Adaptive Output Feedback Stabilization of Nonlinear Systems

by

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For Andy, Kaitlyn and Allison...I Love You

Abstract

Output feedback control design techniques are required in practice due to the limited number of sensors/measurements available for feedback. This thesis focuses on output feedback controller design techniques for nonlinear systems subject to different system restrictions.

The problem of controlling the heart dynamics in a real time manner is formulated as an adaptive learning output-tracking problem. For a class of nonlinear dynamic systems with unknown nonlinearities and non-affine control input , a Lyapunov-based technique is used to develop a control law. An adaptive learning algorithm is exploited that guarantees the stability of the closed-loop system and convergence of the output tracking error to an adjustable neighborhood of the origin. In addition, good approximation of the unknown nonlinearities is also achieved by incorporating a persistent exciting signal in the parameter update law. The effectiveness of the proposed method is demonstrated by an application to a cardiac conduction system modelled by two coupled driven oscillators.

An output feedback design technique is developed to achieve semi-global practical stabilization for a class of non-minimum phase nonlinear systems, subject to parameter uncertainties. This work provides a constructive controller design method for an auxiliary system, whose existence is crucial, but is only assumed in (Isidori, 2000). The control design technique is used to regulate the benchmark van de Vusse reactor. Simulation results demonstrate satisfactory controller performance.

The output feedback control design for a class of non-minimum phase nonlinear systems with unknown nonlinearities is studied. The proposed approach is able to combine the two previous design methods and provide a stabilizing output feedback control law. The performance of the proposed method is demonstrated by simulation results.

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Chapter 1

Introduction

1.1 Overview of Problem

In practice, feedback control design implementations are subject to many physical restrictions. One of the common limitations is the lack of measurement availability. For example, a valve position potentiometer may not be available due to hardware interface limitation or component weight consideration; a temperature sensor may not be available for feedback control due to location constraints, *etc.* For systems with imperfect state measurements, output feedback control design techniques are then required to solve the stabilization or tracking problem, where estimates of the unmeasured states are used in the feedback control design.

When applying output feedback control designs, the so-called minimum-phase property of the dynamic system is highly desirable. Broadly speaking, this property qualifies a class of systems with stable zero-dynamics. If a given system is minimum-phase, an output feedback control design can be constructed in principle without any concern for the stability of the zero dynamics outside of the input-output path. However, non-minimum phase (or inverse response) systems are very common in chemical engineering. Examples are chemical processes with recirculation; liquid level of a distillation column reacting to any increase of steam pressure of the reboiler; chemical reactors (with exothermic reaction) whose temperatures are controlled by the flow of cold inlet streams, etc. The design of an output feedback controller for non-minimum phase nonlinear systems remains a challenging problem.

Another limitation that real-life control systems often imposes on control design is the lack of knowledge of a given system. For example, heart dynamics are so complicated in nature that it is very difficult to extract models of the heart dynamics to describe all the different types of arrhythmias. For systems with unknown nonlinearities, dynamic estimation of the missing system information are required to cancel the effect of the unknown terms as much as possible, so that the overall stability of the closed loop system can be maintained.

1.2 Overview of Thesis

This thesis is structured as follows, Chapter 2 reviews the existing results on output feedback and adaptive control design. Chapter 3 presents a design solution to systems with unknown nonlinearities, with a heart dynamic control application. Chapter 4 proposes an output feedback control design technique for a type of nonlinear non-minimum phase system, and the performance of the design is demonstrated by the van de Vusse reactor. Chapter 5 proposes an output feedback control design method for systems with both non-minimum phase behavior and unknown nonlinearities; Chapter 6 provides the conclusion and recommendations for future research work.

Chapter 2

Literature Review

2.1 Introduction

In this chapter, research work on output feedback control of nonlinear non-minimum phase systems are reviewed. In addition, a historical development review in adaptive control and neural network control are also presented.

2.2 Output Feedback Control Of Non-minimum Phase Systems

For linear systems with unstable zero dynamics, the systems are usually factored into minimum-phase and nonminimum-phase part, with the minimum-phase part inverted for controller design.

2.2.1 Output Feedback Stabilization of Nonlinear Non-minimum Phase Systems

For nonlinear systems, even in the SISO case, such decomposition is still an extremely difficult problem. For a special class of second order SISO nonlinear system, a decomposition was achieved in (Kravaris and Daoutidis, 1990) with an Integral Square Error (ISE)-optimal control law. Factorization for higher order nonlinear systems was proposed in (Ball and van der Schaft, 1996).

However, these approaches were only applicable to a limited class of nonlinear systems. More general and systematic control designs are reviewed in the following section, which cover the research area of output feedback stabilization and output feedback tracking of nonlinear nonminimum-phase systems.

A non-minimum phase compensation structure was developed in (Wright and Kravaris, 1992), based on a synthetic output, which was minimum phase and statically equivalent to the original output. The synthetic output was chosen in an ISE-optimization formulation (Wright and Kravaris, 1992), which led to analytical results only for a limited class of nonlinear systems. Reduced order output feedback controller realizations, arising from this nonlinear nonminimum phase compensation structure, were derived in (Kravaris *et al.*, 1994), and was interpreted in the context of the model-state feedback structure. In (Kravaris *et al.*, 1997), a particular synthetic output with prescribed zeros was then selected in a systematic way. This approach was extended to multivariable nonlinear systems in (Niemiec and Kravaris, 2003).

In (Kanter *et al.*, 2001) and (Kanter *et al.*, 2002), the authors introduced a p-inverse (long-prediction-horizon) control law for nonlinear SISO (MIMO, respectively) non-minimum phase systems. It was shown that the proposed input-output linearizing controllers can place the eigenvalues of the closed-loop jacobian at the eigenvalues of the open-loop jacobian, both evaluated at the nominal equilibrium.

The drawback of the aforementioned approaches is that they are restricted to open-loop stable nonlinear systems. A continuous-time moving horizon state feedback control design was developed in (Panjapornpon *et al.*, 2003), able to handle open-loop unstable nonminimum phase nonlinear systems.

In (Isidori, 2000), a semi-global practical stabilization design tool was proposed for a general class of uncertain non-minimum phase nonlinear systems. This method takes advantage of the zero dynamics terms appearing in $\dot{x}_r$, the rth derivative of the output y, where r is the relative order of the input output. A stabilization scheme is used to make this term available, in a certain sense, for feedback purposes. The author referred to this approach as a method for *unlocking the zero dynamics*. Semi-global stabilization was achieved by a high-gain parameter in the feedback law that corresponds to the magnitude of the initial condition.

This approach was based on the assumption of global stabilizability of an *auxiliary* system. It was shown that this assumption was not restrictive. In fact, the authors pointed out that the assumption in question essentially identified a class of nonlinear systems that were semi-globally stabilizable by means of a feedback driven by functions that are "uniformly completely observable" (UCO) in the sense of (Teel and Praly, 1995).

In (Ilchmann and Isidori, 2002), an adaptive dynamic output feedback stabilization tool was proposed, for a class of polynomially bounded nonlinear systems. The high-gain result in (Isidori, 2000) was extended by considering the adaptation of the gain parameter as a time varying scalar function, which depended on the magnitude of the output and a quantity of the dynamic feedback compensator. The adaptation law used was a time-varying gain driven by an integration in the linear case, and in the nonlinear case an integration coupled with a dead-zone. This adaptive modification overcame the semi-global stabilization limitation, and yielded a global result. In addition, it improved the results of practical stabilzation in (Isidori, 2000) by guaranteeing that the output $y(t)$ tends to any prespecified strip $[0, \lambda], \lambda > 0$.

The regulation problem of a non-minimum phase outputs was solved in (Sira-Ramirez and Fliess, 1998), by using the differential parameterization provided by the system's flatness property. Roughly speaking, a system is flat if one can find a set of outputs (equal in number to the number of inputs) such that all states and inputs can be determined from these outputs without integration. It was shown in (Sira-Ramirez and Fliess, 1998) that, the flat outputs were able to transform the original non-minimum phase system into a minimum phase one by eliminating the unstable zero dynamics. The drawback of this approach is that, no constructive approach has been established to search for the flat outputs for a given nonlinear system.

It was shown in (Marino and Tomei, 2005) that, for a nonlinear non-minimum phase system, if the system is minimum phase with respect to a linear combination of its state variables, and if the system is in output feedback form, then this system can be globally exponentially stabilized by a dynamic output feedback controller.

All of the aforementioned approaches were focused on finding an alternative outputs, with respect to which, the original non-minimum system became minimum phase.

For nonlinear systems in "lower triangular form" or "strict normal form", whose zero dynamics only depend on the zero dynamics states and the output of the nonlinear system, design approaches that utilize the input-to-state stable (ISS) condition were presented in the literature. Assuming the zero-dynamics is ISS, it was shown in (Andrieu and Praly, 2005) that a global stabilizing dynamic output feedback control could be constructed using standard backstepping technique, together with a reduced order observer. In (Karagiannis *et al.*, 2005), the ISS condition was imposed on both zero-dynamics sub-system and the observer dynamics sub-system. The stabilization of the interconnection of the two sub-systems was achieved by satisfying the small-gain condition, with the controller constructed by backstepping technique.

A more unified approach was presented in (Andrieu and Praly, 2008), which recovered and extended the results in (Andrieu and Praly, 2005) and (Karagiannis *et al.*, 2005). The unified approach assumed the knowledge of an observer and, depending on its property, various control laws were designed. Each of the control design approach required different stabilizability assumption on the inverse dynamics, and utilized backstepping design techniques.

2.2.2 Output Feedback Tracking of Nonlinear Non-minimum Phase Systems

There are two major approaches available in the literature for output feedback tracking of nonlinear non-minimum phase systems. The first approach was proposed in (Devasia *et al.*, 1996) (for time-invariant case) and (Devasia and Paden, 1994) (for time-variant case), and further modified in (Tomlin and Sastry, 1998) for singulary perturbed systems. The stabilizing control consists of a feedforward component that generates the zero dynamics trajectory, and a feedback component that stabilizes the whole system.

The computation of the feedforward component consisted in finding the unique bounded solution of the driven zero dynamics to obtain the entire desired state trajectory. To this end, the linearized driven zero dynamics was assumed to be kinematically hyperbolic. The linearization residual of the driven zero dynamics was Lipschitz continuous and locally approximately linear in the zero dynamics. The stablization part of the control was applied not just to the output dynamics as in the case of input-output linearization by state feedback, but rather to the full state.

There are two drawbacks to the above output tracking approach. First, the calculation of the feedforward control was non-causal, in the sense that, if the tracking output (and its derivatives) were not specified ahead of time, an approximation is needed. Second, the domain of validity of the approach was not only restricted to tracking trajectories that were both small in magnitude and also slowly varying, but also limited by the requirement that the linearization of the original system was stabilized.

The second approach was the nonlinear output regulator approach in (Isidori, 1995). For the output of a nonlinear system to track a reference trajectory generated by an exosystem, two assumptions were made for the linearized composite system. First, the linearized closed-loop system was stabilizable. Second, the equilibrium of the exosystem is Poisson stable. Under these assumptions, it was shown that there existed an invariant manifold, which was the center manifold of the composite system, and it also gave necessary and sufficient conditions under which the closed-loop system could be driven to the center manifold contained in the output zeroing manifold. The overall control scheme contained a feedforward component and a feedback component. The feedforward part, once suitably chosen, ensured the existence of the invariant manifold. While the feedback component was designed to make this invariant manifold contained in the output zeroing manifold attractive. Local exponential stability around the equilibrium point was achieved with this approach.

This approach does not assume minimum phase on the part of the open loop system. However, if only the output information is available, it required detectablility of the linearized system, which implied that the system had to be locally minimum-phase.

Inspired by the design method in (Isidori, 2000), the design of a robust regulator problem was presented in (Isidori *et al.*, 2002) by solving a stabilization problem of a suitably-defined auxiliary subsystem by output feedback. The solution of the auxiliary output feedback stabilization problem yielded a dynamic uniform completely observable regulator and solved the output regulator problem with a semi-global domain of attraction.

2.3 Adaptive Control

2.3.1 Introduction

Primarily motivated by the design of autopilots for high-performance aircraft, active research on adaptive control started in the early 1950s. The advances in stability theory and the progress of control theory in the 1960s improved the understanding of adaptive control and brought forth a strong interest in adaptive control in 1970s. By the early 1980s, a wide class of adaptive control schemes with well established stability properties had been developed by several leading researchers (Astrom and Wittenmark, 1989) and (Sastry and Bodson, 1989). In the mid 1980s, a number of redesigns and modifications were proposed and analyzed to improve the robustness of the adaptive controllers in the presence of unmodelled dynamics and /or bounded disturbances. Examples are projection algorithm (Goodwin and Mayne, 1987), (Sastry and Bodson, 1989), dead zone modification (Kreisselmeier and Anderson, 1986), (Peterson and Narendra, 1982), ϵ-modification (Narendra and Annaswamy, 1987), and σ-modifications (Ioannou and Kokotovic, 1983).

An adaptive controller is formed by combining an on-line parameter estimator, which provides estimates of unknown parameters at each instant, with a control law that is motivated from the known parameter case. Depending on the different approaches to combine the parameter estimator and the control law, there are basically two types of adaptive control schemes, indirect adaptive control and direct adaptive control. In the first approach, the plant parameters are estimated on-line and used to calculate the controller parameters. While in the second approach, the plant model is parameterized in terms of the controller parameters that are estimated directly, without intermediate calculations involving plant parameter estimates.

For linear time invariant (LTI) systems, two types of adaptive control approaches are very popular. The first approach is Model Reference Adaptive Control (MRAC), and it was derived from the model reference control (MRC) problem. The second approach is Adaptive Pole Placement Control (APPC), which was derived from the pole placement control approach.

The focus of adaptive control research since the 1990s was on performance properties and extending existing results to nonlinear plants with unknown parameters.

A brief review of the major results in this area are provided in the following section.

2.3.2 Adaptive Control of Nonlinear Systems

Feedback linearization techniques based adaptive control schemes were developed for feedback linearizable nonlinear systems (Sastry and Isidori, 1989). For exponentially minimum-phase systems with globally Lipshitz nonlinearities, (Sastry and Isidori, 1989) suggested the use of parameter adaptive control to help robustify the exact cancellation of nonlinear terms when the uncertainty in the nonlinear terms was parametric. An extension of feedback linearization techniques were presented in (Pomet and Praly, 1992), assuming the system of interest was uniform feedback stabilizable, instead of feedback linearizable. To achieve global stability, some additional assumptions were made; either limiting the growth at infinity of the uncertainties, or not allowing the Lyapunov functions to depend on the unknown parameters.

The above restrictions were finally broken in the work of (Kanellakopoulos *et al.*, 1991), (Krstic *et al.*, 1992), through the introduction of a novel recursive design procedure called adaptive backstepping. This technique required that the nonlinear system be transformable into the so-called parametric-strict-feedback form. The global adaptive regulation and tracking problems for systems in this form was solved, without overparametrization. In addition, the transient performance was guaranteed and explicitly analyzed (Krstic *et al.*, 1995).

Research work on global adaptive output-feedback control of nonlinear systems had been done in (Marino and Tomei, 1993*a*), (Marino and Tomei, 1993*b*), (Khalil, 1996), (Kanellakopoulos *et al.*, 1992), (Jankovic, 1996).

For single-input single-output nonlinear systems with nonlinearities depending on the output only, adaptive, adaptive global stablizable controller was developed, for linearly parameterized (Marino and Tomei, 1993*a*) and nonlinearly parameterized (Marino and Tomei, 1993*b*) nonlinear systems. No growth restriction on the nonlinearities was imposed. For the linearly parameterized case, a solution of the more general tracking problem was also provided. The adaptive control scheme involved a two stage design. First, a robust output feedback compensator is developed. The control did not make use of observers and did not attempt to achieve exact nonlinearity cacellation. It only required a functional bound on the norm of the vector, which may contain unknown parameters. Second, a self-tuning output feedback algorithm was developed which automatically tuned those control parameters and solved the more general problem of set-point regulation.

In the work of (Khalil, 1996), a semi-global adaptive output feedback controller was developed for single-input single-output nonlinear systems, that can be represented globally by an input-output model. The class of systems included the nonlinear systems treated in (Kanellakopoulos *et al.*, 1992), as a special case. The unknown nonlinearities were not required to satisfy any global growth condition, and the model depended linearly on unknown parameters. The design process was as follows. First the output and its derivatives were assumed to be available for feedback and the adaptive controller was designed as a state feedback controller in appropriate coordinates. Then, the controller was saturated outside a domain of interest, and a high gain observer was used to estimate the derivatives of the output. It was proven that if the speed of the high gain observer was sufficiently high, the adaptive output feedback controller recovers the performance achieved under state feedback.

2.3.3 Neural Network Techniques

NN techniques have undergone great developments and have been successfully applied in many fields, such as pattern recognition, signal processing, modelling and system control. The approximating ability of NN has been proven in (Sadegh, 1993). Multilayer NN identification and control techniques have been developed and demonstrated through simulation (Narendra, 1991) and (Narendra and Parthasarathy, 1990), following the popularization of the "backpropagation" algorithm. However, analytical results obtained in (Chen and Khalil, 1992) shows that off-line training is needed, because stability can be guaranteed only when the initial network weights are chosen sufficiently close to the ideal weight. To avoid the above difficulties in constructing stable neural systems, Lyapunov stability theory has been applied in developing control structure and deriving network weight updating laws (Polycarpou, 1996), (Seshagiri and Khalil, 2000) and (Rovithakis and Christodoulou, 1994). Multi-layer NN control has been successfully applied to robotic control (Lewis *et al.*, 1999), (Lewis *et al.*, 1996). In addition, (Ge *et al.*, 1998) provides a systematic treatment of common problems in robotic control by introducing the G-Lee operator.

2.3.4 Parameter Convergence of Nonlinear Adaptive/NN Control

For the various types of adaptive/NN controller design techniques reviewed in the previous sections, the only variables that are not guaranteed to converge are the parameter estimates. If parameter convergence could be guaranteed, it enhances the overall stablility and robustness properties of the closed-loop adaptive system. In other word, the resulting closed-loop system would have a unique solution with the very desirable property of global asymptotic stability.

Parameter convergence properties for linear systems are well established. For any standard adaptive linear scheme, parameter convergence is equivalent to the persistent excitation property of the regressor vector (Anderson, 1977), which, in turn, can be translated into the sufficient richness conditions on the external reference signals (Boyd and Sastry, 1983). For nonlinear systems, the relationships are not available, and it became an active research area in the last decade.

In (Lin and Kanellakopoulos, 1998), it was shown that parameter convergence for a type of linearly parameterized nonlinear systems, which can be transformed via a global diffeomorphism into the output-feedback form, was guaranteed if and only if an appropriate defined signal vector, which did not depend on closed-loop signals, was persistently exciting. In the process of determining whether or not the parameters were converging, the authors showed that the presence of noninearities usually reduced the sufficient richness requirements on the reference signals and hence enhanced parameter convergence. The advantage of this approach is that it can be applied to any adaptive scheme developed for output-feedback noninear systems, as long as its stablility and tracking porperties can be established without relying on parameter convergence.

For a type of nonlinear parameter affine system, a novel parameter estimation routine that allows exact reconstruction of the unknown parameters in finite-time was presented in (Adetola and Guay, 2008), provided a given excitation condition is satisfied. In (Adetola and Guay, 2009) an adaptive compensator that (almost) recover the performance of the finite-time identifier is developed, which guarantees exponential convergence of the parameter estimation error at a rate dictated by the closed-loop system's excitation.

For nonparametric nonlinear systems, the approximator convergence was studied in (Farrell, 1997). The author pointed out that, even though parameter convergence was not necessary for system stability property, tracking error was found to be a direct result of the function approximation error. In addition, for systems with unknown model structure, the convergence of the approximating functions became an issue of primary importance. The analysis in this work showed that as long as a reduced dimension subvector of the regressor vector is persistent exciting, then a specialized form of exponential convergence will be achieved. The author also explicitly defined the region over which the approximator converges when locally supported basis functions are used.

2.4 Summary

This chapter reviews different techniques on output feedback control of nonlinear non-minimum phase systems. It also provides an overview of nonlinear adaptive control, neural network control, and parameter convergence problems. The review of existing techniques serves as the bases of the control design approaches proposed in the rest of the work, in three areas: adaptive neural network output tracking, output feedback stabilization of uncertain non-minimum phase nonlinear systems, and output feedback stabilization of non-minimum phase nonlinear systems with unknown nonlinearities.

Chapter 3

Adaptive Output Tracking of Systems with Unknown Nonlinearities

3.1 Introduction

Heart dynamics are very complicated by nature, and it is widely known that accurate analytical models are difficult to develop for cardiac dynamics and different types of arrhythmias. In addition, real-time control technique is needed because of the fatal nature of cardiac arrhythmias. As a result, real time, model-independent control techniques are needed to control heart dynamics in the presence of cardiac arrhythmias.

The dynamics of cardiac arrhythmias have been closely related to a variety of bifurcations and chaos phenomena. In recent years (Ott *et al.*, 1990), (Garfinkel *et al.*, 1992), (Christini and Collins, 1996) and (Hall *et al.*, 1997), the theory of chaos control has made contributions to a mechanistic understanding of cardiac arrhythmias. Without detailed knowledge of the heart dynamics model structure, chaos control technique is able to regulate the abnormal heart rhythm by stabilizing the system around a desirable yet unstable fixed point. However, this approach is limited because to find a suitable controller, a ”learning stage” must be considered which comprises of pre-control time-series recording and system dynamics estimation. In addition, the “learning stage” based on previous time-series can lead to mis-estimation of system dynamics, because of the evolving nature of biological systems. The available adaptive approach (Christini and Collins, 1997) concentrated on linear chaos control, which is limited because the linear approach can only delay or change the bifurcation location. Nonlinear control is needed in order to modify the stability property. According to the above observation, chaos control can not serve as a real-time control technique to regulate cardiac arrhythmias.

To overcome this difficulty, we propose to apply adaptive control techniques to control cardiac arrhythmias. Rather than treat the unknown heart dynamics as a “black box”, we try to estimate the unknown dynamics using a Neural Network (NN) approach.

The NN approaches reviewed in Chapter 2 are restricted to control-affine nonlinear systems. The problem of adaptively controlling systems with unknown, non-affine input nonlinearities is still open in the literature. Heart dynamics generally falls into the category of non-affine systems, because of the inadequate knowledge of heart dynamics, and limited understanding of how actuators enter the dynamics.

Another limitation of current approaches is that approximation performance of the unknown nonlinearity and parameter estimation convergence are not discussed. Usually a high gain control is employed to dominate the approximation error to ensure good tracking performance. However, in reality, it is often desirable to identify the unknown part of the dynamics.

In this work (Diao and Guay, 2008), we focus on an adaptive learning technique that is applicable to unknown nonlinear dynamic plants with a class of non-affine input uncertainties, that are unknown, but continuous, and satisfy a sector constraint. An external signal, which is designed to be persistent exciting, is imbedded in the parameter update law to ensure good approximation and parameter convergence.

This chapter is organized as follows. Section 3.2 presents the proposed adaptive controller. Application of the adaptive output feedback tracking technique to cardiac dynamics control is provided in Section 3.3. In Section 3.4, brief conclusions of this chapter are given.

3.2 Adaptive Control Design

We consider single-input/single-output (SISO) controllable nonlinear systems of the form

$$\begin{cases} \dot{\xi} &= \phi(\xi, z) \\ \dot{z}_1 &= z_2 \\ &\vdots \\ \dot{z}_\rho &= f(z, \xi, u) \\ y &= z_1 \end{cases} \tag{3.1}$$

where $\xi = [\xi_1, \xi_2, \cdots, \xi_{n-\rho}]^T \in \mathbb{R}^{n-\rho}$ are the state variables of the zero dynamics; $z = [z_1, z_2, \cdots, z_\rho]^T \in \mathbb{R}^\rho$ are the state variables of the main dynamics; $u \in \mathbb{R}$ and $y \in \mathbb{R}$ are the system input and output, respectively. The mapping $f(z, \xi, u)$ is assumed to be an unknown continuous function of z and u, and is assumed to be globally Lipschitz in ξ. Given a reference trajectory y_r, the control objective is to design an output feedback controller for system (3.1), which achieves good tracking performance subject to the unknown nonlinearities in the system.

Let $Y_r = [y_r, \dot{y}_r, \cdots, y_r^{(\rho-1)}]^T$. In this work, it is assumed that $\|Y_r\| \leq c$, $|y_r^{(\rho)}| \leq \bar{\nu}_1$, with known constant $c > 0$, $\bar{\nu}_1 > 0$. Denote ξ_r the "steady state" response of the tracking zero dynamics, governed by the differential equation $\dot{\xi}_r = \phi(\xi_r, Y_r)$, and $\bar{\xi} = \xi - \xi_r$. It is also assumed that $\|\xi_r\| \leq c_\xi$, where $c_\xi > 0$ is a known constant. Denote $e = z - Y_r = [e_1, e_2, \cdots, e_\rho]^T$, $e_c = [e_1, e_2, \cdots, e_{\rho-1}]^T$, and $e_s = \Lambda^T e_c + e_\rho$, where $\Lambda = [\lambda_1, \lambda_2, \cdots, \lambda_{\rho-1}]$ is to be chosen.

In this work, radial basis function (RBF) presented in (Kosmatopoulos *et al.*, 1995) were used to approximate a continuous function $\Psi(x) : \mathbb{R}^p \to \mathbb{R}$

$$\Psi(x) = W^{*T} S(x) + \mu_l(t) \tag{3.2}$$

with approximation error $\mu_l(t)$, and basis function vector

$$\begin{aligned} S(x) &= [s_1(x), s_2(x), \cdots, s_l(x)]^T \\ s_i(x) &= \exp\left[\frac{-(x-\varphi_i)^T(x-\varphi_i)}{\sigma_i^2}\right] \\ & \quad i = 1, 2, \ldots, l \end{aligned}$$

where φ_i is the center of the receptive field, and σ_i is the width of the Gaussian function. The ideal weight W^* in equation (3.2) is defined as

$$W^* := arg \min_{W \in \Omega_w} \left\{ \sup_{x \in \Omega} \left| W^T S(x) - \Psi(x) \right| \right\}$$

where $\Omega_w = \{W \mid \|W\| \leq w_m\}$ with positive constant w_m to be chosen at the design stage, and Ω is a compact set. Universal approximation results stated in (Funahashi, 1989) and (Kosmatopoulos *et al.*, 1995) indicate that, if l is chosen sufficiently large, then $W^{*T} S(x)$ can approximate any continuous function to any desired accuracy on a compact set, given that the centers are chosen close enough.

A number of assumptions are made for system (3.1):

Assumption 3.1 *The sign of $\partial f(z, \xi_r, u)/\partial u$ is known, and there exist a positive* *constant* $\bar{b}_0$ *and a nonzero continuous function* $\bar{b}_1(z)$ *such that*

$$0 < \bar{b}_0 \leq \left|\frac{\partial f(z, \xi_r, u)}{\partial u}\right| \leq \bar{b}_1(z)$$

Assumption 3.2 *The approximation error satisfies* $|\mu_l(x(t))| \leq \bar{\mu}_l$ *with unknown constant* $\bar{\mu}_l > 0$ *over a compact set.*

Assumption 3.2 is implied by the univeral approximation results stated in (Funahashi, 1989) and (Kosmatopoulos *et al.*, 1995).

The design task is achieved in two steps: firstly, a state feedback adaptive tracking controller is designed; secondly, a high gain observer is used together with the state feedback controller to yield an output feedback adaptive tracking control law. We propose the following adaptive controller design.

Given the reference trajectory Y_r and ξ_r, system (3.1) can be rewritten in terms of the tracking error e and $\bar{\xi} = \xi - \xi_r$, as follows,

$$\dot{\bar{\xi}} = \phi(\bar{\xi} + \xi_r, e + Y_r) - \phi(\xi_r, Y_r) \tag{3.3}$$

$$\begin{cases} \dot{e}_c &= A_c e_c + B_c e_\rho \\ \dot{e}_\rho &= f(e + Y_r, \bar{\xi} + \xi_r, u) - y_r^{(\rho)}, \end{cases} \tag{3.4}$$

where

$$A_c = \begin{bmatrix} 0 & 1 & \dots & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & 1 \\ 0 & 0 & \dots & \dots & 0 \end{bmatrix}, \quad B_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

The proof of stability (convergence of the tracking error) is achieved by considering the tracking dynamics (3.3) and the error dynamics (3.4) as an interconnected system. With proper constraint imposed on the interconnected term, the overall stability is guaranteed by the use of small gain theorem (Teel and Praly, 1995), together with a proper choice of control and parameter update law. First, we must make the following assumption concerning system (3.3).

Assumption 3.3 *The tracking dynamics of the system given by (3.3) is input-to-state stable (ISS). That is, there exists a positive definite function* $U(\bar{\xi})$*, such that the following is satisfied,*

$$\begin{aligned} c_1\|\bar{\xi}\|^2 \leq U(\bar{\xi}) \leq c_2\|\bar{\xi}\|^2 \\ \dot{U}(\bar{\xi}) \leq -c_3\|\bar{\xi}\|^2 + c_4\|\bar{\xi}\|\|e\| \end{aligned}$$

where c_1*,* c_2*,* c_3*, and* c_4 *are positive constants, and* $\dot{U}(\bar{\xi})$ *is the time derivative of* $U(\bar{\xi})$ *along the solution of (3.1).*

Considering e as a "disturbance" to the tracking error zero dynamics (3.3), Assumption 3.3 ensures that $\bar{\xi}$ dynamics in (3.3) is *Input-to-State Stable (ISS)* with respect to e.

By $\partial f(z, \xi_r, u)/\partial u \neq 0, \forall z \in \mathbb{R}^\rho, \forall u \in \mathbb{R}$ (Assumption 3.1), it follows from the implicit function theorem (Khalil, 2002), that there exists a continuous function $\alpha(z, \xi_r)$ such that $f(z, \xi_r, \alpha(z, \xi_r)) = 0$. The function $f(z, \xi, u)$ may be re-expressed as

$$\begin{aligned} f(z, \xi, u) &= f(z, \xi_r, u) + f(z, \xi, u) - f(z, \xi_r, u) \\ &= f(z, \xi_r, \alpha(z, \xi_r)) + \int_0^1 \frac{\partial f(z, \xi_r, u_\lambda)}{\partial u_\lambda} d\lambda (u - \alpha(z, \xi_r)) \\ &\quad + [f(z, \xi, u) - f(z, \xi_r, u)] \\ &= b(z, \xi_r, u)(u - \alpha(z, \xi_r)) + [f(z, \xi, u) - f(z, \xi_r, u)] \end{aligned}$$

where $u_\lambda = \lambda u + (1-\lambda)\alpha(z,\xi_r)$, $b(z,\xi_r,u) = \int_0^1 \frac{\partial f(z,\xi_r,u_\lambda)}{\partial u_\lambda} d\lambda$, and the following is assumed,

$$|f(z,\xi,u) - f(z,\xi_r,u)| \leq L_1 \|\bar{\xi}\|$$

where L_1 is a Lipschitz constant.

Approximate the unknown function $\alpha(z)$ as

$$\alpha(z,\xi_r) = W^{*T} S(z,\xi_r) + \mu_1(x(t)). \tag{3.5}$$

Let $\hat{W}$ denote the estimate of W^*. The parameter estimation error is given by $\tilde{W} = \hat{W} - W^*$.

Note that the boundedness of $|\frac{\partial f(z,\xi_r,u)}{\partial u}|$ (Assumption 3.1) implies that $b(z,\xi_r,u)$ is bounded as follows

$$b_0 \leq b(z,\xi_r,u) \leq b_1(z,\xi_r), \tag{3.6}$$

where b_0 is a positive constant and $b_1(z,\xi_r)$ is a nonzero continuous function.

Using equation (3.5), the error dynamics (3.4) can be written as

$$\begin{aligned} \dot{e}_c &= A_c e_c + B_c e_\rho \qquad (3.7) \\ \dot{e}_\rho &= b(z,\xi_r,u)(u - \alpha(z)) - y_r^{(\rho)} + [f(z,\xi,u) - f(z,\xi_r,u)] \end{aligned}$$

The state feedback design and parameter update law are given by

$$u = k(z,\xi_r)e_s + \hat{W}^T S(z,\xi_r) \tag{3.8}$$

$$\dot{\hat{W}} = \gamma_w \text{Proj}\ (\hat{W}, c(t)e_s) \tag{3.9}$$

where $k(z,\xi_r)$ is the controller gain function, γ_w is a positive constant, Proj $(\cdot)$ is a projection algorithm.

The dynamics of $c(t)$ are chosen as follows,

$$\begin{aligned} \dot{c}^T(t) &= -(b_1 k_t - \frac{1}{2} B_c^T P B_c) c^T(t) - b_1 S^T(z, \xi_r) \\ &= -K(t) c^T(t) + B(t), \end{aligned} \tag{3.10}$$

where $K(t) = b_1 k_t - \frac{1}{2} B_c^T P B_c > 0$, and $B(t) = -b_1 S^T(z, \xi_r)$. Note that $K(t)$ can always be made negative by a suitable choice of the gain constant k_t. The matrix P is a positive definite solution of the Riccati-like equation

$$P A_c + A_c^T P + 2\gamma_1 P B_c B_c^T P + \frac{1}{2} k_2 A_c^T A_c + Q = 0$$

for some positive definite symmetric matrix Q and positive $\gamma_1 > 0$ and $k_2 > 0$ chosen as part of the design.

In addition to the above assumptions, we must ensure that a certain persistence of excitation condition is met to ensure that the unknown nonlinearity is estimated correctly.

Assumption 3.4 *There exists positive constant $T > 0$ and $k_N > 0$ such that*

$$\int_t^{t+T} c(\tau) c^T(\tau) d\tau \geq k_N I_N,$$

where $c^T(t)$ is the solution of equation (3.10), I_N is a N-dimensional identity matrix.

The following lemma will be used in the sequel.

Lemma 3.1 *Consider the differential equation*

$$\dot{z}(t) = -\phi(t) \phi^T(t) z(t), \tag{3.11}$$

where $z(t) \in \mathbb{R}_+ \to \mathbb{R}^n$, and $\phi(t) \in \mathbb{R}_+ \to \mathbb{R}^n$ are both column vectors. Assume that there exists a $T > 0$ and a $k > 0$ such that

$$\int_t^{t+T} \phi(\tau)\phi^T(\tau)d\tau \geq kI,$$

then the origin of (3.11) is a globally exponentially stable equilibrium of the system.

The proof of this lemma can be found in (Anderson *et al.*, 1986).

Theorem 3.1 gives the main results for the state feedback controller design.

Theorem 3.1 *Consider the nonlinear system (3.1) in closed-loop with the controller and parameter update law provided in equation (3.8) and (3.9). Assume that the signal $c(t)$ is such that*

$$\int_t^{t+T} c(\tau)c^T(\tau)d\tau \geq k_N I_N$$

for positive constants $T > 0$ and $k_N > 0$, where $c^T(t)$ is a solution of equation (3.10).

Given Assumptions 3.1 to 3.3, all the signals of the closed-loop system are bounded. The parameter estimation errors $\tilde{W}$ converges exponentially to a small neighborhood of the origin.

The mean square tracking error satisfies

$$\frac{1}{t}\int_0^t e_1^2 dt \leq \frac{\alpha_0}{t}V_s(0) + \frac{1}{k}\left(\bar{\mu}_l^2 + \bar{\nu}_1^2 + w_m^2\right) \tag{3.12}$$

where α_0 is a positive constant and $V_s(0)$ is a positive constant depending on system initial conditions.

Furthermore, the tracking error is such that

$$\|e(t)\| \leq \alpha_1 e^{-\beta_1(t-t_0)} + \frac{\alpha_2}{\sqrt{k}}(\bar{\mu}_l^2 + \bar{\nu}_1^2 + w_m^2).$$

where α_1, α_2, β_1 and k are some positive constants.

Proof of Theorem 3.1: Consider the following Lyapunov function candidate for the e_c subsystem (3.7)

$$V_1 = \frac{1}{2} e_c^T P e_c$$

where P is a symmetric, positive definite function.

The derivative of the Lyapunov function V_1 is given by

$$\dot{V}_1 = \frac{1}{2} e_c^T (P A_c + A_c^T P) e_c + e_c^T P B_c e_\rho.$$

Choosing $\Lambda = \frac{1}{2} B_c^T P$, we have

$$e_s = e_\rho + \frac{1}{2} B_c^T P e_c.$$

A candidate Lyapunov function for the error dynamics is

$$V_2 = V_1 + \frac{1}{2} \eta_s^2,$$

where $\eta_s = e_s + c^T(t)\tilde{W}$, and $c^T(t)$ is the state of the filter (3.10) time varying function to ensure persistency of excitation condition. Note that this filter is ISS with respect to the signal $B(t) = -b_1 S(z, \xi_r)$ for any choice of gain k_t large enough.

The time derivative of V_2 is given by

$$\begin{aligned} \dot{V}_2 &= \dot{V}_1 + \eta_s(\dot{e}_s + \dot{c}^T(t)\tilde{W} + c^T(t)\dot{\tilde{W}}) \\ &= \frac{1}{2} e_c^T (P A_c + A_c^T P) e_c + e_c^T P B_c (e_s - \frac{1}{2} B_c^T P e_c) \\ &\quad + \eta_s \left(\dot{e}_\rho + \frac{1}{2} B_c^T P \dot{e}_c + \dot{c}^T(t)\tilde{W} + c^T(t)\dot{\tilde{W}} \right) \end{aligned}$$

Substitution of $e_\rho = e_s - \frac{1}{2}B_c^T P e_c$ yields

$$\begin{aligned}
\dot{V}_2 &= \frac{1}{2}e_c^T(PA_c + A_c^T P)e_c - \frac{1}{2}e_c^T P B_c B_c^T P e_c \\
&\quad + e_c^T P B_c e_s + \eta_s\big(-y_r^{(\rho)} + f(z,\xi,u) - f(z,\xi_r,u)\big) \\
&\quad + \eta_s\Big(b(z,u)(u - \alpha(z,\xi_r)) + \dot{c}^T(t)\tilde{W} + c^T(t)\dot{\tilde{W}}\Big) \\
&\quad + \eta_s\Big(\frac{1}{2}B_c^T P A_c e_c + \frac{1}{2}B_c^T P B_c e_s - \frac{1}{4}B_c^T P B_c B_c^T P e_c\Big).
\end{aligned}$$

Given that $e_s = \eta_s - c(t)^T\tilde{W}$, and

$$\frac{1}{2}e_s^2 \le \eta_s^2 + \tilde{W}^T c(t) c^T(t)\tilde{W}, \tag{3.13}$$

by completing the squares, we have

$$\begin{aligned}
\dot{V}_2 &\le \frac{1}{2}e_c^T(PA_c + A_c^T P)e_c - \frac{1}{2}e_c^T P B_c B_c^T P e_c \\
&\quad + \frac{k_1}{2}e_c^T P B_c B_c^T P e_c + \frac{1}{k_1}\eta_s^2 + \frac{1}{k_1}\tilde{W}^T c(t)c^T(t)\tilde{W} \\
&\quad + \eta_s\big(-y_r^{(\rho)} + f(z,\xi,u) - f(z,\xi_r,u)\big) \\
&\quad + \eta_s\Big(b(z,u)(u - \alpha(z,\xi_r)) + \dot{c}^T(t)\tilde{W} + c^T(t)\dot{\tilde{W}}\Big) \\
&\quad + \frac{1}{2}\eta_s B_c^T P A_c e_c + \frac{1}{2}\eta_s^2 B_c^T P B_c \\
&\quad - \frac{1}{4}\eta_s B_c^T P B_c B_c^T P e_c - \frac{1}{2}\eta_s B_c^T P B_c c^T(t)\tilde{W},
\end{aligned}$$

or

$$\begin{aligned}
\dot{V}_2 &\le \frac{1}{2}e_c^T(PA_c + A_c^T P)e_c + \gamma_1 e_c^T P B_c B_c^T P e_c \\
&\quad + \eta_s\big(-y_r^{(\rho)} + f(z,\xi,u) - f(z,\xi_r,u)\big) \\
&\quad + \eta_s\Big(b(z,u)(u - \alpha(z,\xi_r)) + \dot{c}^T(t)\tilde{W} + c^T(t)\dot{\tilde{W}}\Big) \\
&\quad + \frac{1}{k_1}\tilde{W}^T c(t)c^T(t)\tilde{W} - \eta_s c^T(t)\tilde{W}(\frac{1}{2}B_c^T P B_c)
\end{aligned}$$

$$+\gamma_2\eta_s^2 + \frac{k_2}{4}e_c^T A_c^T A_c e_c \quad (3.14)$$

where k_1, k_2 and k_3 are positive constants and

$$\gamma_1 = \{-\frac{1}{2} + \frac{k_1}{2} + \frac{k_3}{8}\}$$

and

$$\gamma_2 \;=\; \{\frac{1}{k_1} + \frac{1}{4k_2}B_c^T PPB_c + \frac{1}{2}B_C^T PB_c + \frac{1}{8k_3}B_c^T PB_c B_c^T PB_c\}.$$

The constants k_1 and k_3 are chosen such that $\gamma_1 < 0$. If the matrix P is chosen as a positive definite solution of the Riccati-like equation

$$PA_c + A_c^T P + 2\gamma_1 PB_c B_c^T P + \frac{1}{2}k_2 A_c^T A_c + Q = 0$$

for some positive definite symmetric matrix Q, then inequality (3.14) becomes

$$\begin{aligned}\dot{V}_2 \;\leq\; & -\frac{1}{2}e_c^T Q e_c + \gamma_2\eta_s^2 + \eta_s\Big(b(z,u)(u-\alpha(z,\xi_r)) + \dot{c}^T(t)\tilde{W} + c^T(t)\dot{\tilde{W}}\Big) \\ & +\frac{1}{k_1}\tilde{W}^T c(t)c^T(t)\tilde{W} - \eta_s c^T(t)\tilde{W}(\frac{1}{2}B_c^T PB_c) \\ & +\eta_s\big(-y_r^{(\rho)} + f(z,\xi,u) - f(z,\xi_r,u)\big) \quad (3.15)\end{aligned}$$

Substituting equation (3.5), inequality (3.15) becomes

$$\begin{aligned}\dot{V}_2 \;\leq\; & -\frac{1}{2}e_c^T Q e_c + \gamma_2\eta_s^2 + \eta_s\Big(b(z,u)(u-\hat{W}^T S(z,\xi_r) - \mu_1(t)) + \dot{c}^T(t)\tilde{W} + c^T(t)\dot{\tilde{W}}\Big) \\ & +\frac{1}{k_1}\tilde{W}^T c(t)c^T(t)\tilde{W} - \eta_s c^T(t)\tilde{W}(\frac{1}{2}B_c^T PB_c) \\ & -\eta_s y_r^{(\rho)} + \eta_s L_1\|\bar{\xi}\| + \eta_s b(z,u)\tilde{W}^T S(z,\xi_r) \\ \leq\; & -\frac{1}{2}e_c^T Q e_c + \gamma_2\eta_s^2 + \frac{k_\mu}{2}b(z,u)^2\eta_s^2 + \frac{1}{2k_\mu}\mu_1(x(t))^2\end{aligned}$$

$$+\frac{k_d}{2}\eta_s^2 + \frac{1}{2k_d}(y_r^{(\rho)})^2 + \eta_s L_1\|\bar{\xi}\| + \eta_s\Big(b(z,u)(u - \hat{W}^T S(z,\xi_r) + \dot{c}^T(t)\tilde{W} + c^T(t)\dot{\tilde{W}}\Big)$$
$$+\frac{1}{k_1}\tilde{W}^T c(t)c^T(t)\tilde{W} - \eta_s c^T(t)\tilde{W}(\frac{1}{2}B_c^T P B_c) + \eta_s b(z,u)\tilde{W}^T S(z,\xi_r) \qquad (3.16)$$

where k_μ and k_d are positive constants.

Consider the control structure shown in equation (3.8), we choose the following controller

$$\begin{aligned} u &= \hat{W}^T S(z,\xi_r) + e_s\Big(-k_4 - \frac{1}{b_0}\gamma_2 - \frac{k_\mu}{2}b_1(z) - \frac{k_d}{2}\frac{1}{b_0} - \frac{k_w}{2}\frac{b_1(z)^2}{b_0}S(z,\xi_r)^T S(z,\xi_r)\Big) \\ &= \hat{W}^T S(z,\xi_r) + \eta_s\Big(-k_4 - \frac{1}{b_0}\gamma_2 - \frac{k_\mu}{2}b_1(z) - \frac{k_d}{2}\frac{1}{b_0} - \frac{k_w}{2}\frac{b_1(z)^2}{b_0}S(z,\xi_r)^T S(z,\xi_r)\Big) \\ &\quad - c^T(t)\tilde{W}k_t \end{aligned} \qquad (3.17)$$

where $k_4 > 0$ is a constant, b_0 and $b_1(z)$ are the lower and upper bound in the inequality (3.6), and k_t is given by

$$k_t = -k_4 - \frac{1}{b_0}\gamma_2 - \frac{k_\mu}{2}b_1(z) - \frac{k_d}{2}\frac{1}{b_0} - \frac{k_w}{2}\frac{b_1(z)^2}{b_0}S(z,\xi_r)^T S(z,\xi_r)$$

The above control action constitutes filtered tracking error and the approximated nonlinearity.

The weight $\hat{W}$ satisfies $\|\hat{W}\| \le w_m$ where the upper bound, w_m, is guaranteed in the design of an adaptive law (3.9).

Substitution of the controller equation (3.17) in the inequality (3.16) gives

$$\begin{aligned} \dot{V}_2 \le & -\frac{1}{2}e_c^T Q e_c - k_4 b\eta_s^2 + \eta_s\Big(bc^T(t)\tilde{W}k_t + \dot{c}^T(t)\tilde{W} + c^T(t)\dot{\tilde{W}}\Big) \\ & +\eta_s b(z,u)\tilde{W}^T S(z,\xi_r) + \eta_s L_1\|\bar{\xi}\| + \frac{1}{k_1}\tilde{W}^T c(t)c^T(t)\tilde{W} - \eta_s c^T(t)\tilde{W}(\frac{1}{2}B_c^T P B_c) \\ & +\frac{1}{2k_\mu}\mu_1(x(t))^2 + \frac{1}{2k_d}(y_r^{(\rho)})^2 - \frac{k_w}{2}\frac{b_1(z)^2}{b_0}bS(z,\xi_r)^T S(z,\xi_r)\eta_s^2, \end{aligned} \qquad (3.18)$$

because of the inequality (3.6).

The adaptive law (3.9) is chosen such that $\|\hat{W}\| \le w_m$ and

$$-\tilde{W}^T c(t) e_s + \frac{1}{\gamma_w} \tilde{W}^T \dot{\hat{W}} \le 0. \tag{3.19}$$

It takes the following form,

$$\dot{\hat{W}} = \begin{cases} \gamma_w c(t) e_s, & \text{if } \|\hat{W}\| < w_m \text{ or} \\ & \|\hat{W}\| = w_m \text{ and} \\ & \hat{W}^T c(t) e_s \le 0 \\ \gamma_w c(t) e_s - \gamma_w \frac{\hat{W}\hat{W}^T c(t) e_s}{\|\hat{W}\|^2}, & \text{if } \|\hat{W}\| = w_m \text{ and} \\ & \hat{W}^T c(t) e_s > 0 \end{cases} \tag{3.20}$$

Substitution of the controller equation (3.17), the parameter update law inequality (3.19), and the $c(t)$ dynamics (3.10), in the inequality (3.18) gives

$$\begin{aligned} \dot{V}_2 \le\; & -\frac{1}{2} e_c^T Q e_c - k_4 b \eta_s^2 + \gamma_w c^T(t) c(t) \eta_s^2 + \frac{k_w}{2}(b - b_1)^2 \eta_s^2 S^T S - \frac{k_w}{2} \frac{b_1^2}{b_0} b S^T S \eta_s^2 \\ & -\Big((b_1 - b) k_t + \gamma_w c^T(t) c(t)\Big) c^T(t) \tilde{W} \eta_s + \eta_s L_1 \|\bar{\xi}\| + \frac{1}{2k_\mu} \mu_1(t)^2 + \frac{1}{2k_d} (y_r^{(\rho)})^2 \\ & + \frac{1}{2k_w} \tilde{W}^T \tilde{W} + \frac{1}{k_1} \tilde{W}^T c(t) c^T(t) \tilde{W} \end{aligned}$$

where $b_1 = b_1(z)$ $b = b(z, u)$, and $S = S(z, \xi_r)$ due to space limits.

Since $(\frac{b(z,u)}{b_1(z)} - 1)^2 \le 1$, it follows that

$$\begin{aligned} \dot{V}_2 \le\; & -\frac{1}{2} e_c^T Q e_c - k_4 b \eta_s^2 + \gamma_w c^T(t) c(t) \eta_s^2 - \Big((b_1 - b) k_t + \gamma_w c^T(t) c(t)\Big) c^T(t) \tilde{W} \eta_s \\ & + \eta_s L_1 \|\bar{\xi}\| + \frac{1}{2k_\mu} \mu_1(t)^2 + \frac{1}{2k_d} (y_r^{(\rho)})^2 + \frac{1}{2k_w} \tilde{W}^T \tilde{W} + \frac{1}{k_1} \tilde{W}^T c(t) c^T(t) \tilde{W} \end{aligned}$$

Let $V = V_2 + \frac{1}{2}\tilde{W}^T\tilde{W}$, and let $k_c = (b_1 - b)k_t$, we have

$$\begin{aligned}\dot{V} &\leq -\frac{1}{2}e_c^T Q e_c - k_4 b \eta_s^2 - \Big(\gamma_w + \frac{k_w}{2}(k_c + \gamma_w c^T(t)c(t) - \gamma_w)^2\Big)c^T(t)c(t)\eta_s^2 \\ &\quad +(\frac{1}{k_1} - \gamma_w)\tilde{W}^T c(t)c^T(t)\tilde{W} + \frac{1}{2k_w}\tilde{W}^T\tilde{W} + \eta_s L_1\|\bar{\xi}\| + \frac{1}{2k_\mu}\mu_1(x(t))^2 \\ &\quad +\frac{1}{2k_d}(y_r^{(\rho)})^2 \qquad (3.21)\end{aligned}$$

Let $\tilde{k}_4 = k_4 b_0 - \Big(\gamma_w + \frac{k_w}{2}(k_c + \gamma_w c^T(t)c(t) - \gamma_w)^2\Big)c^T(t)c(t) > 0$, inequality (3.21) becomes

$$\begin{aligned}\dot{V} &\leq -\frac{1}{2}e_c^T Q e_c - \tilde{k}_4\eta_s^2 + (\frac{1}{k_1} - \gamma_w)\tilde{W}^T c(t)c^T(t)\tilde{W} + \frac{1}{k_w}\tilde{W}^T\tilde{W} \\ &\quad +\eta_s L_1\|\bar{\xi}\| + \frac{1}{2k_\mu}\mu_1(x(t))^2 + \frac{1}{2k_d}(y_r^{(\rho)})^2\end{aligned}$$

Noting that, by assumption, $|\mu_l(x(t))| \leq \bar{\mu}_l$, $\|y_r^{(\rho)}\| \leq \bar{\nu}_1$, and using the fact that

$$\tilde{W}^T\tilde{W} = (\hat{W} - W^*)^T(\hat{W} - W^*) \leq \|\hat{W}\|^2 + 2|\hat{W}^T W^*| + \|W^*\|^2 \leq 4w_m^2 \qquad (3.22)$$

we obtain

$$\begin{aligned}\dot{V} &\leq -\frac{1}{2}e_c^T Q e_c - \tilde{k}_4\eta_s^2 + \eta_s(L_1\|\bar{\xi}\|) + \frac{1}{2k_d}\bar{\nu}_1^2 + \frac{1}{2k_\mu}\bar{\mu}_l^2 + \frac{1}{k_w}4w_m^2 \\ &\quad +(\frac{1}{k_1} - \gamma_w)\tilde{W}^T c(t)c^T(t)\tilde{W}.\end{aligned}$$

Consider the following composite Lyapunov function for the closed-loop system (3.8), (3.9), (3.3) and (3.4)

$$V_c = V + \alpha U,$$

where α is a positive design parameter. The derivative of the Lyapunov function V_c

is given by

$$\begin{aligned}\dot{V}_c &= \dot{V} + \alpha\dot{U} \\ &\leq -\frac{1}{2}e_c^T Q e_c - \tilde{k}_4\eta_s^2 + \frac{k_5}{2}\eta_s^2 + \frac{1}{2k_5}L_1^2\|\bar{\xi}\|^2 - c_3\alpha\|\bar{\xi}\|^2 \\ &\quad +\Gamma\frac{k_6}{2}e_c^T e_c + \Gamma\frac{k_6}{2}e_s^2 + \frac{1}{2k_6}c_4^2\alpha^2\|\bar{\xi}\|^2 + \frac{1}{2k_d}\bar{\nu}_1^2 + \frac{1}{2k_\mu}\bar{\mu}_l^2 + \frac{1}{k_w}4w_m^2 \\ &\quad +(\frac{1}{k_1} - \gamma_w)\tilde{W}^T c(t)c^T(t)\tilde{W}\end{aligned}$$

where $\Gamma = (1 + B_c^T P P B_c)$, since, by definition, the trajectory error vector is such that

$$e^T e = e_c^T e_c + e_s^2 \leq (1 + B_c^T P P B_c)e_c^T e_c + e_s^2$$

or

$$e^T e \leq (1 + B_c^T P P B_c)(e_c^T e_c + e_s^2).$$

Define

$$\begin{aligned}\lambda = \min\{&\frac{\lambda_{\min}\{Q\}}{2\lambda_{\max}\{P\}} - \Gamma\frac{k_6}{2}, \tilde{k}_4 - \frac{k_5}{2} - \Gamma\frac{k_6}{2}, \\ &c_3 - \frac{L_1^2}{2k_5\alpha c_2} - \frac{c_4^2\alpha}{2k_6c_2}\},\end{aligned}$$

$\Gamma k_6 + \frac{1}{k_1} + \lambda < \gamma_w$, and the gains of the two interconnected system (3.3) and (3.4) are such that $c_3 - \frac{L_1^2}{2k_5\alpha c_2} - \frac{c_4^2\alpha}{2k_6c_2} > 0$, we have

$$\begin{aligned}\dot{V}_c &\leq -\lambda e_c P e_c - \lambda\eta_s^2 - \lambda\alpha U - \lambda\tilde{W}^T c(t)c^T(t)\tilde{W} \\ &\quad +\frac{1}{2k_d}\bar{\nu}_1^2 + \frac{1}{2k_\mu}\bar{\mu}_l^2 + \frac{1}{k_w}4w_m^2 \\ &\leq -\frac{\lambda}{2}e_c P e_c - \frac{\lambda}{2}e_s^2 - \frac{\lambda}{2}\alpha U + \frac{1}{2k_d}\bar{\nu}_1^2 + \frac{1}{2k_\mu}\bar{\mu}_l^2 + \frac{1}{k_w}4w_m^2\end{aligned} \tag{3.23}$$

or

$$\dot{V}_c \;\leq\; -\lambda V_c + \frac{1}{2k_d}\bar{\nu}_1^2 + \frac{1}{2k_\mu}\bar{\mu}_l^2 + \frac{1}{k_w}4w_m^2 \tag{3.24}$$

It follows that the error vector, e, variable η_s, and the parameter estimation errors, $\tilde{W}$, are bounded. Since the tracking trajectory y_r and its first ρ derivatives are bounded, it follows that the states z and ξ are both bounded.

Integration of inequality (3.24) yields an explicit bound for $\|\eta_s\|$ given by

$$\|\eta_s\| \leq \alpha_\eta e^{-\lambda_\eta(t-t_0)} + \frac{1}{\sqrt{k_\eta}}\sqrt{\bar{\mu}_l^2 + \bar{\nu}_1^2 + w_m^2} \tag{3.25}$$

where $\alpha_\eta = \sqrt{2V_c(0)}$, $\lambda_\eta = \frac{\lambda}{2}$, and $k_\eta = \min\,[2k_d, 2k_\mu, k_w/4,]$.

Next, we derive a persistency of excitation condition that guarantees the convergence of the parameter estimates to the ideal weights, W^*.

A solution to equation (3.10) is given by

$$c^T(t) = e^{-K(t)(t-t_0)}c^T(t_0) + \int_{t_0}^{t} e^{-K(t)(t-\tau)}B(\tau)d\tau.$$

It is obvious that the differential equation

$$\dot{c}^T(t) = -K(t)c^T(t) \tag{3.26}$$

is globally exponentially stable.

The element $B(t)$ is a bounded function of time. Note that

$$\|B(t)\|^2 = b_1(z)^2 S^T S.$$

For the particular choice of basis functions proposed in this paper, we have $\|S\| \leq \sqrt{N}$, where N is the number of weights used in the approximation. The boundedness of $b_1(z)$ is obvious, since z is bounded. Therefore, it follows that the norm of $B(t)$ is bounded by some positive number B_M, that is,

$$\|B(t)\| \leq B_M.$$

Using the exponential stability of system (3.26) and the bound on $B(t)$, an explicit bound for the solution of equation (3.10) can be obtained as follows:

$$\|c^T(t)\| \leq Ce^{-\lambda_c(t-t_0)} + C\frac{B_M}{\lambda_c}$$

where $C = \|c^T(t_0)\| > 0$ and $\lambda_c > 0$ is a positive constant.

Next, it is shown that the parameter estimation error $\tilde{W}$ converges to a neighborhood of the origin. In the following, it is shown that, under Assumption 3.4, the unperturbed (i.e., $\eta_s \equiv 0$) dynamics of the parameter estimation errors are exponentially stable.

Lemma 3.1 establishes that the origin of the differential equation

$$\dot{\tilde{W}} = -\gamma_w c(t)c^T(t)\tilde{W}$$

is an exponentially stable equilibrium. In fact, it follows from the proof of Lemma 3.1 that the Lyapunov function $V_w = \frac{1}{2\gamma_w}\tilde{W}^T\tilde{W}$ is such that

$$\dot{V}_w = -\tilde{W}^T c(t)c^T(t)\tilde{W} \leq -c_w\|\tilde{W}\|^2$$

for $c_w > 0$ a positive constant.

It follows from the property of the projection algorithm that the rate of change of V_w along the trajectories of (3.20) is given by

$$\dot{V}_w \leq -\tilde{W}^T c(t)c^T(t)\tilde{W} - \tilde{W}^T c(t)\eta_s.$$

Completing the squares, we get

$$\dot{V}_w \le -\frac{1}{2}\tilde{W}^T c(t)c^T(t)\tilde{W} + \frac{1}{2}\eta_s^2$$

Substitution of equation (3.25) yields

$$\dot{V}_w \le -c_w\gamma_w V_w + \alpha_\eta^2 e^{-2\lambda_\eta(t-t_0)} + \frac{1}{k_\eta}(\bar{\mu}_l^2 + \bar{\nu}_1^2 + w_m^2) \tag{3.27}$$

Integrating, we get

$$\begin{aligned} V_w \quad \le \quad & \max\left\{V_w(t_0), \left|\frac{\alpha_\eta^2}{c_w\gamma_w - 2\lambda_\eta}\right|, \frac{1}{c_w\gamma_w k_\eta}(\bar{\mu}_l^2 + \bar{\nu}_1^2 + w_m^2)\right\} \\ & \exp[-\min\{c_w\gamma_w, 2\lambda_\eta\}(t-t_0)] \\ & +\frac{1}{c_w\gamma_w k_\eta}(\bar{\mu}_l^2 + \bar{\nu}_1^2 + w_m^2). \end{aligned}$$

Consequently, the parameter estimation error is guaranteed to decay exponentially as

$$\|\tilde{W}\| \le \alpha_w e^{-\lambda_w}(t-t_0) + \sqrt{\frac{2}{k_\eta c_w}}(\bar{\mu}_l^2 + \bar{\nu}_1^2 + w_m^2)^{1/2}$$

Taking the limit as $t \to \infty$ confirms that the estimation error converges to a small adjustable neighborhood of the origin given by

$$\lim_{t\to\infty} \|\tilde{W}\| \le \sqrt{\frac{2}{k_\eta c_w}}(\bar{\mu}_l^2 + \bar{\nu}_1^2 + w_m^2)^{1/2}$$

Under the assumption that the persistency of excitation condition is fulfilled, we have demonstrated that the parameter estimation error and the redefined state variable, η_s, converge exponentially fast to an adjustable neighborhood of the origin. The size of the neighborhood can be changed by increasing the size of the controller gain and by reducing the size of the approximation error.

Inequality (3.12) can be deduced from inequality (3.23) as follows. Inequality (3.23) can be written as

$$\dot{V}_c \leq -\frac{1}{2}\lambda_{min}\{P\}e_c^T e_c - \frac{\lambda}{2}e_s^2 - \frac{\lambda}{2}\alpha U + \frac{1}{2k_d}\bar{\nu}_1^2 + \frac{1}{2k_\mu}\bar{\mu}_l^2 + \frac{1}{k_w}4w_m^2 \quad (3.28)$$

where $\lambda_{min}\{Q\}$ is the smallest eigenvalue of Q. Integration of inequality (3.28) gives

$$\begin{aligned} V_c(t) - V_c(0) \leq & -\int_0^t \left(\frac{1}{2}\lambda_{min}\{P\}e_c(\sigma)^T e_c(\sigma) + \frac{\lambda}{2}e_s(\sigma)^2 + \frac{\lambda}{2}\alpha U(\sigma)^2\right) d\sigma \\ & + t\left(\frac{1}{2k_d}\bar{\nu}_1^2 + \frac{1}{2k_\mu}\bar{\mu}_l^2 + \frac{1}{k_w}4w_m^2\right). \end{aligned}$$

This result implies that the following inequality holds

$$\frac{1}{t}\int_0^t e_1(\sigma)^2 d\sigma \leq \frac{2V_c(0)}{t\lambda_{min}\{P\}} + \frac{1}{k}\left(\bar{\mu}_l^2 + \bar{\nu}_1^2 + w_m^2\right)$$

where $k = \frac{1}{\lambda_{min}\{Q\}} min\{k_\mu, k_d, \frac{k_w}{8}\}$. Hence, the boundedness of the mean square error stated in inequality (3.12) is achieved with $\alpha_0 = \frac{2}{\lambda_{min}\{P\}}$ as required.

Integration of inequality (3.24) gives,

$$\begin{aligned} V_c(t) & \leq V_c(0)e^{-\lambda(t-t_0)} + \left(\frac{1}{2k_d}\bar{\nu}_1^2 + \frac{1}{2k_\mu}\bar{\mu}_l^2 + \frac{1}{k_w}4w_m^2 +\right)\int_{t_0}^t e^{-\lambda(t-\tau)}d\tau \\ & \leq V_c(0)e^{-\lambda(t-t_0)} + \left(\frac{1}{2k_d}\bar{\nu}_1^2 + \frac{1}{2k_\mu}\bar{\mu}_l^2 + \frac{1}{k_w}4w_m^2\right) \quad (3.29) \end{aligned}$$

Given (3.13), we have

$$\frac{1}{2p_c}e_s^2 \leq \eta_s^2 + \|\tilde{W}\|^2,$$

where $p_c = \sup_{t>0}\|c(t)\|^2$. With the positive constant

$$p_m = \min\{\lambda_{min}\{P\}, \frac{1}{2p_c}\},$$

inequality (3.29) simplifies to

$$\frac{1}{2}e_c^T e_c + \frac{1}{2}e_s^2 \;\leq\; \frac{V_c(0)}{p_m}e^{-\lambda(t-t_0)} + \frac{1}{p_m}\Big(\frac{1}{2k_d}\bar{\nu}_1^2 + \frac{1}{2k_\mu}\bar{\mu}_l^2 + \frac{1}{k_w}4w_m^2\Big). \qquad (3.30)$$

It follows from inequality (3.30) that

$$e^T e \;\leq\; \Gamma\frac{V_c(0)}{p_m}e^{-\lambda(t-t_0)} + \Gamma\frac{1}{p_m}\left(\frac{1}{2k_d}\bar{\nu}_1^2 + \frac{1}{2k_\mu}\bar{\mu}_l^2 + \frac{1}{k_w}4w_m^2\right)$$

where $\Gamma = 2(1 + B_c^T PPB_c)$. As a result, the adaptive learning tracking control guarantees that the tracking error, e, fulfills the following inequality

$$\|e\| \;\leq\; \alpha_1 e^{-\beta_1(t-t_0)} + \alpha_2\frac{1}{\sqrt{k}}\left(\bar{\mu}_l + \bar{\nu}_1 + w_m\right)$$

where

$$\begin{aligned}
\alpha_1 &= \left(\Gamma\frac{V(0)}{p_m}\right)^{1/2} \\
\beta_1 &= \lambda \\
\alpha_2 &= \left(\frac{\Gamma}{p_m}\right)^{1/2} \\
k &= \min\left[2k_d, 2k_\mu, k_w/4,\right].
\end{aligned}$$

This completes the proof. **Q.E.D.**

Since in practice, only a limited number of measurements can be obtained, one needs to build an observer to estimate the unmeasured states, and implement the state feedback controller with the estimated states. For the tracking error system

(3.4), a high-gain observer (Khalil, 2002) is used, which takes the following form,

$$\begin{cases} \dot{\hat{e}}_1 &= \hat{e}_2 + \frac{l_1}{\epsilon}(e_1 - \hat{e}_1) \\ \dot{\hat{e}}_2 &= \hat{e}_3 + \frac{l_2}{\epsilon^2}(e_1 - \hat{e}_1) \\ &\vdots \\ \dot{\hat{e}}_\rho &= \frac{l_\rho}{\epsilon^\rho}(e_1 - \hat{e}_1) \end{cases} \tag{3.31}$$

where $[l_1, l_2, \ldots, l_\rho]^T$ are the coefficients of a Hurwitz polynomial, ϵ is some small positive constant, and $\hat{e}_i$, $i = 1, \ldots, \rho$ are the estimated tracking errors.

Following (Atassi and Khalil, 1999), we define the scaled estimation errors

$$\eta_i = \frac{e_i - \hat{e}_i}{\epsilon^{\rho-i}} = \frac{\tilde{e}_i}{\epsilon^{\rho-i}} \quad , \quad 1 \leq i \leq \rho \tag{3.32}$$

Using eq.(3.3) and eq.(3.31), the dynamics of the scaled estimation errors are given by

$$\epsilon\dot{\eta} = A_0\eta + \epsilon B_0 \left(f(x,u) - y_r^\rho\right) \tag{3.33}$$

where the matrix $A_0 \in \mathbb{R}^{\rho\times\rho}$ and the vector $B_0 \in \mathbb{R}^\rho$ assume the following form,

$$A_0 = \begin{bmatrix} -\alpha_1 & 1 & \ldots & \ldots & 0 \\ -\alpha_2 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\alpha_{\rho-1} & 0 & \ldots & \ldots & 1 \\ -\alpha_\rho & 0 & \ldots & \ldots & 0 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

We define the Lyapunov function $V_\eta(\eta) = \frac{1}{2}\eta^T P_0 \eta$ where P_0 is the symmetric positive definite matrix solution of

$$P_0 A_0 + A_0^T P_0 = -I$$

Let $V_2 = \frac{1}{2}e_c^T P e_c + \frac{1}{2}e_s^2 + \frac{1}{2\gamma_w}\tilde{W}^T\tilde{W}$ and consider the compact sets, $\Omega = \{e \in \mathbb{R}^\rho, \tilde{W} \in \mathbb{R}^l | V_2 \leq c_1\}$ where $c_1 > \left(\frac{1}{2k_d}\bar{\nu}_1^2 + \frac{1}{2k_\mu}\bar{\mu}_l^2 + \left(\frac{2}{k_w} + \frac{2}{\gamma}\right) w_m^2\right) = c_2$. Define the positive constant b such that $0 < c_2 \leq b < c_1$ and the set $\Omega_b = \{V_2(e, \tilde{W}) \leq b\}$. For the estimation errors, we define the set $\Sigma = \{V_\eta(\eta) \leq \rho\epsilon^2\}$.

In order to apply the result of (Atassi and Khalil, 1999), we must verify that the state-feedback eq.(3.8) and the learning rate for parameter estimation defined in eq.(3.9) are globally bounded. Since they are not, we consider the application of a state-feedback over the compact set Ω. We first compute the constants

$$S_u > \max_{e,\ \tilde{W} \in \Omega} \left(-k(x)e_s + \hat{W}^T S(x)\right) \tag{3.34}$$

and

$$S_\delta > \max_{e,\ \tilde{W} \in \Omega} \left(b_1(x) S(x) e_s\right). \tag{3.35}$$

The maximization is performed over all $(e, \tilde{W}) \in \Omega$, $Y_r \in Y$. It is assumed that the reference trajectories $Y_r = \left[y_r, \dot{y}_r, \ldots, y_r^{(\rho-1)}\right]$ are bounded and evolve in a compact subset Y of $\mathbb{R}^\rho$. This ensures that the state x of the system are also bounded on $\Omega \times Y$. The state-feedback and the adaptive learning rate can then be bounded on Ω by implementing the functions

$$\Psi_s(e, Y_r, \tilde{W}) = S_u \text{ sat}\left(\frac{-k(x)e_s + \hat{W}^T S(x)}{S_u}\right) \tag{3.36}$$

$$\Upsilon_s(e, Y_r, \tilde{W}) = S_\delta \text{ sat}\left(\frac{b_1(x) S(x) e_s}{S_\delta}\right) \tag{3.37}$$

where

$$\text{sat}(w) = \begin{cases} -1 & \text{if} \quad w \leq -1 \\ w & \text{if} \quad -1 < w < 1 \\ 1 & \text{if} \quad w \geq 1 \end{cases} \tag{3.38}$$

The adaptive learning state-feedback is rewritten as

$$\dot{\hat{W}} = \begin{cases} -\gamma\Upsilon_s, & \text{if} \;\; \|\hat{W}\| < w_m \;\; \text{or} \\ & \qquad \|\hat{W}\| = w_m \text{ and } \; \hat{W}^T\Upsilon_s \geq 0 \\ -\gamma\Upsilon_s + \gamma\frac{\hat{W}\hat{W}^T\Upsilon_s}{\|\hat{W}\|^2}, & \text{if} \;\; \|\hat{W}\| = w_m \;\; \text{and} \;\; \hat{W}^T\Upsilon_s < 0 \end{cases} \tag{3.39}$$

$$u = \Psi_s(e, Y_r, \tilde{W}) \tag{3.40}$$

for $e, \tilde{W} \in \Omega$. Having bounded the control and the adaptive learning rate, we pose the output-feedback controller

$$\dot{\hat{e}}_i = \hat{e}_{i+1} + \frac{\alpha_i}{\epsilon^i}\tilde{e}_1, \quad 1 \leq i \leq \rho - 1$$

$$\dot{\hat{e}}_\rho = \frac{\alpha_\rho}{\epsilon^\rho}\tilde{e}_1 \tag{3.41}$$

$$\dot{\hat{W}} = \begin{cases} -\gamma\hat{\Upsilon}_s, & \text{if} \;\; \|\hat{W}\| < w_m \;\; \text{or} \\ & \qquad \|\hat{W}\| = w_m \text{ and } \; \hat{W}^T\hat{\Upsilon}_s \geq 0 \\ -\gamma\hat{\Upsilon}_s + \gamma\frac{\hat{W}\hat{W}^T\hat{\Upsilon}_s}{\|\hat{W}\|^2}, & \text{if} \;\; \|\hat{W}\| = w_m \;\; \text{and} \;\; \hat{W}^T\hat{\Upsilon}_s < 0 \end{cases} \tag{3.42}$$

$$u = \Psi_s(\hat{e}, Y_r, \hat{W}) \tag{3.43}$$

where $\hat{\Upsilon}_s = \Upsilon_s(\hat{e}, Y_r)$.

Theorem 3.2 provides the result for output feedback controller design.

Theorem 3.2 *Consider the nonlinear system eq.(3.1). If assumptions 3.1 to 3.3 are* *met then the dynamic output feedback controller eqs.(3.41)-(3.43) guarantees that for any initial conditions of the closed-loop system starting in* $S \times O$ *there exists* $0 < \epsilon < \epsilon_3^*$ *such that every trajectory of the closed-loop system enters a small neighborhood of the origin in finite-time and it converges exponentially to a small adjustable neighborhood of the origin.*

Proof of Theorem 3.2: We consider the Lyapunov function $V_2 = \frac{1}{2}e_c^T P e_c + \frac{1}{2}e_s^2 + \frac{1}{2\gamma_w}\tilde{W}^T\tilde{W}$. We first note that $\hat{e} = e - D(\epsilon)\eta$ where $D(\epsilon) = \text{diag}\,[\epsilon^{\rho-1}, \ldots, 1]$ and write the closed-loop system as follows

$$\begin{aligned}
\dot{e} &= A_0 e + B_0\left(f(x, \Psi_s(e - D(\epsilon)\eta, \hat{W}, Y_\rho) - y_r^{(\rho)}\right)\\
\dot{\tilde{W}} &= \begin{cases} -\gamma\hat{\Upsilon}_s, & \text{if } \ \|\hat{W}\| < w_m \ \text{ or} \\ & \qquad \|\hat{W}\| = w_m \text{ and } \ \hat{W}^T\hat{\Upsilon}_s \leq 0 \\ -\gamma\hat{\Upsilon}_s + \gamma\frac{\hat{W}\hat{W}^T\hat{\Upsilon}_s}{\|\hat{W}\|^2}, & \text{if } \ \|\hat{W}\| = w_m \ \text{ and } \ \hat{W}^T\hat{\Upsilon}_s > 0 \end{cases}\\
\epsilon\dot{\eta} &= A_0\eta + \epsilon B_0\left(f(x, \Psi_s(e - D(\epsilon)\eta, \hat{W}, Y_r)) - y_r^{(\rho)}\right)
\end{aligned}$$

By Lipschitz continuity of the projection algorithm (Pomet and Praly, 1992), it follows that for a sufficiently small ϵ the inequality

$$\|Proj(\Upsilon_s(e - D(\epsilon)\eta, Y_r), \hat{W}) - Proj(\Upsilon_s(e, Y_r), \hat{W})\| \leq \gamma L_1\|\eta\| \tag{3.44}$$

holds on Ω for some positive nonzero constant L_1 and all $0 < \epsilon < \epsilon_1$.

Similarly, it follows from the continuous differentiability of $b_1(x)$ and $S(x)$ that the following

$$\|\Psi_s(e - D(\epsilon)\eta, Y_r, \hat{W}) - \Psi_s(e, Y_r, \hat{W})\| \leq L_2\|\eta\| \tag{3.45}$$

holds on Ω for some positive nonzero constant L_2 and all $0 < \epsilon \leq \epsilon_2$.

Using eqs.(3.24) and (3.27), the derivative of the Lyapunov function V_2 along the trajectories of the closed-loop system is such that

$$\begin{aligned}\dot{V}_2 &\le -k_5V_2 + c_2 + e_s b(x,u)\left(\Psi_s(e - D(\epsilon)\eta, Y_r, \hat{W}) - \Psi_s(e, Y_r, \hat{W})\right) \\ &\quad + \frac{1}{\gamma}\tilde{W}^T\left(Proj(\Upsilon_s(e - D(\epsilon)\eta, Y_r), \hat{W}) - Proj(\Upsilon_s(e, Y_r), \hat{W})\right)\end{aligned}$$

where $k_5 = \lambda/2$, c_2 is a positive constant arising from the constant terms in eqs.(3.24) and (3.27). In light of the Lipschitz inequalities eq.(3.44) and eq.(3.45), we obtain

$$\begin{aligned}\dot{V}_2 &\le -k_5V_2 + c_2 + L_2L_3\bar{b}\|\eta\| + 2w_mL_1\|\eta\| \\ &= -k_5V_2 + c_2 + k_7\|\eta\|\end{aligned}$$

where $\bar{b} = \max_{e\in\Omega, Y_r\in Y} b_1(e, Y_r)$ and L_3 is the maximum of e_s on Ω. Similarly, the derivative of $V_\eta(\eta)$ along the trajectories of the closed-loop system is given by

$$\begin{aligned}\epsilon\dot{V}_\eta &\le -\eta^T\eta \\ &\quad +2\epsilon\eta^TP_0B_0\left[b(x,u)\left(\Psi_s(e - D(\epsilon)\eta, Y_r, \hat{W}) - (W^{*T}S(e+Y_r) - \mu_l(x(t)))\right) - y_r^{(\rho)}\right]\end{aligned}$$

Since $|\Psi_s| \le S_u$, $\|W^*\| \le w_m$, $\|S(e+Y_r)\| \le \bar{S}$ and $\mu_l(x(t)) \le \bar{\mu}_l$, we obtain

$$\begin{aligned}\epsilon\dot{V}_\eta &\le -\eta^T\eta + 2\epsilon\|\eta\|\lambda_{max}(P_0)\left(\bar{b}(S_u + 2w_m\bar{S} + \bar{\mu}_l) + \bar{\nu}_1\right) \qquad (3.46) \\ &= -\eta^T\eta + 2\epsilon\|\eta\|\lambda\text{max}(P_0)k_6\end{aligned}$$

Then it follows that $\dot{V}_2 \le 0$ on $\{V_2 = c_1\}\times\{V_\eta(\eta) \le \rho\epsilon^2\}$ as long as $\epsilon \le \frac{k_5c_1 - c_2}{k_7\sqrt{\rho/\lambda_{min}(P_0)}} = \epsilon_3 < \min\{\epsilon_1, \epsilon_2\}$. Similarly, $\dot{V}_\eta \le 0$ on $\{V_2 \le c_1\} \times \{V_\eta(\eta) = \rho\epsilon^2\}$ if $\rho \le 4\lambda_{max}(P_0)^3k_6^2$. As a result, we show that the set $\{V_2 \le c_1\} \times \{V_\eta(\eta) \le \rho\epsilon^2\}$ is positively invariant for all $0 < \epsilon \le \epsilon_3$.

Consider initial conditions $\{e(0), \tilde{W}(0)\} \in S$, and $\{\hat{e}(0)\} \in O$ where S is a compact subset $\Omega_b \subset \Omega$ and O is a compact subset of $\mathbb{R}^\rho$. It follows that $\eta(0) \leq \frac{k_4}{\epsilon^{(n-1)}}$, where k_4 is a positive constant which is related to the size of the compact sets S and O.

Note that $f(e, Y_r, \tilde{W}, \mu) = \left[e_2, e_3, \ldots, b(x,u)(\Psi_s(\cdot) - W^{*T} S(e+Y_r) - \mu_l(x(t))\right]^T$ is bounded on Ω. That is, $\|f(x, \Psi_s(\cdot))\| \leq k_5$. By construction, the adaptive learning rate is such that $\|Proj(\Upsilon_s(\cdot), \hat{W})\| \leq k_6$. As a result, the closed-loop trajectories of the system starting in S are such that

$$\|e(t) - e(0)\| \leq k_5 t$$

and

$$\|\tilde{W}(t) - \tilde{W}(0)\| \leq k_6 t.$$

Therefore there is a time T_e at which the closed-loop trajectory $(e(t), \tilde{W}(t))$ escape the set Ω.

From eq.(3.46), it follows that

$$\dot{V}_\eta \leq -\frac{1}{2\epsilon}\eta^T \eta,$$

if $V_\eta \geq \rho\epsilon^2$. Therefore, we have that

$$\dot{V}_\eta \leq -\frac{1}{2\epsilon\lambda_{max}(P_0)} W$$

or

$$V_\eta(t) = e^{-\frac{1}{2\epsilon\lambda}\max(P_0)t} V_\eta(0) \tag{3.47}$$

for $V_\eta \geq \rho\epsilon^2$. As a result, we can find an $\epsilon < \epsilon_3$ and a $T(\epsilon) > 0$ such that $V_\eta(t) \leq \rho\epsilon^2$, $\forall t \geq T(\epsilon)$. Moreover, it is always possible to pick ϵ_3 small enough such that $T(\epsilon_3) \leq$

T_0. As a result, picking $\epsilon_1^* = \min\{\epsilon_1, \epsilon_2, \epsilon_3\}$ ensures that the trajectories of the closed-loop process starting in $S \times O$ enter the compact set $\Lambda = \{V_2 \leq c_1\} \times \{V_\eta \leq \rho\epsilon^2\}$ in finite time $T(\epsilon)$ for all $0 < \epsilon \leq \epsilon_1^*$.

On the set Λ, it is shown that

$$\dot{V}_2 \quad \leq \quad -k_1 V_2 + c_2 + k_3\sqrt{\frac{\rho}{\lambda_{\min}}}\epsilon \tag{3.48}$$

Hence, we have

$$\dot{V}_2 \quad \leq \quad -\frac{k_1}{2}V_2$$

when $\{V_2 \geq 2c_2 + 2k_3\sqrt{\frac{\rho}{\lambda_{\min}}}\epsilon\}$ and, therefore, the trajectories of the system must be such that $\lim_{t\to\infty} V_2(t) \leq 2c_2 + 2\tilde{k}_3\epsilon$. Define $\chi = [e, \tilde{W}]$, if we pick ϵ such that

$$\{V_2 \leq 2c_2 + \tilde{k}_3\epsilon\} \subset \{\|\chi\| \leq \mu/2\}$$

then there is a finite time $\tilde{T}(\mu)$ such that for all $0 < \epsilon \leq \epsilon_4$

$$\|\chi\| \leq \mu/2 \ , \quad \forall t \geq \tilde{T}(\mu)$$

Since η is of order ϵ in the set Λ, it follows that there exists an $\epsilon = \epsilon_5$ and a finite time $\bar{T}(\mu)$ such that $\|\eta\| \leq \mu/2$ for all $t \geq \bar{T}(\mu)$. Therefore, taking $\epsilon_2^* = \min\{\epsilon_4, \epsilon_5\}$ ensures that there is a finite time $T_2 = \max\{\tilde{T}, \bar{T}\}$ such that $\|\chi\| \leq \mu/2$ and $\|\hat{e}\| \leq \mu/2$ for all $t \geq T_2$.

Finally, we can establish the exponential converge of the trajectories of the closed-loop system to a small adjustable neighbourhood of the origin. To do this, we consider the Lyapunov function

$$V = V_2 + V_\eta(\eta) \tag{3.49}$$

Its derivative is such that

$$\dot{V} \leq -k_1 V_2 + c_2 + k_3\|\eta\| - \frac{1}{\epsilon}\|\eta\|^2 + 2\lambda_{\max}(P_0)k_2\|\eta\|$$

By completing the squares, we get

$$\dot{V} \leq -k_1 V_2 - \frac{1}{\epsilon}\|\eta\|^2 + \left(\frac{k_4}{2} + \frac{k_5}{2}\right)\|\eta\|^2 + c_2 + \frac{k_3^2}{2k_4} + \frac{2(\lambda_{\max}(P_0)k_2)^2}{k_5} \quad (3.50)$$

Therefore, there exists $0 < \epsilon \leq \epsilon_3^* \leq \epsilon_2^*$ such that

$$-\frac{1}{\epsilon} + \frac{k_4}{2} + \frac{k_5}{2} \leq -k_6 < 0 \quad (3.51)$$

As a result, we obtain

$$\dot{V} \leq -kV_2 - k_6\|\eta\|^2 + c_2 \quad (3.52)$$

or

$$\dot{V} \leq -k_m V + c_2 \quad (3.53)$$

where $k_m = \min\{k_1, k_6/\lambda_{\max}(P_0)\}$. Using the results above, we guarantee that every trajectory of the system starting in $S \times O$ enters a small neighborhood of the origin in finite time and converges exponentially fast to a small adjustable neighborhood of the origin. This completes the proof. **Q.E.D.**

The above approach is very similar to the one in the state feedback case, except that a saturation function is used to isolate the peaking phenomenon in the estimated state dynamics, so as not to cause instability in the original state dynamics.

Remark 3.1 *In many applications, convergence of the error dynamics to a small neighbourhood of the origin may prove to be a significant limitation. One mechanism* *that is known to reduce the onset of tracking error offsets is the addition of integral action. In the current context, it is straightforward to include the integral term,* $\dot{e}_0 = e_1$*, which guarantees convergence of the tracking error.*

3.3 Application to a Driven Oscillator System

In the literature discussing the problem of regulating cardiac arrhythmias via chaos control, different types of cardiac models have been used. Some are merely constructed to describe a particular type of arrhythmias, such as the irregular interbeat model (Wang *et al.*, 1997). The others are models for different functions in the heart, such as the "black box" models (Hall *et al.*, 1997) and empirical model (Wang *et al.*, 1998) for the AV conduction system, and the mechanistic model that accounts for the action potentials of the ventricular myocardium (Wang *et al.*, 1997).

One distinct physical mechanism in the heart is the pacemaker, consisting of the Sinoatrial (SA) node and the Atrioventricular (AV) node. The idea of considering this system mathematically as a system of coupled nonlinear oscillators is traced back to (van der Pol and van der Mark, 1928). Since then, a number of researchers have tried to study the dynamics of the heartbeat based on limit cycle oscillators (West *et al.*, 1985) and (Keith and Rand, 1984). The model proposed in (Bernardo *et al.*, 1998) describes the overall behavior of SA and AV nodes, captures the essential features of the cardiac conduction system, establishes a correspondence between system parameters and the physiological quantities, and is able to simulate different types of cardiac arrhythmias.

In this section, we apply the proposed adaptive output tracking controller to the four-dimensional coupled driven oscillators model in (Bernardo *et al.*, 1998). The model takes the following form,

$$\dot{x}_1 = \frac{1}{C_1} x_2 \tag{3.54a}$$

$$\dot{x}_2 = -\frac{1}{L_1}[x_1 + g(x_2) + R(x_2 + x_4)] + A\cos(2\pi\omega t) \tag{3.54b}$$

$$\dot{x}_3 = \frac{1}{C_2} x_4 \tag{3.54c}$$

$$\dot{x}_4 = -\frac{1}{L_2}[x_3 + f(x_4) + R(x_2 + x_4)] \tag{3.54d}$$

where x_2 and x_4 describes the action potential of the SA and AV node, and x_1 and x_3 are the voltage corresponding to x_2 and x_4; C_1, C_2, L_1, and L_2 are some constant parameters in the model; R is the constant coupling parameter, A and ω are the amplitude and frequency of the driven signal, which is used to model ectopic pacemakers in some region of the cardiac tissue; and g and f are some nonlinear functions of the following form

$$\begin{aligned} f(x_4) &= -x_4 + \tfrac{1}{3}x_4^3 \\ h(x_2) &= \begin{cases} -x_2^2 - \frac{1}{4} & |x_2| < \frac{1}{2} \\ -x_2 & x_2 > \frac{1}{2} \\ x_2 & x_2 < -\frac{1}{2} \end{cases} \\ g(x_2) &= -x_2 + \tfrac{1}{3}h(x_2). \end{aligned}$$

Systems (3.54a) to (3.54d) are in the form of system (3.1), with $z_1 = x_2$, $\xi = [x_1, x_3, x_4]^T$, $\xi_r = [r_1, r_3, r_4]^T$, where r_1, r_3 and r_4 are equilibrium (invariant) trajectories for the zero dynamics. It is also assumed that the right hand side of equation (3.54b) is unknown. It can be readily shown that this system meets Assumptions 3.1 to 3.3. (Just note that the tracking dynamics are linear in the output variable x_2

and that the resulting system is stable for large values of x_4 and unstable in a small region containing the origin).

The electrical action potential can be measured by an transvenous electrode, which is a common part of artificial pacemakers. Given the following parameter values (Bernardo *et al.*, 1998), $C_1 = 0.25F$, $L_1 = 0.05H$, $C_2 = 0.675F$, $L_2 = 0.027H$, and $R = 0.11\Omega$, the system exhibits a normal 1 : 1 rhythm. By setting $C_1 = 0.15F$, one can generate an arrythmia of 2 : 1 AV block: for every two beats of the SA node, only one beat of AV node is observed. The control objective is to apply the proposed adaptive output tracking controller to make system (3.54a) to (3.54d) track the normal 1 : 1 rhythm, in other words, to suppress the AV block arrythmia.

We choose to use a perturbation to the right hand side of equation (3.54b) as the control,

$$\dot{x}_2 = -\tfrac{1}{L_1}[x_1 + g(x_2) + R(x_2 + x_4)] + A\cos(2\pi\omega t) + u. \tag{3.55}$$

Physically, the control action is an electrical impulse sent to the heart through a transvenous electrode, which enters the system in an affine manner, as shown in equation (3.55).

Remark 3.2 *Other potential control actuators are, perturbation to the intrinsic frequency of the SA node (parameter C_1), and the coupling strengh between the two nodes(parameter R).*

In this example, the unknown nonlinearity $\alpha(z, \xi_r)$ in equation (3.5) is as follows

$$\alpha(x_2, r_1, r_4) = \frac{1}{L_1}(r_1 + g + R(x_2 + r_4)) + A\cos(2\pi\omega t).$$

For the simulation, the number of basis functions is $l = 11$, with $\sigma^2 = 5$, $\phi_i = i - 6$, $i = 1, \ldots, 11$, and $w_m = 4$. The following tuning parameters are used, $k_4 = k_\mu =$

$k_d = k_w = \gamma_2 = 25$, $\gamma_w = 20$.

In the simulation, the controller is not turned on until $t = 5$. Figure 3.1 shows the simulation results of the SA and AV node rhythm, tracking performances and the control action. In the sub-plot of SA and AV node action potential, the dotted line is the SA node action potential, and the solid line is the AV node action potential. It can be seen that before the controller is turned on, the rhythm is $2:1$, with two beats of the SA node, only one beat of AV node is observed. After the controller is turned on at $t = 5$, the rhythm is altered to $1:1$ within one beat. The two sub-plots of tracking show that good tracking performances are achieved within a short time for the SA node and AV node, respectively. Figure 3.2 shows the approximation of the unknown nonlinearity, and parameter estimation. The unknown nonlinearity $\alpha(x_2, r_1, r_4)$ is the solid line, and the approximation $\hat{W}^T S(x_2, r_1, r_4)$ is the dotted line. After running the simulation for 300 second, good approximation is achieved along the r_1 and r_4 direction, and parameter estimations also converge.

3.4 Conclusion

In this chapter, an adaptive output tracking controller is developed for a class of nonlinear systems with unknown nonlinearities, in order to address the heart dynamics control problem in a real time framework. It is proved that the proposed controller is able to make the tracking error converge to a neighborhood of the origin exponentially fast. Simulation results show satisfactory performances can be achieved when applying this technique to regulate irregular heart dynamics. In addition, good approximation of the unknown nonlinearities is also achieved by incorporating a persistent exciting signal in the parameter update law. The proposed technique is an alternative approach to the control of complex chaotic systems with unknown dynamics.

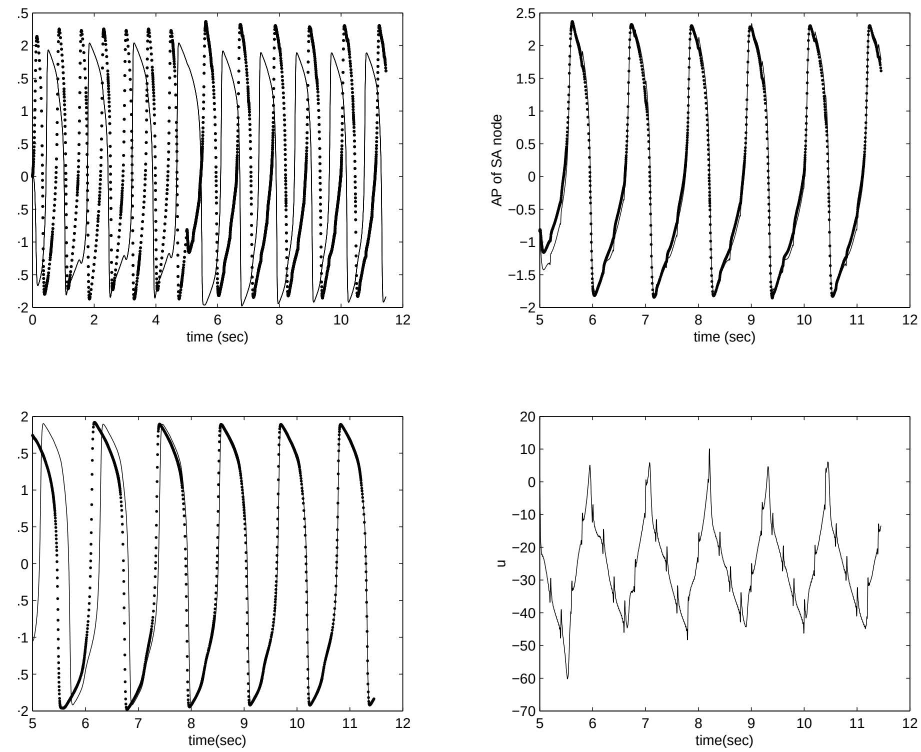

Figure 3.1: AV block suppression, tracking performances and control action

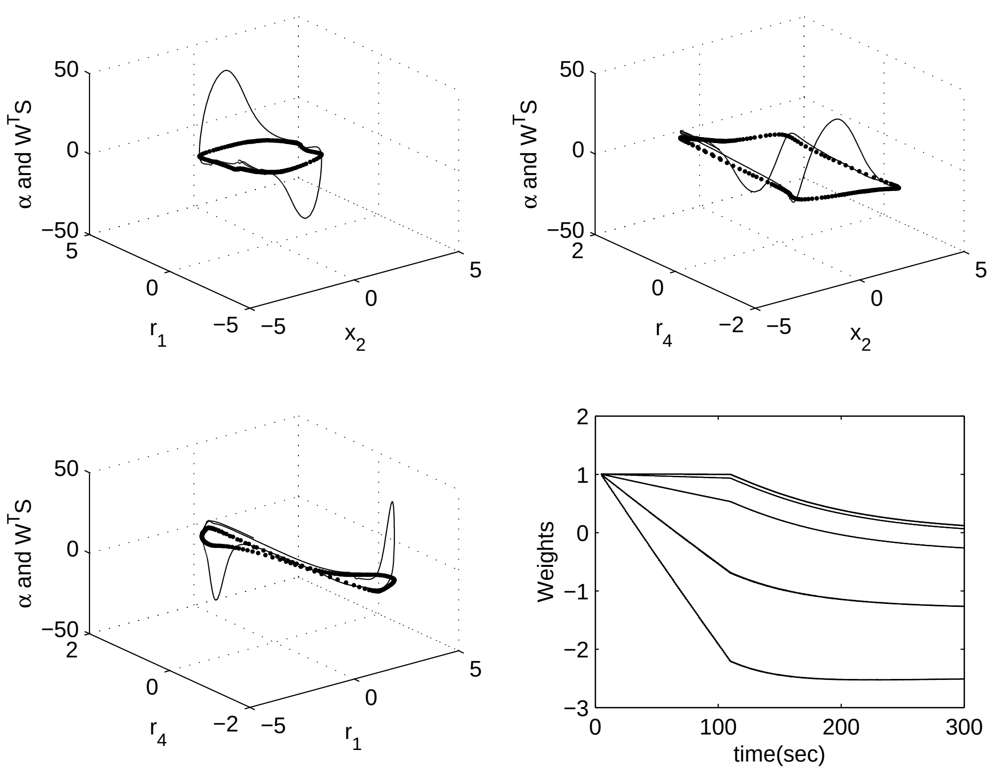

Figure 3.2: Unknown nonlinearities approximation and parameter convergence

Chapter 4

Output Feedback Stabilization of Uncertain Non-minimum Phase Nonlinear Systems

4.1 Introduction

In practice, controller implementation is generally subject to limited number of measurements available for feedback, due to sensor cost and/or availability. In such cases, one relies on the design of controller based on imperfect state measurements, often termed output feedback controller design. Most existing output feedback controller designs require that the zero dynamics of the controlled plant be stable, that is, to be minimum phase. However, a number of processes exhibit non-minimum phase (i.e. inverse response) behavior. This phenomenon can be encountered in the base of a distillation column, where the response of the bottom composition to a change in vapor boilup exhibits an inverse response (Luyben, 1989). It can also be observed in a continuous exothermic reactor, where the inlet stream flowrate is used to control the reactor temperature. In this situation, a positive step change in the inlet flowrate will cause an initial decrease in the reactor temperature (Kravaris *et al.*, 1994).

A review of output feedback control design was provided in Chapter 2. In this work (Diao and Guay, 2005), we examine the assumption from (Isidori, 2000), that "the auxiliary system is globally stabilizable by dynamic output feedback", to provide a constructive controller design procedure and outline the stabilizability requirements for auxiliary systems of relative degree zero, a problem not addressed in (Isidori, 2000).

This chapter is organized as follows, the main results are provided in section 4.2, and simulation results are presented in section 4.3, followed by conclusions in section 4.4.

4.2 Output Feedback Controller Design

4.2.1 Problem Formulation and Motivation

Consider a smooth nonlinear system modelled by equations of the following form

$$\begin{aligned}
\dot{z} &= f_0(z, x_1, \cdots, x_{r-1}, x_r, p) \\
\dot{x}_1 &= x_2 \\
&\vdots \\
\dot{x}_{r-1} &= x_r \\
\dot{x}_r &= h_0(z, x_1, \cdots, x_{r-1}, x_r, p) + b(x_1)u \\
y &= x_1,
\end{aligned} \tag{4.1}$$

where $z \in \mathbb{R}^{n-r}$, and p is a (possibly vector-valued) unknown parameter, ranging over a compact set $\mathcal{P}$.

The following definitions are needed in this work.

Let B_R^k denote the closed cube

$$B_R^k = \{x \in \mathbb{R}^k : |x_i| \leq R, 1 \leq i \leq k\}. \tag{4.2}$$

Definition 4.1 *The equilibrium (z, x) is said to be semi-globally practically stabilizable by dynamic state (respectively, output) feedback if, given any large number $R > 0$, and any small number $\epsilon > 0$, there exists a dynamic state (respectively, output) feedback $u = \bar{u}(z, x, \eta)$, $\dot{\eta} = \bar{\eta}(z, x, \eta)$ (respectively, $u = \bar{u}(z, y, \eta)$, $\dot{\eta} = \bar{\eta}(z, y, \eta)$), such that, in the closed loop system, any initial condition in $B_R^{\tilde{n}}$ produces a trajectory which is captured by the set $B_\epsilon^{\tilde{n}}$, where $\tilde{n}$ is the dimension of the closed loop system.*

Definition 4.2 *Given a system with both inputs and outputs*

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x, u), \end{aligned}$$

the system is 0-detectable if there exist some functions $\beta \in \mathcal{KL}$, and γ_1, $\gamma_2 \in \mathcal{K}_\infty$, such for every $x(0)$ and every u the corresponding solution satisfies the inequality

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma_1(\sup_{[0,t]} \|u\|) + \gamma_2(\sup_{[0,t]} \|y\|) \tag{4.3}$$

as long as it exists.

Now we are ready to give a brief review of the results in (Isidori, 2000). The following is assumed for system (4.1).

Assumption 4.1 *For all $p \in \mathcal{P}$*

$$\begin{aligned} f_0(0, \cdots, 0, p) &= 0 \qquad (4.4) \\ h_0(0, \cdots, 0, p) &= 0 \end{aligned}$$

and $b(x_1) \neq 0$.

Remark 4.1 *Assumption 4.1 ensures that all the uncertain parameters acting as disturbances to the system will not change the equilibrium of the system, and that the gain coefficient is nowhere zero. We do agree that the assumption may be restrictive from a modelling perspective. However, this assumption is not restrictive if steady-state data is used to identify the model parameters. In that case, all parameters which satisfy eq.(4.4) provide a suitable parametrization of the model. One natural way to remove steady-state offsets is to introduce integral action to the controller. It will be shown that one can eliminate the effect of steady-state offset by introducing an adaptive term in the controller design.*

From the control perspective, however, the knowledge of the system gain is crucial, especially when one is dealing with non-minimum phase systems. At this point, we are not aware of techniques that are capable of handling non-minimum phase systems with uncertain gains.

The control objective is to stabilize system (4.1) using a robust output feedback, given that the system is non-minimum phase. The solution of this problem is based on the existence of a dynamic output feedback controller for an auxiliary system associated with system (4.1). The auxiliary system is defined as follows,

$$\begin{aligned} \dot{x}_a &= f_a(x_a, u_a, p) \\ y_a &= h_a(x_a, u_a, p) \end{aligned} \tag{4.5}$$

where

$$
\begin{aligned}
x_a &= \begin{pmatrix} z \\ x_1 \\ \dots \\ x_{r-2} \\ x_{r-1} \end{pmatrix} \\
f_a(x_a, u_a, p) &= \begin{pmatrix} f_0(z, x_1, \cdots, x_{r-1}, u_a, p) \\ x_2 \\ \dots \\ x_{r-1} \\ u_a \end{pmatrix} \\
h_a(x_a, u_a, p) &= h_0(z, x_1, \cdots, x_{r-1}, u_a, p).
\end{aligned}
$$

The basic hypothesis about the auxiliary system (4.5) is the knowledge of a robust global dynamic output feedback stabilizer of the following form ((Isidori, 2000), Assumption 2),

$$
\begin{aligned}
\dot{\eta} &= L(\eta) + My_a \\
u_a &= N(\eta)
\end{aligned}
\tag{4.6}
$$

in which $\eta \in \mathbb{R}^\nu$, $L(0) = 0, N(0) = 0$, and M is a $\nu \times 1$ constant matrix.

Under the above assumption, it is proved in (Isidori, 2000) that system (4.1) can be semi-globally practically stabilized by the following dynamic output feedback law

$$\begin{aligned} \dot{\xi} &= P\xi + Qy \\ \dot{\eta} &= L(\eta) + M\sigma_L(k[\xi_r - N(\eta)]) \\ u &= \frac{1}{b(x_1)}\left[\frac{\partial N}{\partial \eta}(L(\eta) + M\sigma_L(k[\xi_r - N(\eta)]) - \sigma_L(k[\xi_r - N(\eta)])\right] \end{aligned} \tag{4.7}$$

where k is a positive number, $\sigma_L(\cdot)$ is a saturation function

$$\sigma_L(r) = \begin{cases} r, & \text{if} |r| < L \\ sgn(r)L, & \text{if} |r| \geq L, \end{cases}$$

and ξ_r is the estimation of state x_r under the following high gain observer

$$\dot{\xi} = \begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dots \\ \dot{\xi}_{r-1} \\ \dot{\xi}_r \end{pmatrix} = \begin{pmatrix} \xi_2 + gc_{r-1}(y - \xi_1) \\ \xi_3 + g^2 c_{r-2}(y - \xi_1) \\ \dots \\ \xi_r + g^{r-1} c_1 (y - \xi_1) \\ g^r c_0 (y - \xi_1) \end{pmatrix} =: P\xi + Qy,$$

in which $c_0, c_1, \cdots, c_{r-1}$ are the coefficients of some Hurwitz polynomial, g is a positive number.

There are two limitations to the above approach. First, even though it is shown in (Isidori, 2000) that the assumption of global stabilizability of the auxiliary system is not restrictive, how to find such a dynamic output feedback is not a trivial task. Second, it is also observed that, for most applications, the auxiliary output h_a is a function of u_a, in other word, the auxiliary system (4.5) has relative degree zero, which makes the problem even more difficult.

It is suggested in (Isidori, 2000) that the relative degree zero problem be solved by moving the u_a term to the control u, provided that y_a is a linear function of u_a. However, this is not always possible, as shown in the following illustrative example.

Example I

Consider a continuous stirred tank reactor (CSTR), where the series/parallel van de Vusse reaction (van de Vusse, 1964) is taking place:

$$\begin{array}{rcl} A & \xrightarrow{k_1} & B \xrightarrow{k_2} C \\ 2A & \xrightarrow{k_3} & D \end{array}$$

where A is the reactant, B the desired product, C and D are unwanted by-products.

The dynamics of the CSTR can be described in terms of the material balance for species A and B and an energy balance for the reactor as follows:

$$\begin{aligned} \frac{dC_A}{dt} &= -k_1(T)C_A - k_3(T)C_A^2 + (C_{A0} - C_A)u \\ \frac{dC_B}{dt} &= k_1(T)C_A - k_2(T)C_B - C_B u \\ \frac{dT}{dt} &= \frac{1}{\rho C_p}\Big[(-\triangle H_1)k_1(T)C_A + (-\triangle H_2)k_2(T)C_B + (-\triangle H_3)k_3(T)C_A^2 + Q\Big] \\ &\quad +(T_0 - T)u \end{aligned}$$

where C_A, C_B are the concentrations of the species A and B inside the reactor, respectively; T is the temperature inside the reactor; C_{A0} is the concentration of A in the feed stream; T_0 is the feed stream temperature; $k_i(T)$ is the rate coefficient given by the Arrhenius expressions, $k_i(T) = k_{i0}\exp(-E_i/RT)$, $i = 1, 2, 3$; u is the dilution rate, given by $u = F/V$, where F is the inlet flow rate, and V is the reactor volume, which is assumed to be constant; ρ and C_p are the density and specific heat of the reaction mixture, respectively; $-\triangle H_i$, $i = 1, 2, 3$ are the heat of reactions, $-Q$ is the cooling rate per unit volume.

Note that the assumption of constant reactor volume is not restrictive, which can be readily achieved by setting the closed-loop bandwidth of the volume control at a much higher frequency than the bandwidth of the concentration control.

The control objective is to make the output $y = C_B$ track its setpoint, by manipulating the dilution rate, $u = F/V$.

Assuming that $C_B \neq 0$, the following change of variables, $z_1 = \frac{C_{A0}-C_A}{C_B}$, $z_2 = \frac{T_0-T}{C_B}$, $x_1 = C_B$, $y = x_1$, transforms the CSTR dynamics into the normal form:

$$\begin{aligned}
\dot{z}_1 &= \tfrac{1}{x_1}\Big[(1-z_1)k_1(C_{A0}-z_1x_1) + k_3(C_{A0}-z_1x_1)^2 + k_2z_1x_1\Big] \\
\dot{z}_2 &= \tfrac{-1}{\rho C_p x_1}\Big[(-\triangle H_1)k_1(C_{A0}-z_1x_1) + (-\triangle H_2)k_2x_1 \\
&\quad +(-\triangle H_3)k_3(C_{A0}-z_1x_1)^2 + Q\Big] - \tfrac{z_2}{x_1}[k_1(C_{A0}-z_1x_1) - k_2x_1] \\
\dot{x}_1 &= k_1(C_{A0}-z_1x_1) - k_2x_1 - x_1u \\
y &= x_1.
\end{aligned} \tag{4.8}$$

It can be shown later in section 4.3.2 that system (4.8) is locally non-minimum phase around a reference steady state.

The auxiliary system associated with (4.8) is as follows,

$$\begin{aligned}
\dot{z}_1 &= \tfrac{1}{u_a}\Big[(1-z_1)k_1(C_{A0}-z_1u_a) + k_3(C_{A0}-z_1u_a)^2 + k_2z_1u_a\Big] \\
\dot{z}_2 &= \tfrac{-1}{\rho C_p u_a}\Big[(-\triangle H_1)k_1(C_{A0}-z_1u_a) + (-\triangle H_2)k_2u_a \\
&\quad +(-\triangle H_3)k_3(C_{A0}-z_1u_a)^2 + Q\Big] - \tfrac{z_2}{u_a}[k_1(C_{A0}-z_1u_a) - k_2u_a] \\
y_a &= k_1(C_{A0}-z_1u_a) - k_2u_a - u_au_{sp},
\end{aligned} \tag{4.9}$$

where u_{sp} is the nominal control action at the desired set-point to account for nonzero steady state values of the auxiliary output y_a.

It is observed from the last equation in (4.9) that the term multiplying u_a is $-k_1 z_1 - k_2 - u_{sp}$, which depends on the zero dynamics z_1 and z_2. Since the zero dynamics are not observable, we can not move this term to the control u to make the relative degree greater than zero as in (Isidori, 2000).

In order to alleviate the two limitations, we provide in the next section a systematic design procedure to construct a dynamic output feedback for the relative degree zero auxiliary system (4.5). It is shown that the original system (4.1) can still be semi-globally practically stabilized.

4.2.2 Controller Design

To construct a robust dynamic output feedback for the auxiliary system (with relative degree zero), we use a dynamic extension $\dot{u}_a = v$ on the auxiliary input u_a. The auxiliary system (4.5) becomes

$$\begin{aligned}
\dot{z} &= f_0(z, x_1, \cdots, x_{r-1}, u_a, p) \\
\dot{x}_1 &= x_2 \\
&\vdots \\
\dot{x}_{r-1} &= u_a \\
\dot{u}_a &= v \\
y_a &= h_0(z, x_1, \cdots, x_{r-1}, u_a, p).
\end{aligned}$$

where v is the new control. Differentiating y_a, we have

$$\begin{aligned}
\dot{z} &= f_0(z, x_1, \cdots, x_{r-1}, u_a, p) \\
\dot{x}_1 &= x_2 \\
&\vdots \\
\dot{x}_{r-1} &= u_a \\
\dot{y}_a &= \frac{\partial h_0}{\partial x_a} f_a(z, x_1, \cdots, x_{r-1}, u_a, p) + \frac{\partial h_0}{\partial u_a} v
\end{aligned} \tag{4.10}$$

where x_a is the state vector as in (4.5).

The following assumptions are made for the auxiliary system (4.10).

Assumption 4.2 *The zero dynamics of (4.10) is stable with respect to the output* y_a*.*

Assumption 4.3 $\frac{\partial h_0}{\partial x_a} f_a(z, x_1, \cdots, x_{r-1}, u_a, p)$ *is globally Lipschitz in* z *and locally Lipschitz in* $(x_2, \cdots, x_{r-1})$*.*

Assumption 4.4 $\|\frac{\partial h_0}{\partial u_a}\| > \varepsilon > 0$*, and the sign of* $\frac{\partial h_0}{\partial u_a}$ *is known.*

Remark 4.2 *Note that Assumption 4.2 is different from the one proposed in (Isidori, 2000). The stability assumption of the zero dynamics of the auxiliary system (4.10) allows us to construct a dynamic state feedback for the auxiliary system. In addition,, this assumption is not restrictive, either. As shown in (Isidori, 2000), that a memoryless feedback transformation* $u \to u + hy$*, with a proper choice of* h*, can render 0-detectability of the auxiliary system, which in turn, implies minimum phase property (see equation (4.3) in Definition 4.2).*

Under these assumptions, it is guaranteed that there exists a robust dynamic output feedback for the auxiliary system (4.5), as shown in the following lemma.

Lemma 4.1 *There exists a smooth dynamic system of the form*

$$\begin{aligned} \dot{\eta} &= L(x_1, \cdots, x_{r-1}, \eta) + My_a \\ u_a &= \eta \end{aligned} \tag{4.11}$$

in which $\eta \in \mathbb{R}$, $L(0) = 0$, M *is a nonzero constant. In addition, there is a positive definite and proper smooth function* $V(x_a, \eta)$ *whose derivative along the trajectories of the interconnected system (4.5) and (4.11) is negative definite, i.e.,*

$$\frac{\partial V}{\partial x_a} f_a(x_a, \eta, p) + \frac{\partial V}{\partial \eta}[L(x_1, \cdots, x_{r-1}, \eta, p) + My_a] < 0 \tag{4.12}$$

for all $(x_a, \eta) \neq (0, 0)$.

Proof: Under the above assumptions, it is obvious that the following controller

$$\begin{aligned} v &= \frac{1}{\frac{\partial h_0}{\partial u_a}} \left(-k_0 y_a - \frac{\partial h_0}{\partial x_a} f_a(0, x_1, \cdots, x_{r-1}, u_a) \right) \\ &= L(x_1, \cdots, x_{r-1}, u_a) - k_0 (\frac{\partial h_0}{\partial u_a})^{-1} y_a. \end{aligned} \tag{4.13}$$

is able to stabilize the auxiliary system (4.10), provided that k_0 is a large enough positive number. By Assumption 4.4, there exists a positive number m such that $(\|\frac{\partial h_0}{\partial u_a}\|)^{-1} \leq m$. Denoting

$$M = -k_0 m sgn\Big(\frac{\partial h_0}{\partial u_a}\Big),$$

the following controller

$$v = L(x_1, \cdots, x_{r-1}, u_a) + My_a \tag{4.14}$$

is able to stabilize the auxiliary system (4.10) as well.

Given that $\dot{u}_a = v$ and (4.14) stabilizes (4.10), it is implied that the dynamic output feedback (4.11) stabilize the auxiliary system (4.5). In addition, by the converse Lyapunov theorem (Khalil, 2002), there exists a Lyapunov function $V(x_a, \eta)$ such that (4.12) is satisfied. □

Next, consider the dynamic state feedback control

$$\begin{aligned} \dot{\eta} &= L(x_1, \cdots, x_{r-1}, \eta) + Mk[x_r - \eta] \\ u &= \tfrac{1}{b(x_1)}\Big[L(x_1, \cdots, x_{r-1}, \eta) + Mk(x_r - \eta) - k(x_r - \eta)\Big] \end{aligned} \tag{4.15}$$

where k is a positive number. Changing the state variable x_r into the new variable

$$\theta = x_r - \eta,$$

the interconnection of the feedback control (4.15) and system (4.1) becomes

$$\begin{aligned} \dot{x}_a &= f_a(x_a, \theta + \eta, p) \\ \dot{\theta} &= h_a(x_a, \theta + \eta, p) - k\theta \\ \dot{\eta} &= L(x_1, \cdots, x_{r-1}, \eta) + Mk\theta. \end{aligned} \tag{4.16}$$

Consider the positive definite and proper function

$$W(x_a, \eta, \theta) = V(x_a, \eta + M\theta) + \theta^2$$

and let Ω_b denote the set

$$\Omega_b = \{(x_a, \eta, \theta) : W(x_a, \eta, \theta) \leq b\}. \tag{4.17}$$

Then Lemma 2 in (Isidori, 2000) applies to the closed-loop system (4.16) in the same manner, which shows that the original system (4.1) is semi-globally practically stabilizable by (4.15).

Lemma 4.2 *(Lemma 2 in (Isidori, 2000)) For any $R > 0$ and $\epsilon > 0$, and for any $\rho > 0$ and $c > 0$ such that*

$$\Omega_\rho \subset B_\epsilon^{n+1} \subset B_R^{n+1} \subset \Omega_c$$

where B_ϵ^{n+1} and B_R^{n+1} are defined as in (4.2), and Ω_ρ and Ω_c are defined as in (4.17), then there is a number k^* such that, if $k > k^*$, the derivative of the function $W(x_a, \eta, \theta)$ along the trajectories of (4.16) is negative at each point of the set

$$S = \{(x_a, \eta, \theta) : \rho \leq W(x_a, \eta, \theta) \leq c\}.$$

Proof: See (Isidori, 2000).

The dynamic state feedback (4.15) uses the states $(x_2, \cdots, x_r)$, which need to be estimated. A high gain observer together with a saturation element are shown to provide a systematic design approach.

The resulting output feedback controller is as follows,

$$\begin{aligned} \dot{\xi} &= P\xi + Qy \\ \dot{\eta} &= L(y, \xi_2, \cdots, \xi_{r-1}, \eta) + M\sigma_L(k[\xi_r - \eta]) \\ u &= \tfrac{1}{b(y)}\Big[L(y, \xi_2 \cdots, \xi_{r-1}, \eta) + M\sigma_L(k[\xi_r - \eta]) - \sigma_L(k[\xi_r - \eta])\Big]. \end{aligned} \tag{4.18}$$

The control law (4.18) takes similar form with the control law (4.7), which was developed in (Isidori, 2000). The only difference is the presence of the estimated states $(\xi_2, \cdots, \xi_{r-1})$ in (4.18), which does not add any difficulty in the stability proof. Therefore, Theorem 1 in (Isidori, 2000) applies to the closed-loop system (4.18) and

(4.1), which is stated below.

Theorem 4.1 *Suppose Assumptions 4.1 to 4.4 hold and consider system (4.1). Given any arbitrary large number* $R > 0$ *and any arbitrary small number* $\epsilon > 0$*, there are numbers* $k > 0$*,* $g > 0$*,* $L > 0$*,* $k_0 > 0$ *such that, in the closed-loop system (4.18) and (4.1), any initial condition in* B_R^{n+1+r} *produces a trajectory which is captured by the set* B_ϵ^{n+1+r}*.*

Proof: This proof amounts to showing that the presence of the estimated states in (4.18) will not affect the stability conditions imposed in the proof of the Theorem 1 in (Isidori, 2000).

Consider the change of variable $\theta = x_r - \eta$, we have the following closed-loop system

$$\begin{aligned} \dot{x}_a &= f_a(x_a, \theta + \eta, p) \\ \dot{\theta} &= h_a(x_a, \theta + \eta, p) - \sigma_L(k[\xi_r - \eta]) \\ \dot{\eta} &= L(y, \cdots, x_{r-1}, \eta) + M\sigma_L(k[\xi_r - \eta]) \\ \dot{\xi} &= P\xi + Qy. \end{aligned} \tag{4.19}$$

Define the following scaled state estimation error

$$e_i = g^{r-i}(x_i - \xi_i)$$

for $i = 1, \cdots, r$, i.e.,

$$e = D_g(x - \xi)$$

in which $D_g = \text{diag}[g^{r-1}, \cdots, g, 1]$.

Then (4.19) can be written in the following perturbation form,

$$\begin{aligned}
\dot{x}_a &= f_a(x_a, \theta + \eta, p) \\
\dot{\theta} &= h_a(x_a, \theta + \eta, p) - k\theta + \phi_1(\theta, e) \\
\dot{\eta} &= L(y, \cdots, x_{r-1}, \eta) + Mk\theta - M\phi_1(\theta, e) \\
\dot{e} &= gAe + B\phi_2(x_a, \theta, \eta, e, p)
\end{aligned} \tag{4.20}$$

in which A is a Hurwitz matrix, $B = [0, 0, \cdots, 0, 1]^T$, with perturbations ϕ_1 and ϕ_2 as follows,

$$\begin{aligned}
\phi_1(\theta, e) &= k\theta - \sigma_L(k\theta - ke_r) + \Big[L(y, \cdots, \xi_{r-1}, \eta) - L(y, \cdots, x_{r-1}, \eta)\Big] \\
\phi_2(x_a, \theta, \eta, e, p) &= h_a(x_a, \theta + \eta, p) - \sigma_L(k\theta - ke_r) + \dot{\eta}.
\end{aligned}$$

It is shown in (Isidori, 2000) that the key for the stability proof is to prove that the perturbation terms satisfy the following requirements. For all $((x_a, \eta, \theta), e) \in \Omega_{c+1} \times \mathbb{R}^r$

$$\begin{aligned}
|\phi_1(\theta, e)| &\leq \beta_1 \\
|\phi_2(x_a, \theta, \eta, e, p)| &\leq \beta_2 \\
|\phi_1(\theta, e)| &\leq \gamma(\|e\|)
\end{aligned}$$

in which β_1, β_2 are fixed numbers, and $\gamma(\cdot)$ is a continuous function such that $\gamma(0) = 0$.

The only difference between the perturbation in this case with the perturbation term in (Isidori, 2000) is the extra term $L(y, \cdots, \xi_{r-1}, \eta) - L(y, \cdots, x_{r-1}, \eta)$ in ϕ_1. However, the above requirements can be easily verified given Assumption 4.3.

The rest of the proof follows the proof in (Isidori, 2000). □

Remark 4.3 *Note that the assumption of* $\frac{\partial h_0}{\partial x_a} f_a(z, x_1, \cdots, x_{r-1}, u_a, p)$ *being globally*

Lipschitz in z (Assumption 4.3) is rather restrictive. This can be relaxed to a more general locally Lipschitz assumption, which yields a semi-global stability result for the auxiliary system. Moreover, this relaxation does not affect the stability result in Theorem 4.1.

4.2.3 Integral Action

As discussed above, it is generally difficult using the technique considered above to ensure the absence of steady-state error unless Assumption 4.1 is fulfilled. Since this is not the case in general, one may consider the incorporation of integral action into the output-feedback controller equation (4.18). This can be done by using the following integrator

$$\dot{e}_0 = (x_1 - x_{1sp}).$$

with $e_0(0) = 0$. The integrator state provides an estimate of the steady-state value of the control variable given by

$$\hat{u}_{sp} = \mathrm{sign}(b(x_1))k_I e_0 \tag{4.21}$$

where $\mathrm{sign}(b(x_1))$ denotes the sign of the high-frequency gain of the system and k_I is a gain for the integral action.

Using integral action, the controller equation (4.18) is rewritten as

$$\begin{aligned} \dot{\xi} &= P\xi + Qy \\ \dot{\eta} &= L(e_0, y, \xi_2, \cdots, \xi_{r-1}, \eta) + M\sigma_L(k[\xi_r - \eta]) \\ u &= \sigma_P(\hat{u}_{sp}) + \tfrac{1}{b(y)}\Big[L(e_0, y, \xi_2 \cdots, \xi_{r-1}, \eta) + M\sigma_L(k[\xi_r - \eta]) - \sigma_L(k[\xi_r - \eta])\Big]. \end{aligned} \tag{4.22}$$

where $\sigma_P(\hat{u}_{sp})$ is the saturation function with saturation bound P that is used to avoid large values of the estimated correction term $\hat{u}_{sp}$. Note that the term $L(e_0, y, \xi_2, \cdots, \xi_{r-1}, \eta)$ must be changed to account for the incorporation of the integral action. This is due to the fact that the auxiliary output y_a must account for nonzero steady-state values of the control variable. Nominally the dynamics of x_r are given by

$$\dot{x}_r = h_0(z, x_1, x_2, \ldots, x_{r-1}, x_r) + b(x_1)u_{sp} + b(x_1)(u - u_{sp})$$

where u_{sp} is the unknown steady-state value of the control variable that corresponds to the desired set-point, $x_1 = x_{1sp}$. Therefore the nominal auxiliary output is given by,

$$y_a = h_0(z, x_1, x_2, \ldots, x_{r-1}, u_a) + b(x_1)u_{sp}$$

Since the value of u_{sp} is unknown, it is not possible to ensure that y_a reaches the origin, as required. Thus, we replace the value of u_{sp} by its estimate, $\hat{u}_{sp}$, to obtain

$$y_a = h_0(z, x_1, x_2, \ldots, x_{r-1}, u_a) + b(x_1)\hat{u}_{sp} = \hat{h}_s(x_a, u_a, e_0) \qquad (4.23)$$

The design procedure is then repeated using the auxiliary output $\hat{h}_s(x_a, u_a, e_0)$ yielding the term $L(e_0, y, \xi_2, \cdots, \xi_{r-1}, \eta)$ in the output-feedback controller with integral action. The development of the control algorithm is similar to the approach outlined in the previous subsection.

4.3 Applications

4.3.1 Example II

Consider the following system,

$$\begin{aligned}\dot{z} &= pz - x_1 \\ \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_2(1+x_1^2) + (1+x_1^2)(-3x_1+5z) - 3x_1 + 6z + u \\ y &= x_1\end{aligned} \tag{4.24}$$

where p is an uncertain parameter, ranging over $[0, 1]$. It is obvious that the zero dynamics of system (4.24) is unstable. The control objective is to construct an output feedback controller that is able to semi-globally practically stabilize the plant (4.24).

The auxiliary system associated with (4.24) is as follows,

$$\begin{aligned}\dot{z} &= pz - x_1 \\ \dot{x}_1 &= u_a \\ y_a &= -(1+x_1^2)u_a + (1+x_1^2)(-3x_1+5z) - 3x_1 + 6z.\end{aligned}$$

Now we need to check if Assumptions 4.1 to 4.4 are satisfied.

It is obvious that Assumption 4.1 is satisfied. To check Assumption 4.2, we solve u_a from y_a,

$$u_a = -\frac{y_a}{1+x_1^2} + (-3x_1+5z) + \frac{1}{1+x_1^2}(-3x_1+6z)$$

set $y_a = 0$, and get the zero dynamics of the auxiliary system as follows,

$$\begin{aligned}\dot{z} &= pz - x_1 \\ \dot{x}_1 &= (-3x_1 + 5z) + \tfrac{1}{1+x_1^2}(-3x_1 + 6z).\end{aligned} \tag{4.25}$$

Consider a change of variable $\zeta = x_1 - 2z$, and a backstepping Lyapunov function $V = \frac{1}{2}z^2 + \frac{1}{2}\zeta^2$, we can show that V is negative definite along the trajectories of the zero dynamics (4.25), as follows

$$\dot{V} \leq -z^2 - \frac{3\zeta^2}{1 + (\zeta + 2z)^2} - (\zeta - (1-p)z)^2 \leq 0.$$

$\dot{V} = 0$ implies $\zeta = z = 0$, which implies $x_1 = z = 0$. Therefore, Assumption 4.2 holds.

Taking the derivative of y_a as follows,

$$\dot{y}_a = (11 + 5x_1^2)(pz - x_1) + u_a\Big[-2x_1u_a + 2x_1(-3x_1 + 5z) - 6 - 3x_1^2\Big] - (1 + x_1^2)v$$

we can check easily that Assumptions 4.3 and 4.4 are satisfied as well.

In the simulation, controller (4.18) is implemented with the following design parameters $k_0 = 10$, $k = 10$, $c_0 = 1$, $c_1 = 2$, $g = 100$ and $L = 1$. The initial conditions for the states are $z(0) = x_1(0) = x_2(0) = 10$, and $\xi_2(0) = 0$. The uncertain parameter p is setting as $p = |\sin(t)|$. Simulation results in Figure 4.1 show that the proposed controller robustly stabilizes the system to the origin, under the presence of the uncertain parameter.

4.3.2 Example I

Consider again the van de Vusse reaction system in section 4.2.1. An example is the production of cyclopentenol (B) from cyclopentadiene (A) by acid-catalyzed electrophilic addition of water in dilute solution, where cyclopentanediol (C) and dicy-

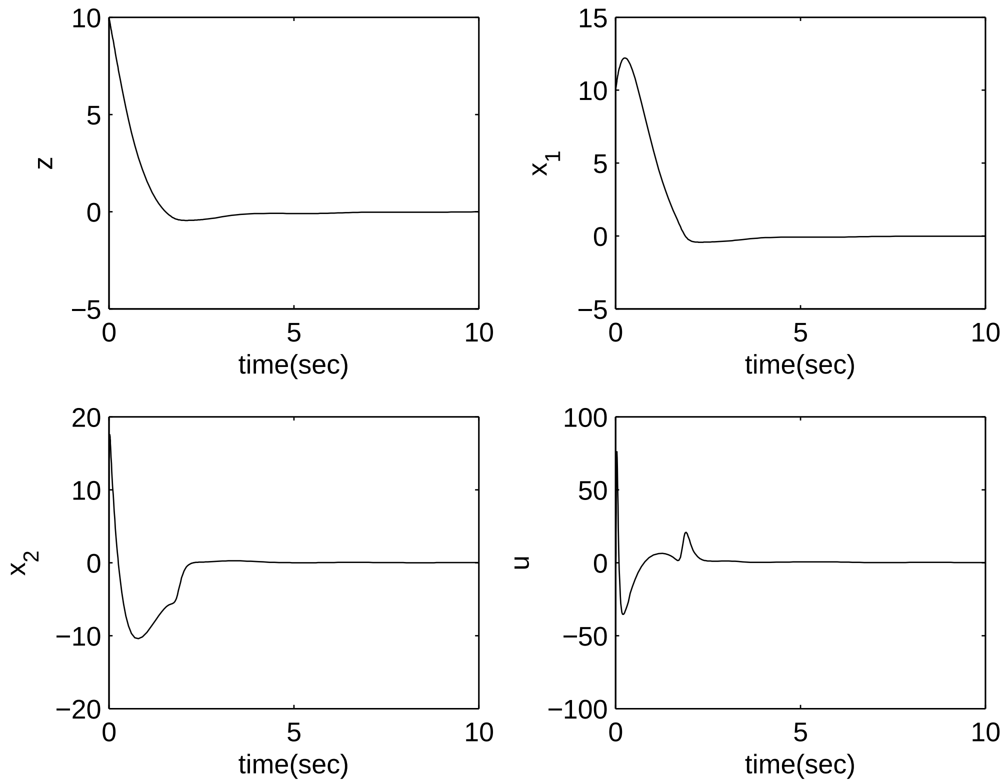

Figure 4.1: State trajectory and controller performance of example II

clopentadiene (D) are also produced as side products (Engell and Klatt, 1993).

The operating condition is $C_{A0} = 5$ gmol $\cdot$ L^{-1}, and $T_0 = 403.15$ K. In addition, the following parameters values are assumed (Engell and Klatt, 1993):

Table 4.1: Parameter Values for the van de Vusse Reactor

$k_{10} = 1.287 \cdot 10^{12}\mathrm{h}^{-1}$	$k_{20} = 1.287 \cdot 10^{12}\mathrm{h}^{-1}$
$k_{30} = 9.043 \cdot 10^{9}\mathrm{L(mol \cdot h)}^{-1}$	$\mathrm{E_1/R} = -9758.3\mathrm{K}$
$E_2/R = -9758.3\mathrm{K}$	$E_3/R = -8560\mathrm{K}$
$\triangle H_1 = 4.2\mathrm{kJ \cdot mol}^{-1}$	$\triangle H_2 = -11\mathrm{kJ \cdot mol}^{-1}$
$\triangle H_3 = -41.85\mathrm{kJ \cdot mol}^{-1}$	$\rho = 0.9342\mathrm{kgL}^{-1}$
$C_p = 3.01\mathrm{kJ(kg \cdot K)}^{-1}$	$Q = -451.509\mathrm{kJ(L \cdot h)}^{-1}$

The control objective is to make the output $y = C_B$ track its setpoint, by manipulating the dilution rate, $u = F/V$. In this work, we would like the output C_B to track a setpoint change to 1.0 $\text{mol} \cdot \text{L}^{-1}$, from the following reference steady-state: $C_{Bs} = 0.9$ $\text{mol} \cdot \text{L}^{-1}$, $C_{As} = 1.25$ $\text{mol} \cdot \text{L}^{-1}$, $T_s = 407.15$ K, which corresponds to $u_s = 19.5218$ hr^{-1}.

To check the stability of the zero dynamics of (4.8), we linearize the zero dynamics around the reference steady state, and get the following eigenvalues: $\lambda_1 = 122.68$, and $\lambda_2 = -11.17$. This shows that system (4.8) is locally non-minimum phase around the reference steady state.

Checking the stability of the zero dynamics of (4.9) around the reference steady state, we get the following eigenvalues: $\lambda = -21.86 \pm 8.93I$. This shows that the auxiliary system is locally minimum phase. Therefore, Assumption 4.2 is satisfied. For Assumption 4.3, only the local Lipschitz condition is satisfied. To verify Assumption 4.4, we check the term $\frac{\partial h_0}{\partial u_a} = -k_2 - u_{sp}$, where $u_{sp} = 30.6015$ hr^{-1}, for the set-point of $C_{Bsp} = 1.0$ $\text{mol} \cdot \text{L}^{-1}$. Since k_2 and u_{sp} are both positive numbers, we know that the sign of $\frac{\partial h_0}{\partial u_a}$ is always negative. In addition, $\left(\|\frac{\partial h_0}{\partial u_a}\|\right)^{-1} < (u_{sp})^{-1} = m$. Assumption 4.4 is therefore satisfied.

In the simulation, controller (4.18) is implemented with the following design parameters $k_0 = 150$, $k = 1$ and $L = 1$. The initial conditions for the states are the reference steady state. The control action is restricted in the range of 5 $\text{h}^{-1} \leq u \leq 50$ h^{-1} (Engell and Klatt, 1993).

We first consider the simulation of the nominal system without integral action. Two sets of simulation results are shown in Figure 4.2. The dotted line demonstrates the performance of control (4.18) using the nominal values of the parameters. It can be seen that the output C_B tracks the new set-point within a short period of time (comparable to the results cited in (Kravaris *et al.*, 1997), in terms of transient response and settling time performance). In contrast, the full line shows the result of the simulation when the parameter estimates used in the controller are different than the nominal values (such that Assumption 4.1 is *not* met). In this case, a 50% error in the value of k_{30} was considered. The steady-state offset can be reduced by increasing the controller gains k, L and k_0. However, this strategy was not employed in this case, due to two factors. Firstly, increasing the controller gain can only reduce, not eliminate, the steady state offset. Secondly, the closed loop stability margin will be sacrificed if the controller gain is increased too much and will cause instability eventually.

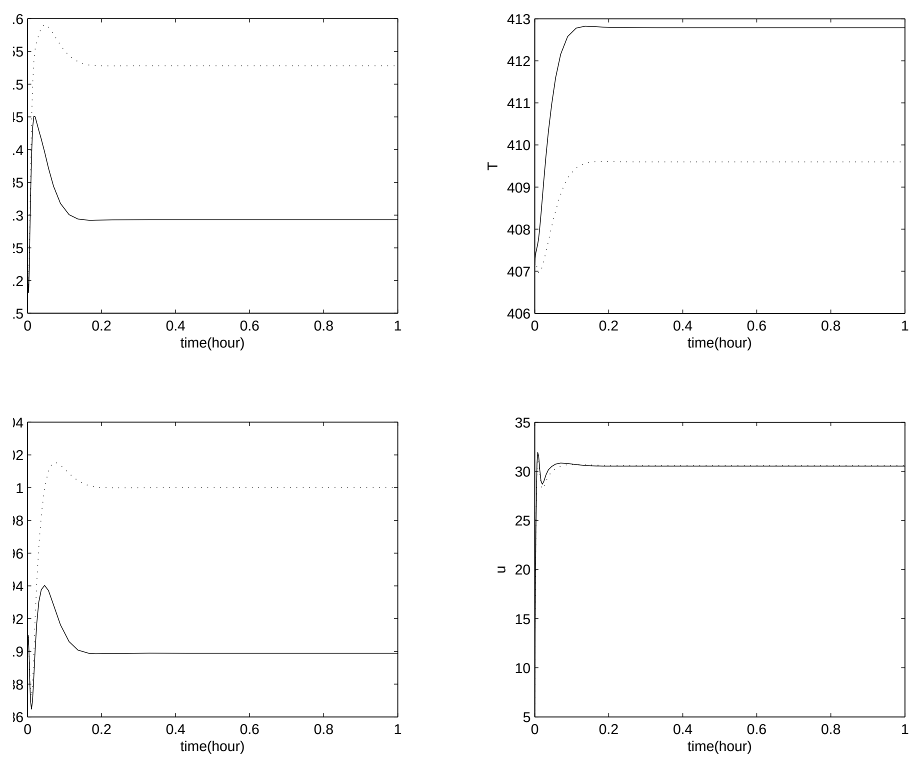

Figure 4.2: State trajectory and controller performance of example I. The dotted line shows results subject to the nominal parameter values. The full line illustrates results subject to changes of model parameters.

To correct the situation, we implement the output-feedback controller with integral action in equation (4.22). The auxiliary output y_a in equation (4.9) becomes

$$y_a = k_1(C_{A0} - z_1 u_a) - k_2 u_a - u_a \hat{u}_{sp} = \hat{h}_s$$

Taking time derivative of y_a, we get

$$\begin{aligned} \dot{y}_a &= \frac{\partial \hat{h}_s}{\partial x_a}\dot{x}_a + \frac{\partial \hat{h}_s}{\partial u_a}\dot{u}_a \\ &= -k_1 u_a \dot{z}_1 - u_a(z_1 \frac{\partial k_1}{\partial z_2} + \frac{\partial k_2}{\partial z_2})\dot{z}_2 - (k_2 + \hat{u}_{sp})v \end{aligned}$$

Given $k_i(T) = k_{i0}\exp(-E_i/RT)$, $i = 1, 2$, and $z_2 = \frac{T_0 - T}{C_B} = \frac{T_0 - T}{u_a}$, we have, for $i = 1, 2$

$$\frac{\partial k_i}{\partial z_2} = \frac{\partial k_i}{\partial T}\frac{\partial T}{\partial z_2} = \frac{\partial k_i}{\partial T}u_a.$$

Therefore,

$$\begin{aligned} \dot{y}_a &= -k_1 u_a \dot{z}_1 - u_a^2(z_1 \frac{\partial k_1}{\partial T} + \frac{\partial k_2}{\partial T})\dot{z}_2 - (k_2 + \hat{u}_{sp})v \\ &= \alpha(z_1, z_2, T, u_a) + \beta(T)v \end{aligned}$$

The controller v in (4.13) is therefore designed as follows,

$$v = -k_0 sign(\beta(\hat{T}))y_a - \frac{\alpha(z_{1sp}, z_{2sp}, \hat{T}, u_a)}{\beta(\hat{T})} \tag{4.26}$$

where $z_{1sp} = \frac{C_{A0} - C_{Asp}}{C_{Bsp}}$, $z_{2sp} = \frac{T_0 - T_{sp}}{C_{Bsp}}$, $\hat{T} = -u_a z_{2sp} + T_0$. C_{Asp}, C_{Bsp} and T_{sp} are the concentrations and temperature at the desired set-point, which, in this case, are $C_{Asp} = 1.25$ mol $\cdot$ L^{-1}, $C_{Bsp} = 1.0$ mol $\cdot$ L^{-1} and $T_{sp} = 407.15$ K.

It is straightforward to check all the assumptions, and the rest of the design is the same as in the nominal case. In the simulation, the same tuning parameters were employed. The integral gain, k_I, was set to 150/0.05. The results are shown in Figure 4.3. It is shown that the controller can effectively recover the required set-point despite the presence of model uncertainties.

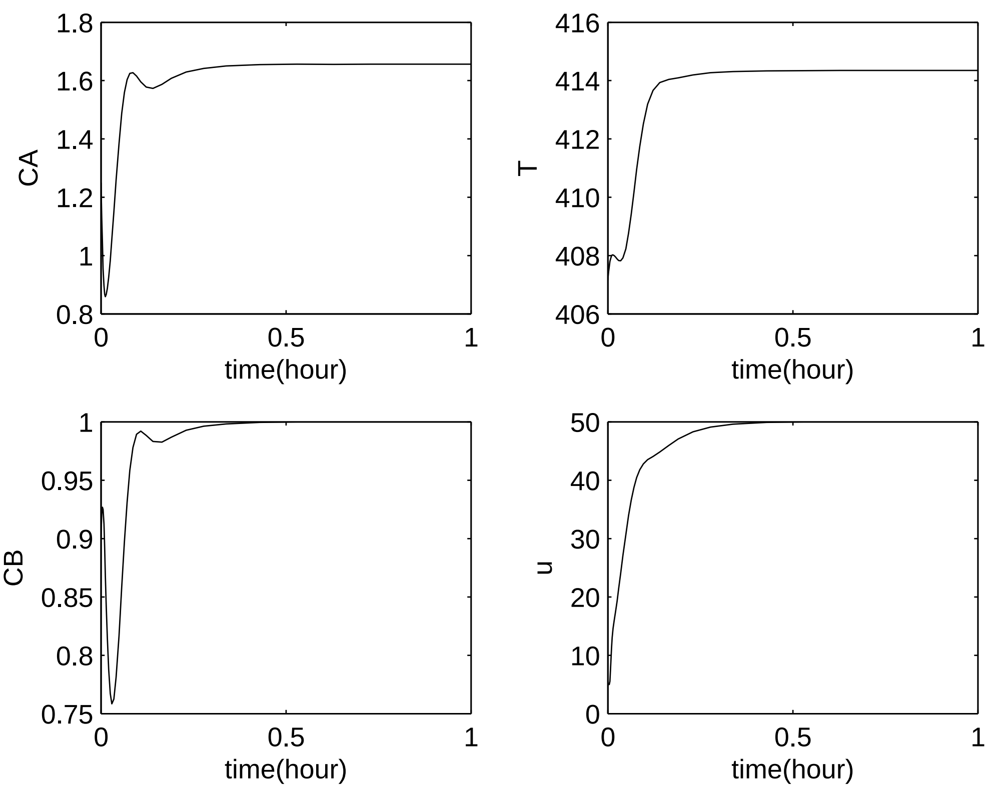

Figure 4.3: State trajectory and controller performance of example I. The full line illustrates results subject to changes of model parameters with the integral controller equation (4.22).

Finally, we consider the effect of set-point changes on the closed-loop system for the integral controller in equation (4.22). We consider the following three set-points for C_B, [0.8,0.9,1.0]. The results are shown in Figure 4.4. In each case, the parameters and steady-state information was taken as the nominal steady-state for a set-point of 1.0 $\mathrm{mol} \cdot \mathrm{L}^{-1}$ of component B. The results demonstrate that the controller performs well at various operating conditions.

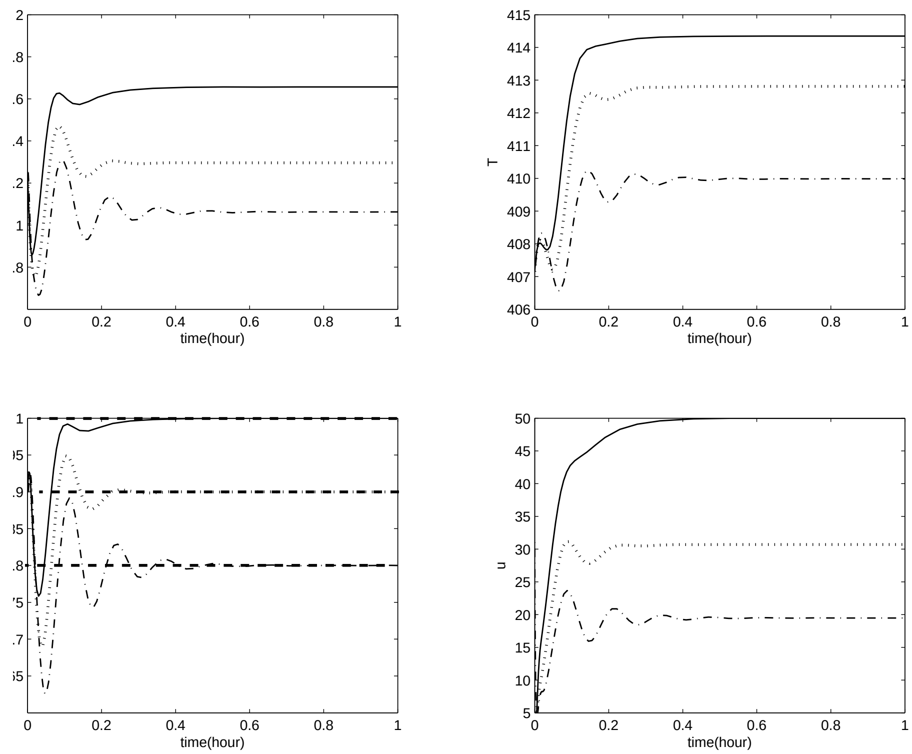

Figure 4.4: State trajectory and controller (4.22) performance of example I at various set-points of C_B, [0.8, 0.9, 1.0].

4.4 Conclusions

In this chapter, we proposed a robust control design that semi-globally practically stabilize a general uncertain non-minimum phase nonlinear system. Simulation results demonstrate that satisfactory controller performance is obtained. In particular, we show that the approach yields excellent performance for the control of the bench mark van de Vusse reactor.

Chapter 5

Output Feedback Stabilization of Non-minimum Phase Nonlinear Systems with Unknown Nonlinearities

5.1 Introduction

In the previous chapters, the following control design problems have been studied, and new design approaches have been proved and verified with simulation examples:

- Output feedback control design for systems with unknown nonliniearities.
- Output feedback control design for non-minimum phase systems with uncertainties.

The focus of this chapter is to study output feedback control design problem for nonlinear systems with both unknown nonlinearities and non-minimum phase behavior. A new design technique is proposed that guarantees both closed-loop stability and functional approximation of the unknown dynamics.

5.2 Problem Formulation and Design Procedures

The system of interest takes the following form,

$$\begin{aligned}
\dot{z} &= f_0(z, x_1, \cdots, x_{r-1}, x_r, p) \\
\dot{x}_1 &= x_2 \\
&\vdots \\
\dot{x}_{r-1} &= x_r \\
\dot{x}_r &= f(z, x_1, \cdots, x_{r-1}, x_r, p) + b(x)u \\
y &= x_1,
\end{aligned} \tag{5.1}$$

where $z \in \mathbb{R}^{n-r}$ are the state variables of the zero dynamics, p is a (possibly vector-valued) unknown parameter, ranging over a compact set $\mathcal{P}$, $x = [x_1, x_2, \cdots, x_r]^T \in \mathbb{R}^r$ are the state variables of the main dynamics; $u \in \mathbb{R}$ and $y \in \mathbb{R}$ are the system input and output, respectively. The mapping $f(z, x_1, \cdots, x_{r-1}, x_r, p)$ is assumed to be an unknown continuous function of z and x. In addition, $b(x)$ is a bounded known nonlinearity, with $0 \leq b_0 \leq b(x) \leq b_1$. It is assumed that the zero dynamics of system (5.1) is unstable. Without loss of generality, it is also assumed that the origin is the equilibrium of (5.1). The control objective is to design an output feedback controller for system (5.1), which semi-globally stabilizes the system subject to the unknown nonlinearities.

In the next two sections, we construct a state feedback as well as an output feedback controller along the following paths: First, we introduce a synthetic output to the system, for which a semi-global output feedback controller, u_1, is available using the results in (Diao and Guay, 2005). Second, we rewrite the unknown nonlinearity, taking into account the synthetic output. Third, we construct a composite controller, which consists of the stabilizing control u_1 and another controller u_2 which dominates the approximation of the unknown dynamics. Closed-loop stability is proven at the end of this section.

5.3 State Feedback Controller Design

5.3.1 Introduction of the Synthetic Output

For system (5.1), we introduce a synthetic output $h_0(z, x, p)$, and consider a system of the form,

$$\begin{aligned} \dot{z} &= f_0(z, x_1, \cdots, x_{r-1}, x_r, p) \\ \dot{x}_1 &= x_2 \\ &\vdots \\ \dot{x}_{r-1} &= x_r \\ \dot{x}_r &= h_0(z, x, p) + u_1 \\ y &= x_1. \end{aligned} \tag{5.2}$$

where u_1 is the control input. $h_0(z, x, p)$ is chosen such that there exists an output feedback control which semi-globally practically stabilize system (5.2).

To show the structure of the controller u_1, and use the knowledge of the Control Lyapunov Function (CLF) associated with it, we need to review the following results in (Diao and Guay, 2005).

In order to construct a controller for (5.2), a dynamic output feedback controller needs to be designed for an auxiliary system of (5.2). The auxiliary system is defined as follows,

$$\begin{aligned} \dot{x}_a &= f_a(x_a, u_a, p) \\ y_a &= h_a(x_a, u_a, p) \end{aligned} \tag{5.3}$$

where

$$\begin{aligned} x_a &= \begin{pmatrix} z \\ x_1 \\ \ldots \\ x_{r-2} \\ x_{r-1} \end{pmatrix} \\ f_a(x_a, u_a, p) &= \begin{pmatrix} f_0(z, x_1, \cdots, x_{r-1}, u_a, p) \\ x_2 \\ \ldots \\ x_{r-1} \\ u_a \end{pmatrix} \\ h_a(x_a, u_a, p) &= h_0(z, x_1, \cdots, x_{r-1}, u_a, p). \end{aligned}$$

The following lemma was proven in (Diao and Guay, 2005).

Lemma 5.1 *There exists a smooth dynamic system of the form*

$$\begin{aligned} \dot{\eta} &= L(x_1, \cdots, x_{r-1}, \eta) + My_a \\ u_a &= \eta \end{aligned} \tag{5.4}$$

in which $\eta \in \mathbb{R}$, $L(0) = 0$, $M = -k_0 m sgn\left(\frac{\partial h_0}{\partial u_a}\right)$, *where* k_0 *is a positive constant, and* m *is a positive number such that* $(\|\frac{\partial h_0}{\partial u_a}\|)^{-1} \leq m$. *In addition, there is a positive definite and proper smooth function* $V(x_a, \eta)$ *whose derivative along the trajectories of the interconnected system (5.3) and (5.4) is negative definite, i.e.,*

$$\frac{\partial V}{\partial x_a} f_a(x_a, \eta, p) + \frac{\partial V}{\partial \eta}[L(x_1, \cdots, x_{r-1}, \eta, p) + My_a] < 0 \tag{5.5}$$

for all $(x_a, \eta) \neq (0, 0)$.$\triangleleft$

Proof: See (Diao and Guay, 2005).

Before presenting the controller structure of u_1, the following definitions are required.

Let $\theta = x_r - \eta$. Let B_R^k denote the closed cube

$$B_R^k = \{x \in \mathbb{R}^k : |x_i| \leq R, 1 \leq i \leq k\}. \tag{5.6}$$

Consider the positive definite and proper function

$$W(x_a, \eta, \theta) = V(x_a, \eta + M\theta) + \theta^2$$

and let Ω_b denote the set

$$\Omega_b = \{(x_a, \eta, \theta) : W(x_a, \eta, \theta) \leq b\}. \tag{5.7}$$

The controller design of u_1 is given by the following Lemma.

Lemma 5.2 *Consider the dynamic state feedback control*

$$\begin{aligned} \dot{\eta} &= L(x_1, \cdots, x_{r-1}, \eta) + Mk[x_r - \eta] \\ u_1 &= L(x_1, \cdots, x_{r-1}, \eta) + Mk(x_r - \eta) - k(x_r - \eta) \end{aligned} \tag{5.8}$$

where k is a positive number, then the following holds. For any $R > 0$ and $\epsilon > 0$, and for any $\rho > 0$ and $c > 0$ such that

$$\Omega_\rho \subset B_\epsilon^{n+1} \subset B_R^{n+1} \subset \Omega_c$$

where B_ϵ^{n+1} and B_R^{n+1} are defined as in (5.6), and Ω_ρ and Ω_c are defined as in (5.7), then there is a number k^* such that, if $k > k^*$, the derivative of the function $W(x_a, \eta, \theta)$ along the trajectories of (5.2) and (5.8) is negative at each point of the set

$$S = \{(x_a, \eta, \theta) : \rho \leq W(x_a, \eta, \theta) \leq c\} \triangleleft .$$

Proof: See (Isidori, 2000).

5.3.2 Rewriting the Nonlinearity

Given the synthetic output $h_0(z, x, p)$, the unknown nonlinearity $f(z, x_1, \cdots, x_{r-1}, x_r, p)$ can be rewritten as follows,

$$\begin{aligned} f(z, x_1, \cdots, x_{r-1}, x_r, p) &= h_0(z, x, p) + \Big[f(z, x_1, \cdots, x_{r-1}, x_r, p) - h_0(z, x, p)\Big] \\ &= h_0(z, x, p) + \tilde{f}(z, x_1, \cdots, x_{r-1}, x_r, p) \\ &= h_0(z, x, p) + \tilde{f}(0, x_1, \cdots, x_{r-1}, x_r, p) \\ &\quad + \Big[\tilde{f}(z, x_1, \cdots, x_{r-1}, x_r, p) - \tilde{f}(0, x_1, \cdots, x_{r-1}, x_r, p)\Big] \end{aligned}$$

Radial basis function (RBF) presented in (Kosmatopoulos *et al.*, 1995) were used to approximate the continuous function $\tilde{f}(0, x_1, \cdots, x_{r-1}, x_r, p) : \mathbb{R}^r \to \mathbb{R}$

$$\tilde{f}(0, x_1, \cdots, x_{r-1}, x_r, p) = W^{*T} S(x) + \mu_l(t) \tag{5.9}$$

with approximation error $\mu_l(t)$, and basis function vector

$$S(x) = [s_1(x), s_2(x), \cdots, s_l(x)]^T \tag{5.10}$$

$$s_i(x) = \exp\left[\frac{-(x-\varphi_i)^T(x-\varphi_i)}{\sigma_i^2}\right] \tag{5.11}$$

$$i = 1, 2, \ldots, l \tag{5.12}$$

where φ_i is the center of the receptive field, and σ_i is the width of the Gaussian function. The ideal weight W^* in equation (5.9) is defined as

$$W^* := arg \min_{W \in \Omega_w} \left\{ \sup_{x \in \Omega} \left| W^T S(x) - \tilde{f}(x) \right| \right\}$$

where $\Omega_w = \{W \mid \|W\| \leq w_m\}$ with positive constant w_m to be chosen at the design stage, and Ω is a compact set.

Therefore, the nonlinearity $f(z, x_1, \cdots, x_{r-1}, x_r, p)$ can be re-written as follows

$$\begin{aligned} f(z, x_1, \cdots, x_{r-1}, x_r, p) &= h_0(z, x, p) + W^{*T} S(x) + \mu_l(t) \\ &\quad + \triangle \tilde{f}(z, x_1, \cdots, x_{r-1}, x_r, p) \end{aligned} \tag{5.13}$$

where $\triangle \tilde{f}(z, x_1, \cdots, x_{r-1}, x_r, p) = \tilde{f}(z, x_1, \cdots, x_{r-1}, x_r, p) - \tilde{f}(0, x_1, \cdots, x_{r-1}, x_r, p)$

5.3.3 Lyapunov Function Construction

The existence of the Lyapunov function in Lemma 5.1 is guaranteed by the converse Lyapunov theorem. However, it is generally a difficult task to search and find such a Lyapunov function. In this section, it is demonstrated that a Lyapunov function can be constructed, based on the Lyapunov function of the zero dynamics of the auxiliary system.

The following assumptions are made for the auxiliary system (5.3).

Assumption 5.1 *u_a enters z dynamics linearly. The auxiliary system (5.3) can be put in the following form:*

$$
\begin{aligned}
\dot{z} &= \tilde{f}_0(z, x_1, \cdots, x_{r-1}, p) + c_z u_a \\
\dot{x}_1 &= x_2 \\
&\vdots \\
\dot{x}_{r-1} &= u_a \\
y_a &= h_0(z, x_1, \cdots, x_{r-1}, u_a, p).
\end{aligned}
$$

where $c_z = [c_{z_1}, \cdots, c_{z_{n-r}}]^T$

Assumption 5.2 *The knowledge of a Lyapunov function V_0 for the zero dynamics of the auxiliary system is available, which is quadratic in $(z, x_1, \cdots, x_{r-1})$ or their linear combinations.*

Assumption 5.3 *$\frac{\partial h_0}{\partial u_a}$ is a function of $(x_1, \cdots, x_{r-1})$ only.*

A three-step approach demonstrates that a Lyapunov function can be constructed for the closed loop auxiliary system, based on the above assumptions.

First, according to the assumption of (4.2) the zero dynamics of the auxiliary output is stable with respect to the auxiliary output y_a. Therefore, there exists a $V_0(x_a, p)$, such that

$$\frac{\partial V_0}{\partial x_a} f_a(x_a, \eta, p) < 0 \tag{5.14}$$

In addition, the knowledge of the Lyapunov function for the zero dynamics are available, based on Assumption 5.2.

Setting y_a to zero, solving for u_a from $h_0(z, x_1, \cdots, x_{r-1}, u_a, p) = 0$, and denoting it as $\bar{u}_a$. Taking Assumption 5.1 into account, the zero dynamics of the auxiliary system becomes

$$
\begin{aligned}
\dot{z} &= \tilde{f}_0(z, x_1, \cdots, x_{r-1}, p) + c_z \bar{u}_a \\
\dot{x}_1 &= x_2 \\
&\vdots \\
\dot{x}_{r-1} &= \bar{u}_a
\end{aligned}
\tag{5.15}
$$

Second, closed loop system (5.3) and (5.4), with control v defined in equation (4.13) can be put into this form:

$$
\begin{aligned}
\dot{z} &= \tilde{f}_0(z, x_1, \cdots, x_{r-1}, p) + c_z u_a \\
\dot{x}_1 &= x_2 \\
&\vdots \\
\dot{x}_{r-1} &= u_a \\
\dot{y}_a &= \tfrac{\partial h_0}{\partial x_a}\big(f_a(z, x_1, \cdots, x_{r-1}, u_a, p) - f_a(0, x_1, \cdots, x_{r-1}, u_a)\big) - k_0 m |\tfrac{\partial h_0}{\partial u_a}|^{-1} y_a.
\end{aligned}
$$

Solving for u_a from $y_a = h_0(z, x_1, \cdots, x_{r-1}, u_a, p)$, it can be shown that $u_a = \bar{u}_a + (\frac{\partial h_0}{\partial u_a})^{-1} y_a$. The closed loop system becomes:

$$
\begin{aligned}
\dot{z} &= \tilde{f}_0(z, x_1, \cdots, x_{r-1}, p) + c_z \bar{u}_a + c_z (\tfrac{\partial h_0}{\partial u_a})^{-1} y_a \\
\dot{x}_1 &= x_2 \\
&\vdots \\
\dot{x}_{r-1} &= \bar{u}_a + (\tfrac{\partial h_0}{\partial u_a})^{-1} y_a \\
\dot{y}_a &= \tfrac{\partial h_0}{\partial x_a}\big(f_a(z, x_1, \cdots, x_{r-1}, u_a, p) - f_a(0, x_1, \cdots, x_{r-1}, u_a)\big) - k_0 m |\tfrac{\partial h_0}{\partial u_a}|^{-1} y_a.
\end{aligned}
$$

Third, the following Lyapunov function is proposed: $V = V_0 + \frac{1}{2}{y_a}^2$. Taking time derivatives of the Lyapunov function:

$$
\begin{aligned}
\dot{V} &= \tfrac{\partial V_0}{\partial x_a} f_a(x_a, \eta, p) + \Big(C_1 z + \sum_{i=1}^{r-1} C_i x_i\Big)(\frac{\partial h_0}{\partial u_a})^{-1} y_a \\
&\quad + y_a \Big[\tfrac{\partial h_0}{\partial x_a}\big(f_a(z, x_1, \cdots, x_{r-1}, u_a, p) - f_a(0, x_1, \cdots, x_{r-1}, u_a)\big) - k_0 m |\tfrac{\partial h_0}{\partial u_a}|^{-1} y_a\Big] \\
&\leq \tfrac{\partial V_0}{\partial x_a} f_a(x_a, \eta, p) + \Big[C_1(\tfrac{\partial h_0}{\partial u_a})^{-1} + L_z\Big]\|z\| y_a + \sum_{i=1}^{r-1} C_i x_i (\frac{\partial h_0}{\partial u_a})^{-1} y_a - k_0 m y_a^2 \qquad (5.16) \\
&\leq -(a_z z + b_z y_a)^2 - \sum_{i=1}^{r-1}(a_i x_i + b_i y_a)^2 - \tilde{k}_0 y_a^2 \\
&\leq 0
\end{aligned}
$$

It can be seen that $V = 0$ implies $[z, x_1, \cdots, x_{r-1}, y_a]^T = 0$, which shows that V is a Lyapunov function of the closed loop auxiliary system.

Therefore, the $\frac{\partial V}{\partial \eta}$ term in the state feedback control law in equation (5.19) is as follows:

$$
\begin{aligned}
\tfrac{\partial V}{\partial \eta}(z, x_1, \cdots, x_{r-1}, \eta, p) &= \tfrac{\partial V}{\partial y_a}\tfrac{\partial y_a}{\partial \eta} \\
&= y_a(z, x_1, \cdots, x_{r-1}, \eta, p)\tfrac{\partial y_a}{\partial \eta}(x_1, \cdots, x_{r-1})
\end{aligned}
\qquad (5.17)
$$

5.3.4 Controller Construction and Stability Proof

The system of interest (5.1) can be put in the following form by substituting the nonlinearity (5.13) into (5.1),

$$\begin{aligned} \dot{z} &= f_0(z, x_1, \cdots, x_{r-1}, x_r, p) \\ \dot{x}_1 &= x_2 \\ &\vdots \\ \dot{x}_{r-1} &= x_r \\ \dot{x}_r &= h_0(z, x, p) + W^{*T} S(x) + \mu_l(t) + \triangle \tilde{f}(z, x_1, \cdots, x_{r-1}, x_r, p) + b(x)u \\ y &= x_1, \end{aligned} \tag{5.18}$$

Compare system (5.2) and (5.18), it can be seen that the differences between these two systems are the presence of the unknown function approximation $W^{*T}S(x)+\mu_l(t)$ and $\triangle\tilde{f}(z, x_1, \cdots, x_{r-1}, x_r, u, p)$. Therefore, if we use u_1 as one sub-controller, which can stabilize system (5.2) with the term $\triangle\tilde{f}(z, x_1, \cdots, x_{r-1}, x_r, u, p)$, then system (5.18) can be stabilized provided that another sub-controller is constructed to take care of $W^{*T}S(x) + \mu_l(t)$.

The following assumption is made.

Assumption 5.4 $\triangle\tilde{f}(z, x_1, \cdots, x_{r-1}, x_r, u, p)$ *is locally Lipschitz in z.*

The following state feedback controller is proposed to stabilize system (5.18),

$$\begin{aligned} \dot{\eta} &= L(x_1, \cdots, x_{r-1}, \eta) + Mk(x_r - \eta) \\ u &= \tfrac{1}{b(x)}\Big[-k(x_r - \eta) + L(x_1, \cdots, \eta) + Mk(x_r - \eta) - k_\eta M \tfrac{\partial V}{\partial \eta} - \hat{W}^T S \Big] \end{aligned} \tag{5.19}$$

where $\hat{W}$ is the estimation of parameter W^*.

Theorem 5.1 summarizes the main result of the state feedback control design solution.

Theorem 5.1 *Suppose Assumption 5.4 holds, and consider system (5.18). Given any arbitrary large number* $R > 0$ *and any arbitrary small number* $\epsilon > 0$*, there are numbers* $k > 0$*,* $k_\eta > 0$ *such that, in the closed-loop system (5.19) and (5.18), any initial condition in* B_R^{n+1} *produces a trajectory which is captured by the set* B_ϵ^{n+1}.

Proof: The closed-loop system (5.19) and (5.18) takes the following form,

$$\begin{aligned}
\dot{z} &= f_0(z, x_1, \cdots, x_{r-1}, x_r, p) \\
\dot{x}_1 &= x_2 \\
&\vdots \\
\dot{x}_r &= h_0(z, x, p) + W^{*T} S(x) + \mu_l(t) + \triangle \tilde{f}(z, x_1, \cdots, x_{r-1}, x_r, p) \\
&\quad -k(x_r - \eta) + L(x_1, \cdots, \eta) + Mk(x_r - \eta) - k_\eta M \tfrac{\partial V}{\partial \eta} - \hat{W}^T S \\
\dot{\eta} &= L(x_1, \cdots, x_{r-1}, \eta) + Mk[x_r - \eta] \\
y &= x_1.
\end{aligned}$$

Changing the state variable x_r into the new variable

$$\theta = x_r - \eta,$$

the interconnection of the feedback control (5.19) and system (5.18) becomes

$$\begin{aligned}
\dot{x}_a &= f_a(x_a, \theta + \eta, p) \\
\dot{\eta} &= L(x_1, \cdots, x_{r-1}, \eta) + Mk\theta \\
\dot{\theta} &= h_0(z, x, p) - k\theta + \triangle \tilde{f}(z, x_1, \cdots, x_{r-1}, x_r, p) \\
& \quad -\tilde{W}^T S - k_\eta M \tfrac{\partial V}{\partial \eta} + \mu_l(t)
\end{aligned} \tag{5.20}$$

where the parameter estimation error $\tilde{W}$ is given by $\tilde{W} = \hat{W} - W^*$.

Consider the Lyapunov function candidate

$$\begin{aligned}
W_t(x_a, \eta, \theta, \tilde{W}) &= W(x_a, \eta, \theta) + \frac{1}{2\gamma_w}\tilde{W}^T\tilde{W} \\
&= V(x_a, \eta + M\theta) + \theta^2 + \frac{1}{2\gamma_w}\tilde{W}^T\tilde{W}
\end{aligned}$$

where γ_w is a positive constant. Taking the time derivative of $W_t(x_a, \eta, \theta, \tilde{W})$ along the trajectory of the closed-loop system (5.20), we have

$$\begin{aligned}
\dot{W}_t &= \frac{\partial V}{\partial x_a}\Big|_{\eta \to \eta + M\theta} f_a(x_a, \theta + \eta, p) + \frac{\partial V}{\partial \eta}\Big|_{\eta \to \eta + M\theta} \Big\{ L(x_1, \cdots, x_{r-1}, \eta) + Mk\theta \\
& \quad + Mh_0 - Mk\theta + M\triangle \tilde{f}(z, x_1, \cdots, x_{r-1}, x_r, p) - M\tilde{W}^T S + M\mu_l(t) \\
& \quad - M^2 k_\eta \frac{\partial V}{\partial \eta} \Big\} + 2\theta \Big\{ h_0(z, x, p) - k\theta + \triangle \tilde{f}(z, x_1, \cdots, x_{r-1}, x_r, p) \\
& \quad - \tilde{W}^T S - k_\eta M \frac{\partial V}{\partial \eta} + \mu_l(t) \Big\} + \frac{1}{\gamma_w}\tilde{W}^T \dot{\tilde{W}}.
\end{aligned}$$

The expression of $\dot{W}_t$ can be put in the following form,

$$\dot{W}_t \leq \frac{\partial V}{\partial x_a} f_a(x_a, \eta, p) + \frac{\partial V}{\partial \eta} \Big\{ L(x_1, \cdots, x_{r-1}, \eta) + M(h_0) \Big\} \tag{5.21a}$$

$$- 2k\theta^2 + \theta R(x_a, \theta, \eta, p) \tag{5.21b}$$

$$- M \frac{\partial V}{\partial \eta} \tilde{W}^T S + \frac{1}{\gamma_w} \tilde{W}^T \dot{\tilde{W}} \tag{5.21c}$$

$$+ M\frac{\partial V}{\partial \eta}\mu_l(t) - k_\eta M^2(\frac{\partial V}{\partial \eta})^2 + \frac{\partial V}{\partial \eta}M\triangle\tilde{f}(z, x_1, \cdots, x_{r-1}, x_r, p). \tag{5.21d}$$

where $R(x_a, \theta, \eta, p)$ is some smooth function.

The sign of line (5.21a) to (5.21d) is analyzed as follows.

It has been shown in Lemma 5.1, equation (5.5), that line (5.21a) is negative. In addition, with the choice of Lyapunov function proposed in Section 5.3.3, this term is explicitly shown in equation (5.16), which will help cancel one term in line (5.21d), as shown later.

Line (5.21b) is negative over a compact set, as shown in Lemma 5.2.

In order to make line (5.21c) negative, we choose the following projection algorithm (Seshagiri and Khalil, 2000). Denoting $\hat{\Omega}$ as the compact set where the parameter estimations are confined within, let $\Omega_\delta = \{W_i \mid a_i - \delta \leq W_i \leq b_i + \delta\}, i = 1, \cdots, l$, where δ is chosen such that $\Omega_\delta \subset \hat{\Omega}$. The adaptive law is chosen such that

$$\tilde{W}^T\Big(\dot{\hat{W}} + (-\gamma_w\frac{\partial V}{\partial \eta}(\eta)MS)\Big) \leq 0,$$

which guarantees that (5.21c) is negative.

Let $\phi_i = -\gamma_w\frac{\partial V}{\partial \eta}(\eta)MS_i$, the projection algorithm $\dot{\hat{W}}_i = [Proj(\hat{W}, \phi)]_i$ is given as follows,

$$[Proj(\hat{W}, \phi)]_i = \begin{cases} \phi_i, & \text{if} \quad a_i \leq \hat{W}_i \leq b_i \text{ or} \\ & \quad\ \hat{W}_i > b_i,\ \phi_i \leq 0 \\ & \quad\ \hat{W}_i < a_i,\ \phi_i \geq 0 \\ \phi_i(1 + \frac{b_i - \hat{W}_i}{\delta}), & \text{if} \quad \hat{W}_i > b_i,\ \phi_i > 0 \\ \phi_i(1 + \frac{\hat{W}_i - a_i}{\delta}), & \text{if} \quad \hat{W}_i < a_i,\ \phi_i < 0 \end{cases} \tag{5.22}$$

Line (5.21d) can be rearranged as follows,

$$
\begin{aligned}
& M\tfrac{\partial V}{\partial \eta}\mu_l(t) - k_\eta M^2(\tfrac{\partial V}{\partial \eta})^2 + \tfrac{\partial V}{\partial \eta} M \triangle \tilde{f}(z, x_1, \cdots, x_{r-1}, x_r, p) \\
\leq\ & \tfrac{k_\mu}{2} M^2 (\tfrac{\partial V}{\partial \eta})^2 + \tfrac{1}{2k_\mu}\mu_l(t)^2 - k_\eta M^2(\tfrac{\partial V}{\partial \eta})^2 + M\tfrac{\partial V}{\partial \eta} L_z \|z\| \\
=\ & -(k_\eta - \tfrac{k_\mu}{2} - 1) M^2 (\tfrac{\partial V}{\partial \eta})^2 + \tfrac{1}{2k_\mu}\mu_l(t)^2 + L_z^2 \|z\|^2 \\
\leq\ & \text{negative term} \quad + \tfrac{1}{2k_\mu}\mu_l(t)^2
\end{aligned}
$$

where k_η is chosen such that, $k_\eta - \frac{k_\mu}{2} > 0$. $L_z^2\|z\|^2$ can be grouped into (5.21a), and be dominated by the $-(a_z z + b_z y_a)^2$ term.

Therefore the derivative of W_t can be written as

$$
\dot{W}_t \ \leq \ \text{negative term} \quad + \frac{1}{2k_\mu}\mu_1(t)^2.
$$

Since the approximation error $\mu(t)$ can be made arbitrary small, we conclude that, for any initial condition in B_R^{n+1}, the closed-loop system (5.19) and (5.18) produces a trajectory which is captured by the set B_ϵ^{n+1}. □

5.4 Output Feedback Controller Design

To implement the state feedback control law in equation (5.19), the unmeasured states need to be estimated. The higher order derivatives of the output, $(x_2, \cdots, x_{r-1})$, can be estimated by using the high gain observer approach, as shown in Chapter 4. The difficulty lies in the z term, which appears in the $\frac{\partial V}{\partial \eta}$ term developed in the previous section. z needs to be estimated and closed loop stability needs to be ensured when using the estimates in the control law (5.19).

To this end, we apply the results in (Karagiannis *et al.*, 2003), and show that there exists an output feedback control law that asymptotically stabilizes the closed loop system (5.1) and (5.19).

5.4.1 Review of Output Feedback Stabilization Results in Literature

In this section, one of the results (Proposition 1) in (Karagiannis *et al.*, 2003) was reviewed, using notations consistent with the rest of this thesis.

The system of interest takes the following form:

$$\begin{aligned} \dot{z} &= A(y,u)z + B(y,u) \\ \dot{y} &= \varphi_0(y,u) + \varphi_1(y,u)z \end{aligned} \tag{5.23}$$

Along with system (5.23), consider a performance output ρ, defined as

$$\rho = h(y,z) \tag{5.24}$$

The following lemma states the results of the Proposition 1 of (Karagiannis *et al.*, 2003).

Lemma 5.3 *Consider a system described by equations of form (5.23) and a performance variable ρ defined as in equation (5.24). Suppose the following assumptions hold.*

1. *There exists a full information control law*

$$u = \alpha(y,z) \tag{5.25}$$

such that all trajectories of the closed loop system (5.23) and (5.25) are bounded and are such that condition (5.24) holds. Moreover, system (5.23) with $u = \alpha(y; z + \tilde{z}(t))$ is globally bounded-input bounded-state stable with respect to the *input $\tilde{z}(t)$.*

2. *There exists a mapping $\beta(y)$ such that the system*

$$\dot{\tilde{z}} = \Big(A(y,u) - \frac{\partial \beta}{\partial y}\varphi_1(y,u)\Big)\tilde{z} \tag{5.26}$$

is uniformly globally stable for any y and u, and $\tilde{z}(t)$ is such that, for any fixed y and z.

$$\lim_{t\to\infty}[\alpha(y, z + \tilde{z}(t))] = \alpha(y,z). \tag{5.27}$$

Then there exists a dynamic output feedback control law, described by equations of (5.29), solving the output feedback regulation problem.

Proof: See (Karagiannis *et al.*, 2003). □

5.4.2 Output Feedback Stabilization

For the output feedback stabilization problem, the system of interest in equation (5.1) is restricted to the following:

$$\begin{aligned} \dot{z} &= \bar{f}_0(x_1, \cdots, x_{r-1}, p)z + c_z x_r \\ \dot{x}_1 &= x_2 \\ &\vdots \\ \dot{x}_{r-1} &= x_r \\ \dot{x}_r &= \bar{f}(x_1, \cdots, x_{r-1}, x_r, p)z + b(x)u \\ y &= [x_1, \cdots, x_r]^T \end{aligned} \tag{5.28}$$

which is in the same format as studied in (Karagiannis *et al.*, 2003), as shown in equation (5.23).

Theorem 5.2 summarizes the main result of this chapter.

Theorem 5.2 *Consider system (5.28). Suppose the two assumptions in Lemma 5.3 hold, with the state feedback control in equation (5.19) chosen as the control law in equation (5.25). Then the dynamic output feedback control law, described by equation (5.29), solves the output feedback stabilization problem.*

Proof: The stability proof follows (Karagiannis *et al.*, 2003), as both assumptions in Lemma 5.3 are satisfied. The proof is repeated here for completeness.

Consider system (5.28) and the dynamic output feedback controller

$$\begin{aligned} \dot{\hat{z}} &= w \\ u &= \alpha(y, N\hat{z} + \beta(y)) \end{aligned} \tag{5.29}$$

where $\alpha(\cdot)$ is as in equation (5.25), N is an invertible matrix, $\beta(y) : \mathbb{R}^p \to \mathbb{R}^n$ is as in Assumption 2 of Lemma 5.3, and w is a new control signal. Let $\tilde{z} = N\hat{z} - z + \beta(y)$.

The closed loop system (5.28) and (5.29) can be written in the z, y, and $\tilde{z}$ coordinates as

$$\begin{aligned} \dot{z} &= A(y, \alpha(y, z + \tilde{z}))z + B(y, \alpha(y, z + \tilde{z})) \\ \dot{y} &= \varphi_0(y, \alpha(y, z + \tilde{z})) + \varphi_1(y, \alpha(y, z + \tilde{z}))z \\ \dot{\tilde{z}} &= Nw - A(y, \alpha(y, z + \tilde{z}))(-\tilde{z} + N\hat{z} + \beta(y)) \\ &\quad -B(y, \alpha(y, z + \tilde{z})) + \tfrac{\partial \beta}{\partial y}[\varphi_0(y, \alpha(y, z + \tilde{z})) \\ &\quad +\varphi_1(y, \alpha(y, z + \tilde{z}))(-\tilde{z} + N\hat{z} + \beta(y))] \end{aligned} \tag{5.30}$$

Picking the new control law w as follows:

$$\begin{aligned}\dot{\hat{z}} &= w \\ &= N^{-1}\Big[A(N\hat{z}+\beta(y)) + B(y,u) - \tfrac{\partial \beta}{\partial y}[\varphi_0 + \varphi_1(N\hat{z}+\beta(y))]\Big]\end{aligned} \tag{5.31}$$

The closed loop system (5.30) becomes:

$$\begin{aligned}\dot{z} &= A(y,\alpha(y,z+\tilde{z}))z + B(y,\alpha(y,z+\tilde{z})) \\ \dot{y} &= \varphi_0(y,\alpha(y,z+\tilde{z})) + \varphi_1(y,\alpha(y,z+\tilde{z}))z \\ \dot{\tilde{z}} &= \Big[A(y,\alpha(y,z+\tilde{z})) - \tfrac{\partial \beta}{\partial y}\varphi_1(y,\alpha(y,z+\tilde{z}))\Big]\tilde{z}\end{aligned} \tag{5.32}$$

As a result, by Assumption 2 of Lemma (5.3), the variable $\tilde{z}$ remains bounded for all t, and it is such that equation (5.27) holds. Hence, by Assumption 1 of Lemma 5.3, y and z are bounded for all t and condition (5.24) holds, which proves the claim.□

5.5 Example

Consider the following system,

$$\begin{aligned}\dot{z} &= pz - x_1 \\ \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -3x_1 + 5z - x_2 + u - \tfrac{x_1 x_2}{1+{x_1}^2} \\ y &= [x_1, x_2]^T\end{aligned} \tag{5.33}$$

where p is an uncertain parameter, ranging over $[0,1]$. The nonlinear term $-\frac{x_1 x_2}{1+{x_1}^2}$ is assumed to be unknown. The zero dynamics of system (5.33) is unstable. The control objective is to construct an output feedback controller that is able to semi-globally practically stabilize the plant (5.33), without the knowledge of the unknown nonlinearity.

The auxiliary system associated with (5.33) is as follows,

$$\begin{aligned} \dot{z} &= pz - x_1 \\ \dot{x}_1 &= u_a \\ y_a &= -3x_1 + 5z - u_a. \end{aligned} \tag{5.34}$$

A Lyapunov function for the auxiliary system is $V_0 = \frac{1}{2}z^2 + \frac{1}{2}\xi^2$, where $\xi = x_1 - 2z$. It can be shown that $\dot{V}_0 \leq -z^2 - \xi^2 \leq 0$ along the trajectory of the zero dynamics of auxiliary system (5.34).

The Lyapunov function for the closed loop auxiliary system is proposed to be $V = V_0 + \frac{1}{2}y_a^2$. It can be shown that $\dot{V} \leq -(z - \frac{5}{2}y_a)^2 - (\xi - \frac{1}{2}y_a)^2 - (k_0 + 3)y_a^2 \leq 0$. Therefore, the $\frac{\partial V}{\partial \eta}$ term in the state feedback control law in equation (5.19) is $\frac{\partial V}{\partial \eta} = -y_a = 3x_1 - 5z + \eta$.

The mapping $\beta(y)$ and state estimation dynamics $\dot{\hat{z}}$are chosen as follows:

$$\begin{aligned} \beta(y) &= x_2 \\ \dot{\hat{z}} &= w = -4\hat{z} + 2x_1 - 3x_2 - u \end{aligned}$$

As a result, the state estimation error dynamics is:

$$\dot{\tilde{z}} = -4\tilde{z}$$

For the simulation, the number of basis functions is $l = 11$, with $\sigma^2 = 2$, $\phi_i = i - 6$, $i = 1, \ldots, 11$, and $w_m = 10$.

In the simulation, controller (5.19) is implemented with the following design parameters $k_0 = 10$, $k = 10$, $L = 5$, $k_\eta = 0.1, k_I = 0.4$, $r_w = 1$. The initial conditions for the states are $z(0) = x_1(0) = x_2(0) = 10$, $\eta(0) = 0$, and $\hat{z} = 0$. The uncertain parameter p is setting as $p = |\sin(t)|$. Simulation results in Figure 5.1 show that the proposed controller robustly stabilizes the system to the origin. Figure 5.2 shows nonlinearity estimation and parameter convergence.

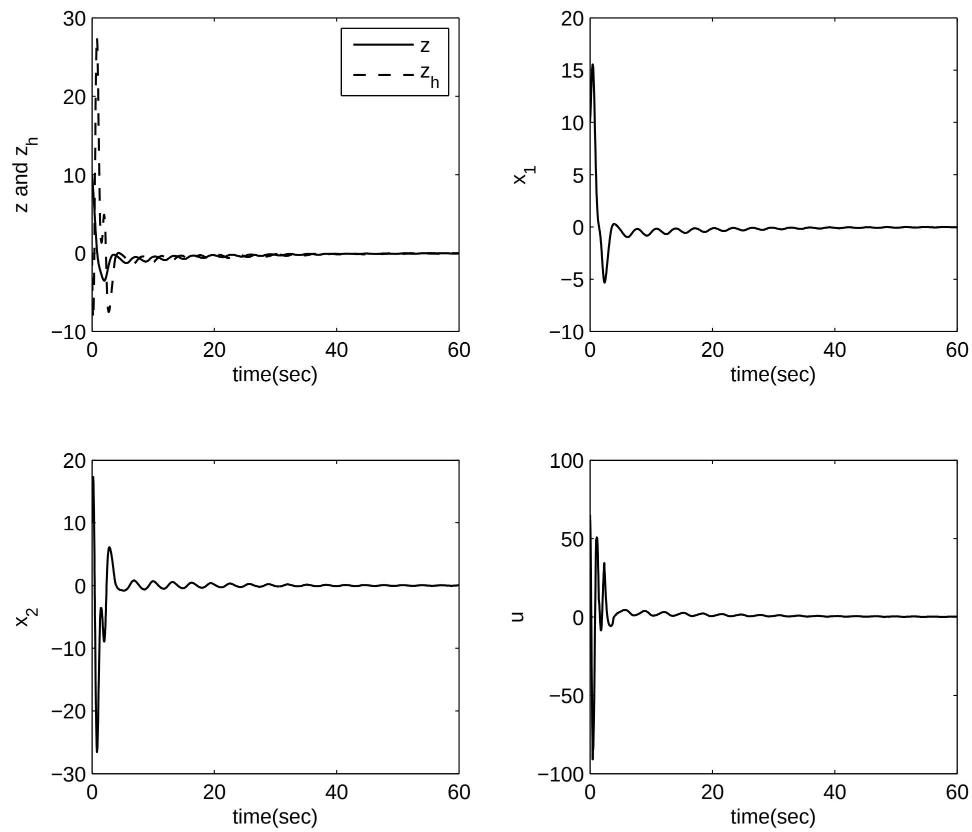

Figure 5.1: State trajectory and controller performance of example1

5.6 CONCLUSIONS

In this chapter, an output feedback stabilizing control design is studied for a class of non-minimum phase nonlinear system, with unknown nonlinearities. The proposed control design consists of two components. The first part is a stabilizing control with

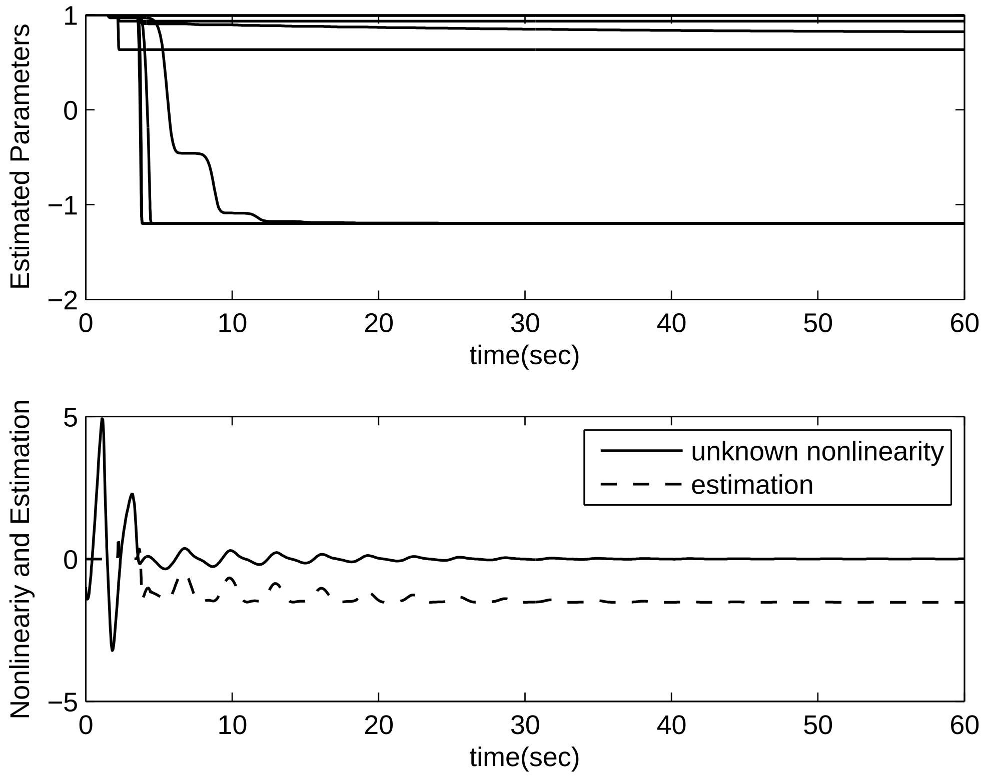

Figure 5.2: Unknown Nonlinearity Estimation and Parameter Convergence

respect to a synthetic output. The second part is a functional estimation control which also takes the synthetic output into account. In addition, the new design technique provides an approach to construct the Lyapunov function for the auxiliary system. The proposed method is demonstrated by simulation.

Chapter 6

Conclusions

The problem of output feedback control of nonlinear systems with system restriction are the research focus of this thesis. For systems with unknown dynamics or non-minimum phase behavior, output feedback control design remains an open research field. This thesis provides systematic approaches for the construction of stabilizing output feedback controllers for nonlinear systems with unknown nonlinearities non-minimum phase behaviors.

6.1 Contributions of Thesis

Output feedback tracking problem for a class of nonlinear systems with unknown nonlinearities was proposed and solved in Chapter 3. The proposed controller design approach is a Lyapunov-based technique. Radial Basis Functions are used to approximate the unknown dynamics. Closed-loop stability are guaranteed by an adaptive learning algorithm. In addition, the adaptive design ensures the convergence of tracking errors to an adjustable neighborhood of the origin. Chapter 3 also provides a solution to improve unknown function estimation, by incorporating a persistent exciting signal in the parameter update law. The benefit of good approximation of the unknown dynamics is that it provides knowledge of the poorly known system dynamics of interest. The additional information on the unknown nonlinearity improves closed-loop performance.

Chapter 4 focuses on output feedback stabilization for nonlinear non-minimum phase systems. The proposed approach removes a restrictive assumption from (Isidori, 2000), and proposes a constructive controller design technique, that achieves semi-global practical stabilization.

For systems with both non-minimum phase behavior and unknown nonlinearities, an output feedback control design technique is presented in Chapter 5. The proposed approach re-writes the unknown nonlinearity, and utilizes the auxiliary output idea from Chapter 4 and functional approximation techniques from Chapter 3. In addition, the explicit form of a candidate Lyapunov function for the system of interest was provided, instead of assuming its existence. To the author's best knowledge, this research work is the first attempt in literature on this challenging problem. It provides a starting point for all following research.

6.2 Future Research Recommendations

The theoretical results presented in this thesis applies to specific groups of nonlinear systems. In particular, more restrictions were placed on the applicable systems for the proposed output feedback control design technique in Chapter 5, due to the difficulty in searching for the Lyapunov function and suitable observer design methods. It will be desirable to extend the results to a wider collection of systems, such as, the more general system proposed at the beginning of Chapter 5.

It is also of interest to investigate the class of non-minimum phase systems studied in (Andrieu and Praly, 2008), with unknown parameters or unknown dynamics. Utilizing the ISS property of the zero-dynamics, adaptive backstepping may be applied to construct a stabilizing control law, with parameter estimation error being fed as disturbances into the zero dynamics. Similarly, functional approximation error may be treated as a bounded disturbance into the zero-dynamics. However, a more complex detectability condition may be needed in order to solve the problem.

Another recommended future research area is the performance comparison study for different functional approximation methods. For example, other linear functional approximation methods (such as polynomials) or nonlinear functional approximation method (such as nonlinear RBF, where the center and the width of the basis functions are adapted). The choice of an appropriate functional approximation remains a very challenging problem for any given class of system. Gaining knowledge of the performance of various approximation approaches on different types of nonlinear systems will help find the most suitable approximation techniques for a certain type of nonlinear system.

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