}))e - \frac{\beta}{2} e^T M e - \frac{1}{2} e^T MQMe. \tag{3.19}$$

Then using the previously shown lemma, we obtain the inequality:

$$\dot{V} \leq L||e||^2 - \frac{\beta}{2} e^T M e - \frac{1}{2} e^T MQMe. \tag{3.20}$$

Since M is positive definite, it follows that $\frac{\beta}{2} e^T M e \geq \frac{\beta\lambda}{2}||e||^2$ where λ is the minimum eigenvalue of M. It then follows that the inequality can be written as:

$$\dot{V} \leq L||e||^2 - (\frac{\beta\lambda}{2} + \frac{\lambda^2 \lambda_Q}{2})||e||^2, \tag{3.21}$$

where λ_Q is the minimum eigenvalue of Q. As a result, for a fixed β and Q that satisfy the conditions of the theorem, $\dot{V} < -\sigma||e||^2$ on S for some positive constant $\sigma > 0$. This proves the asymptotic stability of the error dynamics and the convergence of the metric based observer system. ■

With this proof, we conclude that for a potential expressible system an asymptotically convergent observer can be constructed using the method shown as long as the metric is symmetric, positive definite, and well defined on S for the chosen β and Q.

3.5 Simulation

In this section, we perform a simulation to study the application of the proposed observer design technique. First, the van der Pol oscillator is considered. Then the observer is designed for a three dimensional chaotic Chen attractor. Finally, the problem examined by Sanfelice and Praly in [37] is treated using the potential based method, leading to the discussion of time-invariant metrics the following chapter.

3.5.1 Van der Pol oscillator

The van der Pol oscillator is given as presented:

$$\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= 2x_2(1 - x_1^2) - x_1, \\
y &= x_1.
\end{aligned}$$

This system has a stable focus around the equilibrium focus x^* at $\{x_1 = 0, x_2\}$. The metric tensor is then posed as:

$$G = a_{11}dx_1 \otimes dx_1 + a_{12}dx_1 \otimes dx_2 + a_{22}dx_2 \otimes dx_2$$
$$+a_{11}d\hat{x}_1 \otimes d\hat{x}_1 + a_{12}d\hat{x}_1 \otimes d\hat{x}_2 + a_{22}d\hat{x}_2 \otimes d\hat{x}_2.$$

Figure 3.2 shows how the metric coefficients change with time. The choice of metric yields the error associated one-form:

$$\omega = a_{11}(x_2 - \hat{x}_2)(d\hat{x}_1 - dx_1) + (a_{22}(2x_2(1 - x_1^2)$$
$$- x_1 - 2\hat{x}_2(1 - \hat{x}_1^2) + \hat{x}_1) - a_{12}(x_2 - \hat{x}_2))(d\hat{x}_2 - dx_2).$$

The homotopy operator of ω is taken, $\mathbb{H}\omega$, then the Hessian is calculated with respect to x_1 and x_2.

The choice of tuning matrices are $R = 1$ and:

$$Q = \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix}.$$

The initial conditions are set as:

$$x_1(0) = x_2(0) = 2,\ \hat{x}_1(0) = \hat{x}_2(0) = 0.5,\ a_{11} = a_{22} = 1,\ a_{12} = 0.$$

Figure 3.1 shows the phase portrait of the system states, as well as the observer estimates. The observer asymptotically approaches the unknown system state, x_2, as can be seen in Figure 3.4. Additionally as shown in the proof of Theorem 1, the state

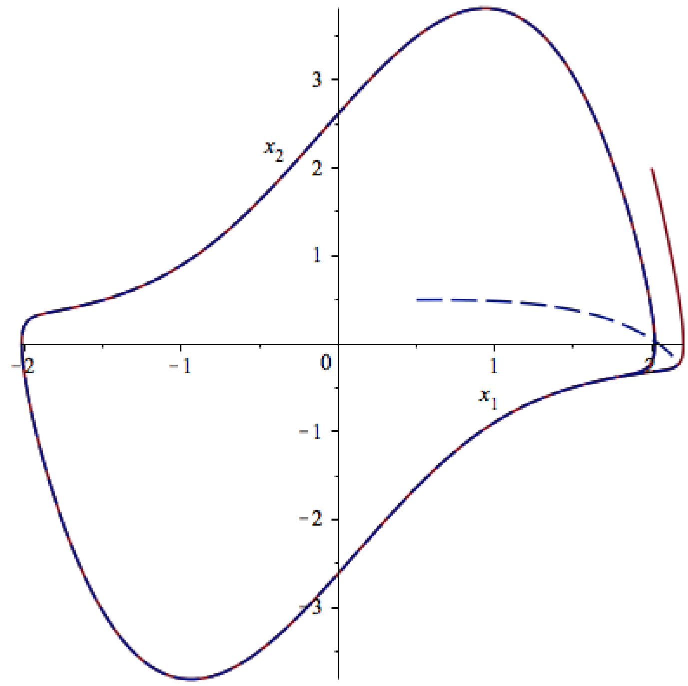

Figure 3.1: A plot of the system about its stable focus (solid) and metric based observer (dashed).

error, e, is asymptotically approaching zero in Figure 3.5.

3.5.2 Chen attractor

The second example is the Chen attractor (a subclass of Chua's chaotic circuit) given in [13]. The system is:

$$\dot{x}_1 = \alpha(x_2 - x_1),$$

$$\dot{x}_2 = (\gamma - \alpha)x_1 - x_1x_3 + \gamma x_2,$$

$$\dot{x}_3 = x_1x_2 - \beta x_3,$$

$$y = x_2.$$

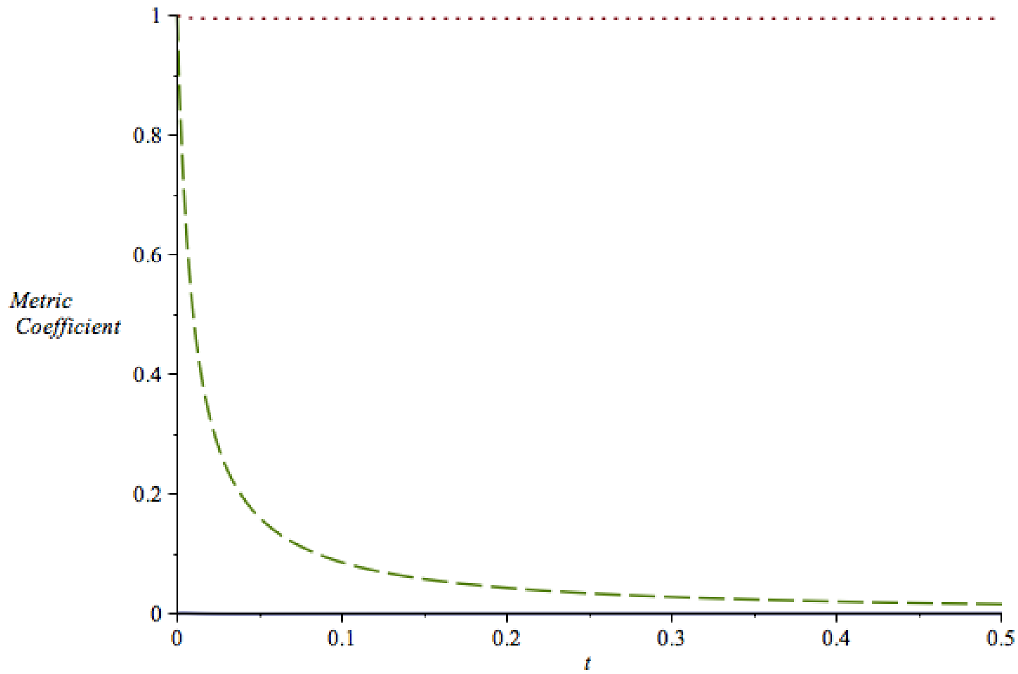

Figure 3.2: A plot of the metric coefficients: a_{11} (dotted), a_{22} (dashed), and a_{12} (solid) versus time. The initial conditions were: $a_{11}(0) = a_{22}(0) = 1$ and $a_{12}(0) = 0$.

For this simulation, the values are taken as: $\alpha = 40$, $\beta = 3$, and $\gamma = 28$. With the initial conditions being $x_1(0) = -0.1$, $x_2(0) = 0.5$, and $x_3(0) = -0.6$. We then note the system has an equilibrium point at $\{x_1 = 0, x_2 = 0, x_3 = 0\}$. Following the proposed technique, we assign the metric:

$$G = a_{11}(dx_1 \otimes dx_1 + d\hat{x}_1 \otimes d\hat{x}_1) + a_{12}(dx_1 \otimes dx_2 + d\hat{x}_1 \otimes d\hat{x}_2)$$
$$+ a_{13}(dx_1 \otimes dx_3 + d\hat{x}_1 \otimes d\hat{x}_3) + a_{22}(dx_2 \otimes dx_2 + d\hat{x}_2 \otimes d\hat{x}_2)$$
$$+ a_{23}(dx_2 \otimes dx_3 + d\hat{x}_2 \otimes d\hat{x}_3) + a_{33}(dx_3 \otimes dx_3 + d\hat{x}_3 \otimes d\hat{x}_3).$$

As before, we set $R = 1$ and $Q = 100\mathbb{I}$. The observer initial conditions are: $\hat{x}_1(0) = \hat{x}_2(0) = \hat{x}_3(0) = 2$, $c_{11}(0) = c_{22}(0) = c_{33}(0) = 1$, and $c_{12}(0) = c_{13}(0) = c_{23}(0) = 0$.

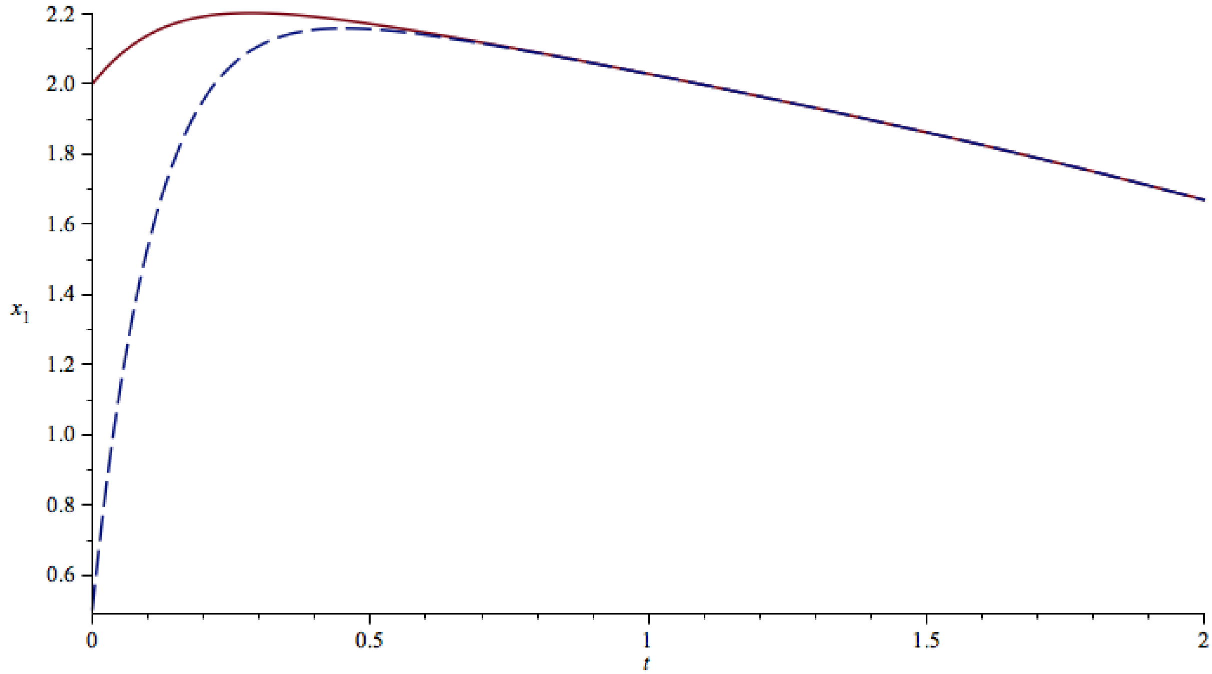

Figure 3.3: A plot of the first state, x_1 (solid), and its observer state, $\hat{x}_1$ (dashed).

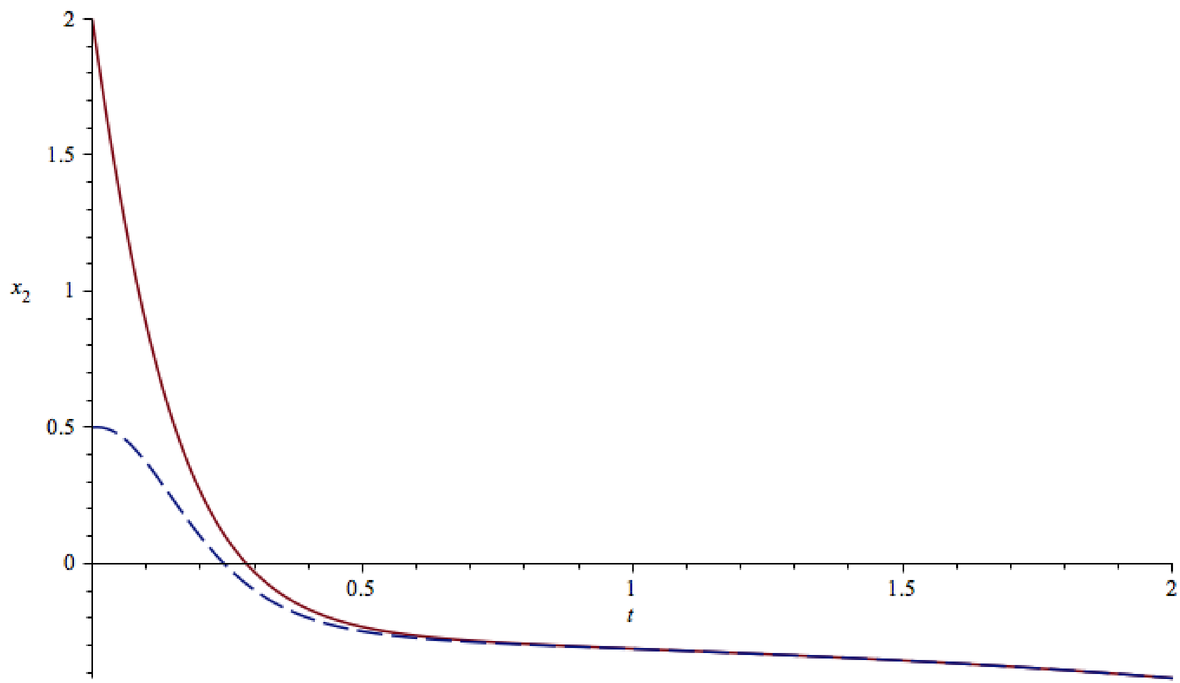

Figure 3.4: A plot of the first state, x_2 (solid), and its observer state, $\hat{x}_2$ (dashed).

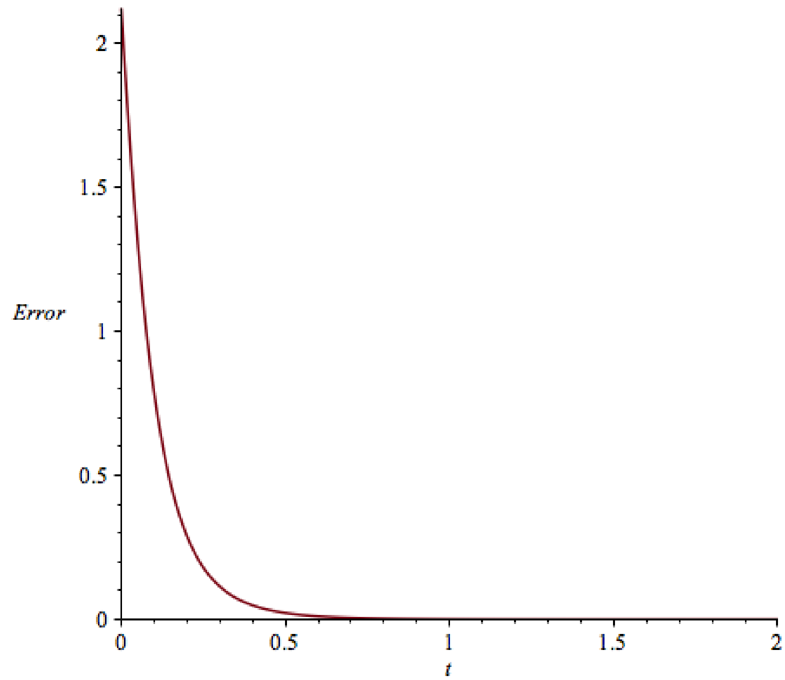

Figure 3.5: A plot of the magnitude of the system error, $||e||$, versus time.

The plots are shown in Figures 3.7 to 3.11.

3.5.3 Motivating example

The last example shows the work of Sanfelice and Praly [37] and shows that, with a time-invariant metric, the potential based observer method can provide interesting and far-reaching results. In equation (35) of the aforementioned paper [37], the following metric tensor is given:

$$M(x) = \begin{bmatrix} 1 & \frac{x_1 x_2}{\sqrt{1+x_1^2}} \\ 0 & \sqrt{1+x_1^2} \end{bmatrix} \tag{3.22}$$

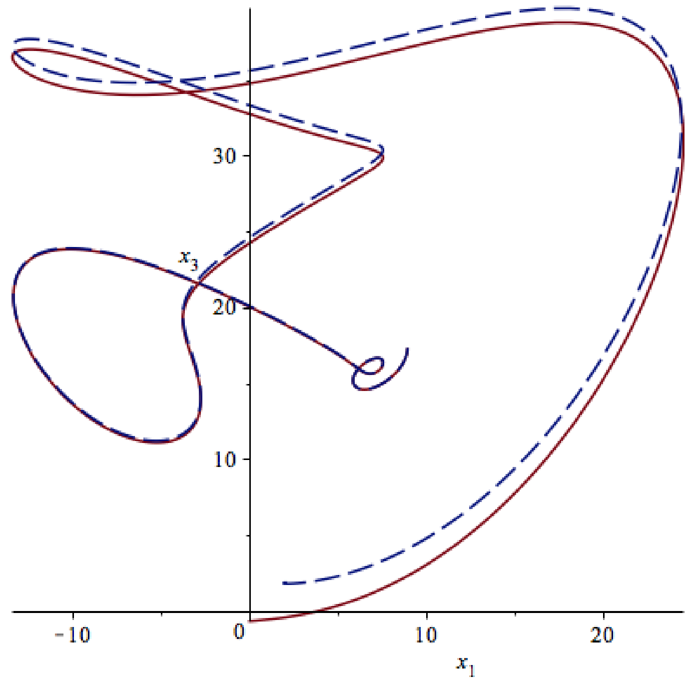

Figure 3.6: A phase plot of the system for the first and third states (the unobserved ones) about its focus (solid) and metric based observer (dashed).

for the dynamical system:

$$\dot{x}_1 = x_2\sqrt{1+x_1^2},\ \dot{x}_2 = -\frac{x_1}{\sqrt{1+x_1^2}}x_2^2,\ y = x_1. \tag{3.23}$$

The observer is subsequently given as:

$$\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} = \begin{bmatrix} \hat{x}_2\sqrt{1+\hat{x}_1^2} \\ -\frac{\hat{x}_1\hat{x}_2^2}{\sqrt{1+\hat{x}_1^2}} \end{bmatrix} - \frac{2k_e(\hat{x})}{1+\hat{x}_1^2+(\hat{x}_1+\hat{x}_2)^2}\begin{bmatrix} 1+\hat{x}_1^2 \\ 1-\hat{x}_1\hat{x}_2 \end{bmatrix}(\hat{x}_1 - y), \tag{3.24}$$

where $k_e(\hat{x})$ is the observer state dependent gain. The observer system is shown to achieve convergence in some region of the equilibrium. We now demonstrate that the same type of problem can be solved if the time-invariant metric can be determined *a*

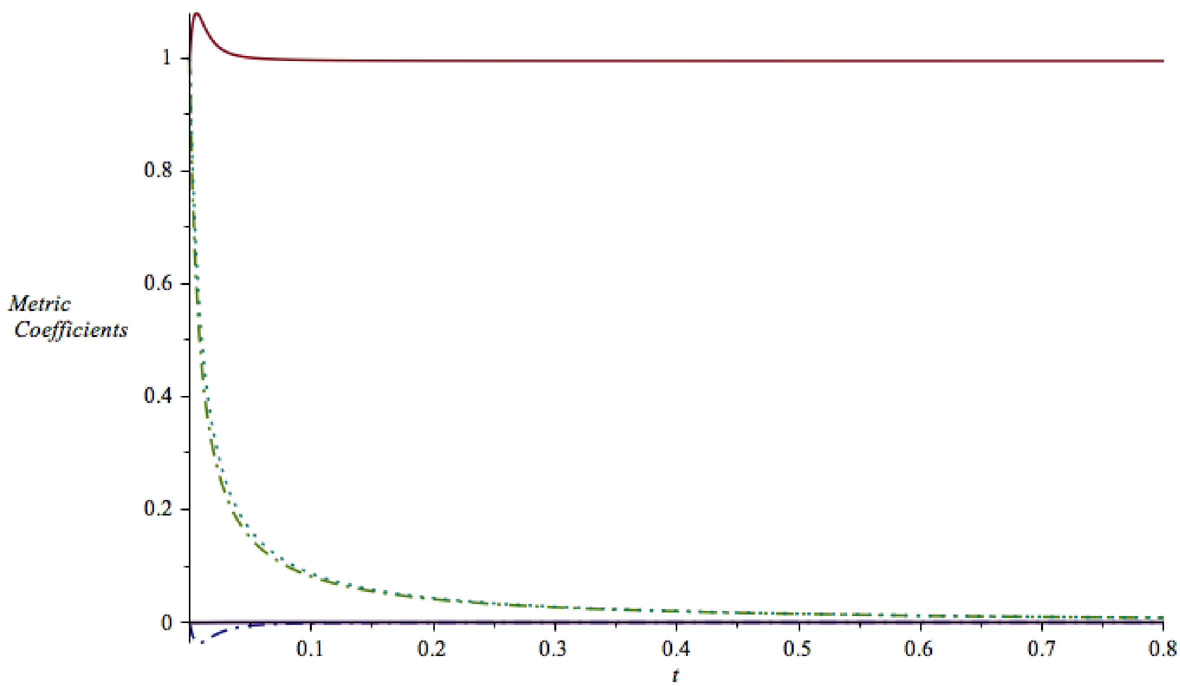

Figure 3.7: A plot of the metric coefficients: a_{11} (solid), a_{12} (dot-dash), a_{13} (line), a_{22} (dashed), a_{23} (line), and a_{33} (dots) versus time. The initial conditions were: $a_{11}(0) = a_{22}(0) = a_{33}(0) = 1$ and $a_{12}(0) = a_{13}(0) = a_{23}(0) = 0$.

priori.

Using the metric based observer approach shown, we can see that for a metric equation of the form:

$$G = \frac{1}{1+x_1^2} dx_1 \otimes dx_1 + \frac{x_1 x_2}{1+x_1^2} dx_1 \otimes dx_2 + dx_2 \otimes dx_2, \tag{3.25}$$

yields the one-form:

$$\omega = -x_2 dx_1. \tag{3.26}$$

This one-form is anti-symmetric. As a result, the system has a convex potential

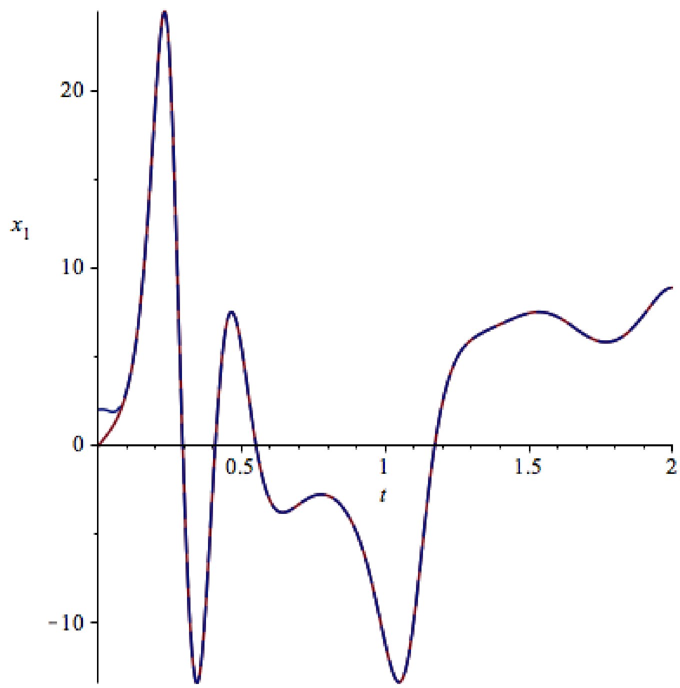

Figure 3.8: A plot of the first state, x_1 (solid), and its observer state, $\hat{x}_1$ (dashed).

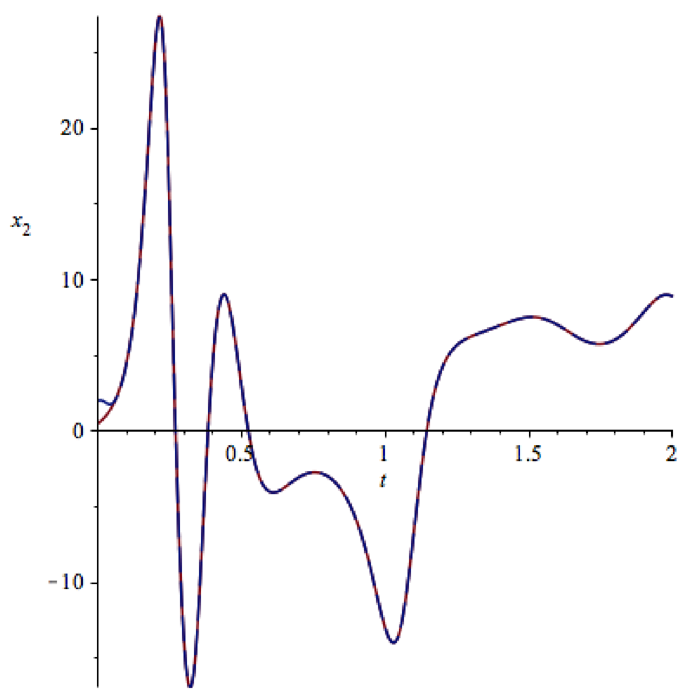

Figure 3.9: A plot of the first state, x_2 (solid), and its observer state, $\hat{x}_2$ (dashed).

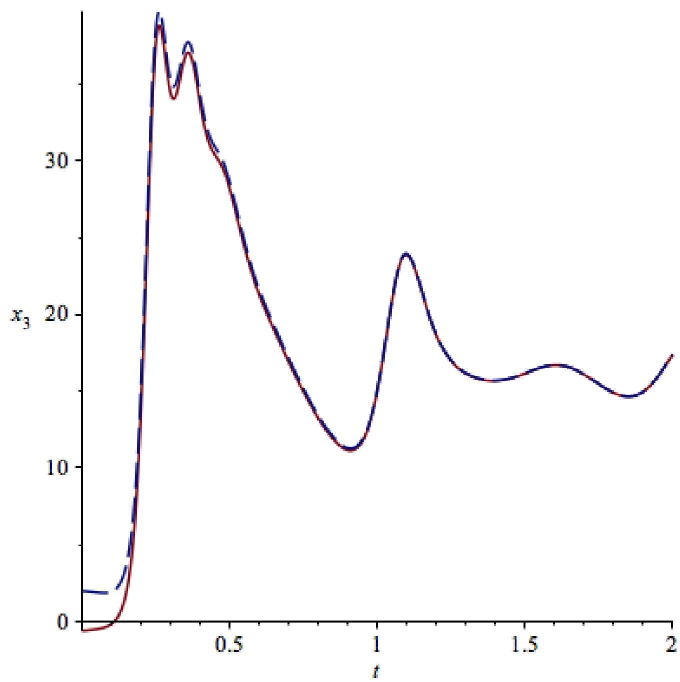

Figure 3.10: A plot of the first state, x_3 (solid), and its observer state, $\hat{x}_3$ (dashed).

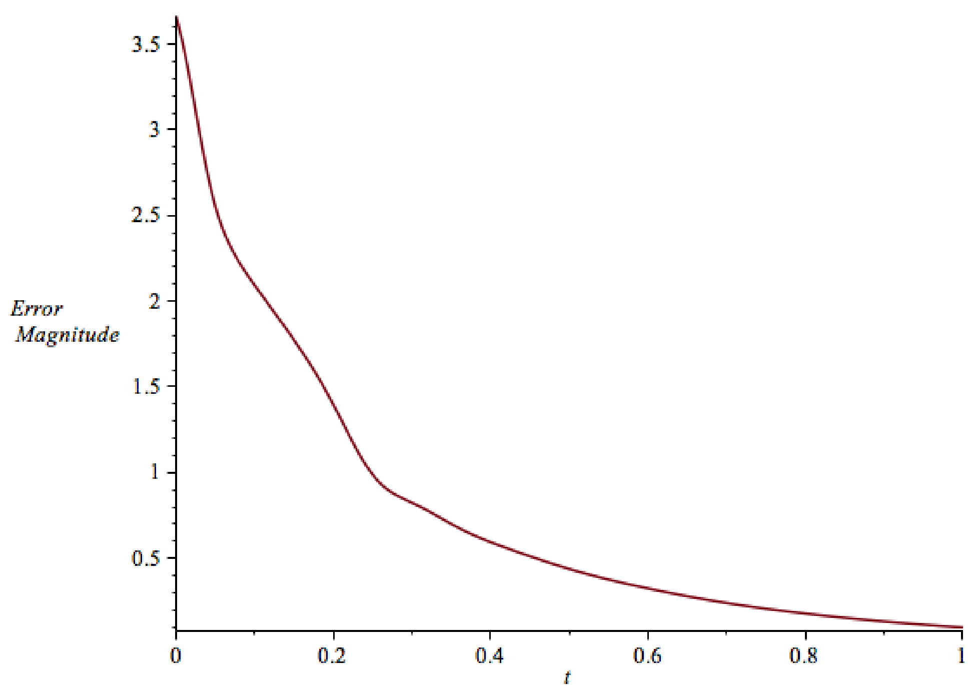

Figure 3.11: A plot of the magnitude of the system error, $||e||$, versus time.

and an asymptotically convergent gain observer. Thus, if we can find a way to obtain the correct time-invariant metric apriori for a given system this result could be generalized, which motivates the discussion in Chapter 4.

3.6 Summary

In this chapter, it was shown that the problem of observer design can be effectively solved using a time-varying metric that defines a dissipative Hamiltonian representation. The potential is obtained using the homotopy operator that is made convex by the choice of a suitable metric differential equation. The observer error dynamics are shown to be asymptotically stable at the origin. The effectiveness of the proposed technique is demonstrated using simulations.

Chapter 4

Time Invariant Metric Observer

4.1 Introduction

Motivated by the example at the end of the last chapter, we design a time-invariant[1] metric that allows for the design of a convergent observer. We consider systems of the type (3.1). The metric is designed with a metric-dependent quadratic Lyapunov function and makes use of the explicit equation for the error associated one-form. The Lyapunov derivative is made negative definite by choice of the observer function.

4.2 Design

Let us consider an arbitrary positive definite metric $G(e)$ that meets the following inequality:

$$e^T G e \geq 0. \tag{4.1}$$

Using the approach proposed in the previous chapter, let this choice of metric be such that the Hessian matrix of the potential is negative semi-definite, that is $e^T \Theta e \leq 0$, in some region of the origin of the error dynamics ($e = 0$). This approach is

[1] Dependant only on the values of the state variables.

problematic in practice since it is generally difficult to guarantee the negative semi-definite property of the Hessian such that the metric is only a function of observer states, $\hat{x}$. For the Lyapunov function, $V = \frac{1}{2}e^T G e$, and the observer dynamics, $\dot{\hat{x}} = f(\hat{x}) + L(\hat{x}, y)$, where $L(\hat{x}, y)$ is a correction term to be designed. The time derivative of V is given as:

$$\dot{V} = e^T \Theta(\hat{x}, e + \hat{x})e - e^T G(e) L(\hat{x}, y) + \frac{1}{2} e^T \dot{G} e. \tag{4.2}$$

In general, there is no obvious solution to ensure that:

$$e^T G(e) L(\hat{x}, y) \geq e^T \Theta e + \frac{1}{2} e^T \dot{G} e, \tag{4.3}$$

even if $e^T \Theta e \leq 0$ and the metric derivative term can be upper bounded.

However, if one relaxes the negative definiteness conditions on the Hessian matrix Θ then it may be possible to remove its dependence on the error vector e. In this chapter, we propose a metric that depends only on $\hat{x}$ and y. As in the previous chapter, this metric is required to be positive definite with an associated Lipschitz continuous Hessian matrix. The metric considered here, is a positive definite observer state dependent metric:

$$x^T G(\hat{x}, y) x \geq 0, \ \forall x \in \mathbb{R}^n. \tag{4.4}$$

Specifically, the two (and analogous three) dimensional metric, whose coefficients are:

$$g_{ij} = \begin{cases} 1 + \hat{x}_i^2 & : i = j \\ 0 & : i \neq j \end{cases},$$

is used for simplicity. Using a form analogous to (4.2) the observer gain of the proposed observer is:

$$L(\hat{x}, y) = G^{-1}(\hat{x}, y)[\Theta(\hat{x}) + \bar{L}(\hat{x}, y)]C^T(h(x) - h(\hat{x})), \tag{4.5}$$

from (4.2), using $h(x) - h(\hat{x}) = C(x - \hat{x}) = Ce$, and assuming without any loss of generality that $C^TC = 1$, one obtains:

$$\dot{V} = e^T\Theta(e, \hat{x} + e)e - e^T(\Theta(\hat{x}) + \bar{L}(\hat{x}, y))e + \frac{1}{2}e^T\dot{G}e. \tag{4.6}$$

Using the relation from Lemma 1, $e^T(\Theta(e, e + \hat{x}) - \Theta(\hat{x}))e \leq K||e||^2$, inequality (4.6) becomes:

$$\dot{V} \leq \hat{K}||e||^2 - e^T\bar{L}(\hat{x}, y)e + \frac{1}{2}e^T\dot{G}e. \tag{4.7}$$

Posing the left hand side as an equality relation ($L.H.S. = 0$):

$$e^T\hat{K}Ie + \frac{1}{2}e^T\dot{G}e = e^T\bar{L}(\hat{x}, y)e, \tag{4.8}$$

where I is the identity matrix of size n, $\hat{K} > K$ and eliminating the error terms we obtain:

$$\hat{K}I + \frac{1}{2}\dot{G} = \bar{L}(\hat{x}, y). \tag{4.9}$$

We then note that the time derivative of $\hat{x}$ depends on $L(\hat{x}, y)$. As a result, one must formulate a system of linear equations to solve for the values of $\bar{L}(\hat{x}, y)$. For our specific choice of metric, the equation for $\bar{L}(\hat{x}, y)$ can be easily derived.

First, we note that the derivatives of the metric are given as:

$$\dot{g}_{ij} = \begin{cases} 2\hat{x}_i\dot{\hat{x}}_i & i = j, \\ 0 & i \neq j \end{cases}.$$

Then since $\dot{\hat{x}} = f(\hat{x}) + L(\hat{x}, y)$, so the diagonal metric derivative coefficients become:

$$\dot{g}_{ii} = 2\hat{x}_i[f_i(\hat{x}) + L_i(\hat{x}, y)]. \tag{4.10}$$

The expression for $L_i(\hat{x}, y)$ is given by:

$$L(\hat{x}, y) = G^{-1}(\hat{x}) \left[\Theta(\hat{x}) + \bar{L}(\hat{x})\right] C^T(h(x) - h(\hat{x})), \tag{4.11}$$

$$= G^{-1}(\hat{x}) \begin{bmatrix} (\Theta_{11} + \bar{\ell}_{11}) & (\Theta_{21} + \bar{\ell}_{21}) & \dots & (\Theta_{n1} + \bar{\ell}_{n1}) \\ (\Theta_{21} + \bar{\ell}_{21}) & (\Theta_{22} + \bar{\ell}_{22}) & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ (\Theta_{n1} + \bar{\ell}_{n1}) & \dots & \dots & (\Theta_{nn} + \bar{\ell}_{nn}) \end{bmatrix} C^T(h(x) - h(\hat{x})). \tag{4.12}$$

Simplifying the metric inverse g_{ii}^{-1} to be the diagonal elements of $G^{-1}(\hat{x})$, the equation becomes:

$$L(\hat{x}, y) = \begin{bmatrix} g_{11}^{-1}(\Theta_{11} + \bar{\ell}_{11}) & g_{22}^{-1}(\Theta_{21} + \bar{\ell}_{21}) & \dots & g_{nn}^{-1}(\Theta_{n1} + \bar{\ell}_{n1}) \\ g_{22}^{-1}(\Theta_{21} + \bar{\ell}_{21}) & g_{22}^{-1}(\Theta_{22} + \bar{\ell}_{22}) & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ g_{nn}^{-1}(\Theta_{n1} + \bar{\ell}_{n1}) & \dots & \dots & g_{nn}^{-1}(\Theta_{nn} + \bar{\ell}_{nn}) \end{bmatrix} C^T(h(x) - h(\hat{x})) \tag{4.13}$$

$$= QC^T(h(x) - h(\hat{x})), \tag{4.14}$$

where Θ_{ij} are the elements of the Hessian matrix, assumed symmetric. The elements of the matrix Q are given by $q_{ij} = g_{kk}^{-1}(\Theta_{km} + \bar{\ell}_{km})$ with:

$$k = \begin{cases} i & : i \geq j \\ j & : j < i \end{cases},$$

$$m = \begin{cases} i & : i \leq j \\ j & : j > i \end{cases}.$$

For the next step, it is assumed that $h(x) \in \mathbb{R}$, so that for $y = Cx$, the vector C is:

$$C_i = \begin{cases} 1 & \text{for } i = p \\ 0 & \text{otherwise} \end{cases}.$$

As a result, the observer gain is given by:

$$L(\hat{x}, y) = \begin{bmatrix} q_{p1}(h(x) - h(\hat{x})) \\ q_{p2}(h(x) - h(\hat{x})) \\ \vdots \\ q_{pn}(h(x) - h(\hat{x})) \end{bmatrix}. \tag{4.15}$$

From (4.9), the coefficients $\bar{\ell}$ are such that:

$$\bar{\ell}_{ij} = \begin{cases} 0 & : i \neq j \\ \hat{K} + \hat{x}_i[f_i(\hat{x}) + g_{kk}^{-1}(\Theta_{km} + \bar{\ell}_{km})] & : i = j \end{cases}.$$

This derivation yields a system of n-equations to be solved for n-$\bar{\ell}_{ii}$ variables. The solutions then depend on the dimension of the system, n, and the measured output vector, C. As an example for a two-dimensional system where $C = [1, 0]$, the function, $\bar{L}$, becomes:

$$\bar{L}(\hat{x}, y) = \begin{bmatrix} \frac{\hat{K} + \hat{x}_1\left[f_1(\hat{x}) + (y - h(\hat{x}))\frac{\hat{x}_1^2}{\hat{x}_1^2\hat{x}_2^2}\Theta_{11}(\hat{x})\right]}{1 - \hat{x}_1(y - h(\hat{x}))\frac{\hat{x}_1^2}{\hat{x}_1^2\hat{x}_2^2}} & 0 \\ 0 & \hat{K} + \hat{x}_1\left[f_2(\hat{x}) + (y - h(\hat{x}))\frac{\hat{x}_2^2}{\hat{x}_1^2\hat{x}_2^2}\Theta_{21}(\hat{x})\right] \end{bmatrix}. \tag{4.16}$$

Then for choice of $\hat{K}$ greater than the Lipschitz constant of the Hessian, and even though the Hessian may not necessarily be negative semi-definite, the Lyapunov derivative can be made negative definite, $\dot{V} < 0$.

4.3 Simulation

In this section, two simulation examples will be used to illustrate the function and design of the metric based observer presented. The systems presented are the two-dimensional Van der Pol oscillator from Chapter 3 and a stable focus system.

4.3.1 Van der Pol oscillator

The van der Pol oscillator is given by:

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= 2x_2(1-x_1^2) - x_1,\\ y &= x_1.\end{aligned}$$

The system exhibits a limit cycle about the origin. Using the metric defined in (4.4) and (4.13), the following observer is obtained:

$$\begin{bmatrix}\dot{\hat{x}}_1\\ \dot{\hat{x}}_2\end{bmatrix} = f(\hat{x}) + \begin{bmatrix}\frac{20+\hat{x}_1\hat{x}_2}{(1+\hat{x}_1^2)\left(1-\frac{\hat{x}_1(y-\hat{x}_1)}{1+\hat{x}_1^2}\right)}\\ \frac{\frac{2}{3}\hat{x}_1\hat{x}_2^2+\hat{x}_1\hat{x}_2+\frac{1}{4}\hat{x}_2^2-\frac{1}{4}\hat{x}_1^2}{1+\hat{x}_2^2}\end{bmatrix}(x_1-\hat{x}_1), \tag{4.17}$$

where the values of $\hat{K}$ is chosen as 20. For the choice of observer this value of K ensures domination of the Hessian Lipschitz constant for $||x_0||^2 < 5.5$. The initial conditions for the system and the observer states are $x_1(0) = x_2(0) = 2$ and $\hat{x}_1(0) = \hat{x}_2(0) = 0.5$, respectively. The results are shown in Figure 4.1. As expected, the observer converges asymptotically to the system state trajectories. Figure 4.4 shows an asymptotically decreasing error magnitude as predicted by the Lyapunov analysis. As in Chapter 3,

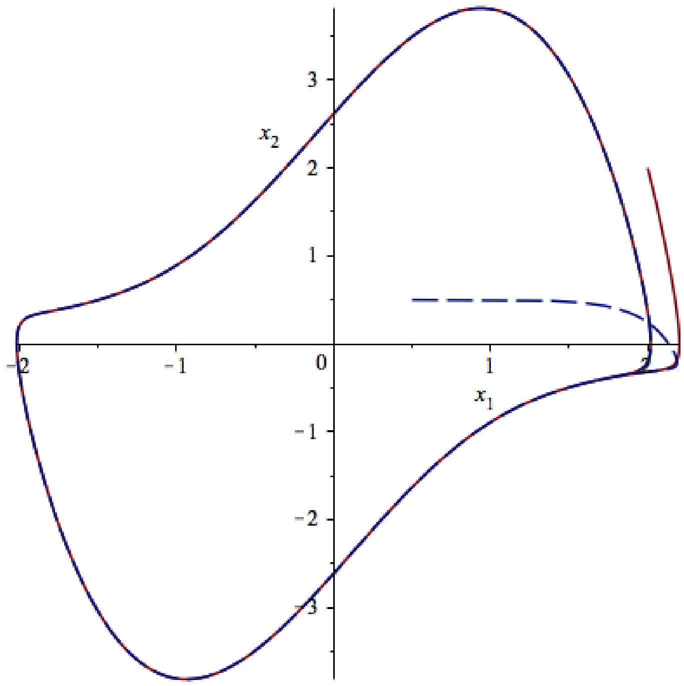

Figure 4.1: A plot of the system about the origin (solid) and metric based observer (dashed).

the time-invariant converges. The rate of convergence of the metric based example from Chapter 3 is more uniform then the time-invariant method due to the dependence on the continuous system dynamics.

4.3.2 Chen attractor

The second example again is the chaotic Chen attractor system from Chapter 3:

$$\dot{x}_1 = 40(x_2 - x_1),$$

$$\dot{x}_2 = -12x_1 - x_1x_3 + 28x_2,$$

$$\dot{x}_3 = x_1x_2 - 3x_3,$$

$$y = x_2.$$

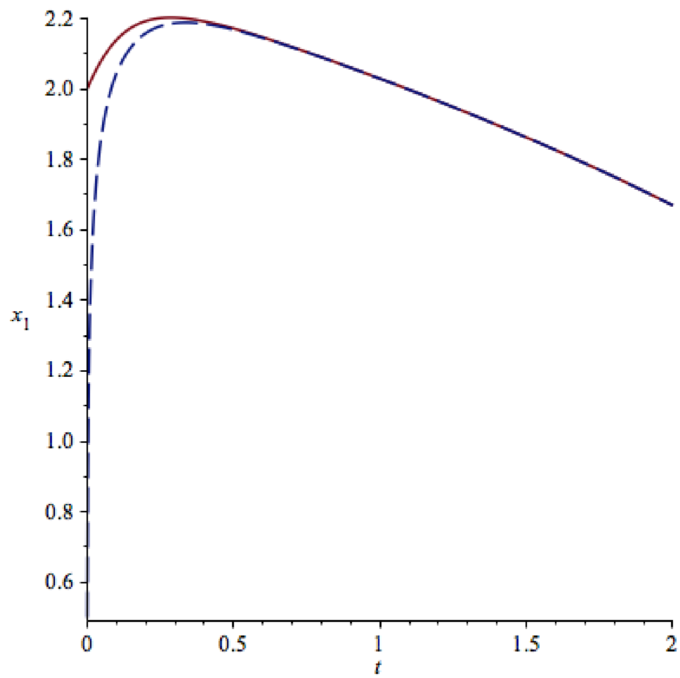

Figure 4.2: A plot of the first state, x_1 (solid), and its observer state, $\hat{x}_1$ (dashed).

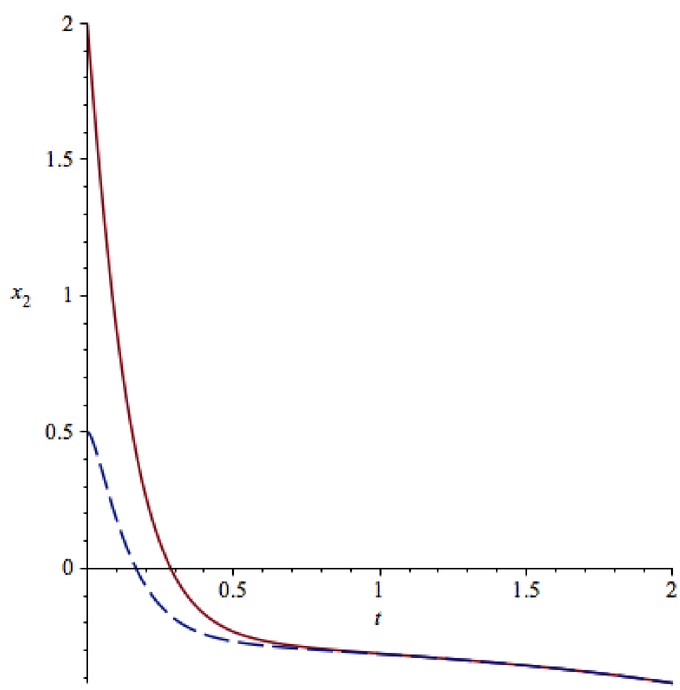

Figure 4.3: A plot of the first state, x_2 (solid), and its observer state, $\hat{x}_2$ (dashed).

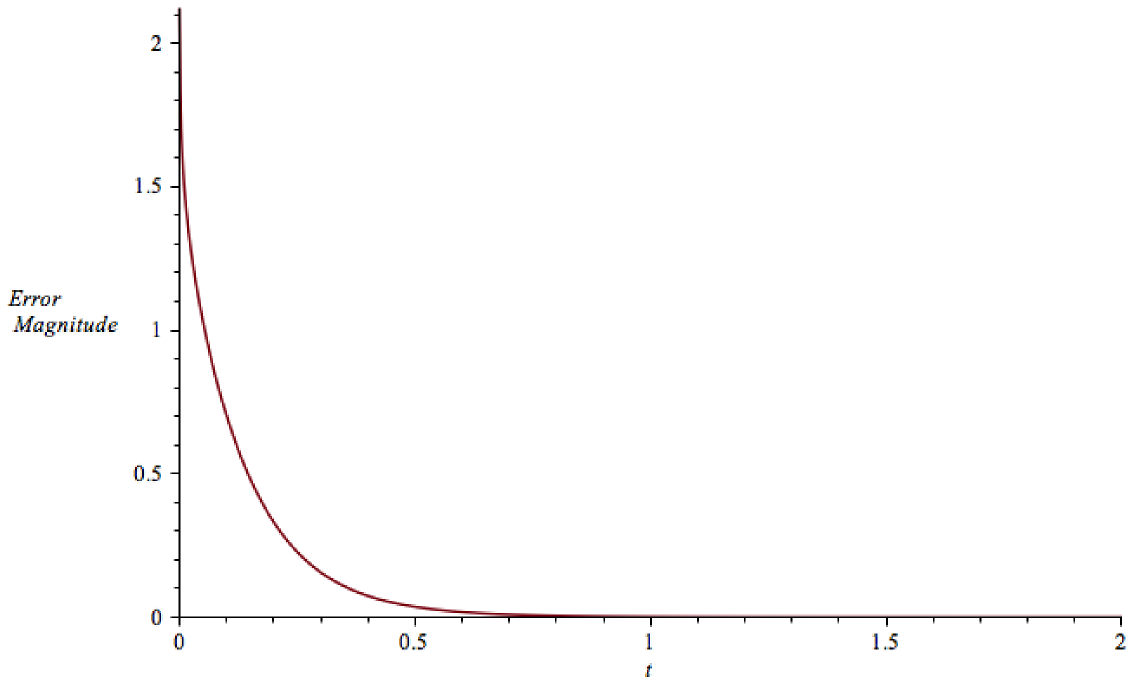

Figure 4.4: A plot of the magnitude of the system error, $||e||$, versus time.

For $C = [0, 1, 0]$, the observer dynamics are given by:

$$\dot{\hat{x}} = f(\hat{x}) +$$
$$\begin{bmatrix} \frac{-14+\frac{1}{5}\hat{x}_2^2\hat{x}_3+3\hat{x}_2^2-10\hat{x}_1^2+\frac{1}{3}\hat{x}_3}{1+\hat{x}_1^2} \\ \frac{1}{1+\hat{x}_2^2}\left(-28-24\hat{x}_2^2+\frac{1}{1-\frac{\hat{x}_2(y-\hat{x}_2)}{1+\hat{x}_2^2}}\left(6x10^4+\hat{x}_2\left(-12\hat{x}_1-\hat{x}_1\hat{x}_3+28\hat{x}_2+\frac{(y-\hat{x}_2)(-28-14\hat{x}_2^2)}{1+\hat{x}_2^2}\right)\right)\right) \\ \frac{\frac{-1}{5}\hat{x}_1\hat{x}_3^2+\frac{1}{5}\hat{x}_1\hat{x}_2^2}{1+\hat{x}_3^2} \end{bmatrix}(x_2 - \hat{x}_2).$$

The initial conditions are $x_1(0) = -0.1$, $x_2(0) = 0.5$, $x_3(0) = -0.6$ and $\hat{x}_i(0) = 2$. The gain value is chosen as $\hat{K} = 6 \times 10^4$. The results are shown in figure 4.5 and 4.6. The observer asymptotically converges to the true state trajectories. The results of the individual states are shown in figures 4.7, 4.8, and 4.9. Unlike the van der Pol example, the rate of convergence for the time-invariant method is similar to the

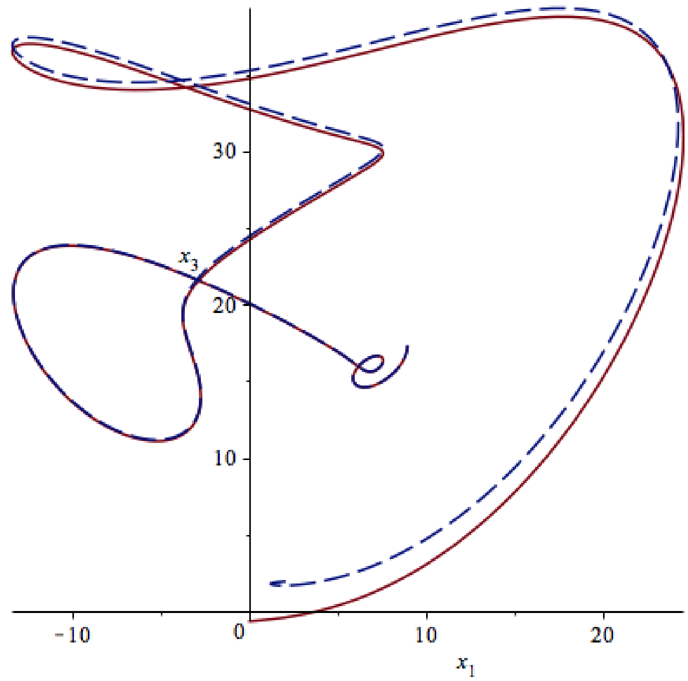

Figure 4.5: A 2-D plot of the Chen attractor system for the first and third states(solid) and their observer (dashed).

metric based method and the behavior is also similar, with the exception that the observer is far more agressive for the x_2 state.

The proposed method can be compared with the deterministic continuous time extended Kalman filter given in [36]. Where for a system and its output:

$$\dot{x} = f(x) + w, \tag{4.18}$$

$$y = Cx + v, \tag{4.19}$$

where w and v are Gaussian noise terms centered at zero with covariance matrices,

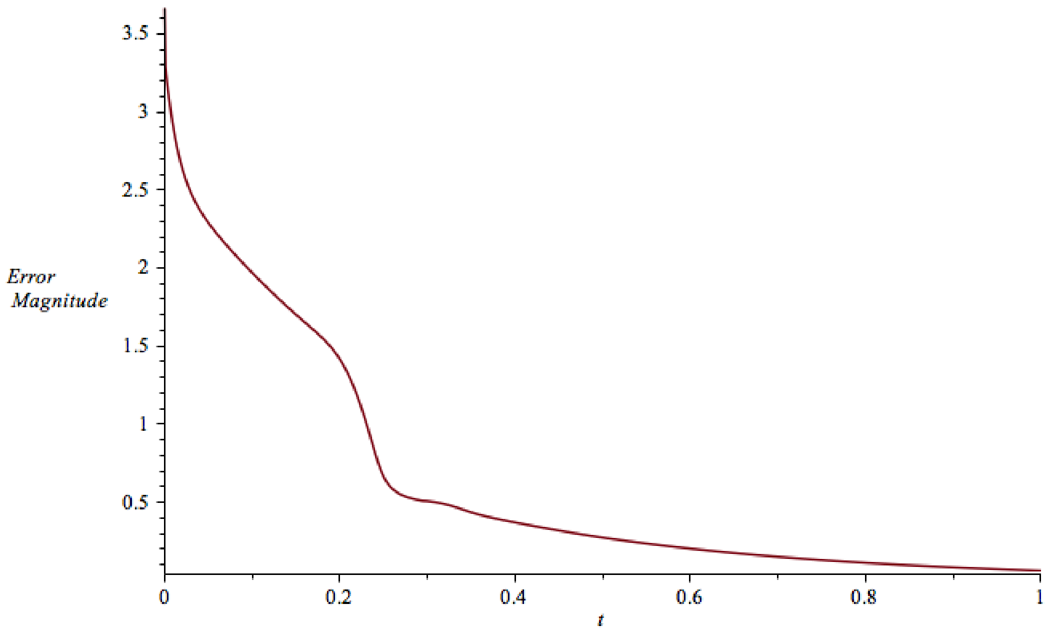

Figure 4.6: A plot of the error magnitude, $||e||$, of the Chen observer system versus time.

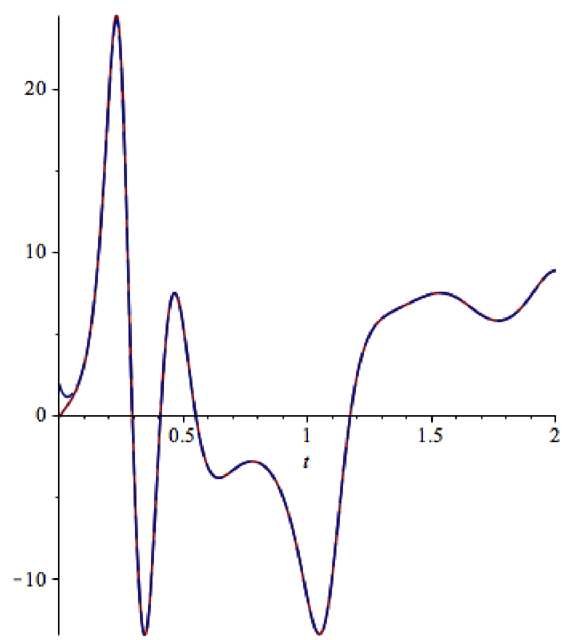

Figure 4.7: A plot of the first Chen attractor state (solid) and its observer (dashed) versus time.

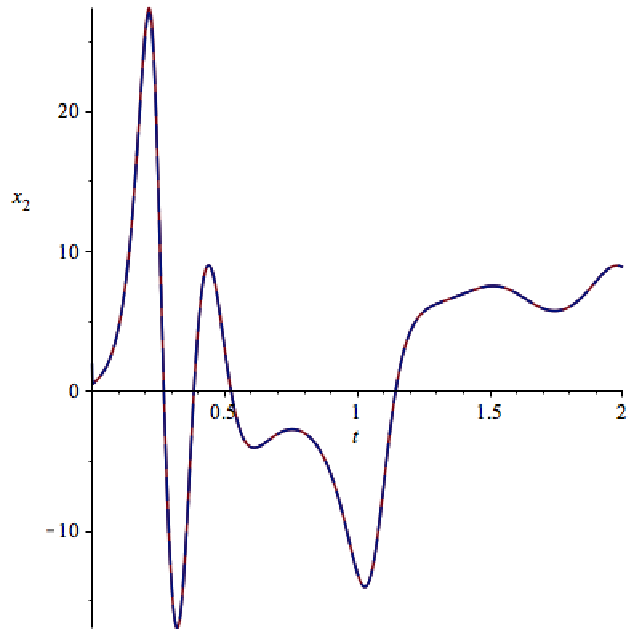

Figure 4.8: A plot of the second Chen attractor state (solid) and its observer (dashed) versus time.

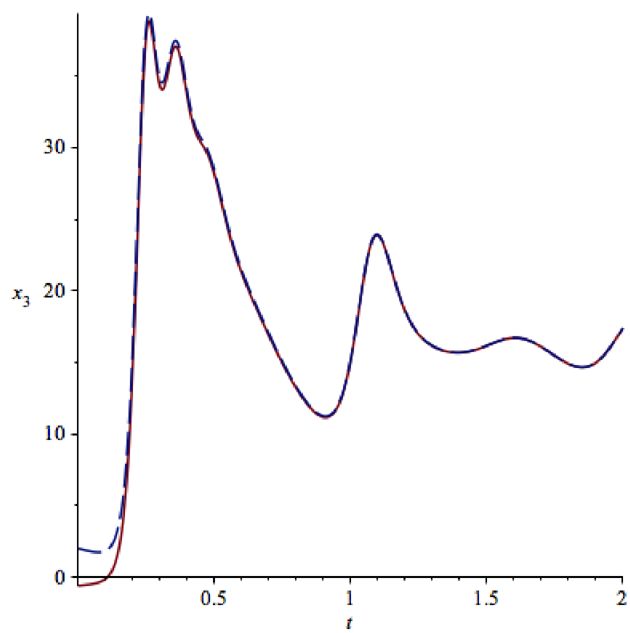

Figure 4.9: A plot of the third Chen attractor state (solid) and its observer (dashed) versus time.

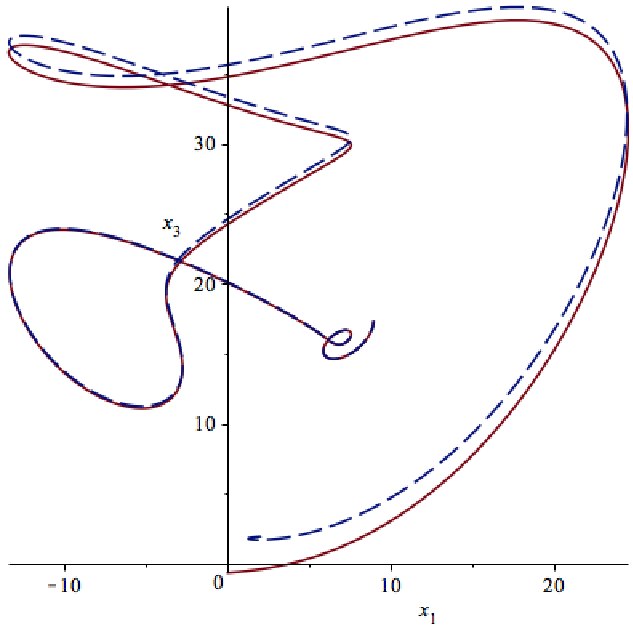

Figure 4.10: A plot of the continuous time extended Kalman filter (dashed) being used to observer the Chen attractor (solid). The two states shown, x_1 and x_3, are the unmeasured states.

Q and R, respectively. The continuous-time extended Kalman filter is given by:

$$\dot{\hat{x}} = f(\hat{x}) + K(y - h(\hat{x})), \tag{4.20}$$

$$K = PC^T R^{-1}, \tag{4.21}$$

$$\dot{P} = (A + \alpha I)P + P(A^T + \alpha I) - PC^T R^{-1} CP + Q, \tag{4.22}$$

with $A = \frac{\partial f}{\partial x}|_{(\hat{x},0)}$ and $F = \frac{\partial f}{\partial w}|_{(\hat{x},0)}$. For the Chen example, we let $Q = 6 \times 10^4 I$, $R = I$ and $\alpha = 0$. The results are shown in Figure 4.10 where the first and third states are shown along the corresponding state estimates. As can be seen in Figure 4.11, the metric-based observer proposed has an error magnitude 60% less than the extended Kalman filter after the initial transient behavior.

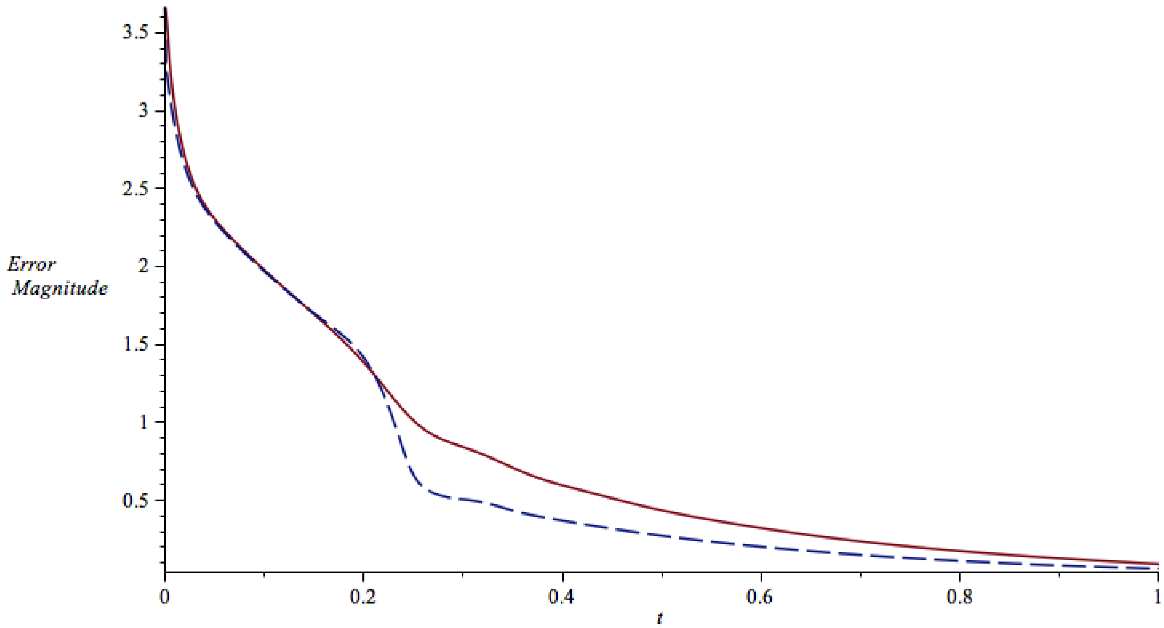

Figure 4.11: A plot of the magnitude of the errors of the extended Kalman filter (solid) and time-invariant metric based (dashed) observers.

4.4 Summary

A time-invariant metric that can be used to design an observer for nonlinear systems approximated by a dissipative potential was proposed. An observer-based on the Hessian of the potential on an observer dependent metric was formulated. Under the assumption that the Hessian matrix resulting for the potential realization is Lipschitz continuous, a Lyapunov analysis confirms that the proposed observer produces asymptotically stable error dynamics. Two simulation examples were presented to demonstrate the effectiveness of the method.

Chapter 5

Summary and Conclusions

This thesis proposed two nonlinear observer design approaches. The first approach applies a time-varying metric dependent on the Hessian of the potential function and the output. In Chapter 4, an observer is proposed based on a time-invariant metric dependent on the observer state and the measured output.

5.1 Summary

In Chapter 2, the homotopy operator was presented and is used to decompose a one-form into its exact (dissipative) and anti-exact (non-dissipative) parts. In Chapter 3 the observer problem was posed as the dynamics of the error between the system and its observer. This error dynamics were made convergent using an arbitrary metric and the operators of exterior calculus. The gradient field of the error dynamics was approximated by the dissipative potential obtained using the homotopy operator. A metric equation dependent on the Hessian of the potential and the measured output was proposed, whose inverse is used to design the observer function. Convergence of the error system was then proven using the metric distance of the error as a Lyapunov function. Two examples demonstrated the effectiveness of the nonlinear the state estimation technique.

The design of a time-invariant metric observer was considered in Chapter 4. The metric considered was chosen to be dependent on the output and the observer states. The design of an observer was proposed that makes the Lyapunov derivative negative definite and which bounds the Lipschitz constant of the Hessian of the potential. A specific metric was chosen and the associated observer equations were derived. Two examples of this observer equation were used, and a comparison was made to the extended Kalman filter.

This work contributes to the nonlinear observer design problem by developing a general physically based method that does not require prior knowledge of a metric function. Techniques of exterior calculus are incorporated to find an alternative to solving partial differential equations that can be impractical. By making use of the homotopy operator a large class of dynamical systems can be expressed as potential driven, and the potential field can be exploited for observer design.

5.2 Future Work

The work leaves many avenues of inquiry unexplored. 'Natural' metric spaces such as the Sasaki or Cheeger-Gromoll metrics have properties such as incompressible geodesic flow, hamiltonian geodesics, and more [2][1] that may be exploited to design a more general class of observers. By manipulating properties of the geodesics similar to the work in [37] improved design methods may be obtainable. Furthermore, the problem of finding a metric that makes the Hessian of the potential negative-definite remains. Finally, in this thesis, the exterior derivative was considered to be metric independent but could be considered as metric dependent with additional implications.

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