Estimation of Time-Varying Parameters and Its Application to Extremum-Seeking Control

by

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To my wonderful parents

for all of their

nagging and support!

Abstract

This dissertation considers the adaptive estimation of time-varying parameters and its use in extremum-seeking control problems. The ability to estimate uncertain time-varying behaviour can have a significant impact on a control system's performance. Hence, the problem of time-varying parameter estimation has been of considerable interest over the last two decades.

The present work provides a formal scheme for time-varying parameter estimation in a class of nonlinear systems. The geometric concept of invariance is the key concept for the parameter estimation techniques developed in this thesis. The techniques use a number of high gain estimators and filters that generate an almost invariant manifold. The almost invariance property allows an implicit mapping and a parameter update law that guarantees exponentially convergence to a small region of the true values of the time-varying parameters. A generalization of the invariant manifold approach is considered to deal with the estimation of periodic parameters with unknown periodicity.

In another step, this thesis seeks to apply the proposed time-varying estimation technique to the solution of extremum-seeking control problems. In extremum-seeking control, a gradient descent algorithm is used to find the optimal value of a measured but unknown cost functions. The contribution of this aspect of the thesis is the

formulation of the extremum-seeking control problem where the unknown gradient of the cost is estimated as a time-varying parameter using the proposed invariance based estimation technique. The proposed approach is extended for the solution of constrained steady-state optimization problems. We establish two methods for finding the optimal points for systems with unknown objective functions that are subject to unknown/uncertain dynamics. For systems with unknown dynamics, a nonlinear proportional-integral controller is designed to find the optimal solution. Then for a class of control affine systems with known high frequency gains, an inverse optimal control technique is used for the direct design of a gradient-based controller.

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Chapter 1

Introduction

This thesis studies the estimation of time-varying parameters in a class of nonlinear dynamical systems. A new parameter estimation technique is proposed for estimation of time varying parameters. The proposed estimation technique is shown to be of interest in a number of applications, especially for extremum-seeking control (ESC) problems. In this introductory chapter, the problem statement and motivations for the study is summarized. A summary of the contributions of the research presented here is provided. Finally, the thesis organization is outlined.

1.1 Problem Statement and Motivations

In many industrial applications, the systems under study are subject to parametric uncertainties. These uncertainties, which can arise from aging, hardware damages and environmental conditions, can degrade system performance. Stability and robustness analysis in the presence of uncertainties is a crucial part of the control system design [18]. Parameter identification is a powerful tool that can enhance the overall stability and robustness properties of closed-loop control systems [51, 2]. Parameter estimation is an essential part of adaptive control techniques that is applied to many application fields including spacecraft, robotics, power systems and process control [74]. The control design for a system with unknown parameters can be divided into two separate steps of identification and control. Figure 1.1 shows a basic block diagram illustrating how parameter identifier can be incorporated into a control scheme.

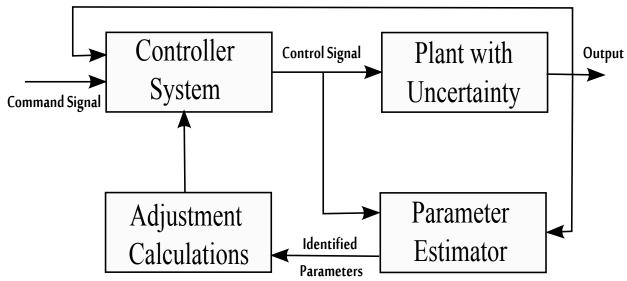

Figure 1.1: Simple block diagram for combination of identifier and controller.

Least-squares and maximum-likelihood are two of the most common techniques for parameter estimation in linear systems. In the former, the unknown parameters are estimated by finding values of the parameter that minimize the sum of squared errors. The accuracy of this approach highly depends on the quality of the measured data, quantization errors and the sampling rate. Inaccurate measurements can introduce bias in the parameter estimates that can reduce the robustness of the optimal solution. As a result, these techniques tend to be inefficient for nonlinear systems with parametric uncertainties [82].

In many instances, estimating parameters in an on-line fashion is advantageous. This approach can improve the control system performance over a wide range of operating conditions [37]. In many industrial applications, mathematical descriptions of processes can contain unknown parameters that are subject to slow or fast time variations. The production output, market demand and trade balance are examples of important economic variables that are subject to seasonal changes [86]. Another example is an induction motor drive, where the rotor and stator resistance values change continuously with different operating and environmental conditions [44, 72]. Bioreactors pose an interesting problem because the kinetic rate is a complex function of the process states. Finding a suitable analytical and experimental model of this function is not easy [26]. In these cases, on-line (real-time) estimation is the only solution for system identification.

On-line parameter estimation can be considered for other purposes such as fault detection, learning, adaptation and real-time optimization (RTO) [10]. In on-line schemes, the uncertain parameters are identified causally at each instant from the current and past measurements whereas offline methods utilize complete information-rich data sets. Despite the well established approaches for estimation of uncertain constant parameters, the estimation of time-varying parameters remains challenging.

Extremum-seeking control can be viewed as a dual control methodology having optimization and control features. ESC provides a mechanism to implement on-line optimization in a control system. The technique can be used to determine the optimal process set-points that optimize an uncertain performance function. Provided that the performance function is locally convex (concave), this method obtains a local minimum (maximum). However, the time-varying uncertainties associated with the function make it necessary to use adaptation to search for the unknown optimum. The standard ESC considers a measurable objective function that has unknown mathematical structure. Averaging analysis and singular perturbations are the principal mathematical tools for solving the uncertain optimization problem. Although many researches have been done to improve the performance of ESC algorithm, the schemes may still suffer from major disadvantages due to the limitations associated with the corresponding mathematical analysis.

The two principal objectives of this thesis are:

1. Provide a new formal scheme for real-time estimation of uncertain time-varying parameters in nonlinear dynamical systems.

2. Implement the estimation algorithm to present an alternative ESC approach to improve the performance of the real-time optimization problem.

1.2 Statement of Contributions

The following list summarizes the contributions of this thesis to the body of knowledge in parameter estimation and extremum-seeking control.

1. Section 3.3 establishes an alternative adaptive estimation technique based on the geometric concept of almost invariant manifolds. The proposed parameter update law converges exponentially to a small region of the true values of the time-varying parameters. This work has been published in [64].

2. Section 3.4 presents an application of the geometric-based estimation scheme for the estimation of specific growth rate in bioreactors. Since it is assumed that the biomass concentration is not available for measurement, an asymptotic observer is considered for the estimation of the unmeasured state variables. This work has been published in [62].

3. Chapter 4 introduces a generalization of the almost invariant manifold to estimate the periodic time-varying parameters with unknown periodicity. This approach has the advantage of providing estimation of an upper bound to all higher order time derivatives of the time-varying parameters. This work has been accepted for publication in [66].

4. Section 5.3 establishes a novel ESC technique to compensate for averaging analysis in conventional ESC. The proposed estimation technique is used to estimate the gradient of the unknown static cost function. Then a controller is designed to yield a gradient descent algorithm. This work has been published in [65].

5. Section 5.3 describes an extension of the time-varying ESC approach for the solution of constrained optimization problems. An augmented barrier function formulation is proposed to transform the constrained optimization problem to an unconstrained problem. A part of this work has been published in [35].

6. Section 6.2 presents a black-box ESC for nonlinear systems with unknown cost functions that are subject to the unknown system's dynamics. A nonlinear proportional-integral (PI) controller is designed to compensate for time-scale separation in conventional ESC. This work has been published in [63].

7. Section 6.3 introduces a grey-box ESC for nonlinear system with unknown cost function subject to the uncertain dynamics with a known high frequency gain. An inverse optimal control technique is used for the direct design of the control law. The proposed ESC results in an improvement of the transient performance and fast convergence to the optimal solution. The work has been accepted for publication in [67].

1.3 Thesis Organization

This dissertation is organized as follows. A review of the current literature that is relevant to the proposed research work is presented in Chapter 2, followed by a brief mathematical description of "Immersion and Invariance".

Chapter 3 presents an adaptive algorithm for estimation of the unknown time-varying parameters. The technique considers the idea of almost invariant manifolds to generate an implicit mapping that relate the known variables and the unknown variables. A robust parameter update law is designed through an adaptive compensator. The application of this method is proposed for the estimation of the specific growth rate parameter in bioprocesses.

Chapter 4 generalizes the idea of almost invariant manifold to provide an exact invariant manifold for periodic time-varying parameters. The proposed invariant manifold provide an update law for estimation of the unknown periodic signals and the upper bound values of their rate of change, simultaneously.

Chapters 5 and 6 focus on the application of the proposed time-varying estimation technique in ESC problems. In Chapter 5, we establish a time-varying ESC technique for the solution of unknown static optimization problem. The time-varying parameter estimation technique is used in a gradient descent algorithm to find the optimal solution. A similar idea is considered for the solution of constrained steady-state optimization problems. The augmented barrier function is the key tool to convert a constrained optimization problem to an unconstrained problem.

Chapter 6 presents a new approach for unknown cost function subject to unknown/uncertain nonlinear dynamics. In the case that the nonlinear dynamics are completely unknown, a nonlinear PI controller is designed to find the optimal value of the input at the steady-sate. In the case that the high frequency gain of the system dynamics is available, the inverse optimal control technique is used to find the optimal values of the state variables. Both of these algorithms provide fast convergent of the closed-loop ESC system that operates within the time-scale of the system dynamics.

Finally, Chapter 7 summarizes the contributions of the thesis and outlines directions for future research.

Chapter 2

Literature Review

The relevant literature for the proposed work is reviewed in this chapter. Special attention is paid to parameter estimation, extremum-seeking control and the geometric approach of immersion and invariance (I&I).

2.1 Time-Varying Parameter Estimation

2.1.1 On-line versus Offline Identification

One of the key challenges in industrial applications is to achieve reliable real-time estimation of unknown parameters. A simple way for on-line estimation of the parameters is to modify the offline techniques in a recursive form. To illustrate the recursive identification, least-squares and recursive least-squares (RLS) techniques are discussed in this section.

Consider a linear time invariant (LTI) model to approximate the output $y_{(k)}$ of a control system with input variable $u_{(k)}$:

$$y_m(k) = \sum_{i=1}^{n} [a_i y_m(k-i) + b_i u(k-i)] \tag{2.1}$$

where y_m is the model output, a_i and b_i are constant unknown parameters. The prediction error is given by

$$e(k) = y(k) - y_m(k) - \sum_{i=1}^{n} a_i[y(k-i) - y_m(k-i)] \tag{2.2}$$

substituting (2.1) in (2.2) results in

$$y(k) = e(k) + \sum_{i=1}^{n} a_i y(k-i) + b_i u(k-i). \tag{2.3}$$

The equation (2.3) is linear in parameters. If we let

$$\theta = [a_1 \cdots a_n \; b_1 \cdots b_n]^T$$

and

$$\phi(k-1) = [y(k-1) \cdots y(k-n) \; u(k-1) \cdots u(k-n)]^T$$

then we can write (2.3) in the standard regression form as

$$y(k) = \phi(k-1)^T \theta + e(k), \quad k = 1, \ldots, N \tag{2.4}$$

or in vector form as

$$Y = \Phi^T \theta + e \tag{2.5}$$

where Y is the vector of output measurements, Φ is the matrix containing past outputs and inputs, θ is the unknown constant parameter vector and e is the error vector. The identification problem is solved as an unconstrained optimization problem by introducing the objective function $V(\theta) = e^T e$. This system is identifiable if the optimization problem has a unique solution. The minimum of the cost function can be obtained by solving the system of equations $\frac{\partial V}{\partial \theta} = 0$ for $\hat{\theta}$. The solution is easily obtained as

$$\hat{\theta} = (\Phi\Phi^T)^{-1}\Phi Y. \tag{2.6}$$

To extend this method for on-line estimation, we need to show that the parameter estimation can be achieved recursively. Consider the equations (2.4)-(2.6) for the first sequence of N data. The following recursive equation holds between the current and the next parameter estimations

$$\hat{\theta}(N+1) = \hat{\theta}(N) + K(N)[Y(N+1) - \Phi(N)^T\hat{\theta}(N)] \tag{2.7}$$

where the second term on the right hand side of (2.7) is called the correction term. The matrix $K(N)$ is a weighting matrix which will be discussed more in the next subsections. Looking closely at (2.6), it is desirable that the information matrix $(\Phi\Phi^T)$ be non-singular. This leads to the idea of persistence of excitation (PE). A unique solution is guaranteed if the information matrix is invertible and bounded [32].

One major application of on-line parameter identification in dynamical systems is closed-loop identification and control (CLIC). The main advantage of this method is its ability for the identification of systems with unstable open loop dynamics. Adaptive dual control (ADC) systems are a specific class of CLIC techniques, where the control signal is designed in such a way that the control goal and improvement of parameter estimation are achieved at the same time [5]. Parameter convergence is an important issue as it enhances the overall stability and robustness properties of the closed-loop adaptive systems [51]. In both linear and nonlinear adaptive systems, parameter convergence is related to the satisfaction of the PE condition. This conditions stipulates that there exist constants c_1, c_2 and T_0 such that the trajectory of the system satisfy the following inequality [74]:

$$c_1 I \leq \frac{1}{T_0} \sum_{i=k}^{k+T_0-1} \phi_i \phi_i^T \leq c_2 I, \quad \forall k > T_0 \tag{2.8}$$

where $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^P$ is a regressor vector. The matrix I is the identity matrix with suitable dimension.

Although the matrix $\phi(\tau)\phi^T(\tau)$ is singular for all τ, the PE condition requires that ϕ rotates sufficiently in space to ensure that the summation term is uniformly positive definite over any interval of some length T_0. This condition is unfortunately difficult to check because of its dependency on closed loop signals. Fortunately, this shortcoming has been remedied for linear systems.

In adaptive linear systems, the PE condition can be converted to a sufficient richness (SR) condition on the reference input signal. Necessary and sufficient conditions for parameter convergence are then expressed in terms of the reference signal. A popular result implies that exponential convergence is achieved whenever the reference signal contains enough frequencies. Otherwise, convergence to a subspace in the parameter space is achieved [74]. A brief description of SR concept and its relation to PE conditions is provided in Appendix B.

The first attempts to relate a closed loop PE condition to a SR condition on the external reference signal for nonlinear adaptive control systems are presented in [51]. This work provides a procedure for determining *a priori* whether or not a specific reference signal is sufficiently rich for a specific output feedback nonlinear system, and hence whether or not parameter estimates is expected to converge. However, this technique is not directly applicable to ESC problems because the reference signal in this class of problem is not known in advance. In the area of adaptive extremum-seeking control, the efficiency of the scheme depends on the convergence properties of the adaptive algorithm utilized [36]. Hence, it is crucial to define an excitation signal that generates acceptable compromise between the objectives of identification and control. This problem has been addressed in [3], where a perturbation signal is designed in closed-loop fashion to satisfy the sufficient richness condition on the desired set-point. The perturbation signals are sinusoidal functions with time-varying amplitudes that are added to the command signal. The minimum values of the coefficients have been achieved at the steady-state by solving a quadratic optimization problem. The PE condition is appended to the optimization problem through constraints. Although the advantage of this work is in the identification part, it has been shown that parameter convergence is achieved with a minimum loss of control performance.

Finite-time parameter estimation has been proposed in [4] for a class of nonlinear systems subject to a given excitation condition. The sufficient excitation in this finite-time approach is the invertibility of the integral of a filtered regressor matrix, which is equivalent to a standard PE condition. The main advantage of this finite-time approach is the independence of the invertibility conditions to the control input u, the parameter identifier $\hat{\theta}$ and the velocity state vector $\dot{x}$. A modification of this technique has been proposed in [5] where an adaptive compensator is used to circumvent the need for the real-time inversion of the regression matrix.

2.1.2 Adaptive Control of Time-Varying Parameters

Numerous results have been reported for the adaptive control of linear and nonlinear systems with uncertain time-varying parameters. Time-varying adaptive control problems are generally difficult, and no general theory exists for the stability and convergence analysis of such system. One practical way [89] is to classify the time-varying parametric uncertainties into different categories. The classification can be made by considering *a priori* knowledge of the unknown parameters, such as periodicity, linearity and the rate of changes in the parameters. However, classification over time-scale separations of the uncertain parameters is the most common approach.

In [57], it was shown that global stability can be achieved for time-varying linear systems, without the need for a PE condition. An indirect adaptive pole placement control is considered in [85] for linear systems with bounded but fast variations of the unknown parameters. The robust adaptive regulation [55], and tracking [56] problems have also been proposed for linear systems with time-varying (LTV) parameters with unknown and bounded additive disturbances. The algorithm achieves disturbance rejection in closed-loop. It has also been shown that robustness with respect to exogenous disturbances, parameter estimation errors and parameter variations from their nominal values can be achieved. Recently, a direct adaptive control approach based on linear sampled-data periodic controller was proposed to solve a model reference adaptive control problem for linear systems [58]. The proposed controller can handle rapid changes of the plant parameters with adequate transient behaviour and acceptable control effort.

Over the past few years, adaptive control design techniques for nonlinear systems with time-varying parameters have been developed. In [29], a robust adaptive control technique is employed for the control of nonlinear systems with unknown control direction, uncertain parameters and unknown disturbance in strict feedback form. It is shown that in the case of time-varying uncertainties, global uniform ultimate boundedness (GUUB) of the system can be achieved.

Adaptive stabilization of a more general nonlinear system with arbitrarily fast time-variations of the unknown parameters is provided in [24]. It is proven that local and global stability can be obtained for both unknown and known control matrices, respectively. In [76], adaptive stabilization of a nonlinear system with slowly varying parameters by an extension of the control Lyapunov function is considered. Another important class of time-varying systems are chaotic systems. Since the chaotic systems have severe nonlinearities with specific behaviour, their synchronization is not an easy task in systems with unknown and time-varying parameters. To address these synchronization problems, different adaptive schemes for chaotic systems have been presented [73, 7].

2.1.3 Identification of Time-Varying Parameters in Linear Systems

The conventional LTI identification strategies are not valid for the LTV systems, due to the changes of the operating conditions. Many of the algorithms for the identification of time-varying parameters are obtained by modifying the designs developed for constant parameters such as RLS. The key idea behind these modifications is to use a smaller data set by excluding the old measurement data. However, these techniques are suitable for slowly varying parameters, and the arbitrarily fast variations of the parameters results in ill-conditioning [94].

Linear parameter-varying (LPV) systems are a subclass of LTV systems, where the linear system matrices depend on the time-varying parameters. LPV systems can also be realized as the weighted combination of multiple linear models. The weighting functions can consist of certain signals such as the system's states and outputs, or arbitrary time-varying functions [15].

If the time-varying parameters appear in linear fractional structure, it is possible to simplify the system to a least-squares problem. Consider the following LPV model as

$$A(\delta, \theta(k))y(k) = B(\delta, \theta(k))u(k) \tag{2.9}$$

where y is the output, u is the input, δ is the delay operator and $\theta(k)$ is the time-varying parameter. $A(\cdot)$ and $B(\cdot)$ are described as

$$A(\delta, \theta(k)) = 1 + a_1(\theta(k))\delta + \cdots + a_n(\theta(k))\delta^n$$
$$B(\delta, \theta(k)) = b_0(\theta(k)) + b_1(\theta(k))\delta + \cdots + b_m(\theta(k))\delta^m$$

a_i and b_j are linear combinations of known basis functions, which are given by

$$a_i(\theta(k)) = a_i^1 f_1(\theta(k)) + \cdots + a_i^N f_N(\theta(k))$$
$$b_j(\theta(k)) = b_j^1 f_1(\theta(k)) + \cdots + b_j^N f_N(\theta(k))$$

where a_i^l and b_j^l are constant parameters. There are many options for the basis functions such as polynomial functions, periodic functions (for physical system with periodicity) and the sigmoidal functions which are used in neural networks [11].

All the constant coefficients are denoted by the $(n + m + 1) \times N$ matrix

$$\Theta = \begin{bmatrix} a_1^1 & \cdots & a_n^1 & b_0^1 & \cdots & b_m^1 \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ a_1^N & \cdots & a_n^N & b_0^N & \cdots & b_m^N \end{bmatrix}^T \tag{2.10}$$

and the regressor matrix, which consists of past information and basis functions, is given as follows:

$$\Phi(k) = \begin{bmatrix} -y(k-1) \\ \vdots \\ -y(k-n) \\ u(k) \\ \vdots \\ u(k-m) \end{bmatrix} \begin{bmatrix} f_1(\theta(k)) & \cdots & f_N(\theta(k)) \end{bmatrix}. \tag{2.11}$$

Considering (2.9)-(2.11) and the measurement noise, the model can be written in the standard regression form as

$$Y(k) = \Phi^T(k)\Theta + \nu(k). \tag{2.12}$$

By (2.5) and (2.12), it is clear that these two equations are similar, where (2.5) is in vector form while (2.12) is in matrix form. The only difference with (2.5) is that the uncertainty in (2.5) shows up as a difference between reference model and the true model, while the uncertainty in this situation is a sequence of white noise with zero mean. Hence, all least-squares methods are implementable and the PE conditions can be assigned in a similar manner. For LPV systems with linear fractional dependence of the parameters [49], gradient based nonlinear optimization methods have been used for parameter estimation in a recursive algorithm.

In many cases, the input-output identification of LPV systems is not suitable and a state-space representation is necessary for control design purpose [84]. Consider the state-space structure of an LPV system

$$\begin{aligned} x(k+1) &= A(\theta(k))x(k) + B(\theta(k))u(k) \\ y(k) &= C(\theta(k))x(k) + D(\theta(k))u(k) \end{aligned} \tag{2.13}$$

where x, y, u and θ are the state variables, output variables, input variables and the scheduling parameter, respectively. The identification goal is to estimate the matrices A, B, C and D using measurements of the output, input and the scheduling parameter. Many identification techniques deal with the LPV systems in state-space representations [87, 17].

2.1.4 Identification of Time-Varying Parameters in Nonlinear Systems

Nonlinearity is an inherent characteristic of physical systems. The nonlinear terms in the dynamic systems may cause specific phenomena, even weak nonlinear systems can display extremely complex behaviour. As a result, nonlinear system identification is a much more demanding task and unlike, the linear case, there are no general analysis methods that can be applied to all systems [42]. The focus in this section is on the estimation of time-varying parameters in nonlinear dynamical systems.

Observer-based Identification

Consider a nonlinear time-varying and deterministic system

$$\begin{aligned} \dot{x} &= f(x, u(t), \theta(t)) \\ y(t) &= h(x) \end{aligned} \tag{2.14}$$

where $u(t) \in \mathbb{R}$ is the manipulated input, $y(t) \in \mathbb{R}$ is the measurable output, $x(t) \in \mathbb{R}^n$ is a state vector, $\theta(t) \in \mathbb{R}^p$ are unknown time-varying parameters, $f(\cdot)$ is a smooth vector valued function and $h(\cdot)$ is a smooth function. An observer can be designed to estimate the state space variables and the unknown parameters. Using a standard approach, the unknown parameters are combined in the vector of state variables to solve the joint state and parameter estimation problem. One such technique is the extended Kalman filter (EKF). The EKF is briefly introduced in the following. To simplify the presentation, it is assumed that the unknown parameters are constant. First consider the augmented state $z = [x^T, \theta^T]^T$ with dynamics

$$\dot{z} = \widehat{f}(z, u) = \begin{bmatrix} f^T(x, u, \theta) & 0 \end{bmatrix}^T. \tag{2.15}$$

The general state estimation can be written simply as

$$\begin{aligned} \dot{\hat{z}} &= \widehat{f}(\hat{z}, u) + K(\hat{z})(y - \hat{y}) \\ \hat{y} &= h(\hat{z}) \end{aligned} \tag{2.16}$$

where $\hat{x}$ and $\hat{y}$ are the estimates of the states and the output, $K(\cdot)$ is the observer gain vector. The state estimation error $e = z - \hat{z}$ has dynamics given by

$$\dot{e} = \widehat{f}(\hat{z} + e, u) - \widehat{f}(\hat{z}, u) - K(\hat{z})(h(\hat{z} + e) - h(\hat{z})). \tag{2.17}$$

Linearization of (2.17) at the point $e = 0$, results in

$$\dot{e} = (A(\hat{z}) - K(\hat{z})H(\hat{z}))e \tag{2.18}$$

with $A(\hat{z}) = \frac{\partial \widehat{f}}{\partial z}|_{z=\hat{z}}$ and $H(\hat{z}) = \frac{\partial h}{\partial z}|_{z=\hat{z}}$. The observer gain $K(\hat{z})$ should be designed such that the objective function

$$J = \int_0^t e^T(\tau)e(\tau)d\tau$$

is minimized. Since the system is deterministic, the observer gain can be easily obtained by solving of the following dynamic Riccati equation [23]:

$$\dot{R} = -RH^T HR + RA^T + AR \tag{2.19}$$

where $R \in \mathbb{R}^{n\times n}$ is a symmetric matrix. The final gain is obtained as $K(\hat{z}) = R(\hat{z})H^T(\hat{z})$.

Despite the advantages of the EKF, it has some drawbacks [12]. First, there is no guarantee of the convergence and the stability of this algorithm (especially for time-varying nonlinear systems [92]). An adaptive observer has been designed in [12], which can improve the performance of the EKF. The key feature of this approach is a smooth transformation of the system to a time-varying observable canonical form. The main benefit of this canonical form is that the system is expressed linearly with respect to the unknown variables.

Simultaneous state and parameter estimation can also be achieved by designing a high gain adaptive observer. In [25], a high gain observer is proposed for a class of nonlinear systems with Lipschitz nonlinearities. The main advantage of this observer-based estimator is ease of implementation and tuning. A high gain observer-based estimator is introduced in [21], where the unknown time-varying parameters are appended to the state vector. The observer is shown to provide estimates of the states and time-varying parameters.

An alternative approach to observer-based estimation is to decouple the state and parameter estimation problems [23]. Consider the nonlinear dynamical system given by:

$$\begin{aligned} \dot{x}_p &= F_p(x)\theta + g(x,u) \\ \dot{x}_q &= f_q(x,u) \end{aligned} \tag{2.20}$$

where $x_p \in \mathbb{R}^p$, $x_q \in \mathbb{R}^{n-p}$ and $\theta \in \mathbb{R}^p$ are system states and unknown (constant) parameter vector. Assume that the states are measurable, $g(\cdot)$, $f_q(\cdot)$ are smooth vector-valued functions and $F_p(\cdot)$ is a smooth matrix-valued function. The Luenberger design technique results in the following adaptive observer [23]:

$$\begin{aligned} \dot{\hat{x}}_p &= F_p(x)\hat{\theta} + G(x,u) - K(x_p - \hat{x}_p) \\ \dot{\hat{\theta}} &= F_p(x)^T P(x_p - \hat{x}_p) \end{aligned} \tag{2.21}$$

where the constant matrices K and P should satisfy the following linear Lyapunov equation:

$$K^T P + PK = -Q \tag{2.22}$$

where K, P and Q are positive definite matrices. The matrix $F_p(\cdot)$ is a regressor matrix, that should satisfy the PE condition.

Sliding-mode observer (SMO) is a common approach for decoupled state and parameter estimation (see [16] and the references therein). The main advantages of SMO are simplicity of design, finite-time convergence and robustness against nonlinear uncertainty. The sliding-mode concept consists of two main parts. The first step is to select a sliding (switching) surface that is independent of the uncertainties. In the second step, a variable structure algorithm is applied to force the system's variables to reach the desired surface in finite-time, followed by a sliding motion for robust performance. A general procedure for time-varying parameter estimation of a nonlinear systems is provided in [41]. An extension of this approach to systems with partially known states is presented in [6]. The main disadvantage of sliding-mode approaches is the reliability to estimate the sign of the parameter estimation error which may be very difficult to obtain in practice. Indeed, due to the discontinuous nature of the input (control law), a chattering phenomena can occur that can cause problematic behaviours in the actuators of the system. To address the chattering problem, high-order sliding-mode approach was introduced in [50]. It is shown that higher order sliding-modes can improve the performance of the standard (first order) sliding-mode where finite-time convergence is preserved [60]. In general, the second order sliding-mode (and in particular, the super-twisting) algorithm is the most popular technique among higher order sliding-mode techniques [71]. The finite-time convergence and the stability analysis of the work in [50] has been proved by geometrical methods.

The first Lyapunov stability proof for the super-twisting approach is presented in [61] where the design gains are achieved by solving an algebraic Lyapunov equation.

Set-based Identification

Set-based estimation is an alternative approach for systems with bounded unknown parameters. This method considers a feasible solution set which is compatible with model uncertainties and tries to minimize the size of the set at each sampling instant.

In [22], a set-based adaptive estimation is used to estimate time-varying parameters in nonlinear systems along with an uncertainty set. The proposed method is such that the uncertainty set update is guaranteed to contain the true value of the parameters. This technique does not require a functional representation of the time-varying behaviour nor the polynomial approximations of the time-varying parameters. The application of this method is presented for estimation of unknown heat loads and heat sinks in building systems [22].

2.2 Extremum-Seeking Control

Extremum-seeking (also known as peak seeking or self optimizing) control is a class of adaptive control that deals with regulation to unknown set points. The control algorithm finds the operating set-points that optimize a performance function. ES is implementable to physical nonlinear systems and/or systems with nonlinear control objective functions, where the nonlinearities have local extrema. Thus, ES can be used for both adjusting a set point to obtain an optimal value for the output, or for tuning parameters of a feedback control law [8]. Although ESC technique was initiated and used for decades, the first precise stability proof for general nonlinear systems has been provided in the early 2000s [47].

In the conventional ESC, the objective function is assumed to be available for on-line measurement but has no known mathematical description. The function is described in terms of unknown variables and it is assumed that there exists an adaptive law that can find the optimal value of the cost function over a range of the system's variables. The most popular schemes in this category are the methods based on averaging and singular perturbation techniques. Figure 2.1 shows a basic extremum-seeking loop for a static map subject to a sinusoidal perturbation. In this diagram, the uncertain static map, between the measurable output y and the input variable θ, is described by the unknown function $f(\cdot)$. The variable $\hat{\theta}$ is estimation of the unknown optimal input θ^*. The main idea is to find an adaptive update law for $\hat{\theta}$ to reach the value θ^* that results in an optimal value of output $y^* = f(\theta^*)$.

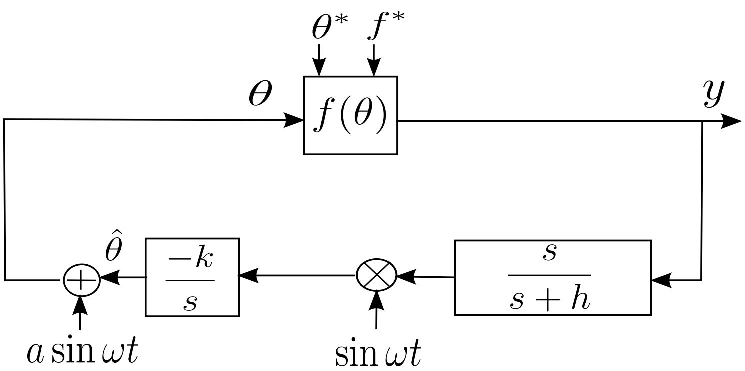

Figure 2.1: Basic extremum-seeking scheme [8].

This class of ESC uses an excitation signal to obtain information about the gradient of the unknown function. A known perturbation signal is added to the input of the controlled system to induce a cyclic response in the performance function. The functional signal response is then passed through various filter(s) to determine the sign of the unknown gradient (derivative of the function with respect to the parameter) and the decision variable θ is adjusted following a gradient descent algorithm. A successful application of the scheme is based on an appropriate choice of the filter parameters, the perturbation signal parameters and the adaptation gain.

The first precise stability proof of feedback ESC, based on averaging and singular perturbations techniques was provided in [47]. Over the last few years, many researchers have considered various approaches to overcome the limitations of ESC. In [45], some of these conditions were removed by using dynamic compensators, while the measurement noise rejection was also achieved.

The first proof of non-local and semi-global stability analysis of ESC was presented in [78]. The main contribution of this work was to provide an explicit description of the domain of the attraction for the closed-loop ESC. In [79], the effect of choosing different dither signals on the closed-loop performance was investigated. The performance indices considered include the speed of convergence, the domain of convergence and the ultimate bound on the trajectories. The results show that the choice of dither signal (amplitude and frequency) has a significant impact on the performance of ESC scheme. In [80], it was shown that global optimization can be achieved by allowing the amplitude of the dither to be adjusted in an adaptive fashion. If these sufficient conditions hold, the convergence to a small neighborhood of the global extremum is obtained from a large set of initial conditions.

In most extremum-seeking approaches, the gradient of the output map is estimated by using a combination of a high-pass filter and a low-pass filter. An EKF can be used [30] as an alternative to the low-pass and high-pass filters. The main advantage of the EKF is a faster response compared to classical filters and the potential to extend the algorithm to multivariable ESC problems.

Recently, non-gradient approaches have also been proposed for the design of ESC systems [31, 69]. A Newton-based ESC was developed in [31]. The Newton-based ESC techniques provides superior transient performance. It also provides an effective mechanism to solve multi-input ESC problems. A periodic sampled-data controller based on the discrete-time Shubert algorithm to a continuous-time system is proposed in [69]. This sampling method does not require information of the objective function gradient. It can be shown to reach the global optimum. It is therefore ideal for application in non-convex problems. It does require considerable sampling which must be obtained near the steady-state maps. As a result, it can be very slow in typical applications.

2.3 Immersion and Invariance

This section briefly reviews elements of the immersion and invariance technique. For the sake of demonstration and comparison with future developments in this thesis, an application of I&I for parameter identification is presented. The invariance principle has been widely used for mathematical description of physical systems [88]. The following is a basic definition of invariant sets (manifolds) [43].

Definition 2.1. *Consider a nonlinear dynamical system of the form*

$$\dot{x} = \mathscr{F}(x),\ x(t_0) = x_0 \tag{2.23}$$

where $\mathscr{F} : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth vector-valued function with an arbitrary initial condition x_0 at the time t_0. Then, a set $S \subset R^n$ is called an invariant set for (2.23) if, for any $x_0 \in S \Rightarrow x(t) \in S,\ \forall t \in \mathbb{R}$. In addition, S is called an invariant manifold *if S is defined as a manifold.*

2.3.1 Main Algorithm of I&I

A brief discussion of the I&I methodology is given in this section. The presentation is based on the work of Astolfi and Ortega [9].

Consider a nonlinear dynamics with a general structure as,

$$\dot{x} = f(x, u) \tag{2.24}$$

where $f(\cdot, \cdot)$ is a smooth vector-valued function, with state $x \in \mathbb{R}^n$ and control input $u \in \mathbb{R}^m$. The objective is to find a state feedback control law $u = u(x)$ such that the closed-loop system is locally (globally) asymptotically converges to the equilibrium point $x^* \in \mathbb{R}^n$. For this purpose, a locally (globally) asymptotically stable target dynamical system $\dot{\zeta} = \alpha(\zeta)$ with equilibrium point ζ^*, an immersion mapping $x = \pi(\zeta)$ and a control function $u = \omega(x)$ are defined, such that

$$\frac{\partial \pi(\zeta)}{\partial \zeta} \alpha(\zeta) = f(\pi(\zeta), \omega(\pi(\zeta)))$$

where $\zeta \in \mathbb{R}^p$ with $p \leq n$. In the next step, an implicit manifold $\varphi(x) = 0$ should be found such that

$$\{x \in \mathbb{R}^n \mid \varphi(x) = 0\} = \{x \in \mathbb{R}^n \mid x = \pi(\zeta) \text{ for some } \zeta \in \mathbb{R}^{\mathrm{p}}\}. \tag{2.25}$$

If the off-the-manifold coordinate variable z is defined as $z = \varphi(x)$, then a control law $u = \eta(x, z)$ can be designed such that

$$\begin{aligned} \dot{z} &= \frac{\partial \varphi}{\partial x}[f(x, \eta(x, z))] \\ \dot{x} &= f(x, \eta(x, z)) \end{aligned} \tag{2.26}$$

are bounded and satisfy

$$\lim_{t \to \infty} z(t) = 0. \tag{2.27}$$

In other words, the control law renders the manifold $x = \pi(\zeta)$ attractive and invariant while keeping the closed-loop trajectories bounded. Therefore, the closed-loop system will reach the trajectories of the desired target system where the steady-state $x^* = \pi(\zeta^*)$ is a locally (globally) asymptotically stable equilibrium.

Remark 2.1. *The choice of target dynamical system has a significant impact on the ease to find an analytic solution for the I&I problem. This is not a trivial task for general nonlinear systems. However, in some applications (see [1], [9] and the references therein) there exists a suitable selection of the target dynamics candidate to solve the I&I-stabilization problem.*

Remark 2.2. *Since there exists an alternative choice for $\varphi(x)$, a wide range of control laws can be designed for the closed-loop system. If there exists a partition of $x = [x_1^T, x_2^T]^T$, with $x_1 \in \mathbb{R}^p$ and $x_2 \in \mathbb{R}^{n-p}$ with a corresponding partition of $\pi = [\pi_1^T, \pi_2^T]^T$ such that π_1 is a global diffeomorphism then $z = \varphi(x) = x_2 - \pi_2(\pi_1^{-1}(x_1))$ is an obvious choice for the invariant manifold which results in $u = \eta(x, x_2 - \pi_2(\pi_1^{-1}(x_1)))$.*

Remark 2.3. *A similar idea of attraction of the system's trajectories to an invariant manifold is exploited in sliding-mode control. The sliding surface has a desired target dynamics that is made attractive by a discontinuous control action. In general, the I&I approach has two main advantages over sliding-mode control. In I&I, stabilization can be achieved without reaching the desired manifold. Moreover, the "reduction property" (a control law that asymptotically immerses the full system dynamics into the reduced order one) in the design of the control law cannot be implied by sliding-mode approach.*

2.3.2 I&I Parameter Estimation

Consider a nonlinear system with parametric uncertainty given by

$$\dot{x} = f(x) + G(x)\theta \tag{2.28}$$

where $x \in \mathbb{R}^n$ is the measured state vector, $f(x)$ and $G(x)$ are known vector-valued and matrix-valued functions, respectively. The vector $\theta \in \mathbb{R}^p$ represents the constant unknown parameters. The objective is to find a parameter update law $\hat{\theta}$ that estimates the unknown parameter θ in real-time. In the context of I&I, this is equivalent to the design of a parameter update law $\hat{\theta}$ and a vector-valued function $\beta(x)$ such that the implicit manifold

$$z = \varphi(x, \hat{\theta}) = \hat{\theta} - \theta + \beta(x) \tag{2.29}$$

is invariant and asymptotically converges to zero [53]. Following the development in [53], the following technical definition and assumption are stated.

Definition 2.2. *Given a symmetric positive definite matrix $Q \in \mathbb{R}^{p\times p}$. A mapping $\mathcal{M}(q) : \mathbb{R}^p \to \mathbb{R}^p$ is said to be Q-monotone if and only if , $\forall a, b \in \mathbb{R}^p$,*

$$(a - b)^T Q(\mathcal{M}(a) - \mathcal{M}(b)) \geq 0.$$

If the left-hand side is strictly larger than zero, then we have a strictly Q-monotone property.

Assumption 2.1. *There exists a vector-valued function $\beta(x)$, and a compact set $\mathcal{X} \subset \mathbb{R}^n$ such that the linear in parameter mapping $\mathscr{L}(x, \theta) = \frac{\partial \beta(x)}{\partial x} G(x)\theta$ is (strictly) Q-monotone for all $x \in \mathcal{X}$. Moreover, $\mathscr{L}(x, \theta) \equiv 0 \Rightarrow \theta = 0$ for a fixed initial condition $x(0) \in \mathbb{R}^n$.*

If Assumption 2.1 holds then the following proposition can be achieved, based on the monotonicity property of the map $\mathscr{L}(x, \theta)$ [52].

Proposition 2.1. *Consider system (2.28) and the estimator*

$$\dot{\hat{\theta}} = -\frac{\partial \beta(x)}{\partial x}[f(x) + G(x)(\hat{\theta}(t) + \beta(x))] \tag{2.30}$$

where the function $\beta(x)$ satisfies Assumption 2.1. Suppose $\forall (x(0), \hat{\theta}(0)) \in \mathcal{X} \times \mathbb{R}^p$, we have $x(t) \in \mathcal{X}$, $\forall t \geq 0$. Then, $\lim_{t\to\infty}[\hat{\theta}(t) + \beta(x)] = \theta$.

2.3.3 Simple Example

Consider the following dynamical system

$$\begin{aligned} \dot{x}_1 &= -2x_1 - x_2^2 - \theta_1 x_1 - 1 \\ \dot{x}_2 &= -2x_2 - x_2\theta_2 + 1 \end{aligned} \tag{2.31}$$

where the unknown constant parameters are $\theta_1 = 4$ and $\theta_2 = 5$. The objective is to use the I&I parameter update law (2.30) for estimation of the unknown parameters. First, we need to find a suitable function $\beta(x)$ that satisfies the Assumption 2.1. An obvious choice of $\beta(x)$ is the solution of a set of partial differential equations (PDE), $\frac{\partial \beta(x)}{\partial x} = G^T(x)$ [52]. The solutions of the PDE for this example is given as

$$\frac{\partial \beta(x)}{\partial x} = \begin{bmatrix} -x_1 & 0 \\ 0 & -x_2 \end{bmatrix} \Rightarrow \beta(x) = \begin{bmatrix} \frac{-x_1^2}{2} \\ \frac{-x_2^2}{2} \end{bmatrix}.$$

Following the design procedure, a parameter update law can be designed by (2.30). For simulation purpose, the initial conditions of the state variables and the estimator are all set to zero. Figure 2.2 shows that the parameter estimate trajectories asymptotically converge to their true values.

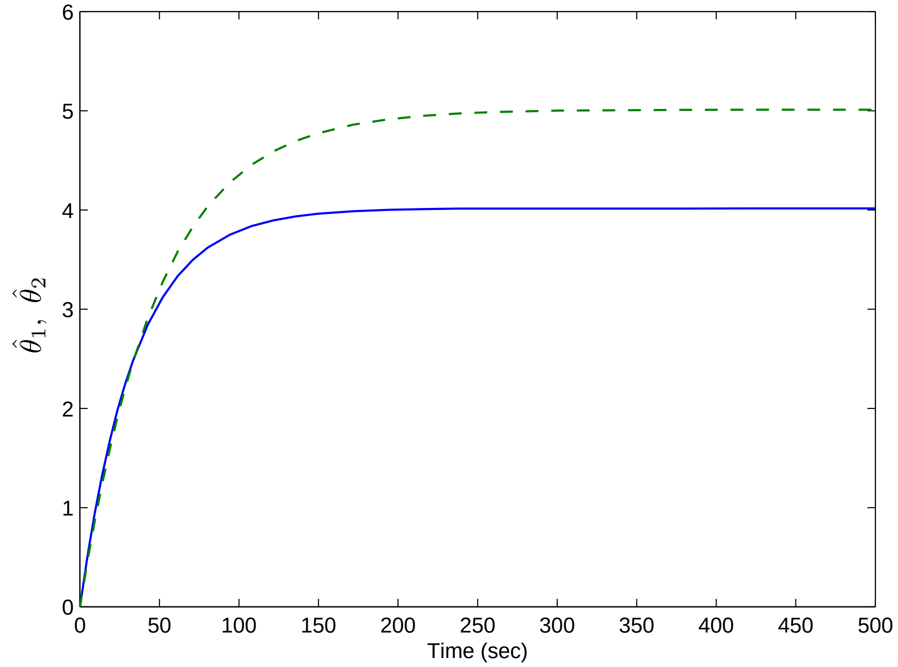

Figure 2.2: Unknown parameter estimates: $\hat{\theta}_1$ (——) and $\hat{\theta}_2$ $(- - -)$.

2.4 Summary

In this chapter, we presented an overview of literatures that are relevant to our proposed thesis. We highlight the following:

1. A comprehensive review of identification and control approaches for a wide range of systems with parametric uncertainties.

2. A brief introduction to conventional extremum-seeking control problem and recent developments to improve its performance.

3. Emphasizing the importance of invariance principle in control and estimation through study of immersion and invariance techniques.

Chapter 3

Estimation of Time-Varying Parameters

3.1 Introduction

The vast majority of the literature on adaptive estimation and adaptive control is limited to systems with slowly time-varying or constant parameters. In practice however, the time-varying behaviour of the process parameters may be of significant importance. The ability to estimate such time-varying behaviour can have a significant impact on control system's performance. Hence, the problem of time-varying parameter estimation has been of considerable interest over the last two decades.

In this chapter, an alternative technique based on invariant manifolds is proposed for a class of nonlinear systems with parametric uncertainties. The concept of invariance has been widely used in nonlinear control theory [9]. The proposed method utilizes a number of high gain estimators and filters leading to the generation of an almost invariant manifold. The almost invariance property provides explicit mappings that relate the known variables and the unknown variables implicitly. A parameter update law is designed that guarantees exponentially convergence of the estimated parameters to a small region of the true values of the time-varying parameters [64]. A similar idea of invariant manifolds for parameter estimation has been used in the design of sliding mode-based techniques. In these approaches, the sliding surfaces are the invariant manifolds [9], [90].

Bioprocesses are now commonplace in a wide range of industries [54]. Despite significant advances, the study of bioprocess dynamics and control still faces considerable challenges in the development of effective and reliable techniques. Bioprocesses typically display a high degree of nonlinearity and their dynamics are often subject to significant sources of uncertainties. These uncertainties originate, in part, from the limited knowledge of the growth kinetics [34]. In most applications, the description of growth kinetics is limited to simple analytical expressions for the growth rate functions which have been shown to achieve some success both empirically and/or experimentally. Several studies have been published on the estimation of unknown parameters for bioprocesses governed by popular growth kinetics model structures such as the Haldane and Monod models (see, e.g., [40, 93, 27, 39]). Due to their empirical nature, such growth rate kinetic models can often be unreliable in practice. Growth kinetics can often be subject to changes related to changes of operating conditions and cell metabolism that cannot be captured by simple models. As a result, it is necessary to develop algorithms for precise on-line estimation growth rate in bioprocesses. Here, an invariant manifolds approach is proposed for the estimation of specific growth rate. An asymptotic observer is considered for state estimation, and convergence conditions are established.

3.2 Problem Description

The system considered has the following nonlinear parameter-affine form

$$\dot{x} = f(x,u) + G(x,u)\theta(t) \tag{3.1}$$

where $x \in \mathbb{R}^n$ is the vector of state variables, $u \in \mathbb{R}^m$ is the vector of input variables, $\theta(t) \in \mathbb{R}^p$ is the vector of unknown time-varying and bounded parameters. The entries of $\theta(t)$ may represent physically meaningful unknown model parameters, uncertain disturbances or could be associated with any finite set of universal basis functions. It is assumed that $\theta(t)$ is uniquely identifiable and lie within an initially known compact set Θ^0. The mapping $f : \mathbb{R}^{n\times m} \to \mathbb{R}^n$ is a smooth vector-valued function and $G : \mathbb{R}^{n\times m} \to \mathbb{R}^{n\times p}$ is a smooth matrix-valued function. We assume that all states of the system are available for measurement. Also, there is a known and bounded input $u \in \mathcal{U}$ such that the state trajectories evolve on a known compact set $\mathbb{X} \subset \mathbb{R}^n$. The unknown parameters $\theta(t)$ are sufficiently smooth functions of time and such that $\|\dot{\theta}(t)\| \leq \gamma$. It is assumed that the value of γ is an unknown positive bounded constant. It will be shown in Chapter 4 that an estimation of γ can be achieved for unknown periodic parameters.

Remark 3.1. *Identifiability is a standard requirement in adaptive estimation. It states that for any two output trajectories $y(t,\theta_1)$ and $y(t,\theta_2)$ corresponding to two parameter values θ_1 and θ_2, respectively, the following property can be guaranteed:*

$$y(t,\theta_1) = y(t,\theta_2),\ \forall t \geq t_0 \ \Rightarrow\ \theta_1 = \theta_2.$$

The main objective is to design an adaptive update law to estimate $\theta(t)$ using the available information that includes the measurement of x, u and the known nonlinearities $f(\cdot)$ and $G(\cdot,\cdot)$.

3.3 Geometric-Based Approach

In this section, the proposed invariant manifold design is presented. The basic idea is to construct a mapping from known variables to the unknown variables that has an almost invariance property for sufficiently large value of the design gain. This implicit mapping can be used as an alternative to estimate the unknown parameters [64].

3.3.1 Almost Invariant Manifold Design

The estimator model for (3.1) is defined as

$$\dot{\hat{x}} = -k(\hat{x} - x), \qquad \hat{x}(t_0) = x(t_0) \tag{3.2}$$

where $k > 0$ is a tuning gain to be assigned. Two additional filters are required in the design. They are given as follows:

$$\begin{aligned} \dot{\psi} &= -k(\psi - f(x,u)), \quad \psi(t_0) = 0 \\ \dot{\Phi} &= -k(\Phi - G(x,u)), \quad \Phi(t_0) = 0. \end{aligned} \tag{3.3}$$

Remark 3.2. *The filter inputs of $f(\cdot)$ and $G(\cdot,\cdot)$ are bounded by assumption of continuity and boundedness of $x \in \mathbb{X}$ and $u \in \mathcal{U}$, $\forall t \geq t_0 \geq 0$ where $\mathbb{X}$ and $\mathcal{U}$ are compact sets. Accordingly, the boundedness of the filter outputs ψ, Φ, can be established by the BIBO stability of the filter dynamics.*

Next, we define the concept of an attractive almost invariant manifold as it is applied in this thesis.

Definition 3.1. *Consider a non-autonomous ordinary differential equation*

$$\dot{x} = F(x,t) \tag{3.4}$$

where $F : \mathcal{M} \times \mathbb{R} \to \mathbb{R}^n$ is a smooth vector field, and $\mathcal{M} \subset \mathbb{R}^n$ is a compact set.

Let $\mathscr{M}_1(x,t)$ and $\mathscr{M}_2(x,t)$ be continuous and differentiable functions. Then the non-empty and time-varying mapping $\mathscr{M}_1(x,t) = \mathscr{M}_2(x,t)$, $\forall x \in \mathcal{M}$ is called an attractive and almost invariant manifold with respect to (3.4) if there exists a class $\mathcal{KL}$ function β and for any initial state $x(t_0) \in \mathcal{M}$, there is a non-negative constant ϵ, independent of $t_0 \geq 0$, such that the variable $z(t) = \mathscr{M}_1 - \mathscr{M}_2$ satisfies

$$\|z(t)\| \leq \beta(\|z(t_0)\|, t - t_0) + \epsilon, \quad \forall\ t \geq t_0. \tag{3.5}$$

If (3.5) is satisfied for $\mathcal{M} = \mathbb{R}^n$ and any initial state $x(t_0)$, then we have globally attractive almost invariant manifold.

Proposition 3.1. *Consider the estimator (3.2) and filters (3.3), the following implicit expression*

$$k(\hat{x} - x) + \psi + \Phi\theta(t) = 0 \tag{3.6}$$

defines an attractive and almost invariant manifold. The off-the-manifold coordinate variables

$$z(t) = k(\hat{x} - x) + \psi + \Phi\theta(t) \tag{3.7}$$

are bounded and enter an order $\mathcal{O}(k^{-1})$ neighbourhood of the origin.

Proof. The derivative of $z(t)$ along the trajectories of the system is given by

$$\dot{z}(t) = k(\dot{\hat{x}} - \dot{x}) + \dot{\psi} + \dot{\Phi}\theta(t) + \Phi\dot{\theta}(t). \tag{3.8}$$

It follows from (3.1)-(3.3) that

$$\dot{z}(t) = -kz(t) + \Phi\dot{\theta}(t) = -k\left(z(t) - \frac{\Phi\dot{\theta}(t)}{k}\right). \tag{3.9}$$

By the boundedness of time-varying matrix Φ and the time-varying parameters, it follows that $\Phi\dot{\theta}(t)$ is bounded. Consider the Lyapunov function candidate

$$V_z = \frac{1}{2} z^T(t) z(t).$$

By differentiation of V_z with respect to time, we have

$$\dot{V}_z = -k z^T(t) z(t) + z^T(t) \Phi \dot{\theta}(t) \leq -k \|z\|^2 + M\|z\|$$

where $\|\Phi\dot{\theta}(t)\| \leq M < \infty$. To ensure that $-k\|z\|^2$ is a dominant term in the above inequality a constant value $0 < \rho < 1$ is used as

$$\dot{V}_z = -k(1-\rho)\|z\|^2 - k\rho\|z\|^2 + M\|z\| \leq -k(1-\rho)\|z\|^2, \ \forall \|z\| \geq \frac{M}{k\rho}.$$

This shows the GUUB of time-varying dynamics (3.9). Similar to the approach in [43], if we introduce $\omega_1(\|z\|) = \omega_2(\|z\|) = \frac{1}{2}\|z\|^2$, the ultimate bound ϵ in (3.5) can be achieved as

$$\epsilon = \omega_1^{-1}\left(\omega_2\left(\frac{M}{k\rho}\right)\right) = \frac{M}{k\rho}.$$

where $\omega_1(\cdot)$ and $\omega_2(\cdot)$ are class $\mathcal{K}_\infty$ functions. If k is bounded and sufficiently large, the manifold normal coordinate variables $z(t)$ enter a small neighbourhood of the origin. The size of this neighbourhood depends on the choice of the high gain k with order of $\mathcal{O}(\frac{1}{k})$. □

Remark 3.3. *For unknown constant parameters ($\dot{\theta} = 0$), it follows that the manifold normal coordinates $z(t)$ are invariant and internally exponentially stable.*

3.3.2 Parameter Estimation

The almost invariant manifold (3.6) can be re-written as

$$k(\hat{x} - x) + \psi = -\Phi\theta(t). \tag{3.10}$$

This equation provides an implicit relationship between the known variables (Φ, ψ, $\hat{x}$, x) and the unknown variable $\theta(t)$. This mapping provides direct information about the time-varying parameters without the need for any *a priori* information. This information can be obtained by defining the auxiliary variables p and q with dynamics

$$\begin{aligned} \dot{p} &= -kp - \Phi^T \Phi \hat{\theta} \\ \dot{q} &= -kq + \Phi^T (k(\hat{x} - x) + \psi). \end{aligned} \tag{3.11}$$

Based on (3.10) and (3.11), the proposed parameter update law is given by

$$\dot{\hat{\theta}}(t) = k(\Phi^T \Phi)^{-1}[\dot{\delta} + k\delta] \tag{3.12}$$

where $\delta = p - q$. The inverse operator in (3.12) requires one to test the invertibility condition of the matrix $\Phi^T \Phi$ in real-time. Moreover, the computation of the inverse matrix can cause difficulties when the matrix is ill-conditioned. To avoid this problem and enhance the implementability of the algorithm, an adaptive estimator can be used to compensate for the inverse operator. The following dynamic update is used

$$\dot{\Sigma}(t) = k[bI - \Phi^T \Phi(\Sigma(t))], \quad \Sigma(t_0) = bI \tag{3.13}$$

where I is the identity matrix and $b > 1$. Based on (3.12) and (3.13) the parameter update law is given by

$$\dot{\hat{\theta}}(t) = k\Sigma(t)[\dot{\delta} + k\delta]. \tag{3.14}$$

Assumption 3.1. *There exists constants $\alpha > 0$ and $T > 0$ such that*

$$\int_t^{t+T} \Phi^T(\tau)\Phi(\tau)d\tau \geq \alpha I, \quad \forall t > t_0. \tag{3.15}$$

The condition of the Assumption 3.1 is equivalent to the standard persistency of excitation condition required for parameter convergence [74]. The matrix inequality (3.15) is such that $\Omega(t) = \int_t^{t+T} \Phi^T(\tau)\Phi(\tau)d\tau - \alpha I$ is a positive definite matrix, i.e.,

$$\varphi^T \Omega(t)\varphi > 0,\ \forall \varphi \neq 0,\ \forall t > t_0.$$

Before stating the first result of this paper, we state the positive definiteness of the matrix $\Sigma(t)$ through the solution of (3.13) and Assumption 3.1, as

$$\begin{aligned}
\Sigma(t) =& \exp\left[\int_{t_0}^{t} -k\Phi^T(\tau)\Phi(\tau)d\tau\right]\Sigma(t_0) + kb\int_{t_0}^{t}\exp\left[\int_{\tau}^{t} -k\Phi^T(\xi)\Phi(\xi)d\xi\right]d\tau \\
&> kb\int_{t_0}^{t}\exp\left[\int_{\tau}^{t} -k\Phi^T(\xi)\Phi(\xi)d\xi\right]d\tau \geq \left(kb\int_{t_0}^{t} e^{-k\lambda(t-\tau)}d\tau\right) I \\
&= \frac{b(1-e^{-k\lambda(t-t_0)})}{\lambda} I
\end{aligned} \tag{3.16}$$

where the last inequality is achieved from the boundedness of the regressor matrix as $\|\Phi^T\Phi\| \leq \lambda$. Moreover, we have

$$\begin{aligned}
\Sigma(t) \leq& \Sigma(t_0) + kb\int_{t_0}^{t}\exp\left[\int_{\tau}^{t} -k\Phi^T(\xi)\Phi(\xi)d\xi\right]d\tau \\
\leq& \Sigma(t_0) + \left(kb\int_{t_0}^{t} e^{-k\alpha(t-\tau)}d\tau\right) I \leq \frac{b\alpha + b(1-e^{-k\alpha(t-t_0)})}{\alpha} I.
\end{aligned} \tag{3.17}$$

Theorem 3.1. *The parameter update law (3.14) is such that the estimation error $\tilde{\theta}(t) = \theta(t) - \hat{\theta}(t)$ is bounded and converges to a small neighbourhood of the origin. The size of this neighborhood is adjustable by the gain k.*

Proof. From (3.9), it is clear that $\frac{\Phi\dot{\theta}(t)}{k}$ is a small perturbation term for sufficiently large value of k. Hence, the almost invariant manifold

$$k(\hat{x} - x) + \psi = -\Phi\theta(t) + \frac{\Phi\dot{\theta}(t)}{k} \tag{3.18}$$

is obtained, and the parameter update law (3.14) can be expressed implicitly in the form

$$\dot{\hat{\theta}}(t) = k\Sigma(t)\Phi^T\Phi\tilde{\theta}(t) - \Sigma(t)\Phi^T\Phi\dot{\theta}(t). \tag{3.19}$$

The quadratic Lyapunov function is defined as

$$V_{\tilde{\theta}} = \frac{1}{2}\tilde{\theta}^T(t)\tilde{\theta}(t). \tag{3.20}$$

By differentiating of (3.20) along (3.19), we have

$$\dot{V}_{\tilde{\theta}} = -k\tilde{\theta}^T(t)(\Sigma(t)\Phi^T\Phi)\tilde{\theta}(t) + \tilde{\theta}^T(t)(\Sigma(t)\Phi^T\Phi)\dot{\theta}(t) + \tilde{\theta}^T(t)\dot{\theta}(t).$$

By applying Young's inequality on the second term, one obtains

$$\begin{aligned}\dot{V}_{\tilde{\theta}} \leq & - k\tilde{\theta}^T(t)(\Sigma(t)\Phi^T\Phi)\tilde{\theta}(t) + \frac{k}{2}\tilde{\theta}^T(t)(\Sigma(t)\Phi^T\Phi)\tilde{\theta}(t) \\ & + \frac{1}{2k}\dot{\theta}^T(t)(\Sigma(t)\Phi^T\Phi)\dot{\theta}(t) + \tilde{\theta}^T(t)\dot{\theta}(t).\end{aligned}$$

Next we claim the boundedness of the matrix $\Sigma(t)\Phi^T\Phi$ as follows,

$$\begin{aligned}&\frac{d}{dt}(\Sigma(t)\Phi^T\Phi) = \dot{\Sigma}(t)\Phi^T\Phi + \Sigma(t)\dot{\Phi}^T\Phi + \Sigma(t)\Phi^T\dot{\Phi} \\ &= -(2kI + k\Phi^T\Phi)\Sigma(t)\Phi^T\Phi + kb\Phi^T\Phi + k\Sigma(t)G^T\Phi + k\Sigma(t)\Phi^T G.\end{aligned} \tag{3.21}$$

By integration, one gets

$$\begin{aligned}\Sigma(t)\Phi^T\Phi = & \exp\left[\int_{t_0}^{t} -k(2I + \Phi^T(\tau)\Phi(\tau))d\tau\right]\Sigma(t)\Phi^T\Phi\mid_{t=t_0} + \int_{t_0}^{t}(kb\Phi^T(\tau)\Phi(\tau) \\ & + k\Sigma(\tau)G^T(\tau)\Phi(\tau) + k\Sigma(\tau)\Phi^T(\tau)G(\tau))\exp\left[\int_{\tau}^{t} -k(2I + \Phi^T(\xi)\Phi(\xi))d\xi\right]d\tau.\end{aligned} \tag{3.22}$$

Since the matrix $\Phi^T\Phi$ is such that

$$\Phi^T\Phi = \int_{t_0}^{t} k(\Phi^T(\tau)G(\tau) + G^T(\tau)\Phi(\tau))\exp[-2k(t-\tau)]d\tau \geq 0, \tag{3.23}$$

and $\Sigma(t)$ is a positive definite matrix, then

$$\begin{aligned}&\Sigma(t)\Phi^T\Phi \geq \int_{t_0}^{t}(kb\Phi^T\Phi + k\Sigma G^T\Phi + k\Sigma\Phi^T G)\exp\left[\int_{\tau}^{t} -k(2I + \Phi^T(\xi)\Phi(\xi))d\xi\right]d\tau \\ &\geq \int_{t_0}^{t}(kb\Phi^T(\tau)\Phi(\tau))\exp\left[\int_{\tau}^{t} -k(2I + \Phi^T(\xi)\Phi(\xi))d\xi\right]d\tau \geq \\ &\int_{t_0}^{t}(kb\Phi^T(\tau)\Phi(\tau))e^{-k(\lambda+2)(t-\tau)}d\tau \geq \int_{t_0+NT}^{t}(kb\Phi^T(\tau)\Phi(\tau))e^{-k(\lambda+2)(t-\tau)}d\tau\end{aligned} \tag{3.24}$$

where N is an integer. It follows from (3.16) and (3.24) that

$$\frac{b\alpha(1-e^{-k(\lambda+2)(t-t_0-NT)})}{\lambda+2}I \leq \Sigma(t)\Phi^T\Phi \leq \frac{b\alpha+b(1-e^{-k\alpha(t-t_0)})}{\alpha}\Phi^T\Phi. \tag{3.25}$$

Hence $\forall t \subseteq [t_i, t_i+T]$,

$$\eta_1 = \frac{\alpha(1-e^{-k(\lambda+2)T})}{\lambda+2} \leq \left\|\Sigma\Phi^T\Phi\right\| \leq \frac{b(1+\alpha)}{\alpha}\lambda = \eta_2. \tag{3.26}$$

Consider the function $\dot{V}_{\tilde{\theta}}$ and inequality (3.26). We apply Young's inequality on the last term such that

$$\begin{aligned}\dot{V}_{\tilde{\theta}} \leq &- k\tilde{\theta}^T(t)(\Sigma(t)\Phi^T\Phi)\tilde{\theta}(t) + \frac{k}{2}\tilde{\theta}^T(t)(\Sigma(t)\Phi^T\Phi)\tilde{\theta}(t) + \frac{1}{2k}\dot{\theta}^T(t)(\Sigma(t)\Phi^T\Phi)\dot{\theta}(t)\\ &+ \frac{k\eta_1}{2}\tilde{\theta}^T(t)\tilde{\theta}(t) + \frac{1}{2k\eta_1}\dot{\theta}^T(t)\dot{\theta}(t).\end{aligned}$$

It follows that, after collecting the similar terms, one can write the inequality:

$$\begin{aligned}\dot{V}_{\tilde{\theta}} &\leq -\frac{k}{2}\tilde{\theta}^T(t)(\Sigma(t)\Phi^T\Phi - \eta_1 I)\tilde{\theta}(t) + \frac{1}{2k}\dot{\theta}^T(t)(\Sigma(t)\Phi^T\Phi + \eta_1^{-1}I)\dot{\theta}(t)\\ &\leq \frac{-k\eta_1(b-1)}{2}V_{\tilde{\theta}} + \frac{1}{2k}(\eta_2+\eta_1^{-1})\gamma^2,\end{aligned}$$

$\forall t \in [t_i,\, t_{i+1}]$ on an interval of length T. Thus,

$$\begin{aligned}V_{\tilde{\theta}}(t_{i+1}) &\leq e^{\frac{-k\eta_1(b-1)}{2}T}V_{\tilde{\theta}}(t_i) + \int_{t_i}^{t_{i+1}} e^{\frac{-k\eta_1(b-1)}{2}(t_{i+1}-\tau)}\frac{1}{2k}(\eta_2+\eta_1^{-1})\gamma^2 d\tau\\ &= e^{\frac{-k\eta_1(b-1)}{2}T}V_{\tilde{\theta}}(t_i) + [1-e^{\frac{-k\eta_1(b-1)}{2}T}](\frac{\gamma^2}{(b-1)\eta_1 k^2})(\eta_2+\eta_1^{-1}).\end{aligned} \tag{3.27}$$

By recursion, for $i = 0,\, 1,\, \ldots, N$, one obtains:

$$V_{\tilde{\theta}}(t_N) = e^{\frac{-k\eta_1(b-1)}{2}NT}V_{\tilde{\theta}}(t_0) + [1-e^{\frac{-k\eta_1(b-1)}{2}NT}]\left(\frac{(\eta_2+\eta_1^{-1})\gamma^2}{(b-1)\eta_1 k^2}\right).$$

Taking the limit as $N \to \infty$, we have

$$\lim_{N\to\infty}\left\|\tilde{\theta}(t_N)\right\| \leq \sqrt{\frac{2(\eta_2+\eta_1^{-1})\gamma^2}{(b-1)\eta_1 k^2}}. \tag{3.28}$$

It follows that the parameter estimation error converges exponentially to a small neighbourhood of the origin. It is easy to see that as $k \to \infty$, the parameter estimation error $\tilde{\theta}(t) \to 0$. Then, $\tilde{\theta}(t) = 0$ is globally exponentially stable. □

Remark 3.4. *If the unknown parameters are constant or vanish in finite time, then global exponential convergence to the true values can be achieved without any priori knowledge of the parameters.*

Remark 3.5. *It has been shown that $\tilde{\theta}(t)$ can be made arbitrarily small by choosing a sufficiently large value of the gain. However, large values of the gain reduce the robustness of the algorithm in the presence of high frequency signals such as measurement noise. Hence, a trade-off is required between response accuracy and sensitivity to noise.*

Remark 3.6. *All the results of this work are stated in a way that is independent of the choice of the control input. However, it is important to note that the input variable can be used to enhance the PE condition. This can be achieved by the design of a suitable controller or by ensuring a sufficiently rich set of input signals.*

Remark 3.7. *As a model based technique, the proposed approach requires an accurate model of $f(x, u)$ and $G(x, u)$. Any model uncertainty would impact the accuracy of the estimation routine. In practice, one approach is to lump uncertain terms. That is, if $G(x, u)$ is uncertain then one can lump the uncertainties using a reparameterized model with parameter $\beta(t) = G(x, u)\theta(t)$. The same would apply if $f(x, u)$ was uncertain. In the absence of the exact state measurements, an adaptive observer approach would be required to estimate the unknown state variables.*

A consequence of Theorem 3.1 can be stated as follows.

Corollary 3.1. *Assume a positive constant μ such that $\left\|\ddot{\theta}(t)\right\| \leq \mu$. Then dynamic* *trajectories of the system can be obtained implicitly as*

$$\begin{aligned}\dot{z}(t) &= -kz(t) - \Phi\frac{\ddot{\theta}(t)}{k} \\ \dot{\hat{\theta}}(t) &= k\Sigma(t)\Phi^T\Phi\tilde{\theta}(t) - \Sigma(t)\Phi^T\Phi\dot{\theta}(t) + \frac{1}{k}\Sigma(t)\Phi^T\Phi\ddot{\theta}(t)\end{aligned} \tag{3.29}$$

where $z(t) = k(\hat{x} - x) + \psi + \Phi\theta(t) - \Phi\frac{\dot{\theta}(t)}{k}$*. Thus, a value of* k *can be assigned such that the variables* $z(t)$ *and* $\theta(t)$ *are bounded and approach a small neighbourhood of the origin.*

3.4 Application to Bioreactors

Consider the following microbial growth process

$$\dot{x} = \mu(s)x - ux \tag{3.30}$$

$$\dot{s} = -\frac{1}{Y}\mu(s)x + (s_0 - s)u \tag{3.31}$$

where $x \in [0, \infty)$ and $s \in [0, \infty)$ denote biomass and substrate concentrations, respectively, $u > 0$ is the dilution rate, s_0 denotes the concentration of the substrate in the feed, and $Y > 0$ is the yield coefficient. The inlet substrate is fed into the tank with a constant concentration s_0. It is assumed that the inlet flow rate and the output flow rate are the same values, so the overall volume V of the tank is kept constant. Also, the solution in the mixing tank is assumed to be well-mixed. A simple schematic representation of the bioreactor is given in Figure 3.1.

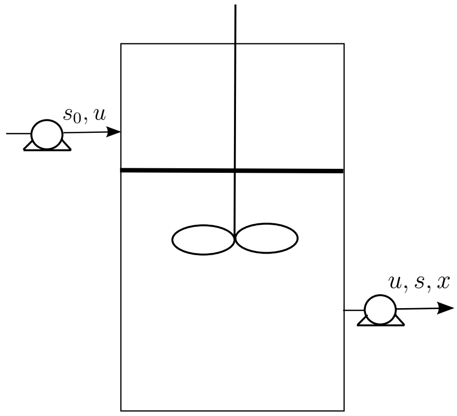

Figure 3.1: A simple schematic representation of the bioreactor.

The variables x and s are the state variables and u is the input variable for control system model (3.30) and (3.31). The nonlinear function $\mu(s)$ is the growth rate of the process. Many different choices of $\mu(s)$ have been used in the literature to describe large classes of biochemical processes [93]. In general, the development of suitable analytic expressions can be difficult due to the inherent complexity of the $\mu(s)$ models [25].

In this work, we treat the growth rate model as an unknown time-varying parameter $\mu(t)$, to be estimated. For this purpose, the concepts of invariant manifold and adaptive parameter update law are used for the estimation of $\mu(t)$. We assume that the biomass concentration, x, is not available for measurement. A nonlinear observer is proposed to estimate the biomass concentration [62].

The following assumptions are required.

Assumption 3.2. *The state trajectories $y = [x, s]^T$ evolve on a known compact subset of $\mathbb{Y} \in \mathbb{R}^2$.*

Assumption 3.3. *The physical nature of the problem is such that the growth rate $\mu(s)$ and its first order derivative with respect to time are inherently bounded. It is* *assumed that* $|\dot{\mu}(t)| \leq \gamma, \ \forall t \geq t_0$.

Assumption 3.4. *The known input dilution rate is such that* $u \geq u_0$ *with known lower bound* $u_0 > 0$.

In this section, it is assumed that only the substrate concentration is available for measurement. In fed-batch bioprocess, the biomass concentration cannot be measured easily, especially in high biomass concentrations and complex substrate containing applications [38]. An adaptive observer is proposed to provide estimates of the biomass concentration.

Here, the product of the unknown growth rate and the biomass concentration, $\theta(t) = \mu(t)x$, is considered as an unknown time-varying parameter. The unknown parameter $\theta(t)$ is estimated from (3.31) using the manifold-based approach described in the previous section. The manifold design can be summarized using the following equations

$$\begin{aligned}\dot{\hat{s}} &= -k^2(\hat{s} - s)\\ \dot{\psi}_1 &= -k^2(\psi_1 - (s_0 - s)u)\\ \dot{\phi}_1 &= -k^2(\phi_1 + \tfrac{1}{Y}).\end{aligned} \tag{3.32}$$

The auxiliary variables and estimator are expressed as

$$\begin{aligned}\dot{p}_1 &= -kp_1 - {\phi_1}^T \phi_1 \hat{\theta}(t)\\ \dot{q}_1 &= -kq_1 + {\phi_1}^T (k^2(\hat{s} - s) + \psi_1)\\ \dot{\sigma}_1(t) &= k[b - (\phi_1^T \phi_1)\sigma_1(t)]\end{aligned} \tag{3.33}$$

and for parameter estimation, we have

$$\dot{\hat{\theta}}(t) = k^2 \sigma_1(t)[\dot{\delta}_1 + k\delta_1], \quad \delta_1 = p_1 - q_1. \tag{3.34}$$

Since the filter output ϕ_1 converges to the scalar value $-\frac{1}{Y}$, the PE condition of Assumption 3.1 is fulfilled with $\alpha = \frac{1}{Y^2} > 0$ and $\forall T > t_0$. The following observer structure is used for the state prediction

$$\dot{\hat{x}} = \mathrm{Proj}\{\hat{\theta} - \hat{x}u, \hat{x}\} \tag{3.35}$$

where $\hat{x}$ is estimate of the biomass concentration x, and $\mathrm{Proj}\{\cdot, \cdot\}$ denotes a projection algorithm [46]. The projection algorithm is given by

$$\dot{\hat{x}} = \begin{cases} \xi & \text{if } \hat{x} > 2\epsilon \text{ or } \xi \geq 0 \\ \xi \max\{0, \frac{\hat{x}-\epsilon}{\epsilon}\} & \text{otherwise} \end{cases} \tag{3.36}$$

where $\xi = \hat{\theta} - \hat{x}u$, and ϵ is a small value. The estimation of the unknown growth rate model can be obtained as $\hat{\mu}(t) = \frac{\hat{\theta}(t)}{\hat{x}}$. The convex function $P(\hat{x}) = 2\epsilon - \hat{x}$ and small value ϵ of the projection algorithm are such that, for any $\hat{x}(t_0) \in \chi_\epsilon \Rightarrow \hat{x}(t) \in \chi_\epsilon$, $\forall t \geq t_0$. The convex set χ_ϵ is defined as $\chi_\epsilon = \{\hat{x} \in \mathbb{R} \geq 0 \mid P(\hat{x}) \leq 2\epsilon\}$. Thus, the projection algorithm in the adaptive observer (3.35), guarantees nonzero and Lipschitz continuous state estimation.

Theorem 3.2. *Assume that the substrate concentration s is measurable and let Assumptions 3.2 to 3.4 hold for systems (3.30) and (3.31). The parameter update law (3.34) and the observer (3.35) are such that the parameter estimation error $\tilde{\theta}(t) = \theta(t) - \hat{\theta}(t)$ and the state estimation error $e_x = x - \hat{x}$ converge to a small neighbourhood of zero.*

Proof. Consider the following Lyapunov function candidate

$$W = \frac{1}{2}\tilde{\theta}^2 + \frac{1}{2}e_x^2.$$

Its time derivative is given by

$$\dot{W} = \tilde{\theta}(t)[\dot{\theta}(t) - \dot{\hat{\theta}}(t)] + e_x[\dot{x} - \dot{\hat{x}}].$$

If ones substitutes the implicit form of (3.34) and invoking the properties of projection operator from (3.35), it follows that

$$\dot{W} \leq -k^2(\sigma_1(t)\phi_1^T\phi_1)\tilde{\theta}^2(t) + \tilde{\theta}(t)(\sigma_1(t)\phi_1^T\phi_1)\dot{\theta}(t) + \tilde{\theta}(t)\dot{\theta}(t) + \tilde{\theta}(t)e_x - ue_x^2.$$

The boundedness of $\sigma_1(t)\phi_1^T\phi_1$ can be obtained similar to (3.26). Applying Young's inequality to all indefinite terms, one can rewrite the above inequality as follows

$$\begin{aligned}\dot{W} &\leq -k^2(\sigma_1(t)\phi_1^T\phi_1)\tilde{\theta}^2(t) + \frac{k^2}{2}(\sigma_1(t)\phi_1^T\phi_1)\tilde{\theta}^2(t) + \frac{1}{2k^2}(\sigma_1(t)\phi_1^T\phi_1)\dot{\theta}^2(t)\\ &+ \frac{k^2\eta_1}{4}\tilde{\theta}^2(t) + \frac{2}{k^2\eta_1}\dot{\theta}^2(t) + \frac{k^2\eta_1}{4}\tilde{\theta}^2(t) + \frac{2}{k^2\eta_1}e_x^2 - ue_x^2.\end{aligned}$$

By Assumption 3.4 and collecting the similar terms, we obtain

$$\dot{W} \leq -\frac{k^2}{2}(\sigma_1(t)\phi_1^T\phi_1 - \eta_1)\tilde{\theta}^2(t) - (u_0 - \frac{2}{k^2\eta_1})e_x^2 + \frac{1}{2k^2}(\sigma_1(t)\phi_1^T\phi_1 + \eta_1^{-1})\dot{\theta}^2(t).$$

For the given k, there exist a strictly positive constant $k' = \min\{\frac{k^2\eta_1(b-1)}{2}, u_0 - \frac{2}{k^2\eta_1}\}$ such that

$$\dot{W} \leq -k'W + \left(\frac{\eta_2 + \eta_1^{-1}}{2k^2}\right)\gamma^2. \tag{3.37}$$

It follows that $\tilde{\theta}$ and e_x converge exponentially to a small neighbourhood of the origin. The size of this neighbourhood depends on the choice of gain k. This completes the proof of Theorem 3. □

Remark 3.8. *The convergence rate of the proposed adaptive asymptotic observer depends on the values of input. It can be concluded from k' that the higher the amplitude of the dilution rate is, the faster the convergence of the estimates.*

3.5 Simulation Examples

Here, we consider bioprocesses with two of the most commonly used growth kinetics. The first specific growth rate function is the Monod model:

$$\mu(s) = \frac{\mu_m s}{K_s + s} \tag{3.38}$$

and the second one is the Haldane model:

$$\mu(s) = \frac{\mu_m s}{K_i s^2 + s + K_s} \tag{3.39}$$

where $\mu_m > 0$ is the maximum value of the specific growth rate, K_s denotes the saturation constant, and K_i is an inhibition constant. It is important to note that the models (3.38) and (3.39) are assumed to be unknown. They are simply specified for the purpose of the simulation study. Since all the constant parameters are bounded and nonzero, Assumption 3.3 is satisfied for (3.38) and (3.39).

We consider the conditions specified in [93], where $\mu_m = 0.4$ (h^{-1}), $K_s = 0.4$ (g/l), $K_i = 0.5$ (l/g), $Y = 1/0.3636$, and $s_0 = 2$ (g/l). The initial conditions for the system are $x(0) = 2$ (g/l), and $s(0) = 0.4$ (g/l). The estimation gain is $k = \sqrt{10}$. The initial conditions for the parameter update law and the state observer are $\hat{\theta}(0) = 0$ and $\hat{x}(0) = 1$, respectively. In the simulation, we consider the plant with a square wave dilution rate $u(t)$ shown in Figure 3.2.

The simulation results are shown in Figures 3.3-3.5. The unknown parameter $\theta(t)$ and its estimation are depicted in Figure 3.3. The trajectories of the biomass concentration and its corresponding adaptive observer are illustrated in Figure 3.4. The static estimation of the growth kinetic from dynamic estimations $\hat{\theta}$, and $\hat{x}$ are shown in Figure 3.5.

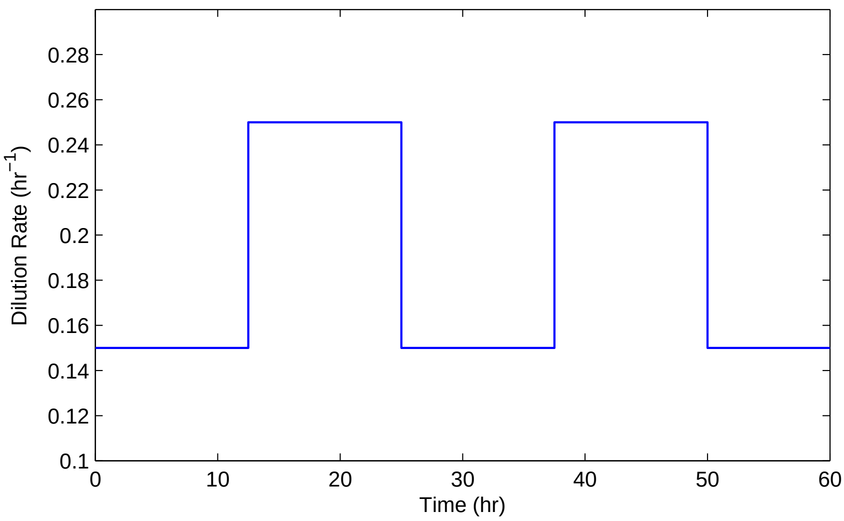

Figure 3.2: System input $u(t)$.

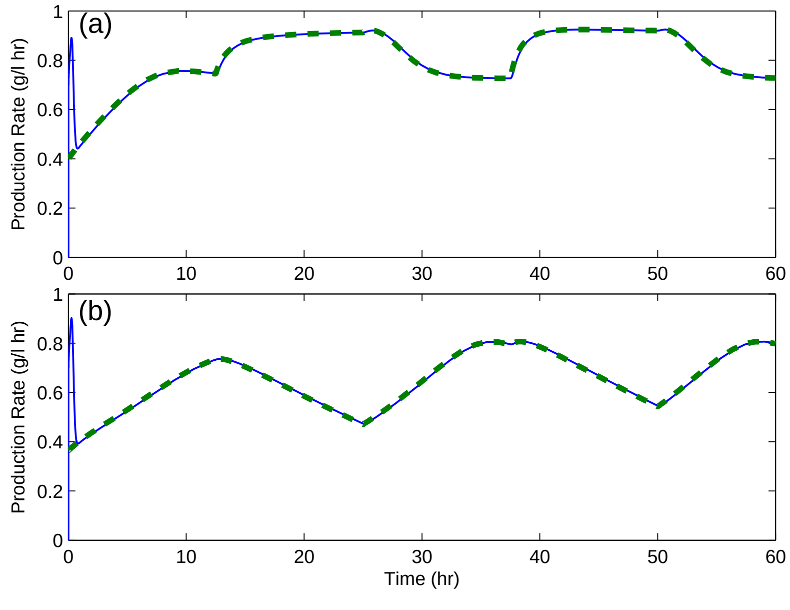

Figure 3.3: Parameter $\theta(t)(- - -)$ and its estimate $\hat{\theta}(t)$ (——). (a) Monod Model; (b) Haldane Model.

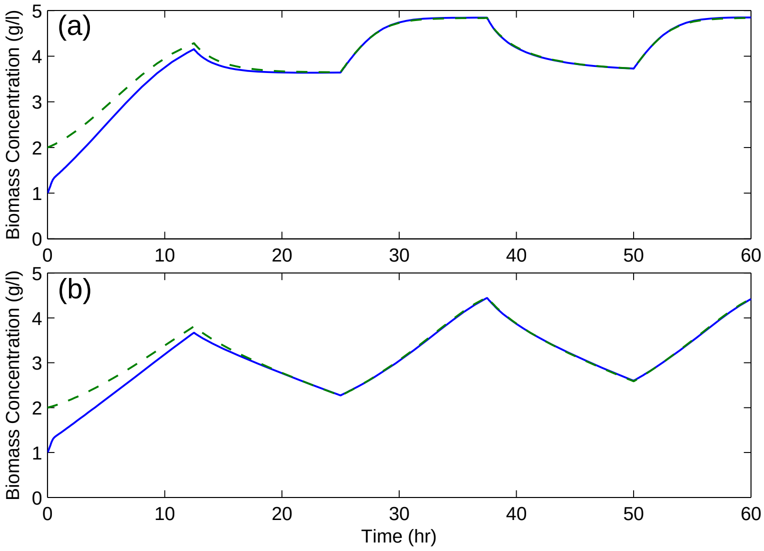

Figure 3.4: Trajectory $x(t)(---)$ and its estimate $\hat{x}(t)$ (——). (a) Monod Model; (b) Haldane Model.

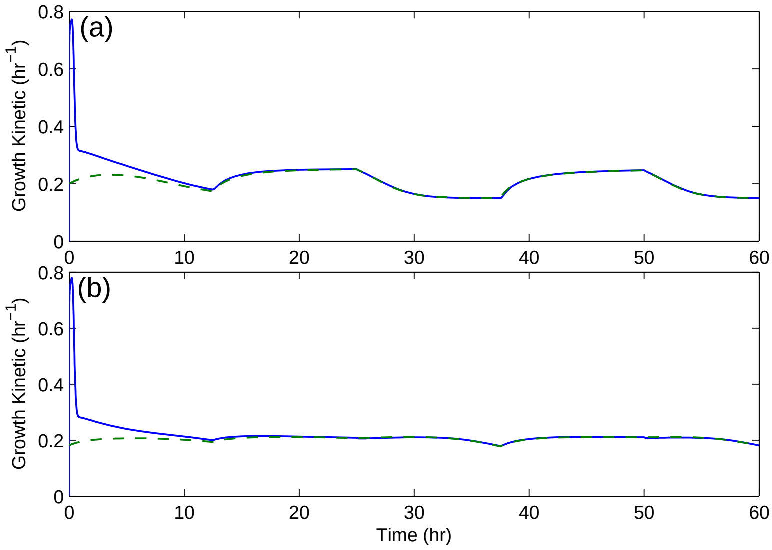

Figure 3.5: Parameter $\mu(t)(---)$ and its estimate $\hat{\mu}(t)$ (——). (a) Monod Model; (b) Haldane Model.

As predicted by the theoretical development, the parameter estimation error can be made arbitrary small by choosing a sufficiently large value of the gain k. However, large values of the gain may reduce the robustness of the algorithm in the presence of high frequency signals such as measurement noise. To ensure the practical viability of the proposed parameter estimation framework, we verify its performance in the presence of measurement noise with the same gain value. The measurements of substrate concentration s is subject to a white noise with zero mean and an amplitude equivalent to 8% of the maximum changes in the state trajectory. Figure 3.6 shows the noisy measurements of substrate concentration.

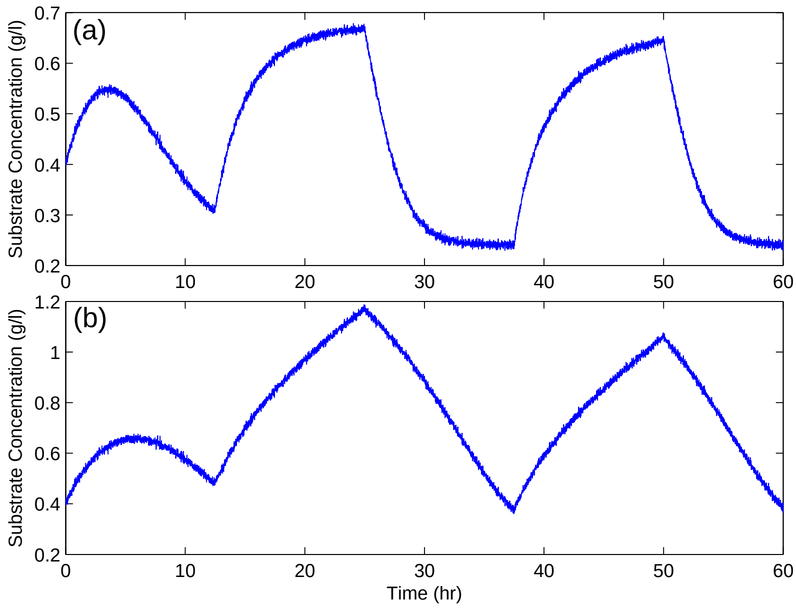

Figure 3.6: Substrate concentration in the presence of measurement noise. (a) Monod Model; (b) Haldane Model.

The parameter estimates along with the true value is shown in Figure 3.7. Although the noise has a significant effect on the parameter estimates, they remain close to the true parameter values in both cases. It is interesting to note that the noise has a more important effect on the growth rate estimate for the Haldane model.

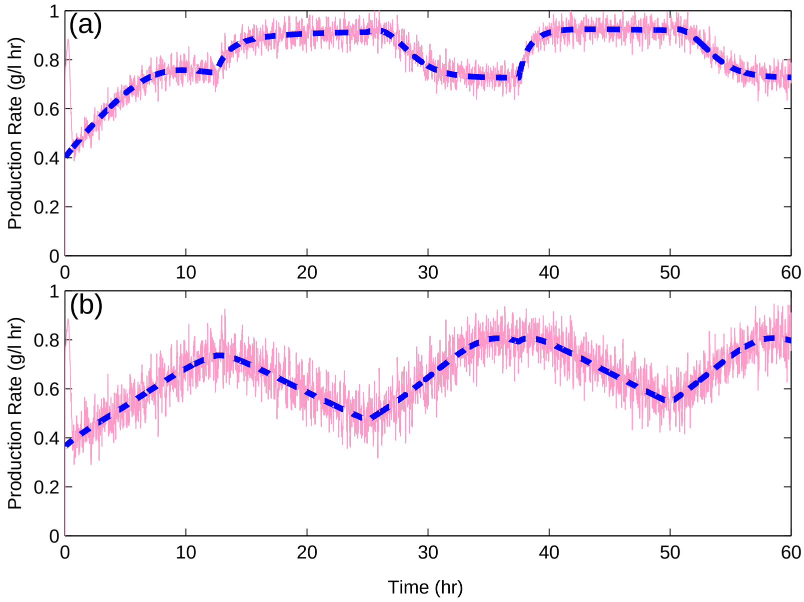

Figure 3.7: Parameter $\theta(t)(---)$ and its estimate $\hat{\theta}(t)$ (——) from noisy measurements. (a) Monod Model; (b) Haldane Model.

Figure 3.8 depicts the estimation of biomass concentration x in the presence of noisy measurements. If one compares Figures 3.4 and 3.8, similar results have been achieved despite the presence of noisy measurements. It should be noted that the asymptotic observer (3.35) integrates the parameter estimate $\hat{\theta}(t)$. As a result, some degree of attenuation of high frequency noise should be expected. This property demonstrates the robustness of the estimation algorithm to measurement noise. Overall, the simulation results show that the proposed technique also performs well in the presence of measurement disturbances. Although a smaller estimation error can be achieved by higher gain value, a trade-off is required between response accuracy and sensitivity to noise.

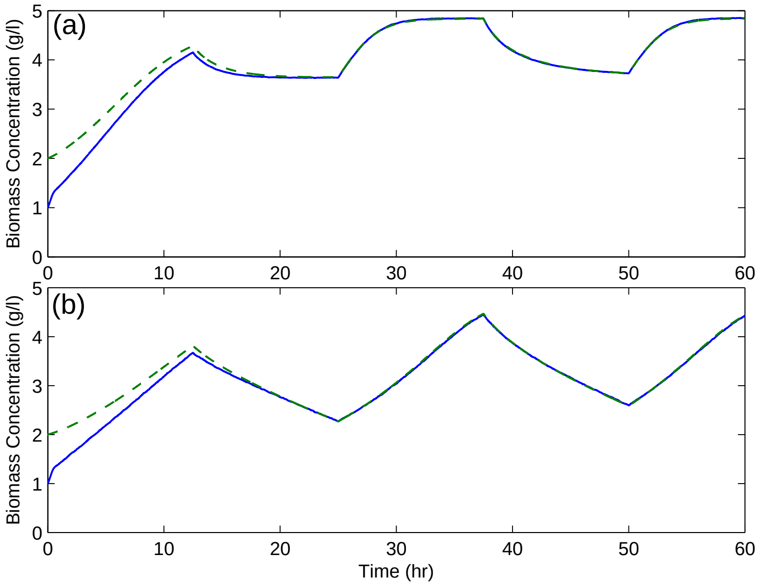

Figure 3.8: Trajectory $x(t)(---)$ and its estimate $\hat{x}(t)$ (——) from noisy measurements. (a) Monod Model; (b) Haldane Model.

3.6 Summary

Despite the well established approaches for estimation of uncertain constant parameters, the estimation of time-varying parameters is still a challenging problem. In this chapter, an alternative approach for the estimation of time-varying parameters in nonlinear systems is proposed. The approach provides a formal scheme that relies on the definition of an almost invariant manifold. The boundedness of the parameter estimation error is achieved for sufficiently large value of the gain. This method is effective and easy to implement with only one tuning parameter. The application of this method is proposed for the estimation of the specific growth rate in bioprocesses. An asymptotic observer is designed for the case where the biomass concentration is not available for measurement. The joint parameter and state estimation algorithm has been implemented to two simulation examples to demonstrate its effectiveness. The results also demonstrate the robustness of the method in the presence of measurement noise.

Chapter 4

Estimation of Uncertain Periodic Parameters

4.1 Introduction

Unknown periodic signals are also classified as a subclass of the time-varying parametric uncertainties [89]. This class of uncertainties are essential in control systems as most disturbance signals can be modelled as mixtures of periodic signals. When the periodic disturbances are not available for measurements, only limited performance can be achieved and output regulation is not practically possible. Knowledge of the characteristics of periodic disturbances can drastically improve the performance of closed-loop control systems (see [13] and the references therein). Although many studies have been reported for systems with periodic uncertain disturbances [13, 81, 48, 19, 89], few approaches focus on the direct estimation of the unknown disturbances. Estimation of these time-varying parameters are of great importance for disturbance rejection, fault detection and monitoring [48]. Unfortunately, most techniques proposed for the estimation of unknown inputs for nonlinear systems require restrictive assumptions on the uncertainties such as boundedness of the uncertainties in a known compact set, and Lipschitz condition of the nonlinear uncertainties.

In this chapter, the results of the previous chapter are generalized to deal with the estimation of periodic time-varying parameters with unknown periodicity. This class of uncertain periodic signals is often used in practice to model periodic exogenous disturbances [83]. Their accurate estimation can be used in the design of regulatory control systems. Here, the algorithm can be interpreted as a generalization of the invariant manifold approach where the invariant manifold depends on higher order derivatives of the unknown parameters.

4.2 Extension of the Main Algorithm

In the following, the qth time derivative of $\theta(t)$ with respect to time is denoted by $\theta^{(q)}(t) = \frac{d\theta^q(t)}{dt^q}$. Let $\theta_i(t)$, be uncertain periodic parameters as

$$\theta_i(t) = a_{i0} + \sum_{\ell=1}^{L_i} \left(a_{i\ell} \cos \frac{2\pi}{T_{i\ell}} t \right) + \sum_{\ell'=1}^{L'_i} \left(b_{i\ell'} \sin \frac{2\pi}{T'_{i\ell'}} t \right), \quad i = 1, ..., p, \tag{4.1}$$

where L_i and L'_i are the known number of distinct frequencies for each unknown parameter $\theta_i(t)$. $T_{i\ell}$ and $T'_{i\ell'}$ are unknown time periods, and $a_{i\ell}$, $b_{i\ell'}$ are unknown constant coefficients. In order to guarantee that the problem is well-posed the following assumption is made for the unknown parameters (4.1).

Assumption 4.1. *$T_{i\ell}, T'_{i\ell'} \geq T$, and the gain is chosen such that $\sqrt{k}T \geq 2\pi + 1$, for a positive constant T in (3.15).*

Theorem 4.1. *Let Assumptions 3.1 and 4.1 hold for the nonlinear dynamics (3.1) with periodic time-varying parameters (4.1) and known constants α and T. The parameter update law (3.14) is such that the estimation error $\tilde{\theta}(t)$ is bounded and converges to an $\mathcal{O}(k^{-1})$ neighbourhood of the origin. An upper bound approximation for the rate of changes of the unknown time-varying parameters is achieved as $\|\dot{\theta}(t)\| \leq \frac{2\pi\epsilon}{T}\sum_{i=1}^{p}(L_i + L_i')$ where ϵ is a positive constant that depends on the gain k and magnitude of the estimator (3.2) and filters (3.3).*

Proof. The first derivative of $\theta_i(t)$ with respect to time is given by

$$\begin{aligned}\dot{\theta}_i(t) = &-a_{i1}\frac{2\pi}{T_{i1}}\sin(\frac{2\pi}{T_{i1}}t) - \cdots - a_{iL_i}\frac{2\pi}{T_{iL_i}}\sin(\frac{2\pi}{T_{iL_i}}t) \\ &+ b_{i1}\frac{2\pi}{T_{i1}'}\cos(\frac{2\pi}{T_{i1}'}t) + \cdots + b_{iL_i'}\frac{2\pi}{T_{iL_i'}'}\cos(\frac{2\pi}{T_{iL_i'}'}t).\end{aligned}$$

Under Assumption 4.1, the following inequality can be obtained

$$\left|\frac{\dot{\theta}_i(t)}{k^{\frac{1}{2}}}\right| \leq \frac{2\pi}{\sqrt{k}T}\left(\sum_{\ell=1}^{L_i}|a_{i\ell}| + \sum_{\ell'=1}^{L_i'}|b_{i\ell'}|\right), \quad i = 1, ..., p. \tag{4.2}$$

Following a similar analysis, we obtain:

$$\left|\frac{\theta_i^{(n)}(t)}{k^{\frac{n}{2}}}\right| \leq \left(\frac{2\pi}{\sqrt{k}T}\right)^n\left(\sum_{\ell=1}^{L_i}|a_{i\ell}| + \sum_{\ell'=1}^{L_i'}|b_{i\ell'}|\right) \leq \left|\frac{\dot{\theta}_i(t)}{k^{\frac{1}{2}}}\right|, \quad i = 1, ..., p \;\; \& \;\; n = 2, 3, ... \tag{4.3}$$

Based on the higher order derivatives of the vector $\theta(t)$, the almost invariant manifold (3.16) can be modified to yield:

$$k(\hat{x} - x) + \psi = -\Phi\theta(t) + \frac{\Phi\dot{\theta}(t)}{k} - \frac{\Phi\ddot{\theta}(t)}{k^2} + \cdots \tag{4.4}$$

Based on the inequality (4.3), the following geometric series can be established:

$$\lim_{n\to\infty}\left(\left|\frac{\dot{\theta}_i(t)}{k^{\frac{1}{2}}}\right| + \left|\frac{\ddot{\theta}_i(t)}{k}\right| + \cdots + \left|\frac{\theta_i^{(n)}(t)}{k^{\frac{n}{2}}}\right|\right) \leq \frac{1}{1-\frac{2\pi}{\sqrt{k}T}}\left|\frac{\dot{\theta}_i(t)}{k^{\frac{1}{2}}}\right|$$
$$= \frac{\sqrt{k}T}{\sqrt{k}T-2\pi}\left|\frac{\dot{\theta}_i(t)}{k^{\frac{1}{2}}}\right|. \tag{4.5}$$

Without loss of generality, we assume that $\sqrt{k}T = 2\pi+1$. Then, $(2\pi+1)\left|\frac{\dot{\theta}(t)}{k^{\frac{1}{2}}}\right|$ is the largest perturbation term for the off-the-manifold coordinate variable $z(t)$. For the invariant manifold (4.4), the parameter update law is defined implicitly as

$$\dot{\hat{\theta}}(t) = k\Sigma(t)\Phi^T\Phi\tilde{\theta}(t) - \Sigma(t)\Phi^T\Phi\dot{\theta}(t) + \frac{1}{k}\Sigma(t)\Phi^T\Phi\ddot{\theta}(t) + \cdots \tag{4.6}$$

From the Lyapunov function (3.18) and update law (4.6), we have

$$\dot{V}_{\tilde{\theta}} = \tilde{\theta}^T(t)\dot{\theta}(t) - k\tilde{\theta}^T(t)(\Sigma(t)\Phi^T\Phi)\tilde{\theta}(t) + \tilde{\theta}^T(t)(\Sigma(t)\Phi^T\Phi)\dot{\theta}(t)$$
$$- \frac{1}{k}\tilde{\theta}^T(t)(\Sigma(t)\Phi^T\Phi)\ddot{\theta}(t) + \cdots$$

Application of Young's inequality on the third and higher terms yields the following inequality:

$$\dot{V}_{\tilde{\theta}} \leq \tilde{\theta}^T(t)\dot{\theta}(t) - k\tilde{\theta}^T(t)(\Sigma(t)\Phi^T\Phi)\tilde{\theta}(t) + \frac{k}{2}\tilde{\theta}^T(t)(\Sigma(t)\Phi^T\Phi)\tilde{\theta}(t)$$
$$+ \frac{1}{2k}\dot{\theta}^T(t)(\Sigma(t)\Phi^T\Phi)\dot{\theta}(t) + \frac{1}{2}\tilde{\theta}^T(t)(\Sigma(t)\Phi^T\Phi)\tilde{\theta}(t) + \frac{1}{2k^2}\ddot{\theta}^T(t)(\Sigma(t)\Phi^T\Phi)\ddot{\theta}(t) + \cdots$$

Based on the geometric series (4.5) and the maximum perturbation term, the inequality becomes:

$$\dot{V}_{\tilde{\theta}} \leq \tilde{\theta}^T(t)\dot{\theta}(t) - k\tilde{\theta}^T(t)(\Sigma(t)\Phi^T\Phi)\tilde{\theta}(t) + \frac{k^2}{2(k-1)}\tilde{\theta}^T(t)(\Sigma(t)\Phi^T\Phi)\tilde{\theta}(t)$$
$$+ \frac{(2\pi+1)^2}{2}\left|\frac{\dot{\theta}(t)}{\sqrt{k}}\right|^T(\Sigma(t)\Phi^T\Phi)\left|\frac{\dot{\theta}(t)}{\sqrt{k}}\right|.$$

Applying Young's inequality on the first term and collecting the terms, one obtains:

$$
\begin{aligned}
\dot{V}_{\tilde{\theta}} \leq & -\frac{k(k-2)}{2(k-1)}\tilde{\theta}^T(t)(\Sigma(t)\Phi^T\Phi - \eta_1 I)\tilde{\theta}(t) \\
& + \frac{(2\pi+1)^2}{2k}\dot{\theta}^T(t)\left(\Sigma(t)\Phi^T\Phi + \frac{(k-1)}{(2\pi+1)^2\eta_1(k-2)}I\right)\dot{\theta}(t) \\
\leq & \frac{-k\eta_1(b-1)}{4}V_{\tilde{\theta}} + \frac{(2\pi+1)^2}{2k}\left(\eta_2 + \frac{4}{(2\pi+1)^2\eta_1}\right)\gamma^2.
\end{aligned} \tag{4.7}
$$

By choosing the gain such that $k \gg (2\pi+1)^2$, exponential convergence to a small neighbourhood of the origin is achieved.

The next step focuses on providing an estimate of the upper bound $\|\dot{\theta}(t)\|$. By exploiting the orthogonality feature of the trigonometric functions for the mapping (4.4), one obtains

$$
\begin{aligned}
& \frac{2}{T_{i1}}\int_{\tau}^{\tau+T_{i1}} \left[\Phi^T\left(k(\hat{x}-x)+\psi\right)\right]\cos(\frac{2\pi}{T_{i1}}t)dt = \\
& \qquad \frac{2}{T_{i1}}\int_{\tau}^{\tau+T_{i1}} \Phi^T\Phi\left(-\theta(t) + \frac{\dot{\theta}(t)}{k} - \frac{\ddot{\theta}(t)}{k^2} + \cdots\right)\cos(\frac{2\pi}{T_{i1}}t)dt.
\end{aligned} \tag{4.8}
$$

Under Assumptions 3.1, 4.1 and the above inequality, we have

$$
2\|\Phi\|\|k(\hat{x}-x)+\psi\| \geq \alpha\left|\frac{1}{1+(\frac{2\pi}{kT_{i1}})^2}a_{i1}\right|. \tag{4.9}
$$

From (4.9), we have

$$
\begin{aligned}
|a_{i1}| & \leq \left(\frac{1}{1+(\frac{2\pi}{kT_{i1}})^2}\right)^{-1}\left(\frac{2\|\Phi\|\|k(\hat{x}-x)+\psi\|}{\alpha}\right) \\
& = \frac{(kT_{i1})^2+4\pi^2}{(kT_{i1})^2}\left(\frac{2\|\Phi\|\|k(\hat{x}-x)+\psi\|}{\alpha}\right) \\
& \leq \frac{(kT)^2+4\pi^2}{(kT)^2}\left(\frac{2\|\Phi\|\|k(\hat{x}-x)+\psi\|}{\alpha}\right) = \epsilon.
\end{aligned} \tag{4.10}
$$

If $\sin(\frac{2\pi}{T'_{i1}}t)$ is used instead of $\cos(\frac{2\pi}{T_{i1}}t)$ in (4.8), then

$$
|b_{i1}| \leq \frac{(kT'_{i1})^2+4\pi^2}{(kT'_{i1})^2}\left(\frac{2\|\Phi\|\|k(\hat{x}-x)+\psi\|}{\alpha}\right) \leq \epsilon. \tag{4.11}
$$

Moreover, the same value of ϵ can be obtained for second and higher order coefficients of the periodic time-varying parameters (4.1). Substitution of this upper bound in (4.2) yields

$$|\dot{\theta}_i(t)| \leq \frac{2\pi}{T}(L_i + L_i')\epsilon, \quad i = 1, .., p,$$

and finally,

$$\|\dot{\theta}(t)\| \leq \sum_{i=1}^{p} |\dot{\theta}_i(t)|. \tag{4.12}$$

This completes the proof of Theorem 4.1. □

Remark 4.1. *For the case where the $T_{i\ell}s$ and $T'_{i\ell'}s$ are known, a less conservative upper bound can be obtained for $\|\dot{\theta}(t)\|$. Even, a simple knowledge on one of time periodicities may result in a more precise value of the upper bound estimation.*

Remark 4.2. *According to (4.4) and the ϵ value, the bound on $\theta_i(t)$ can be approximated explicitly. Without loss of generality, let T_i be an unknown fundamental period for $\theta_i(t)$. The upper bound on the constant term a_{i0} is specified as*

$$\frac{1}{T_i}\int_{\tau}^{\tau+T_i} (\|\Phi\|\|k(\hat{x} - x) + \psi\|)\, dt \geq$$
$$\frac{\alpha}{T_i}\left|\int_{\tau}^{\tau+T_i}\left(-\theta_i(t) + \frac{\dot{\theta}_i(t)}{k} - \frac{\ddot{\theta}_i(t)}{k^2} + \cdots\right)dt\right| = \alpha\,|a_{i0}|\,.$$

Therefore, the bound is given by

$$\begin{aligned}|\theta_i(t)| &\leq |a_{i0}| + \left(\sum_{\ell=1}^{L_i}|a_{i\ell}| + \sum_{\ell'=1}^{L_i'}|b_{i\ell'}|\right) \leq |a_{i0}| + (L_i + L_i')\epsilon \\ &\leq \left(\frac{0.5(kT)^2}{(kT)^2 + 4\pi^2} + L_i + L_i'\right)\epsilon = \epsilon_i.\end{aligned} \tag{4.13}$$

The projection of the derive bound (4.13) to parameter estimation (3.14) yields

$$\bar{\theta}_i(t) = \begin{cases} \hat{\theta}_i(t) & |\hat{\theta}_i(t)| \leq \epsilon_i \\ \epsilon_i sign(\hat{\theta}_i(t)) & |\hat{\theta}_i(t)| > \epsilon_i \end{cases} \tag{4.14}$$

where $\bar{\theta}_i(t)$ is a projection of $\hat{\theta}_i(t)$, for $i = 1, ..., p$.

4.3 Simulation Example

Consider the following dynamical system

$$\dot{x}_1 = -0.2x_1 - x_2^2 + \theta_1(t)$$
$$\dot{x}_2 = u(t) + \theta_2(t)x_1. \tag{4.15}$$

The time-varying parameters are defined as

$$\theta_1(t) = \cos^6(t)$$
$$\theta_2(t) = \begin{cases} 1 + \cos(4t) & 2(j-1)\pi \le t < (2j-1)\pi \quad j = 1, 2, \cdots \\ 2 & (2j-1)\pi \le t < 2j\pi \end{cases}$$

where $T_1 = \pi$ and $T_2 = 2\pi$. The objective is to estimate the time evolution of $\theta_1(t)$ and θ_2(t) from the information of the nonlinear model. This will be achieved using the almost invariant manifold and the parameter update law (3.14).

For simulation purposes, it is assumed that $u(t) = -2x_2$ and $b = 2$. The initial conditions are chosen as $x_1(0) = x_2(0) = 0$ and $\hat{\theta}_1(0) = \hat{\theta}_2(0) = 0$. Figure 4.1 shows the measured state trajectories used in the simulation.

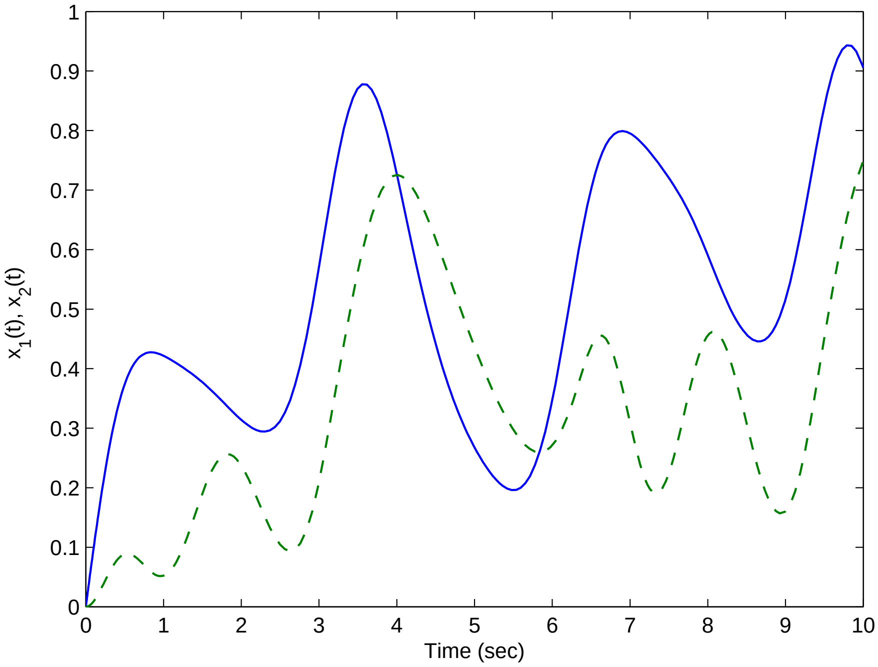

Figure 4.1: Trajectories of the states: $x_1(t)$ (——) and $x_2(t)$ $(- - -)$.

As previously mentioned, the parameter estimation error can be made as small as desired by using larger values of the gain k. To evaluate the performance of the proposed scheme in dealing with time-varying parameters, simulations are carried at two different values of k. For the first simulation, the gain is chosen as $k = 20$. The parameter estimates along with their true values are shown in Figure 4.2.

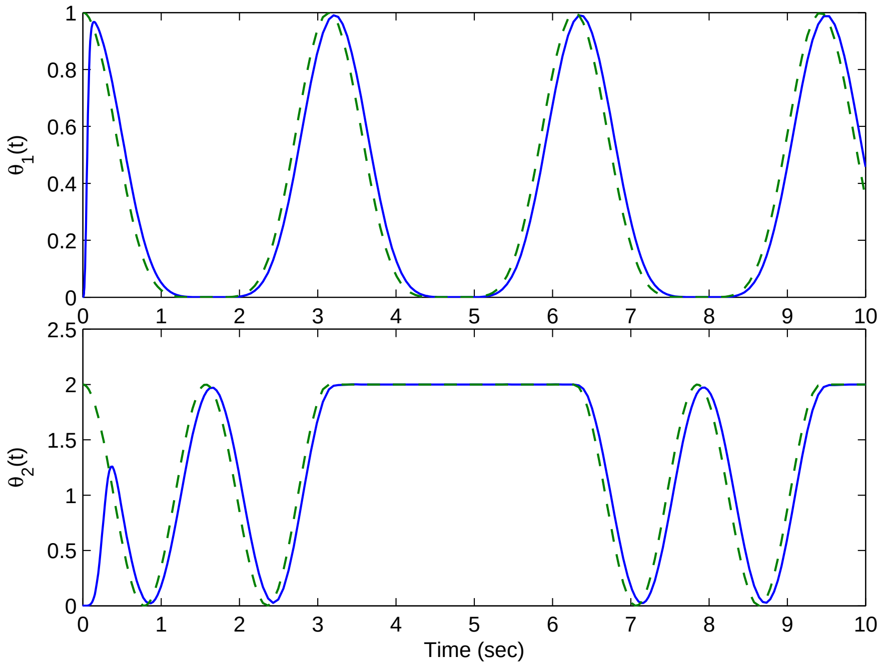

Figure 4.2: Trajectories of unknown parameter estimates ($k = 20$): parameter $\theta(t)$ $(- - -)$ and its estimate $\hat{\theta}(t)$ (——).

The normal manifold coordinate variables $z(t)$ are shown in Figure 4.3.

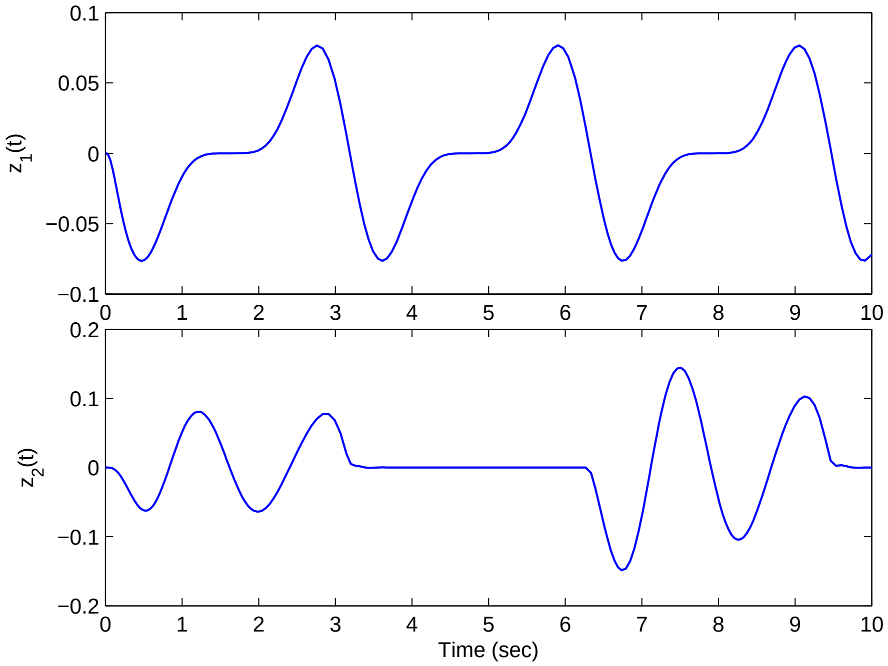

Figure 4.3: Trajectories of the implicit manifold ($k = 20$).

It is important to point out that the value of the variables, $z_1(t)$ and $z_2(t)$, are not known. As confirmed in this figure, the implicit manifold variables $z_1(t)$ and $z_2(t)$ converge to the small neighbourhood of zero. However, since the parameter $\theta_2(t)$ has no variations at $(2j-1)\pi \leq t < 2j\pi$, for $j = 1, 2, ...$, the exact values $\theta_2(t) = 2$ and $z_2(t) = 0$ are achieved for this time range.

To test the effect of a higher gain value on the performance of the adaptive estimator, the simulation results for the same problem with a gain of $k = 200$ are shown in Figures 4.4 and 4.5.

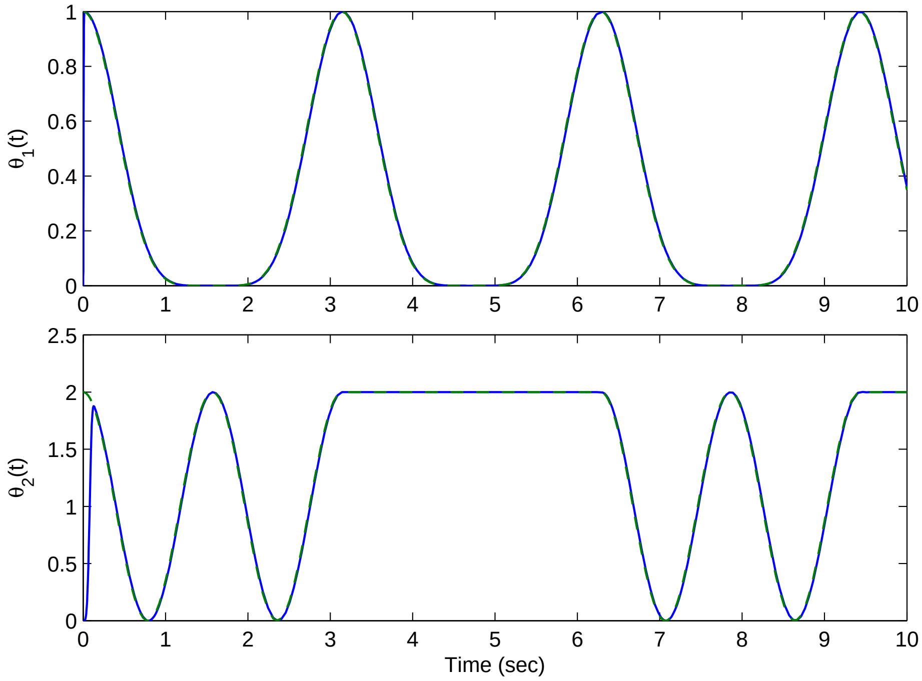

Figure 4.4: Trajectories of unknown parameter estimates ($k = 200$): parameter $\theta(t)$ ($- - -$) and its estimate $\hat{\theta}(t)$ (——).

As expected, by comparing Figures 4.2 and 4.4, the higher gain leads to the better performance and the parameter estimate tracking the parameter variations more effectively. The large gain value not only affect the estimation accuracy, but also results in faster convergence. For $\theta_2(t)$ at $(2j-1)\pi \leq t < 2j\pi$, for $j = 1, 2, ...$, the performances of both gains are similar, and precise parameter estimates are obtained.

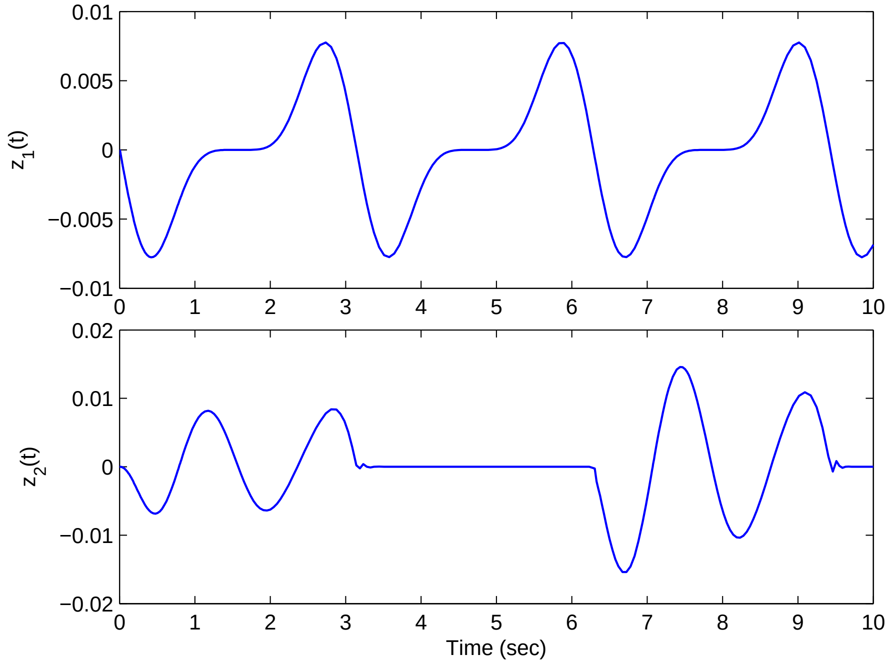

Figure 4.5: Trajectories of the implicit manifold ($k = 200$).

Figure 4.5 depicts the simulations results for the gain $k = 200$. The simulation shows that the higher gain manifold significantly improves the performance of the parameter estimation scheme. The simulations confirm that a satisfactory performance can be achieved without requiring that the system state reaches the reference manifold.

In order to estimate $\|\dot{\theta}(t)\|$, we need to write $\theta_1(t)$ as

$$\theta_1(t) = \cos^6(t) = \frac{5}{16} + \frac{15}{32}\cos(2t) + \frac{6}{32}\cos(4t) + \frac{1}{32}\cos(6t).$$

The number of distinct frequencies for $\theta_1(t)$ and $\theta_2(t)$ are $L_1 = 3$ and $L_2 = 1$, respectively. Choose $T = 2$ and $\alpha = 0.25$, the conditions of Assumptions 3.1 and 4.1 will be satisfied $\forall t \geq 0$. Based on (4.12), $\|\dot{\theta}(t)\| \leq 33.8$. However, this bound remains large compared to the unknown true values of the time-varying parameters with $\|\dot{\theta}(t)\| \leq 4.29$. A more precise estimation for $\|\dot{\theta}(t)\|$ can be achieved by choosing more reliable values of α and T.

4.4 Summary

The geometric-based estimation technique is extended to nonlinear systems with periodic uncertainties. The proposed adaptive algorithm provide an update law for estimation of unknown time-varying parameters. We showed how an invariant manifold can be constructed based on the periodic uncertainties. The proposed invariant manifold enables us to establish a geometric series to predict the upper bound values of the periodic parameters and their rate of changes. A less conservative prediction of these values can be achieved by having a good knowledge of PE condition's constants.

Chapter 5

ESC for Static Optimization

5.1 Introduction

Extremum-seeking control is a powerful tool to track the optimum of an unknown but measurable objective function which attracts considerable attention in many applications (see [91] and the references therein).

The standard ESC scheme is very effective and considers mild assumptions of the process dynamics, but it suffers from a lack of good transient performance. Over the last few years, there have been numerous attempts to address the limitations of ESC. Of particular interest in this study are attempts to extend ESC to the solution of constrained steady-state optimization problems.

One of the main challenges of perturbation-based ESC techniques is the close relationship that exists between averaging analysis and the convergence to the unknown optimum. This relationship dictates the speed at which one achieves the optimum and the size of the neighbourhood of the optimizer that can be guaranteed. Hence, the amplitude and frequency of the dither signal must be chosen very carefully as highlighted in [79].

In this chapter, we provide an alternative extremum-seeking technique to compensate for averaging analysis in conventional ESC. A time-varying parameter estimation technique which is derived from the definition of almost invariant manifolds is used to estimate elements of the unknown gradient of the cost function. The ESC then uses the estimated gradient in a gradient descent algorithm [65]. The main advantage of the time-varying estimation approach is to remove the need for averaging to establish the convergence of the extremum-seeking controller to the unknown steady-state optimum of a measured output function. Correspondingly, the time-varying ESC provides an improvement in transient performance with a comparatively minimal impact of the dither signal. Also, the proposed approach is extended for the solution of constrained steady-state optimization problems. A barrier function formulation is proposed to transform the constrained optimization problem to an unconstrained problem using an augmented cost. The time-varying ESC technique estimates the gradient of the augmented cost and proceeds as a gradient descent formulation. Barrier functions must be handled with care because of their potential for singularity at the boundaries of the feasible region. The proposed ESC formulation provides an effective and reliable mechanism to solve constrained optimization problems [35]. It avoids problems associated with singularities while ensuring feasibility of the system trajectories.

5.2 Problem Description

Consider an unknown and nonlinear static optimization problem

$$\min \; y = \ell(u) \tag{5.1}$$

where u is the input taking values in $\mathcal{U} \subset \mathbb{R}^m$ and $y \in \mathbb{R}$ is the unknown but measurable cost function to be minimized. The function $\ell(u)$ is assumed to be C^∞ in its argument.

The objective of ESC is to bring the input to the unknown minimizer u^* that minimizes the cost function y. The following assumptions are required to ensure that this problem is well-posed.

Assumption 5.1. *The equilibrium cost is such that*

$$\frac{\partial \ell(u^*)}{\partial u} = 0, \; \frac{\partial \ell(u)}{\partial u}(u - u^*) \geq \alpha_1 \left\| u - u^* \right\|^2$$

$\forall u \in \mathcal{U}$ *with strictly positive constant* α_1.

In practice, there is a limit on the input magnitude. Hence, the input space $\mathcal{U}$ is defined as $\mathcal{U} = \{u \mid \|u\| \leq r_u\}$ where r_u is a positive constant that identifies the upper limit on the size of the norm of the input u. Based on the Assumption 5.1 only a local convexity of the static optimization is required. Since the minimization of y is performed in real-time, the input u is taken as a time-varying signal. That is,

$$y(t) = \ell(u(t)) \tag{5.2}$$

The differentiation of (5.2) with respect to time results in $\dot{y} = \frac{\partial \ell(u)}{\partial u}\dot{u}$. The unknown gradient is defined as:

$$\theta(t) = \left(\frac{\partial \ell(u)}{\partial u} \right)^T$$

and the cost dynamics is described by the following expression:

$$\dot{y} = \theta(t)^T \dot{u}(t). \tag{5.3}$$

5.3 Time-Varying ESC

The design of the extremum-seeking scheme is based on the unknown dynamics (5.3). The first step consists in the estimation of the time-varying parameter $\theta(t)$. In the second step, we define a suitable adaptive controller that accomplishes the extremum-seeking task [65].

5.3.1 Parameter Estimation

The time-varying gradient estimation uses an approach similar to the one described in Chapter 3. An estimator model for (5.3) is defined as

$$\dot{\hat{y}} = -k^2(\hat{y} - y), \tag{5.4}$$

where the value of the tuning gain $k > 0$ will be assigned in the design. A filter is described along with the structure of the system by

$$\dot{\phi} = -k^2(\phi - \dot{u}) \tag{5.5}$$

where an almost invariant manifold is defined as

$$k^2(\hat{y} - y) + \phi^T\theta(t) = 0.$$

The auxiliary variables and estimator are expressed as

$$
\begin{aligned}
\dot{p} &= -k^2 p - \phi\phi^T \hat{\theta}(t) \\
\dot{q} &= -k^2 q + \phi(k^2(\hat{y} - y)) \\
\dot{\Sigma}(t) &= k^2[bI - \phi\phi^T(\Sigma(t))].
\end{aligned} \tag{5.6}
$$

Let $\Theta \triangleq B(0, r_\theta)$, the unit ball centered at the origin with radius r_θ. By Assumption 5.1, the uncertainty set radius r_θ can be set to L_1. Based on (5.6) the gradient update law is generated by

$$
\dot{\hat{\theta}}(t) = \mathrm{Proj}(k^2\Sigma(t)[\dot{\delta} + k\delta], \hat{\theta}), \ \delta = p - q, \tag{5.7}
$$

with $\hat{\theta}(0) = \theta_0 \in \Theta$. The operator $\mathrm{Proj}(\eta_1, \hat{\theta})$ with $\eta_1 = k^2\Sigma(t)[\dot{\delta} + k\delta]$, denotes a Lipschitz projection operator [46] such that

$$
\begin{aligned}
&-\tilde{\theta}^T \mathrm{Proj}(\eta_1, \hat{\theta}) \leq -\tilde{\theta}^T \eta_1 \\
&\hat{\theta}(0) \in \Theta \implies \hat{\theta}(t) \in \Theta, \ \forall t \geq 0
\end{aligned}
$$

where $\tilde{\theta}(t) = \theta(t) - \hat{\theta}(t)$ is a parameter estimation error. A standard projection algorithm can be defined as follows. Let us consider the function $\mathcal{P}(\hat{\theta}) = \|\hat{\theta}\|^2 - L_1^2$. Its gradient is given by $\nabla_{\hat{\theta}}\mathcal{P}(\hat{\theta}) = 2\hat{\theta}^T$. The positive constant L_1 can be achieved from the smoothness of the function $\ell(u)$ and boundedness of the input u such that $\|\nabla_u \ell(u)\| \leq L_1$. The projection algorithm (5.7) can be given as follows:

$$
\dot{\hat{\theta}} = \begin{cases} \eta_1 & \text{if } \mathcal{P}(\hat{\theta}) < 0 \text{ or } \nabla_{\hat{\theta}}\mathcal{P}(\hat{\theta})\eta_1 \leq 0) \\ \left(I - \frac{\nabla_{\hat{\theta}}\mathcal{P}(\hat{\theta})^T \nabla_{\hat{\theta}}\mathcal{P}(\hat{\theta})}{\|\nabla_{\hat{\theta}}\mathcal{P}(\hat{\theta})\|^2}\right)\eta_1 & \text{otherwise.} \end{cases} \tag{5.8}
$$

The following assumption is required to ensure convergences of the parameter estimates to their true values.

Assumption 5.2. *There exists constants $\alpha > 0$ and $T > 0$ such that*

$$\int_{t}^{t+T} \phi(\tau)\phi^T(\tau)d\tau \geq \alpha I, \quad \forall t > t_0. \tag{5.9}$$

This assumption is a standard PE condition that must be met by the closed-loop ESC.

5.3.2 Controller Design

Let $P(u) = \|u\|^2 - r_u^2$ and define $\eta_2 = -k_g\hat{\theta}(t) + d(t)$, where $d(t)$ is a small bounded dither signal with $\|d(t)\| \leq D$, and $k_g > 0$ is an optimization gain. The dither signal $d(t)$ can be chosen arbitrarily but it is required to satisfy the PE condition of the Assumption 5.2. A common choice of $d(t)$ signals are sinusoidal waves, because of their orthogonality feature [79].

Remark 5.1. *Unlike the perturbation based ESC, the stability analysis of our proposed algorithm is not sensitive to the choice of amplitude and frequency of the sinusoidal signal.*

If the boundedness of u is enforced via a Lipschitz projection algorithm, then we have the extremum-seeking control update law

$$\dot{u} = \begin{cases} \eta_2 & \text{if } P(u) < 0 \text{ or } (P(u) = 0 \text{ and } \nabla_u P(u)\eta_2 < 0) \\ \left(I - \frac{\nabla P^T \nabla P}{\|\nabla P\|^2}\right)\eta_2 & \text{otherwise.} \end{cases} \tag{5.10}$$

Note that, since $\hat{\theta}$ and $d(t)$ are assumed to be bounded, and since $\|I - \frac{\nabla P^T \nabla P}{\|\nabla P\|^2}\| \leq 1$, then the controller (5.10) is such that $\|\dot{u}\| \leq k_g L_1 + D = L_u$.

Theorem 5.1. *Let Assumptions 5.1 and 5.2 hold. There exist positive gains, k and k_g such that the parameter update law (5.8) and the control law (5.10) are such that the closed-loop extremum-seeking control system converges to an $[\mathcal{O}(k_g/k), \mathcal{O}(D/k_g)]$ neighborhood of the minimizer u^* of the static nonlinear optimization problem.*

Proof. The parameter update law (5.7) can be written implicitly in the form

$$\dot{\hat{\theta}}(t) = \text{Proj}(k^2\Sigma(t)\phi\phi^T\tilde{\theta}(t) - \Sigma(t)\phi\phi^T\dot{\theta}(t), \hat{\theta}). \tag{5.11}$$

Let $\tilde{u} = u - u^*$ with a quadratic Lyapunov function as

$$V = \frac{1}{2}\tilde{\theta}^T(t)\tilde{\theta}(t) + \frac{1}{2}\tilde{u}^T\tilde{u}. \tag{5.12}$$

By differentiating of (5.12) along (5.10) and (5.11), we have

$$\begin{aligned}\dot{V} \leq &- k^2\tilde{\theta}^T(t)(\Sigma(t)\phi\phi^T)\tilde{\theta}(t) + \tilde{\theta}^T(t)(\Sigma(t)\phi\phi^T)\dot{\theta}(t)\\ &+ \tilde{\theta}^T(t)\dot{\theta}(t) - k_g\tilde{u}^T\theta(t) + k_g\tilde{u}^T\tilde{\theta}(t) + \tilde{u}^T d(t).\end{aligned}$$

By Assumption 5.1, one can write the following inequality

$$\begin{aligned}\dot{V} \leq &- k^2\tilde{\theta}^T(t)(\Sigma(t)\phi\phi^T)\tilde{\theta}(t) + \tilde{\theta}^T(t)(\Sigma(t)\phi\phi^T)\dot{\theta}(t)\\ &+ \tilde{\theta}^T(t)\dot{\theta}(t) - k_g\alpha_1\tilde{u}^T\tilde{u} + k_g\tilde{u}^T\tilde{\theta}(t) + \tilde{u}^T d(t).\end{aligned}$$

Applying Young's inequality to all indefinite terms of the last inequality, there exists a positive constant k_1 such that

$$\begin{aligned}\dot{V} \leq &- k^2\tilde{\theta}^T(t)(\Sigma(t)\phi\phi^T)\tilde{\theta}(t) + \frac{k^2}{2}\tilde{\theta}^T(t)(\Sigma(t)\phi\phi^T + \frac{1}{k}I)\tilde{\theta}(t)\\ &- k_g\alpha_1\tilde{u}^T\tilde{u} + \frac{1}{2k^2}\dot{\theta}^T(t)(\Sigma(t)\phi\phi^T + kI)\dot{\theta}(t) + \frac{kk_g}{2}\tilde{\theta}^T(t)\tilde{\theta}(t)\\ &+ \frac{k_g}{2k}\tilde{u}^T\tilde{u} + \frac{k_1}{2}\tilde{u}^T\tilde{u} + \frac{1}{2k_1}d^T(t)d(t).\end{aligned}$$

Next we claim the boundedness of the vector-valued function ϕ and the matrix $\Sigma(t)$ as follows. The control input update law (5.10) is bounded, due to the Lipschitz projection algorithm property and the boundedness of the dither signals. Accordingly, the boundedness of the filter output ϕ, can be established by the BIBO stability of the filter dynamics. Hence it can be concluded that $||\phi|| \leq \gamma_1 = \max\{\phi(t_0), L_u\}$.

Similar to (3.24), one gets

$$\Sigma(t)\phi\phi^T \geq \int_{t_0}^{t}(k^2b\phi(\tau)\phi^T(\tau))\exp\left[\int_{\tau}^{t} -k^2(2I+\phi(\xi)\phi^T(\xi))d\xi\right]d\tau \geq$$
$$\int_{t_0}^{t}(k^2b\phi(\tau)\phi^T(\tau))e^{-k^2(\gamma_1^2+2)(t-\tau)}d\tau \geq \int_{t_0+NT}^{t}(k^2b\phi(\tau)\phi^T(\tau))e^{-k^2(\gamma_1^2+2)(t-\tau)}d\tau,$$

whit an arbitrary integer N. As a result of Assumption 5.2, we obtain

$$\Sigma(t) \leq \Sigma(t_0) + k^2b\int_{t_0}^{t}\exp\left[\int_{\tau}^{t} -k^2\phi(\xi)\phi^T(\xi)d\xi\right]d\tau \leq \Sigma(t_0)$$
$$+ (k^2b\int_{t_0}^{t} e^{-k^2\alpha(t-\tau)}d\tau)I \leq \frac{b\alpha + b(1-e^{-k^2\alpha(t-t_0)})}{\alpha}I.$$

Assuming the above inequalities, one gets the following bounds:

$$\frac{b\alpha(1-e^{-k^2(\gamma_1^2+2)(t-t_0-NT)})}{\gamma_1^2+2}I \leq \Sigma(t)\phi\phi^T \leq \frac{b\alpha+b(1-e^{-k^2\alpha(t-t_0)})}{\alpha}\phi\phi^T, \tag{5.13}$$

$\forall t \subseteq [t_i, t_i+NT]$. Therefore,

$$\gamma_2 = \frac{b\alpha(1-e^{-k^2(\gamma_1^2+2)T})}{\gamma_1^2+2} \leq \left\|\Sigma\phi\phi^T\right\| \leq \frac{b(1+\alpha)}{\alpha}\gamma_1^2 = \gamma_3. \tag{5.14}$$

With collecting the similar terms, we can have the following inequality

$$\dot{V} \leq -\left(\frac{k^2\gamma_2 - k(k_g+1)}{2}\right)\|\tilde{\theta}\|^2 - \left(k_g\alpha_1 - \frac{k_g}{2k} - \frac{k_1}{2}\right)\|\tilde{u}\|^2$$
$$+\left(\frac{\gamma_3+2k}{2k^2}\right)\|\dot{\theta}\|^2 + \frac{1}{2k_1}D^2.$$

By choosing k and k_g such that

$$k\gamma_2 - (k_g+1) > 0$$
$$k_g\left(\alpha_1 - \frac{1}{2k}\right) > \frac{k_1}{2}$$

then for the given gains, there exist strictly positive constants k_a, k_b and $k' =$

$\min\{k_a, k_b\}$ such that

$$\begin{aligned}\dot{V} &\leq -k_a\|\tilde{\theta}\|^2 - k_b \|\tilde{u}\|^2 + \left(\frac{\gamma_3 + 2k}{2k^2}\right)\|\dot{\theta}\|^2 + \frac{1}{2k_1}D^2 \\ &\leq -2k'V + \left(\frac{\gamma_3 + 2k}{2k^2}\right)\|\dot{\theta}\|^2 + \frac{D^2}{2k_1}.\end{aligned}$$

Next recall that $\theta = \frac{\partial \ell}{\partial u}^T$. Since $\dot{u} = \mathrm{Proj}(\eta_2, u)$, it follows that the rate of change of the unknown gradient θ is given by, $\dot{\theta} = \frac{\partial^2 \ell(u)}{\partial u \partial u^T}\mathrm{Proj}(\eta_2, u)$. Since $\|\frac{\partial^2 \ell(u)}{\partial u \partial u^T}\| \leq L_2$, where L_2 is a positive constant, it follows that $\|\dot{\theta}\| \leq L_2(k_g L_1 + D) = L_\theta$. Upon substitution, one obtains

$$\dot{V} \leq -2k'V + \left(\frac{\gamma_3 + 2k}{2k^2}\right) L_\theta^2 + \frac{D^2}{2k_1}. \tag{5.15}$$

Hence, it follows that $\tilde{\theta}$ and $\tilde{u}$ converge locally exponentially to a small neighborhood of the origin. The size of this neighborhood depends on the choice of gains k, k_g and the magnitude of the dither signal. □

Remark 5.2. *Since, the rate of change of the parameters are proportional to the optimization gain, the convergence to a small neighborhood of the origin is achieved by ensuring that $k \ggg k_g$. The proof of the last theorem confirms that one can adjust the speed of the response by increasing the optimization gain k_g. The increase in k_g leads to a corresponding increase in k. It is important to note that one cannot increase the optimization gain arbitrarily. Increasing the gain reduces the effect from the dither signal on the control variable $u(t)$, which also minimizes its impact on the performance of the estimation routine. In general, there exists a maximal value of the gain k_g that can be achieved. This value depends on many factors such as the choice of dither and the nonlinearity of the unknown function.*

5.4 Constrained ESC

Let us consider the optimization problem (5.1). Assume that there exists an additional smooth vector-valued function $\beta(u) = [\beta_1(u), \cdots, \beta_{n_c}(u)]^T \in \mathbb{R}^{n_c}$. Further assume that these functions are unknown but available for measurement. Each element of β encodes an inequality constraint to be met by the nonlinear system. The objective is to steer the system to the unknown value u^* that solves the constrained optimization problem

$$\begin{aligned} &\min\ y = \ell(u) \\ &\text{subj. to} \\ &\qquad \beta(u) \geq 0. \end{aligned} \tag{5.16}$$

We only consider inequality constraints. Strict equality constraints are not treated. It will be assumed that the optimization problem (5.16) is convex. This is formally stated as follows.

Assumption 5.3. *The cost function $\ell(u)$ is a strictly convex function, and the constraints β_i, $i = 1, \cdots, n_c$, are strictly concave functions. This guarantees that the optimization problem (5.16) has a unique global minimizer u^*.The second order sufficient conditions for a global minimum at u^* is stated as follows*

1. *There exists a unique Lagrange multiplier vector $\nu^* \in \mathbb{R}^+_{n_c}$ such that*
$$\nabla_u L(u^*, \nu^*) = \nabla_u \ell(u^*) - \sum_{i=1}^{n_c} \nu_i^* \nabla_u \beta_i(u^*) = 0.$$
2. *The Hessian matrix*
$$\nabla_u^2 L(u^*, \nu^*) = \nabla_u^2 \ell(u^*) - \sum_{i=1}^{n_c} \nu_i^* \nabla_u^2 \beta_i(u^*),$$

is positive definite on the affine subspace tangent to the feasible at u^*:

$$\omega^T \nabla_u^2 L(u^*, \nu^*)\omega > 0,$$

for all $\omega \in W \subset \mathbb{R}^m$, where $W = \{\omega : \omega^T \nabla_u \beta_i(u^*)^T = 0,\ i = 1, ..., n_c\}$.

By the smoothness of the function $\ell(u)$, vector-valued function $\beta(u)$ and boundedness of the input $u \in \mathcal{U}$, it follows that $\ell(u)$ and $\beta(u)$ and their derivatives are bounded over $\mathcal{U}$. Hence, the constraints inequalities are added to the second part of the Assumption 5.1 as stated in the Assumption 5.4.

Assumption 5.4. *The constraints β_i, $i = 1, \cdots, n_c$, are such that*

$$\|c\| \leq C, \ \left\| \frac{\partial \beta_i(u)}{\partial u} \right\| \leq L_{1i}, \ \left\| \frac{\partial^2 \beta_i(u)}{\partial u \partial u^T} \right\| \leq L_{2i}$$

$\forall u \in \mathcal{U}$ with positive constants C, L_{1i} and L_{2i}.

5.4.1 Augmented Barrier Functions

In this section, we consider an interior-point approach in which the cost y is augmented with a barrier function as

$$\bar{y} = L(u, \mu) = \ell(u) - \mu \sum_{i=1}^{n_c} \psi(\beta_i(u)), \tag{5.17}$$

where μ is a positive constant to be assigned. The function $\psi(\cdot)$ is the barrier function. The most widely applied class of barrier functions are logarithmic, $\psi(a) = \ln(a)$. The minimization of $L(u, \mu)$ approximates the solution of the constrained problem as $u^*(\mu)$ corresponding to a given value of the penalty parameter μ. Following the property of the barrier functions [68], the constrained optimums can be approached by reducing the value of μ. It recovers the solution of the original problem as $\mu \to 0$.

Since, the gradient of the augmented cost with respect to u must be estimated, strict feasibility of the trajectories of u may not be enforced by the control. This is problematic when one considers log barrier functions as even a very small violation in the constraints can render the augmented cost undefined. As a result, one must adapt the choice of barrier function to avoid such situations. Following [68], a logarithmic barrier function can be modified as follows

$$\psi(a) = \begin{cases} \ln(a) & \text{if } a > \epsilon/2 \\ s(a) & \text{if } a \leq \epsilon/2 \end{cases}, \tag{5.18}$$

where a represents the argument of the barrier function, $\epsilon > 0$ is a small positive number that measures the distance of the argument of ψ to the constraint. The function $s(a)$ is a quadratic expression chosen such that $s(\epsilon/2) = \ln(\epsilon/2)$, $s'(\epsilon/2) = \ln'(\epsilon/2)$ and $s''(\epsilon/2) = \ln''(\epsilon/2)$. This choice of penalty/barrier function penalizes the constraints while removing difficulties associated with small violations of the constraints. It also removes problems associated with infeasible initial values of the input, $u(0)$.

Following the time-varying approach proposed in Section 5.3, we define the gradient as

$$\bar{\theta}(t) = \left(\frac{\partial L}{\partial u}\right)^T$$

then the rate of change of the augmented cost dynamics can be written as

$$\dot{\bar{y}} = \bar{\theta}(t)^T \dot{u}. \tag{5.19}$$

Consider the augmented cost with $\psi(\cdot)$ given by (5.18). If one defines the set N_c, as indices of active constraints. That is, constraints i such that $\beta_i(u) \leq \epsilon/2$. Then the gradient of $\bar{y}$ is given by

$$\frac{\partial L}{\partial u} = \begin{cases} \frac{\partial \ell}{\partial u} - \mu \sum_{i \notin N_c} \frac{1}{\beta_i(u)} \frac{\partial \beta_i(u)}{\partial u} & \text{if } i \notin N_c \\ \frac{\partial \ell}{\partial u} - \mu \sum_{i \in N_c} \frac{\partial s_i(u)}{\partial u} - \mu \sum_{j \notin N_c} \frac{1}{\beta_j(u)} \frac{\partial \beta_j(u)}{\partial u} & \text{if } i \in N_c,\ j \notin N_c \\ \frac{\partial \ell}{\partial u} - \mu \sum_{i \in N_c} \frac{\partial s_i(u)}{\partial u} & \text{if } i \in N_c. \end{cases} \tag{5.20}$$

The corresponding Hessian matrix is given by

$$\frac{\partial^2 L}{\partial u \partial u^T} = \begin{cases} \frac{\partial^2 \ell}{\partial u \partial u^T} - \mu \sum_{i \notin N_c} \frac{-1}{\beta_i^2} \frac{\partial \beta_i}{\partial u}^T \frac{\partial \beta_i}{\partial u} + \frac{1}{\beta_i} \frac{\partial^2 \beta_i}{\partial u \partial u^T} & \text{if } i \notin N_c \\ \frac{\partial^2 \ell}{\partial u \partial u^T} - \mu \sum_{i \in N_c, j \notin N_c} \frac{\partial^2 s_i}{\partial u \partial u^T} - \frac{1}{\beta_j^2} \frac{\partial \beta_j}{\partial u}^T \frac{\partial \beta_j}{\partial u} + \frac{1}{\beta_j} \frac{\partial^2 \beta_j}{\partial u \partial u^T} & \text{if } i \in N_c, j \notin N_c \\ \frac{\partial^2 \ell}{\partial u \partial u^T} - \mu \sum_{i \in N_c} \frac{\partial^2 s_i}{\partial u \partial u^T} & \text{if } i \in N_c. \end{cases} \tag{5.21}$$

The following lemma indicates that the gradient descent $\dot{u} = -k_g \bar{\theta}(t)$ with known gradient $\bar{\theta}(t)$ of the augmented cost (5.17) converges to the minimizer $u^*(\mu)$. This lemma is used in the statement of the convergence of the proposed extremum-seeking controller.

Lemma 5.1. *Consider the augmented cost (5.17). The gradient descent update $\dot{u} = -k_g \bar{\theta}(t)$ is such that the augmented cost decreases monotonically and reaches the minimizer of $L(u, \mu)$, $u^*(\mu)$.*

Proof. If one considers the expressions (5.20), (5.21) and Assumption 5.3, it is easy to show that the Hessian of the augmented cost meets the following inequality:

$$\frac{\partial^2 L(u, \mu)}{\partial u \partial u^T} > \rho_1 I, \ \forall u \in \mathcal{U} \tag{5.22}$$

uniformly in $\mu \in [0, \mu^*]$, with $\rho_1 > 0$, a strictly positive number. By existence and uniqueness of the solution of the gradient descent algorithm for the augmented cost, the following result can be obtained.

$$(u - u^*(\mu))^T \frac{\partial L(u, \mu)}{\partial u}^T \geq \alpha_2 \|u - u^*(\mu)\|^2, \ \forall u \in \mathcal{U}. \tag{5.23}$$

The last inequality is similar to the first part of Assumption 5.1, where α_2 is a strictly positive constant. Let $\tilde{u} = u - u^*(\mu)$ and consider the following Lyapunov function candidate for the input dynamics:

$$\mathcal{V} = \frac{1}{2} \tilde{u}^T \tilde{u}. \tag{5.24}$$

Differentiation with respect to t yields the following inequality:

$$\dot{\mathcal{V}} = -k_g \tilde{u}^T \left(\frac{\partial \ell(u)}{\partial u} - \mu \sum_{i=1}^{n_c} \psi'(\beta(u)) \frac{\partial \beta(u)}{\partial u} \right) \leq -k_g \alpha_2 \|\tilde{u}\|^2.$$

As a result, the system converges to the unknown minimizer $u^*(\mu)$. □

Remark 5.3. *The variable μ is usually chosen larger than ϵ. It can be slowly decreased to an arbitrary minimum value μ^* to approximate the unknown minimizer u^*. In this work, we consider the simple exponential decrease*

$$\dot{\mu} = \begin{cases} -\epsilon_0 \mu & \mu > \mu^* \\ 0 & \textit{otherwise}, \end{cases} \tag{5.25}$$

where $\epsilon_0 \geq 0$ is a small non-negative number. Note that by the continuity of $\mu(t)$, the continuity properties of the gradient and Hessian of the augmented cost with respect to u are preserved and the gradient algorithm remains well defined.

Since the time-varying parameter $\bar{\theta}(t)$ is unknown, one must consider an adaptive control approach. The design of the extremum-seeking routine is based on the dynamics (5.19). The first step consists in the estimation of the time-varying parameters. In the second step, we define a suitable controller that achieves the extremum-seeking task. To summarize, the proposed time-varying gradient estimation algorithm and controller design are given as follows

$$
\begin{aligned}
\dot{\hat{\bar{y}}} &= -k^2(\hat{\bar{y}} - \bar{y}) \\
\dot{\bar{\phi}} &= -k^2(\bar{\phi} - \dot{\bar{u}}) \\
\dot{\bar{p}} &= -k\bar{p} - \bar{\phi}\bar{\phi}^T\hat{\bar{\theta}}(t) \\
\dot{\bar{q}} &= -k\bar{q} + \bar{\phi}(k^2(\hat{\bar{y}} - \bar{y})) \\
\dot{\bar{\Sigma}}(t) &= k[bI - \bar{\phi}\bar{\phi}^T(\bar{\Sigma}(t))] \\
\dot{\hat{\bar{\theta}}}(t) &= \mathrm{Proj}(k^2\bar{\Sigma}(t)[\dot{\bar{\delta}} + k\bar{\delta}], \hat{\bar{\theta}}), \ \bar{\delta} = \bar{p} - \bar{q}, \\
\dot{\bar{u}} &= \mathrm{Proj}(-k_g\hat{\bar{\theta}}(t) + d(t), \bar{u}),
\end{aligned} \tag{5.26}
$$

with estimation gain k, optimization gain k_g and a bounded dither signal $d(t)$.

Theorem 5.2. *Let all the assumptions hold. Then there exist gains k and k_g such that the closed-loop ESC system (5.26) converges to a neighbourhood of the minimizer $u^*(\mu)$ of the augmented unconstrained cost (5.17) that approximates to an $\mathcal{O}(\mu)$ neighbourhood of the minimizer of the static nonlinear optimization problem (5.16).*

Proof. The proof of this theorem is analogous to the proof of Theorem 5.1. An alternative proof of this theorem with set-based adaptive estimation technique [22] can also be found in [35]. □

5.5 Simulation Examples

5.5.1 Case 1

The unknown static input-output map is given by [31] as

$$y = 100 + \frac{1}{2}\left(u - \begin{bmatrix} 2 \\ 4 \end{bmatrix}\right)^T \begin{bmatrix} 100 & 30 \\ 30 & 20 \end{bmatrix} \left(u - \begin{bmatrix} 2 \\ 4 \end{bmatrix}\right) \tag{5.27}$$

This static map meets all the assumptions in the optimization problem. The objective is to minimize the output y with respect to input u. We apply the proposed extremum-seeking control algorithm with $b = 1.1$, $k = \sqrt{10}$, $k_g = 0.02$, and arbitrary dither signal $d(t) = [0.02\sin(10t),\ 0.02\sin(20t)]^T$. The initial conditions are chosen as $u(0) = [0, 0]^T$ and $\hat{\theta}(0) = [10, 10]^T$, where $\hat{\theta}(t)$ is estimation of the unknown gradient.

The simulation results are shown in Figures 5.1- 5.3. The extremum-seeking control input trajectories are shown in Figure 5.1. The changes of the unknown cost function y is depicted in Figure 5.2. In Figure 5.3, the corresponding input trajectories are shown on a contour plot of the unknown objective function. The results demonstrate that the proposed extremum-seeking control algorithm provides a rapid progression to the unknown minimizer of the optimization problem. In addition, the control system provides satisfactory transient behaviour for both the inputs and the objective function. To show the effect of the dither signal on the closed-loop behaviour, the dither is changed as $d(t) = [0.2\sin(100t),\ 0.2\sin(2t)]^T$. The simulation results are shown in Figures 5.4 and 5.5. The impact of the dither signal is shown to preserve feasibility of the closed-loop trajectories while achieving the minimizer of the static optimization problem.

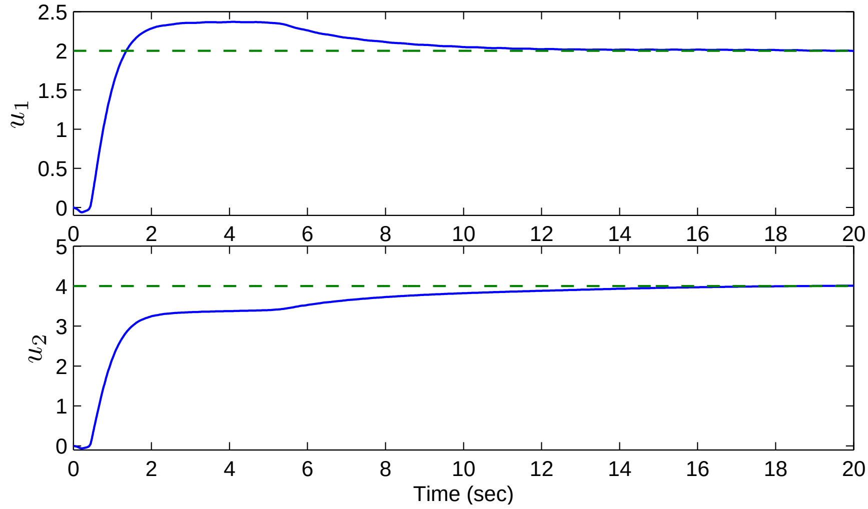

Figure 5.1: Control inputs: optimal value (— — —) and its estimate (——).

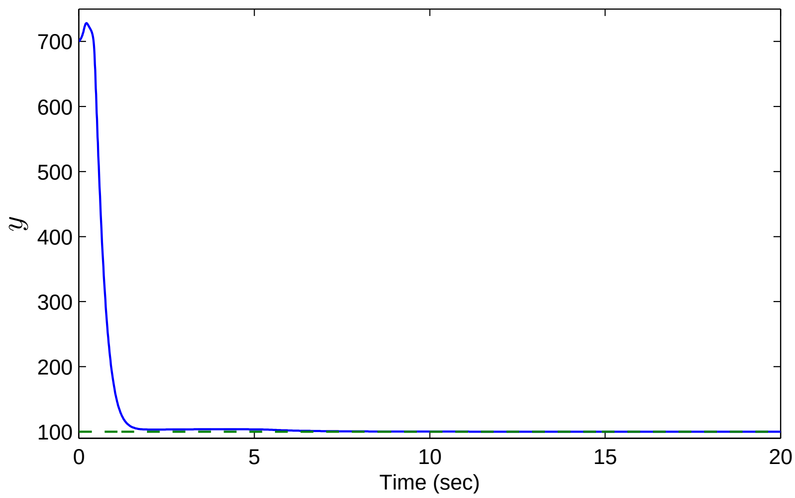

Figure 5.2: Unknown cost function: optimal value (— — —) and its estimate (——).

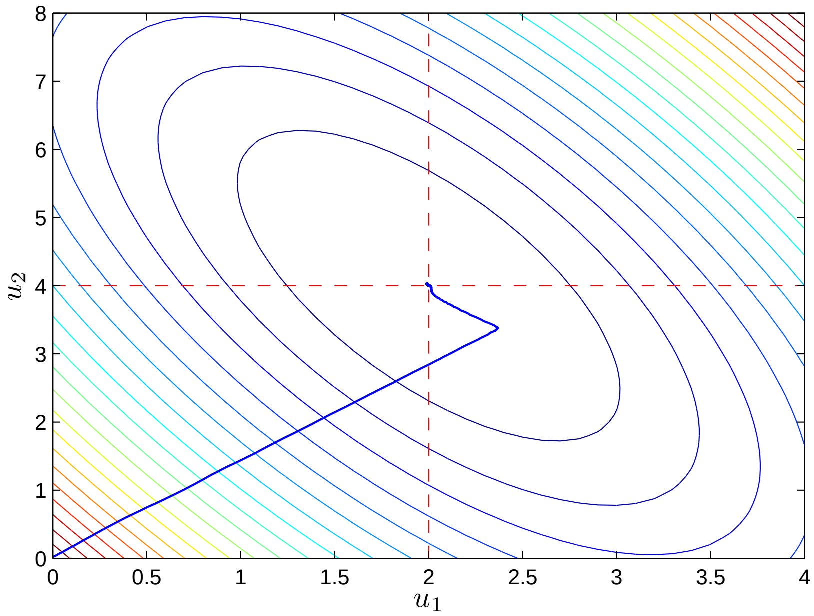

Figure 5.3: Contour plot of the unknown cost with ESC input trajectories.

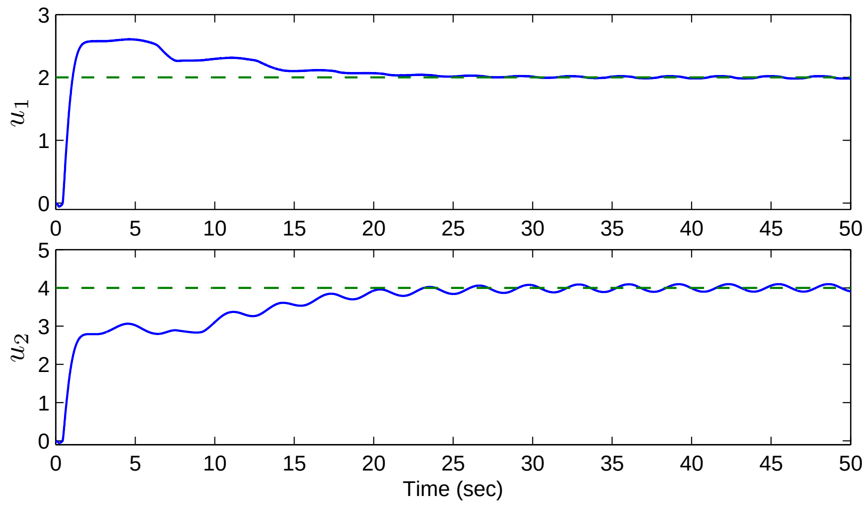

Figure 5.4: Control inputs with a different dither signal: optimal value (– – –) and its estimate (——).

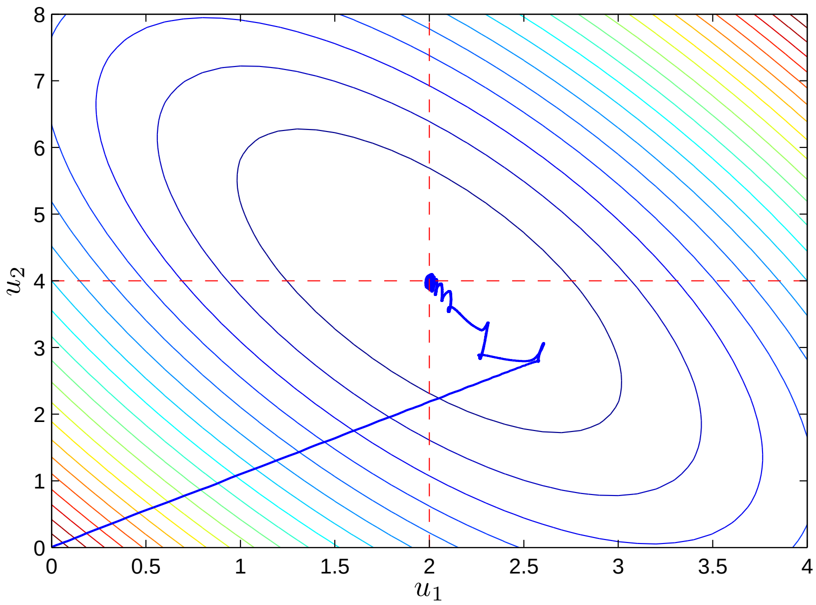

Figure 5.5: Contour plot of the unknown cost with ESC input trajectories for a different dither signal.

5.5.2 Case 2

In this simulation, we demonstrate the effectiveness of the proposed constrained ESC approach to solve a quadratic programming problem. We consider the two-input problem given in (5.27), but subject to the following constraints:

$$u_1 \leq 2, \ \frac{1}{2}u_1 + u_2 \leq 4.$$

Again, this static optimization problem is convex. For the simulation purpose, the algorithm is used with the same tuning parameters, initial conditions, and dither signal of the Case 1. The parameters that are associated with barrier function are chosen as $\epsilon = 0.0001$, $\epsilon_0 = 0.001$, $\mu(0) = 0.001$ and $\mu^* = 0.0001$. It is easy to see that the unknown constrained optimum is located at $u_1^* = 2$, $u_2^* = 3$ with $y^* = 110$.

Figures 5.6 and 5.7 show the inputs and output ESC trajectories, respectively. In

Figure 5.8, the corresponding constrained input trajectories are shown on a contour plot of the unknown objective function. The results demonstrate the effectiveness of the constrained ESC to locate the unknown constrained optimum. The constrained ESC converges to a neighbourhood of the correct optimum.

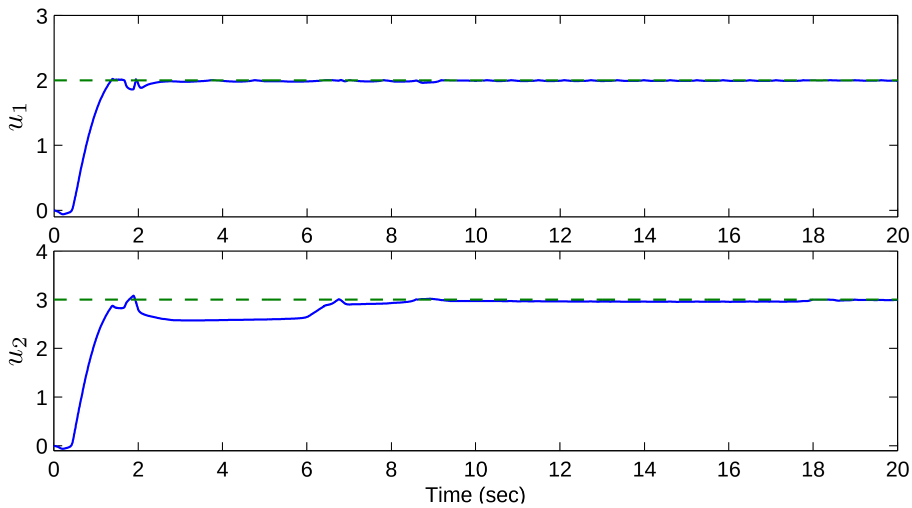

Figure 5.6: Constrained control inputs: optimal value (– – –) and its estimate (——).

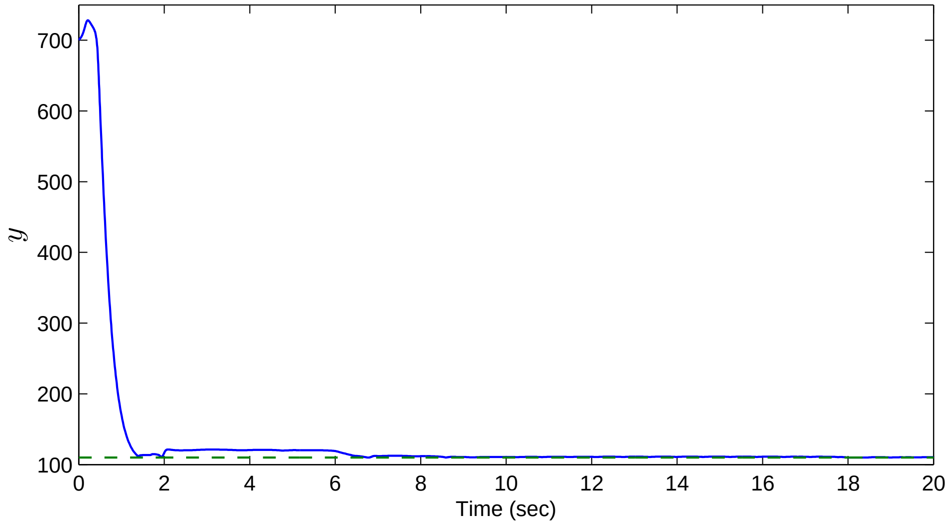

Figure 5.7: Unknown cost function with unknown constraints: optimal value (– – –) and its estimate (——).

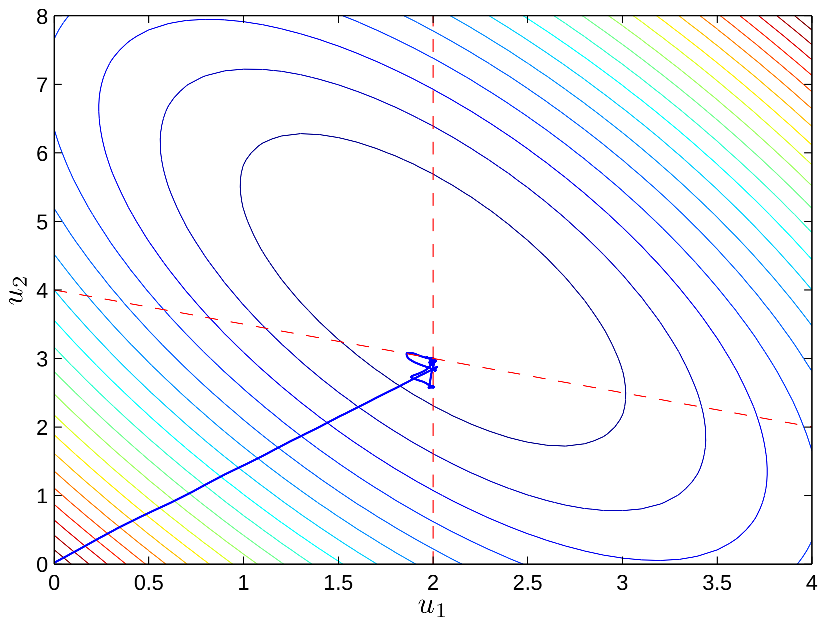

Figure 5.8: Contour plot of the constrained unknown cost with ESC input trajectories.

5.6 Summary

In this chapter, an alternative ESC technique was proposed for the solution of unknown static optimization problem. The technique relies on the time-varying estimation of the unknown gradient. The estimation scheme is based on the geometric concepts of almost invariant manifolds. The ESC algorithm is shown to provide local exponential convergence of the closed-loop system to the unknown optimum. The extension of the algorithm to the solution of constrained optimization problems is also provided. The technique simplifies the tuning of the designed gains by avoiding the limitations associated with choice of dither signal.

Chapter 6

ESC for Dynamical Systems

6.1 Introduction

The approach proposed in Chapter 5 is shown to avoid the need for averaging analysis to solve the unknown static optimization problem. This contribution minimizes the sensitivity of the algorithm to the choice of dither signal and improves the transient performance of the closed-loop system.

On the other hand, most optimization problems in industrial applications are subject to dynamic changes that are caused by uncertain nonlinear dynamics. Finding the optimal operating conditions for systems with uncertain dynamics and/or uncertain objective functions is a challenging area in real-time optimization problems. For these cases, a singular perturbation analysis of the persistently perturbed ESC loop has a significant role in establishing the stability analysis of conventional ESC problem. This technique yields a time-scale separation between fast transients of the system dynamics and the slow quasi steady-state condition. The time-scale separation of the closed-loop system may result in slow performance for the optimization of dynamical systems [59].

In the case where a model is available, one can use an adaptive extremum-seeking technique as proposed in [36] to stabilize a nonlinear system to the unknown optimum of a known but unmeasured cost function. In the case where a model is not available, multi-unit extremum-seeking control techniques (such as the one proposed in [77]) can be used to find the unknown optimum. This technique can solve uncertain steady-state optimization problems without the requirement for a time-scale separation. However, the need for a second nearly identical physical system seriously limits its applicability in practice. Recently, Lie bracket averaging techniques are considered to stabilize unknown dynamical systems using ESC [75]. This approach does not explicitly rely on the need for time-scale separation, but it requires a known control Lyapunov function of the unknown control system. Sliding-mode ESC is an alternative technique that can be used for optimization problem with unknown system dynamics. The stability of this ESC approach is guaranteed by introducing an auxiliary time-varying parameter and find its extremum point using sliding-modes [70]. This technique uses a similar idea of time-scale separation where it is assumed that the system dynamics are much faster than the dynamics of the auxiliary parameter. This approach provides a trade-off between convergence rate and precise performance of the optimizer. A numerical optimization algorithm was proposed in [91] to solve uncertain optimization problems in the absence of a time-scale separation. It is shown in [91] that the method is robust to input disturbances and uncertainties in the plant. Although this algorithm is highly robust, the application is limited due to the need for sporadic gradient measurements of the unknown objective function.

In this chapter, we provide an alternative extremum-seeking technique to compensate for time-scale separation in conventional ESC. Two different cases are considered in this chapter. First, it is assumed that the cost function and the system dynamics are unknown. In Section 6.2, we provide a nonlinear proportional-integral ESC (PI-ESC) design technique. The proportional action can be shown to provide instantaneous decrease of the cost toward the optimum. The integral action is shown to act like a standard ESC algorithm to identify the correct value of the optimal steady-state value of the input variable. The combined effect of the two controller modes provides fast transient performance of the closed-loop ESC that operates within the time-scale of the process dynamics.

In the second case, we assume the knowledge of the high frequency gain of the system dynamics. An inverse optimal control technique is used for the direct design of the control law. The proposed ESC results in an improvement of the transient performance and fast convergence to the optimal solution. This grey-box ESC technique, presented in Section 6.3, can be interpreted as a generalization of the ideas introduced in Section 5.3.

All the results of this chapter are based on the geometric concept of almost invariant manifolds introduced in Section 3.3 to estimate the unknown time-varying parameters. The proposed ESC schemes use the estimated parameters in a gradient descent algorithm to solve the unknown/uncertain optimization problems.

6.2 Black-Box ESC

6.2.1 Problem Description and Preliminaries

We consider a control-affine nonlinear system of the form:

$$\dot{x} = f(x) + G(x)u \tag{6.1}$$

$$y = h(x) \tag{6.2}$$

where $x \in \mathbb{R}^n$ is the vector of state variables, and u is the vector of input variables taking values in $\mathcal{U} \subset \mathbb{R}^m$. $y \in \mathbb{R}$ is an unknown but measurable cost function to be minimized. The vector-valued function $f(x)$, the matrix-valued function $G(x)$ and the scalar function $h(x)$ are assumed to be C^∞.

The objective is to design a controller u such that the system converges to the unknown equilibrium x^* and u^* that minimizes y. The derivative of the cost function $y = h(x)$ is given by

$$\dot{y} = \frac{\partial h(x)}{\partial x} f(x) + \frac{\partial h(x)}{\partial x} G(x)u = L_f h + L_G h u \tag{6.3}$$

where $L_f h$ and $L_G h$ are the Lie derivatives of $h(x)$ with respect to $f(x)$ and $G(x)$, respectively. Consider a control input

$$u_1 = -k_g (L_G h)^T + \hat{u} \tag{6.4}$$

where $k_g > 0$ and $\hat{u}$ is a constant vector in $\mathcal{U}$. From the control law (6.4) and the nonlinear dynamics (6.1), we have

$$\dot{x} = f(x) - k_g G(x)(L_G h)^T + G(x)\hat{u} \tag{6.5}$$

The equilibrium map is the n dimensional vector-valued function $x = \pi(\hat{u})$ which is such that $f(\pi(\hat{u})) - k_g G(\pi(\hat{u}))(L_G h(\pi(\hat{u})))^T + G(\pi(\hat{u}))\hat{u} = 0$. We assume that the controller (6.4) with a proper choice of the gain k_g satisfies the stability condition of the closed-loop system as stated in the following assumption.

Assumption 6.1. *Define the deviation variable $\tilde{x} = x - \pi(\hat{u})$. There exists a gain $k_g > k^*$ and a positive definite function $W(\tilde{x})$ such that*

$$\beta_1\|x - \pi(\hat{u})\|^2 \leq W(\tilde{x}) \leq \beta_2\|x - \pi(\hat{u})\|^2$$

$$\begin{aligned}\frac{\partial W}{\partial \tilde{x}}(f + G\hat{u}) - k_g \frac{\partial W}{\partial \tilde{x}} G G^T \frac{\partial h}{\partial x}^T \\ \leq -\beta_3\|x - \pi(\hat{u})\|^2 - (k_g - k^*)\frac{\partial W}{\partial \tilde{x}} G G^T \frac{\partial W}{\partial \tilde{x}}^T \end{aligned} \tag{6.6}$$

$$\left\|\frac{\partial W}{\partial \tilde{x}}\right\| \leq \beta_4\|x - \pi(\hat{u})\|, \quad \left\|\frac{\partial h}{\partial x}\right\| \leq \beta_5\|x - \pi(\hat{u})\|$$

$\forall \tilde{x} \in \mathcal{D}(\hat{u})$ and $\forall \hat{u} \in \mathcal{U}$, where $\mathcal{D}(\hat{u})$ represents a neighbourhood of the equilibrium map $x = \pi(\hat{u})$. β_i, $i = 1, \cdots, 5$ are positive constants and k^* is a non-negative constant.

Remark 6.1. *Assumption 6.1 describes a class of nonlinear systems with possibly unstable open-loop dynamics that can be stabilized by the state-feedback (6.4) with a suitable choice of k_g. The static controller (6.4) guarantees convergence of the closed-loop system (6.5) to the steady-state manifold $x = \pi(\hat{u})$.*

Remark 6.2. *One direct consequence of Assumption 6.1 is that the angle between the gradients ∇h and ∇W is $|\angle(\nabla h, \nabla W)| < \frac{\pi}{2}$. In other words, the vector-valued function $\frac{\partial W}{\partial \tilde{x}}$ is such that*

$$\beta_6 \leq \frac{\frac{\partial W}{\partial \tilde{x}}\left(\frac{\partial h}{\partial x}\right)^T}{\left\|\frac{\partial W}{\partial \tilde{x}}\right\|\left\|\frac{\partial h}{\partial x}\right\|} \leq \beta_7 \tag{6.7}$$

with positive constants β_6, $\beta_7 \in (0, 1]$.

The steady-state cost function is given by $y = h(\pi(\hat{u})) = \ell(\hat{u})$. Thus, at steady-state, the problem is reduced to finding the minimizer u^* of the $y = \ell(\hat{u})$. An additional assumption is required concerning the steady-state cost function $y = \ell(\hat{u})$.

Assumption 6.2. *The steady-state cost function is such that*

$$\frac{\partial \ell(u^*)}{\partial \hat{u}} = 0, \ \frac{\partial \ell(\hat{u})}{\partial \hat{u}}(\hat{u} - u^*) \geq \alpha_1 \left\| \hat{u} - u^* \right\|^2, \quad \forall \hat{u} \in \mathcal{U}$$

where α_1 is a strictly positive constant.

Assumption 6.3. *The matrix-valued function $G(x)$ is full rank $\forall x \in \mathcal{D}(\hat{u})$, $\forall \hat{u} \in \mathcal{U}$.*

Assumption 6.2 is a common assumptions in standard ESC problem [78] that provide local stabilization and local feasible solution of the cost function at the steady-state. Finally, Assumption 6.3 states that the cost function is relative order one in a neighbourhood of the unknown optimum. By the relative order assumption it follows that $L_G h \neq 0$ in a neighbourhood of the unknown optimum $x^* = \pi(u^*)$.

If one has access to the Lie derivatives, then by following the approach in [63], the adaptive state-feedback controller

$$u = -k_g (L_G h)^T + \hat{u} \tag{6.8}$$

solves the optimization problem, where $\hat{u}$ is a steady-state bias term with the dynamics

$$\dot{\hat{u}} = -\frac{1}{\tau_I}(L_G h)^T \tag{6.9}$$

where $k_g > 0$ and $\tau_I > 0$ are optimization gains.

Lemma 6.1. *Consider the nonlinear system (6.1) and (6.2) subject to the Assumptions 6.1 to 6.3. Then there exists a τ_I^* such that for all $\tau_I > \tau_I^*$ the nonlinear system in closed-loop with the ESC controller (6.8) and (6.9) converges to the equilibrium $x^* = \pi(u^*)$ that minimizes the cost function $h(x)$.*

Proof. It is shown in the following that the cost will decrease until a value of u is reached such that the gradient of the cost be zero at the steady-state. Let $\tilde{x} = x - \pi(\hat{u})$ and $\mathcal{D}(\hat{u}) = \{\tilde{x} \in \mathbb{R}^n \mid \|\tilde{x}\| \leq r\}$ for $r > 0$, a positive constant. As a result of Assumptions 6.1, there exist a Lyapunov function $W(\tilde{x})$ such that

$$\dot{W} \leq -\beta_3 \|x - \pi(\hat{u})\|^2 - k_g^* \left\| \frac{\partial W}{\partial \tilde{x}} G \right\|^2 - \frac{\partial W}{\partial \tilde{x}} \frac{\partial \pi}{\partial \hat{u}} \dot{\hat{u}} \tag{6.10}$$

where $k_g^* = k_g - k^*$. We define a new deviation variable $\tilde{u} = u^* - \hat{u}$, and an extended Lyapunov function candidate

$$V = W + \frac{1}{2} \tilde{u}^T \tilde{u}. \tag{6.11}$$

By considering (6.9) and (6.10), differentiation of V with respect to time, yields

$$\dot{V} \leq -\beta_3 \|x - \pi(\hat{u})\|^2 - k_g^* \left\| \frac{\partial W}{\partial \tilde{x}} G \right\|^2 + \frac{1}{\tau_I} \frac{\partial W}{\partial \tilde{x}} \frac{\partial \pi}{\partial \hat{u}} (L_G h)^T + \frac{1}{\tau_I} (L_G h) \tilde{u}.$$

Therefore, it follows that

$$\begin{aligned} \dot{V} \leq & -\beta_3 \|x - \pi(\hat{u})\|^2 - k_g^* \left\| \frac{\partial W}{\partial \tilde{x}} G \right\|^2 + \frac{1}{\tau_I} \frac{\partial W}{\partial \tilde{x}} \frac{\partial \pi}{\partial \hat{u}} \left(L_G h - \frac{\partial h(\pi(\hat{u}))}{\partial x} \frac{\partial \pi}{\partial \hat{u}} \right)^T \\ & + \frac{1}{\tau_I} \frac{\partial W}{\partial \tilde{x}} \frac{\partial \pi}{\partial \hat{u}} \frac{\partial \pi}{\partial \hat{u}}^T \frac{\partial h}{\partial x}^T + \frac{1}{\tau_I} \left(L_G h - \frac{\partial h(\pi(\hat{u}))}{\partial x} \frac{\partial \pi}{\partial \hat{u}} \right) \tilde{u} + \frac{1}{\tau_I} \frac{\partial h(\pi(\hat{u}))}{\partial x} \frac{\partial \pi}{\partial \hat{u}} \tilde{u}. \end{aligned} \tag{6.12}$$

The quasi steady-state dynamics of the system (6.5) describe the dynamics of the system along the equilibrium manifold $x = \pi(\hat{u})$. To model the corresponding quasi steady-state output response, we consider the modified output

$$\begin{aligned} Y = H(x) &= h(x) + \frac{\partial h}{\partial x} (f(x) - k_g G(x)(L_G h)^T + G(x)\hat{u}) \\ &= h(x) + \frac{\partial h}{\partial x} (F(x) + G(x)\hat{u}). \end{aligned} \tag{6.13}$$

Clearly $H(\pi(\hat{u})) = h(\pi(\hat{u}))$. Let us define the new time-scale $d\tau = \epsilon dt$ where $\epsilon > 0$ is a small positive number. The dynamics of Y for the new time-scale is given as

$$\begin{aligned}\frac{dY}{d\tau} =&\epsilon\frac{\partial h}{\partial x}(F(x)+G(x)\hat{u})+\epsilon\frac{\partial h}{\partial x}\left(\frac{\partial F}{\partial x}+\frac{\partial G}{\partial x}\hat{u}\right)(F(x)+G(x)\hat{u})\\&+\epsilon(F(x)+G(x)\hat{u})^T\frac{\partial^2 h}{\partial x\partial x^T}(F(x)+G(x)\hat{u})+\frac{\partial h}{\partial x}G(x)\frac{d\hat{u}}{d\tau}.\end{aligned}\tag{6.14}$$

The quasi steady-state dynamics are the dynamics obtained in the limit as $\epsilon\to 0$. Thus taking $\epsilon = 0$, one obtains

$$\frac{dy}{d\tau}=\frac{\partial h(\pi(\hat{u}))}{\partial x}G(\pi(\hat{u}))\frac{d\hat{u}}{d\tau}.\tag{6.15}$$

Since at the steady-state $y = h(\pi(\hat{u})) = \ell(\hat{u})$, it follows that

$$\frac{\partial h(\pi(\hat{u}))}{\partial x}G(\pi(\hat{u}))=\frac{\partial h(\pi(\hat{u}))}{\partial x}\frac{\partial\pi(\hat{u})}{\partial\hat{u}}=\frac{\partial\ell(\hat{u})}{\partial\hat{u}}\tag{6.16}$$

which is the gradient of the steady-state cost. By substituting (6.16) in (6.12), we have

$$\begin{aligned}\dot{V}\leq&-\beta_3\|x-\pi(\hat{u})\|^2-k_g^*\left\|\frac{\partial W}{\partial\tilde{x}}G\right\|^2+\frac{1}{\tau_I}\frac{\partial W}{\partial\tilde{x}}\frac{\partial\pi}{\partial\hat{u}}\left(L_Gh(x)-L_Gh(\pi(\hat{u}))\right)^T\\&+\frac{1}{\tau_I}(L_Gh(x)-L_Gh(\pi(\hat{u})))\tilde{u}+\frac{1}{\tau_I}\frac{\partial W}{\partial\tilde{x}}\frac{\partial\pi}{\partial\hat{u}}\frac{\partial\pi}{\partial\hat{u}}^T\frac{\partial h}{\partial x}^T+\frac{1}{\tau_I}\frac{\partial\ell(\hat{u})}{\partial\hat{u}}\tilde{u}.\end{aligned}\tag{6.17}$$

By Assumption 6.2, the last term of (6.17) can be upper bounded to yield

$$\begin{aligned}\dot{V}\leq&-\beta_3\|\tilde{x}\|^2-k_g^*\left\|\frac{\partial W}{\partial\tilde{x}}G\right\|^2-\frac{\alpha_1}{\tau_I}\|\tilde{u}\|^2+\frac{1}{\tau_I}\frac{\partial W}{\partial\tilde{x}}\frac{\partial\pi}{\partial\hat{u}}\left(L_Gh(x)-L_Gh(\pi(\hat{u}))\right)^T\\&+\frac{1}{\tau_I}(L_Gh(x)-L_Gh(\pi(\hat{u})))\tilde{u}+\frac{1}{\tau_I}\frac{\partial W}{\partial\tilde{x}}\frac{\partial\pi}{\partial\hat{u}}\frac{\partial\pi}{\partial\hat{u}}^T\frac{\partial h}{\partial x}^T.\end{aligned}\tag{6.18}$$

Given that $h(x)$ and $G(x)$ are smooth, it follows that there exists a Lipschitz constant L_h such that

$$\begin{aligned}\dot{V}\leq&-\beta_3\|\tilde{x}\|^2-k_g^*\left\|\frac{\partial W}{\partial\tilde{x}}G\right\|^2-\frac{\alpha_1}{\tau_I}\|\tilde{u}\|^2+\frac{1}{\tau_I}L_h\|x-\pi(\hat{u})\|\|\tilde{u}\|\\&+\frac{1}{\tau_I}L_h\left\|\frac{\partial W}{\partial\tilde{x}}\right\|\left\|\frac{\partial\pi}{\partial\hat{u}}\right\|\|x-\pi(\hat{u})\|+\frac{1}{\tau_I}\left\|\frac{\partial W}{\partial\tilde{x}}\right\|\left\|\frac{\partial\pi}{\partial\hat{u}}\right\|^2\left\|\frac{\partial h}{\partial x}\right\|.\end{aligned}$$

By Assumption 6.1, and the boundedness of $\left\|\frac{\partial\pi(\hat{u})}{\partial\hat{u}}\right\|$, $\forall\hat{u}\in\mathcal{U}$, we can write the following inequality

$$\begin{aligned}\dot{V} \leq &- \beta_3\|\tilde{x}\|^2 - k_g^* \left\| \frac{\partial W}{\partial \tilde{x}} G \right\|^2 - \frac{\alpha_1}{\tau_I}\|\tilde{u}\|^2 + \frac{1}{\tau_I} L_h \|x - \pi(\hat{u})\| \|\tilde{u}\| \\ &+ \frac{1}{\tau_I} L_h L_\pi \beta_4 \|x - \pi(\hat{u})\|^2 + \frac{1}{\tau_I} L_\pi^2 \beta_4 \beta_5 \|x - \pi(\hat{u})\|^2. \end{aligned} \tag{6.19}$$

The inequality (6.19) can be written in matrix form as

$$\dot{V} \leq -k_g^* \left\| \frac{\partial W}{\partial \tilde{x}} G \right\|^2 - [\|\tilde{x}\|,\ \|\tilde{u}\|] \begin{bmatrix} \beta_3 - \frac{L_\pi \beta_4 (L_h + L_\pi \beta_5)}{\tau_I} & -\frac{L_h}{2\tau_I} \\ -\frac{L_h}{2\tau_I} & \frac{\alpha_1}{\tau_I} \end{bmatrix} \begin{bmatrix} \|\tilde{x}\| \\ \|\tilde{u}\| \end{bmatrix}. \tag{6.20}$$

The minimum eigenvalue of the matrix

$$\Lambda_1 = \begin{bmatrix} \beta_3 - \frac{L_\pi \beta_4 (L_h + L_\pi \beta_5)}{\tau_I} & -\frac{L_h}{2\tau_I} \\ -\frac{L_h}{2\tau_I} & \frac{\alpha_1}{\tau_I} \end{bmatrix}$$

is positive if

$$\tau_I > \frac{1}{\beta_3} \left(\frac{L_h^2}{4\alpha_1} + L_\pi \beta_4 (L_h + L_\pi \beta_5) \right) = \tau_I^*. \tag{6.21}$$

Thus, based on (6.20) and (6.21), the system exponentially converges to the origin which occurs at the input u^* with corresponding state variable x^*. □

Remark 6.3. *The proposed controller is similar to a conventional PI controller. The proportional component $k_g(L_G h)^T$ is such that the system is forced to an equilibrium where $L_G h$ is close to zero. The integral action given by $\hat{u}$ is used to identify the unknown optimum steady-state input u^*.*

Since the dynamics of the system are unknown, one must consider an adaptive control approach to implement the PI-ESC control law (6.8) and (6.9). Defining $\theta_0 = L_f h$ and $\theta_1 = (L_G h)^T$, the dynamics (6.3) can be re-written as

$$\dot{y} = \theta_0 + \theta_1^T u = \theta^T \zeta \tag{6.22}$$

where $\zeta = [1, u^T]^T$ and $\theta = [\theta_0, \theta_1^T]^T$. The design of the extremum-seeking scheme is based on the unknown dynamics (6.22). The first step consists in the estimation of the time-varying parameters $\theta(t)$. In the second step, we define a suitable adaptive controller that accomplishes the extremum-seeking task [63].

6.2.2 Adaptive PI-ESC

In this section, an extremum-seeking control approach is proposed to minimize the measured cost y in the absence of specific knowledge about the dynamics and the cost function. The approach utilizes the time-varying parameter estimation proposed in Chapter 3 to implement the ESC in a proportional-integral form.

Let $\hat{y}$ represent the estimator model for (6.22), and the filter output ϕ is described along with the regressor vector ζ. To do so, we consider the following dynamics

$$\begin{aligned}\dot{\hat{y}} &= -k^2(\hat{y} - y)\\ \dot{\phi} &= -k^2(\phi - \zeta).\end{aligned} \tag{6.23}$$

The auxiliary variables p and q, along with adaptive estimator Σ are described as

$$\begin{aligned}\dot{p} &= -kp - \phi\phi^T\hat{\theta}(t)\\ \dot{q} &= -kq + \phi(k^2(\hat{y} - y))\\ \dot{\Sigma}(t) &= k[bI - \phi\phi^T(\Sigma(t))],\ b > 1\end{aligned} \tag{6.24}$$

where $\hat{\theta}$ is a parameter estimation with adaptive update law

$$\dot{\hat{\theta}}(t) = \mathrm{Proj}(k^2\Sigma(t)[\dot{\delta} + k\delta], \hat{\theta}),\ \delta = p - q. \tag{6.25}$$

The value of estimation gain k is a positive constant to be assigned.

The input space $\mathcal{U}$ is defined as $\mathcal{U} = \{u \mid \|u\| \leq r_u\}$ where r_u is a positive constant that identifies the upper limit on the size of the norm for the control input u. The extremum-seeking controller is given by

$$\begin{aligned} u &= -k_g\hat{\theta}_1 + \hat{u} + d(t) \\ \dot{\hat{u}} &= -\frac{1}{\tau_I}\hat{\theta}_1, \end{aligned} \tag{6.26}$$

where $d(t)$ is a bounded dither signal with $\|d(t)\| \leq D$. The main task of dither signal is to improve the PE condition of estimation scheme. The Lipschitz projection algorithm of (6.25) and the PE assumption are defined similar to (5.8) and the Assumption 5.2 of the Section 5.3.

Theorem 6.1. *Let Assumptions 6.1-6.3 and Assumption 5.2 be fulfilled. Consider the parameter estimation algorithm (6.25) with a gain k, and the extremum-seeking controller (6.26) with a proportional gain k_g and integral constant gain τ_I. Then there exists a τ_I^* such that for all $\tau_I > \tau_I^*$, the system converges locally exponentially to an $[\mathcal{O}(k_g/k), \mathcal{O}(D/k_g)]$ neighbourhood of the minimizer $x^* = \pi(u^*)$ of the measured cost function y.*

Proof. The parameter update law (6.25) can be expressed implicitly in the form (5.11), with the parameter estimation error $\tilde{\theta}(t) = \theta(t) - \hat{\theta}(t)$. The extension of the Lyapunov function (6.11) is defined as

$$\mathcal{V} = \frac{1}{2}\tilde{\theta}^T\tilde{\theta} + V. \tag{6.27}$$

Taking the derivative of $\mathcal{V}$, yields

$$\begin{aligned} \dot{\mathcal{V}} \leq &- k^2\tilde{\theta}^T(\Sigma(t)\phi\phi^T)\tilde{\theta} + \tilde{\theta}^T(\Sigma(t)\phi\phi^T)\dot{\theta} + \tilde{\theta}^T\dot{\theta} + \frac{\partial W}{\partial \tilde{x}}(f + G\hat{u}) \\ &- k_g\frac{\partial W}{\partial \tilde{x}}G\hat{\theta}_1 + \frac{1}{\tau_I}\frac{\partial W}{\partial \tilde{x}}\frac{\partial \pi}{\partial \hat{u}}\hat{\theta}_1 + \frac{1}{\tau_I}\tilde{u}^T\hat{\theta}_1 + \frac{\partial W}{\partial \tilde{x}}Gd(t). \end{aligned}$$

Noting that $\hat{\theta}_1 = \theta_1 - \tilde{\theta}_1$, and follow the approaches (6.11) to (6.19), we have

$$\begin{aligned}\dot{\mathcal{V}} \leq & -k^2\tilde{\theta}^T(\Sigma(t)\phi\phi^T)\tilde{\theta} + \tilde{\theta}^T(\Sigma(t)\phi\phi^T)\dot{\theta} + \tilde{\theta}^T\dot{\theta} - \beta_3\|\tilde{x}\|^2 - k_g^*\left\|\frac{\partial W}{\partial \tilde{x}}G\right\|^2 - \frac{\alpha_1}{\tau_I}\|\tilde{u}\|^2 \\ & + \frac{1}{\tau_I}L_h\|\tilde{x}\|\|\tilde{u}\| + \frac{1}{\tau_I}L_\pi\beta_4(L_h + L_\pi\beta_5)\|\tilde{x}\|^2 + k_g\left\|\frac{\partial W}{\partial \tilde{x}}G\right\|\left\|\tilde{\theta}_1\right\| \\ & + \frac{1}{\tau_I}L_\pi\beta_4\left\|\tilde{\theta}_1\right\|\|\tilde{x}\| + \frac{1}{\tau_I}\left\|\tilde{\theta}_1\right\|\|\tilde{u}\| + \left\|\frac{\partial W}{\partial \tilde{x}}G\right\|\|d(t)\|. \end{aligned} \tag{6.28}$$

Following Section 5.3 and Assumption 5.2, there exist positive constants γ_1, γ_2 and γ_3 such that $\|\phi\| \leq \gamma_1$, and $\gamma_2 \leq \|\Sigma\phi\phi^T\| \leq \gamma_3$. Hence,

$$\begin{aligned}\dot{\mathcal{V}} \leq & -\frac{\gamma_2 k^2}{2}\tilde{\theta}^T\tilde{\theta} + \frac{\gamma_3}{2k^2}\dot{\theta}^T\dot{\theta} + \tilde{\theta}^T\dot{\theta} - \beta_3\|\tilde{x}\|^2 - k_g^*\left\|\frac{\partial W}{\partial \tilde{x}}G\right\|^2 - \frac{\alpha_1}{\tau_I}\|\tilde{u}\|^2 \\ & + \frac{1}{\tau_I}L_h\|\tilde{x}\|\|\tilde{u}\| + \frac{1}{\tau_I}L_\pi\beta_4(L_h + L_\pi\beta_5)\|\tilde{x}\|^2 + k_g\left\|\frac{\partial W}{\partial \tilde{x}}G\right\|\left\|\tilde{\theta}_1\right\| \\ & + \frac{1}{\tau_I}L_\pi\beta_4\left\|\tilde{\theta}_1\right\|\|\tilde{x}\| + \frac{1}{\tau_I}\left\|\tilde{\theta}_1\right\|\|\tilde{u}\| + \left\|\frac{\partial W}{\partial \tilde{x}}G\right\|\|d(t)\|. \end{aligned}$$

Since $\theta = [\theta_0, \theta_1^T]^T$, applying Young's inequality and collecting the similar terms lead to the following inequality

$$\begin{aligned}\dot{\mathcal{V}} \leq & -\frac{k}{2}(\gamma_2 k - 1)\|\tilde{\theta}_0\|^2 - \frac{k}{2}\left(\gamma_2 k - k_g^2 - \frac{L_\pi\beta_4 + 1}{\tau_I} - 1\right)\|\tilde{\theta}_1\|^2 \\ & - [\|\tilde{x}\|,\ \|\tilde{u}\|]\begin{bmatrix}\beta_3 - \frac{L_\pi\beta_4(L_h + L_\pi\beta_5 + 0.5k^{-1})}{\tau_I} & -\frac{L_h}{2\tau_I} \\ -\frac{L_h}{2\tau_I} & \frac{1}{\tau_I}(\alpha_1 - \frac{1}{2k})\end{bmatrix}\begin{bmatrix}\|\tilde{x}\| \\ \|\tilde{u}\|\end{bmatrix} \\ & - \left(\frac{k_g^*}{2} - \frac{1}{2k}\right)\left\|\frac{\partial W}{\partial \tilde{x}}G\right\|^2 + \frac{\gamma_3 + k}{2k^2}\|\dot{\theta}\|^2 + \frac{1}{2k_g^*}D^2. \end{aligned}$$

If a fixed value of the estimation gain is chosen as $k > \max\{\frac{1+L_\pi\beta_4+\tau_I^*}{\gamma_2\tau_I^*}, \frac{1}{2\alpha_1}\}$, then the controller gains should be chosen such that

$$\frac{1}{k} + k^* < k_g < (\gamma_2 k - \frac{1 + L_\pi\beta_4}{\tau_I} - 1)^{\frac{1}{2}},$$

$$\tau_I > \frac{1}{\beta_3}\left(\frac{L_h^2}{4\alpha_1'} + L_\pi\beta_4(L_h + L_\pi\beta_5 + 0.5k^{-1})\right) = \tau_I^*.$$

If $\tau_I > \tau_I^*$, then the matrix

$$\Lambda_2 = \begin{bmatrix} \beta_3 - \frac{L_\pi \beta_4 (L_h + L_\pi \beta_5 + 0.5k^{-1})}{\tau_I} & -\frac{L_h}{2\tau_I} \\ -\frac{L_h}{2\tau_I} & \frac{1}{\tau_I}(\alpha_1 - \frac{1}{2k}) \end{bmatrix}$$

is positive definite. Thus, for the given gains, there exist positive constants k_a, k_b and λ_2 such that

$$\dot{\mathcal{V}} \leq -k_a \|\tilde{\theta}\|^2 - k_b \left\| \frac{\partial W}{\partial \tilde{x}} G \right\|^2 - \lambda_2 \|\tilde{x}\|^2 - \lambda_2 \|\tilde{u}\|^2 + \frac{\gamma_3 + k}{2k^2} \|\dot{\theta}\|^2 + \frac{1}{2k_g^*} D^2, \qquad (6.29)$$

where $\lambda_2 = \lambda_{\min}(\Lambda_2)$. Therefore, $\tilde{\theta}$, $\tilde{u}$ and $\frac{\partial W}{\partial \tilde{x}} G$ converge to a neighbourhood of the origin. As $\hat{u}$ approaches a neighbourhood of u^*, the state x enters a neighbourhood of the steady-state optimum $x^* = \pi(u^*)$. This is achieved by using an estimation gain k that is larger than the proportional gain k_g to ensure that all constants multiplying the corresponding norms are negative. Based on the dynamics (6.5), there exists constant $\mathcal{X}_0$ and $\mathcal{X}_1$ such that $\|\dot{x}\| \leq \mathcal{X}_0 + k_g \mathcal{X}_1$, $\forall x \in \mathcal{D}(\hat{u})$ and $\forall \hat{u} \in \mathcal{U}$. Since the magnitude of $\dot{\theta}$ is proportional to the magnitude of the velocity vector $\dot{x}$, we can bound the size of $\dot{\theta}$ by

$$\|\dot{\theta}\| \leq \mathcal{X}_2(\mathcal{X}_0 + k_g \mathcal{X}_1)$$

where $\mathcal{X}_2$ is a positive constant. The term $\frac{\gamma_3 + k}{2k^2}$ is such that it can be minimized by increasing k. As a result, the ESC system approaches an $\mathcal{O}(\|\dot{\theta}\|/k)$ neighbourhood of the origin that can be written as $\mathcal{O}(k_g/k)$. The contribution of the dither signal $d(t)$ is $\mathcal{O}(D/k_g)$ from the origin. This completes the proof. □

Remark 6.4. *The main design limitation is the choice of k_g that is required to be large enough to stabilize the unstable and unknown nonlinear systems. By assumption, we must have $k_g^* < k_g \ll k$. Therefore, based on the Assumption 6.1, we must choose k_g such that $k^* \ll k_g$.*

6.3 Grey-Box ESC

6.3.1 Problem Description and Preliminaries

In this section, we consider a class of uncertain nonlinear control-affine systems of the form:

$$\dot{x}_q = f_q(x) \tag{6.30}$$

$$\dot{x}_p = f_p(x) + G_p(x)u \tag{6.31}$$

$$y = h(x_p) \tag{6.32}$$

where $x = [x_q^T, x_p^T]^T \in \mathbb{R}^n$ are the measured state variables, and u are the control inputs taking values in $\mathcal{U} \subset \mathbb{R}^m$. $y \in \mathbb{R}$ is the unknown but measurable cost function to be minimized. The vector-valued function $f(x) = [f_q^T, f_p^T]^T$ and the scalar function $h(x_p)$ are assumed to be unknown and C^∞ in their argument. The matrix-valued function $G_p(x)$ is known and bounded for bounded x.

The objective is to design a state-feedback controller such that the system converges to the unknown equilibrium x_p^* that minimizes y. The following assumptions are made about the system.

Assumption 6.4. *The equilibrium cost is such that*

$$\frac{\partial h(x_p^*)}{\partial x_p} = 0, \ \frac{\partial h(x_p)}{\partial x_p}(x_p - x_p^*) \geq \alpha_1 \left\|x_p - x_p^*\right\|^2$$

$\forall x_p \in \mathbb{R}^m$*, where* α_1 *is a strictly positive constant.*

Assumption 6.5. *The control gain matrix* $G_p(x)$ *is invertible* $\forall x \in \mathbb{R}^n$*.*

Assumption 6.6. *The state* $x_q \in \mathbb{R}^{n-m}$ *has dynamics (6.30) that are input-to-state stable when* x_p *is considered as the input [36].*

The feasibility of the uncertain optimization problem is guaranteed by Assumption 6.4. The purpose of Assumption 6.5 is to allow for a direct design of the gradient-based controller. This is not a restrictive assumption since many mechanical and electrical systems satisfy this property of the control gain matrix [14]. The Assumption 6.6 states that the x_q dynamics are stable for $x_p = x_p^*$.

The time derivative of the cost function $y = h(x_p)$ is

$$\dot{y} = \frac{\partial h(x_p)}{\partial x_p}\dot{x}_p = \frac{\partial h(x_p)}{\partial x_p}\left(f_p(x) + G_p(x)u\right). \tag{6.33}$$

If one has access to the cost function $h(x_p)$ and the drift term $f_p(x)$, then it follows that the controller

$$u = G_p^{-1}(x)\left(-f_p(x) - k_g\left(\frac{\partial h(x_p)}{\partial x_p}\right)^T\right) \tag{6.34}$$

with $k_g > 0$, solves the real-time optimization problem. Let $\tilde{x}_p = x_p - x_p^*$ and consider the Lyapunov function candidate

$$\mathcal{W} = \frac{1}{2}\tilde{x}_p^T\tilde{x}_p.$$

By Assumption 6.4 and the control input (6.34), differentiation with respect to t, yields

$$\dot{\mathcal{W}} = -k_g\frac{\partial h(x_p)}{\partial x_p}\tilde{x}_p \le -k_g\alpha_1\|\tilde{x}_p\|^2.$$

Hence, the system converges globally exponentially to the unknown minimizer x_p^*.

Since the dynamics of the system are uncertain, one must consider an adaptive control approach to implement the control law (6.34). Defining $\theta_1(t) = f_p(x)$ and $\theta_2(t) = \left(\frac{\partial h(x_p)}{\partial x_p}\right)^T$, the dynamics (6.31) and (6.33) can be re-written as

$$\dot{x}_p = \theta_1(t) + G_p(x)u \tag{6.35}$$

$$\dot{y} = \theta_2^T(t)\dot{x}_p. \tag{6.36}$$

The design of the extremum-seeking scheme is based on the uncertain dynamics (6.35) and (6.36). First, we provide an adaptive update law for estimation of the time-varying parameters $\theta_1(t)$ and $\theta_2(t)$. Then, a direct gradient-based controller is designed to find the optimum points of the state variables [67].

6.3.2 Direct Design of Adaptive ESC

To estimate the time-varying parameters, the following steps are required. First, the geometric-based approach of Section 3.3 is utilized to estimate the nonlinear drift term $\theta_1(t)$. Then, an update law is provided for the velocity state vector $\dot{x}_p$, based on the drift term estimation. Note that throughout this section, it is assumed that the state of the system $x(\cdot)$ is available for measurement but the measurement of the velocity state vector $\dot{x}(\cdot)$ is not required. Finally, from the estimated velocity state vector, we develop an adaptive update law for the time-varying gradient $\theta_2(t)$. The above steps can be summarized as follows.

The estimator model for (6.35) is defined as

$$\dot{\hat{x}}_p = -k^2(\hat{x}_p - x_p), \tag{6.37}$$

where the value of the tuning gain $k > 0$ is assigned in the design. The filters are described along with the structure of the system by

$$\begin{aligned}\dot{\psi} &= -k^2(\psi - G_p(x)u)\\ \dot{\Phi} &= -k^2(\Phi - I).\end{aligned} \tag{6.38}$$

Consider the estimator (6.37) and filter (6.38), an almost invariant manifold can be defined for a sufficiently large value of gain k, as

$$k^2(\hat{x}_p - x_p) + \psi = -\Phi\theta_1(t) + \frac{\Phi\dot{\theta}_1(t)}{k^2}. \tag{6.39}$$

The mapping (6.39) is similar to the manifold (3.16). From the auxiliary variables p_1 and q_1 with the dynamics

$$\begin{aligned}\dot{p}_1 &= -kp_1 - \Phi^T\Phi\hat{\theta}_1 \\ \dot{q}_1 &= -kq_1 + \Phi^T(k^2(\hat{x}_p - x_p) + \psi),\end{aligned} \tag{6.40}$$

the parameter update law is achieved as

$$\dot{\hat{\theta}}_1(t) = \text{Proj}(k^2(\Phi^T\Phi)^{-1}[\dot{\delta}_1 + k\delta_1], \hat{\theta}_1)\ \delta_1 = p_1 - q_1. \tag{6.41}$$

In order to estimate $\theta_2(t)$, we define a velocity state predictor as

$$\dot{\hat{x}}_p = \hat{\theta}_1 + G_p(x)u. \tag{6.42}$$

Consider the dynamics (6.36) and (6.42), an output estimator and a filter can be generated as

$$\begin{aligned}\dot{\hat{y}} &= -k(\hat{y} - y) \\ \dot{\phi} &= -k(\phi - \dot{\hat{x}}_p).\end{aligned} \tag{6.43}$$

Following the similar procedure of Section 3.3, a modified almost invariant manifold

$$k(\hat{y} - y) = -\phi^T\theta_2(t) + \frac{\phi^T\dot{\theta}_2(t)}{k} - (\theta_1(t) - \hat{\theta}_1(t))^T\theta_2(t) \tag{6.44}$$

is obtained that provides an implicit relationship from known variables to the unknown variables. The auxiliary variables and estimator are expressed as

$$\begin{aligned}\dot{p}_2 &= -kp_2 - \phi\phi^T\hat{\theta}_2 \\ \dot{q}_2 &= -kq_2 + \phi(k(\hat{y} - y)) \\ \dot{\Sigma}(t) &= k[bI - \phi\phi^T(\Sigma(t))], \quad \Sigma(t_0) = bI\end{aligned} \tag{6.45}$$

where $b > 1$ is a constant. Therefore, the gradient update law is generated by

$$\dot{\hat{\theta}}_2(t) = \text{Proj}(k\Sigma(t)[\dot{\delta}_2 + k\delta_2], \hat{\theta}_2),\ \ \delta_2 = p_2 - q_2. \tag{6.46}$$

The Lipschitz projection operator in (6.41) and (6.46) are defined similar to the one presented in (5.8). The convex sets over the unknown parameters can be assigned, with compact balls centered at the origin as $\Theta_i = B(0, r_{\theta_i}),\ i = 1, 2$. The uncertainty set radius r_{θ_i} can be determined from available information and limitations of the physical system.

Moreover, we consider a similar assumption of (5.9) for PE condition. Hence, a convergence of $\hat{\theta}_2$ to the true time-varying gradient is ensured.

The input space $\mathcal{U}$ is defined as in Section 6.2. The proposed direct extremum-seeking control is given by

$$u = G_p^{-1}(x_p)\left(-\hat{\theta}_1(t) - k_g\hat{\theta}_2(t) + d(t)\right) \tag{6.47}$$

where $d(t)$ is a dither signal with $\|d(t)\| \leq D$.

Theorem 6.2. *Let Assumptions 6.4-6.6 and 5.2 hold. Consider the extremum-seeking controller (6.47) and the parameter update laws (6.41) and (6.46). Then there exist positive gains k and k_g such that the closed-loop extremum-seeking control system converges to an $[\mathcal{O}(k_g/k), \mathcal{O}(D/k_g)]$ neighborhood of the minimizer x_p^* of the measured cost function y.*

Proof. Defining $\theta(t) = [\theta_1^T(t), \theta_2^T(t)]^T$, the parameter estimation error can be constructed as $\tilde{\theta}(t) = \theta(t) - \hat{\theta}(t)$. Let $\tilde{x}_p = x_p - x_p^*$, the quadratic Lyapunov function is defined as

$$V = \frac{1}{2}\tilde{\theta}^T(t)\tilde{\theta}(t) + \frac{1}{2}\tilde{x}_p^T\tilde{x}_p. \tag{6.48}$$

Taking the derivative of V yields

$$\dot{V} = \tilde{\theta}^T(t)(\dot{\theta}(t) - \dot{\hat{\theta}}(t)) + \tilde{x}_p^T\dot{x}_p.$$

The parameter update law for $\theta(t)$ can be expressed implicitly in the form

$$\dot{\hat{\theta}}_1(t) = \mathrm{Proj}(k^2\tilde{\theta}_1(t) - \dot{\theta}_1(t), \hat{\theta}_1)$$
$$\dot{\hat{\theta}}_2(t) = \mathrm{Proj}(k\Sigma(t)\phi\phi^T\tilde{\theta}_2(t) - \Sigma(t)\phi\phi^T\dot{\theta}_2(t) + k\Sigma(t)\phi\tilde{\theta}_1^T(t)\theta_2(t), \hat{\theta}_2). \tag{6.49}$$

Substituting (6.49), we get

$$\begin{aligned}\dot{V} \leq & -k^2\tilde{\theta}_1^T(t)\tilde{\theta}_1(t) - k\tilde{\theta}_2^T(t)(\Sigma(t)\phi\phi^T)\tilde{\theta}_2(t) + 2\tilde{\theta}_1^T(t)\dot{\theta}_1(t) + \tilde{\theta}_2^T(t)\dot{\theta}_2(t) \\ & + \tilde{\theta}_2^T(t)(\Sigma(t)\phi\phi^T)\dot{\theta}_2(t) - k\tilde{\theta}_2^T(t)(\Sigma(t)\phi\theta_2^T(t))\tilde{\theta}_1(t) + \tilde{x}_p^T\dot{x}_p.\end{aligned}$$

Based on the fact $\tilde{\theta}_2^T\Gamma(t)\dot{\theta}_2 \leq \frac{k}{2}\tilde{\theta}_2^T\Gamma(t)\tilde{\theta}_2 + \frac{1}{2k}\dot{\theta}_2^T\Gamma(t)\dot{\theta}_2$, and the adaptive control law (6.47), $\dot{V}$ can be written as

$$\begin{aligned}\dot{V} \leq & -k^2\tilde{\theta}_1^T(t)\tilde{\theta}_1(t) - \frac{k}{2}\tilde{\theta}_2^T(t)(\Sigma(t)\phi\phi^T)\tilde{\theta}_2(t) + 2\tilde{\theta}_1^T(t)\dot{\theta}_1(t) + \frac{1}{2k}\dot{\theta}_2^T(t)(\Sigma(t)\phi\phi^T)\dot{\theta}_2(t) \\ & - k\tilde{\theta}_2^T(t)(\Sigma(t)\phi\theta_2^T(t))\tilde{\theta}_1(t) + \tilde{\theta}_2^T(t)\dot{\theta}_2(t) + \tilde{x}_p^T(\tilde{\theta}_1(t) - k_g\hat{\theta}_2(t) + d(t))\end{aligned}$$

where $\Gamma(t)$ can be any positive (semi)definite matrix. Considering $\hat{\theta}_2(t) = \theta_2(t) - \tilde{\theta}_2(t)$ and Assumption 6.4, the following inequality holds:

$$\begin{aligned}\dot{V} \leq & -k^2\tilde{\theta}_1^T(t)\tilde{\theta}_1(t) - \frac{k}{2}\tilde{\theta}_2^T(t)(\Sigma(t)\phi\phi^T)\tilde{\theta}_2(t) + 2\tilde{\theta}_1^T(t)\dot{\theta}_1(t) \\ & + \frac{1}{2k}\dot{\theta}_2^T(t)(\Sigma(t)\phi\phi^T)\dot{\theta}_2(t) - k\tilde{\theta}_2^T(t)(\Sigma(t)\phi\theta_2^T(t))\tilde{\theta}_1(t) \\ & + \tilde{\theta}_2^T(t)\dot{\theta}_2(t) - k_g\alpha_1\tilde{x}_p^T\tilde{x}_p + \tilde{x}_p^T\tilde{\theta}_1(t) + k_g\tilde{x}_p^T\tilde{\theta}_2(t) + \tilde{x}_p^Td(t).\end{aligned} \tag{6.50}$$

The estimated velocity state vector $\dot{\hat{x}}_p = -k_g\hat{\theta}_2(t) + d(t)$ is bounded, due to the Lipschitz projection algorithm property and the boundedness of the dither signals. Accordingly, the boundedness of the filter output ϕ, can be established by the BIBO stability of the filter dynamics. It can be concluded that $||\phi|| \leq \gamma_1 = \max\{\phi(t_0), k_g r_{\theta_2} +$

$D\}$. Hence, there exist positive constants γ_2 and γ_3 such that

$$\begin{aligned}\|\Sigma\| &\leq \frac{b(1+\alpha)}{\alpha} = \gamma_2 \\ \left\|\Sigma\phi\phi^T\right\| &\geq \frac{b\alpha(1-e^{-k(\gamma_1^2+2)T})}{\gamma_1^2+2} = \gamma_3.\end{aligned}$$

Applying Young's inequality to all indefinite terms of $\dot{V}$, yields

$$\begin{aligned}\dot{V} \leq &- k^2\tilde{\theta}_1^T(t)\tilde{\theta}_1(t) - \frac{k\gamma_3}{2}\tilde{\theta}_2^T(t)\tilde{\theta}_2(t) + \frac{k^2}{4}\tilde{\theta}_1^T(t)\tilde{\theta}_1(t) + \frac{1}{k^2}\|\dot{\theta}_1(t)\|^2 \\ &+ \frac{k\gamma_3}{12}\tilde{\theta}_2^T(t)\tilde{\theta}_2(t) + \frac{3}{k\gamma_3}\|\dot{\theta}_2(t)\|^2 + \frac{3k\left\|\Sigma(t)\phi\theta_2^T(t)\right\|^2}{\gamma_3}\tilde{\theta}_1^T(t)\tilde{\theta}_1(t) \\ &+ \frac{\gamma_2\gamma_1^2}{2k}\|\dot{\theta}_2(t)\|^2 + \frac{k\gamma_3}{12}\tilde{\theta}_2^T(t)\tilde{\theta}_2(t) - k_g\alpha_1\tilde{x}_p^T\tilde{x}_p + \frac{k^2}{4}\tilde{\theta}_1^T(t)\tilde{\theta}_1(t) + \frac{1}{k^2}\tilde{x}_p^T\tilde{x}_p \\ &+ \frac{k\gamma_3}{12}\theta_2^T(t)\tilde{\theta}_2(t) + \frac{3k_g^2}{k\gamma_3}\tilde{x}_p^T\tilde{x}_p + \frac{k_g\alpha_1}{2}\tilde{x}_p^T\tilde{x}_p + \frac{1}{2k_g\alpha_1}\left\|d(t)\right\|^2.\end{aligned}$$

By collecting the similar terms, one can rewrite the above inequality as

$$\begin{aligned}\dot{V} \leq &-\left(\frac{k^2}{2} - \frac{3k\left\|\Sigma(t)\phi\theta_2^T(t)\right\|^2}{\gamma_3}\right)\|\tilde{\theta}_1\|^2 - \left(\frac{k_g\alpha_1}{2} - \frac{1}{k^2} - \frac{3k_g^2}{k\gamma_3}\right)\|\tilde{x}_p\|^2 \\ &- \frac{k\gamma_3}{4}\|\tilde{\theta}_2\|^2 + \frac{1}{k^2}\|\dot{\theta}_1\|^2 + \left(\frac{3}{k\gamma_3} + \frac{\gamma_2\gamma_1^2}{2k}\right)\|\dot{\theta}_2\|^2 + \frac{1}{2k_g\alpha_1}\left\|d(t)\right\|^2.\end{aligned} \tag{6.51}$$

Based on (6.51), the gains k and k_g, should be chosen such that

$$\begin{aligned}k &> \frac{6\left\|\Sigma\phi\theta_2^T\right\|^2}{\gamma_3} \\ \frac{3}{k\gamma_3}k_g^2 &- \frac{\alpha_1 k_g}{2} + \frac{1}{k^2} > 0.\end{aligned}$$

Equivalently, this yields:

$$\begin{aligned}&k > \max\left\{\frac{6\left\|\Sigma\phi\theta_2^T\right\|^2}{\gamma_3}, \left(\frac{48}{\gamma_3\alpha_1^2}\right)^{\frac{1}{3}}\right\} \\ &\frac{k\gamma_3(\frac{\alpha_1}{2} - \sqrt{\frac{\alpha_1^2}{4} - \frac{12}{\gamma_3 k^3}})}{6} < k_g < \frac{k\gamma_3(\frac{\alpha_1}{2} + \sqrt{\frac{\alpha_1^2}{4} - \frac{12}{\gamma_3 k^3}})}{6}.\end{aligned} \tag{6.52}$$

It follows that for the given gains, there exist strictly positive constants k_a, k_b, k_c and

$k' = \min\{k_a, k_b, k_c\}$ such that

$$\dot{V} \leq -k_a\|\tilde{\theta}_1\|^2 - k_b\|\tilde{\theta}_2\|^2 - k_c\|\tilde{x}_p\|^2 + \frac{1}{k^2}\|\dot{\theta}_1\|^2 + \left(\frac{3}{k\gamma_3} + \frac{\gamma_2\gamma_1^2}{2k}\right)\|\dot{\theta}_2\|^2 + \frac{1}{2k_g\alpha_1}\|D\|^2$$
$$\leq -2k'V + \frac{1}{k^2}\|\dot{\theta}_1\|^2 + \left(\frac{3}{k\gamma_3} + \frac{\gamma_2\gamma_1^2}{2k}\right)\|\dot{\theta}_2\|^2 + \frac{1}{2k_g\alpha_1}\|D\|^2. \quad (6.53)$$

The rate of change of the parameters are

$$\|\dot{\theta}_i\| \leq \max\left\{\left\|\frac{\partial f_p}{\partial x_p}\right\|, \left\|\frac{\partial^2 h}{\partial x_p \partial x_p^T}\right\|\right\}\left(\|\tilde{\theta}_1\| + k_g\|\hat{\theta}_2\| + D\right), \ i = 1, \ 2.$$

Since, the rate of change of the parameters are proportional to the optimization gain, the convergence to a small neighborhood of the origin is achieved by ensuring that $k \gg k_g$. Hence, $\tilde{\theta}$ and $\tilde{x}_p$ converge locally exponentially to an $\mathcal{O}(k_g/k)$ and $\mathcal{O}(D/k_g)$ neighborhood of the origin. □

Remark 6.5. *Theoretically, this algorithm can be used to optimize an objective function subject to fast unstable dynamics. By considering a fixed value of estimation gain k, one can increase the size of the domain of attraction by reducing the optimization gain k_g and/or increase the radius r_{θ_1} and r_{θ_2} of the time-varying uncertainty balls. However, these may cause a slow convergence rate of the closed-loop system to the optimum values, and input saturation, respectively.*

6.4 Simulation Examples

6.4.1 Case 1

The unknown dynamical system with unknown cost function are given as

$$\dot{x}_1 = 0.1x_1 + u$$
$$y = 1 + 2(x_1 - 1.2)^2. \quad (6.54)$$

This system has a pole at 0.1 which is a slow unstable pole. The objective is to minimize the output y. To evaluate the performance of the proposed scheme in dealing with integral action, simulations are carried for three different cases. We apply the proposed extremum-seeking control algorithm with $b = 1.1$, $k = \sqrt{10}$, $k_g = 0.1$ and dither signal $d(t) = 0.5\sin(10t)$. The initial conditions are chosen as $x(0) = \hat{u}(0) = 0$ and $\hat{\theta}(0) = [0,\, 0]^T$, where $\hat{u}$ and $\hat{\theta}$ are estimation of the optimal value of input and unknown time varying parameters. The optimal value of the objective function is $y^* = 1$ that can be achieved at $x_1^* = 1.2$ and optimal steady-state input $u^* = -0.12$.

We start with a proportional only ESC. The changes of the unknown cost function y and the proportional only ESC trajectory is shown in Figure 6.1.

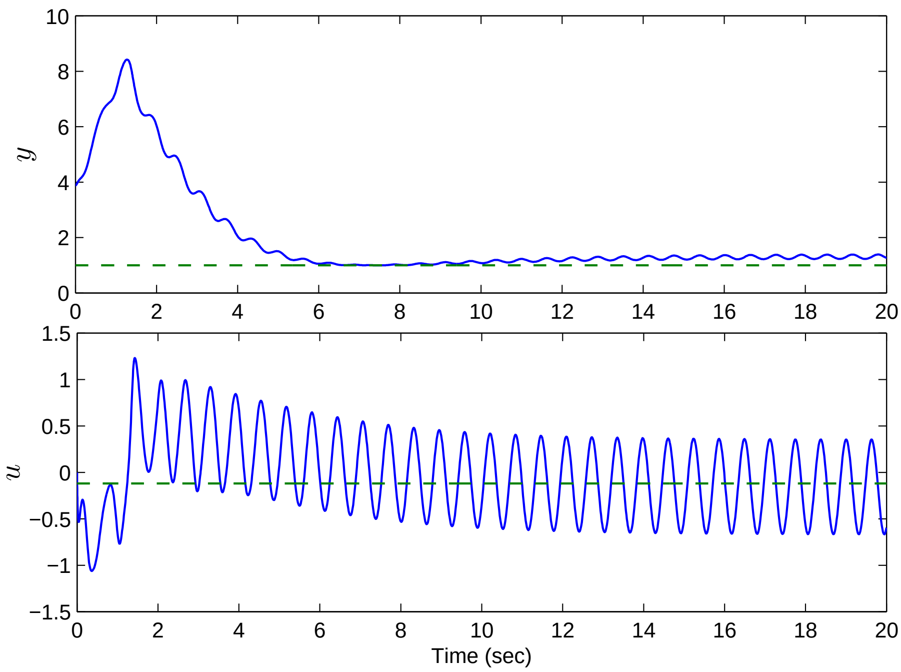

Figure 6.1: The cost function and the corresponding control input (P-ESC, $k_g = 0.1$): optimal value (— — —) and its estimate (——).

As confirmed in this figure, there is an offset between the optimal value and the cost function trajectory. To test the effect of higher proportional gain on the performance of the algorithm, the simulation results with $k_g = 1$ is provided in the absence of integral action for the same problem. The results for the new gain is shown in Figure 6.2.

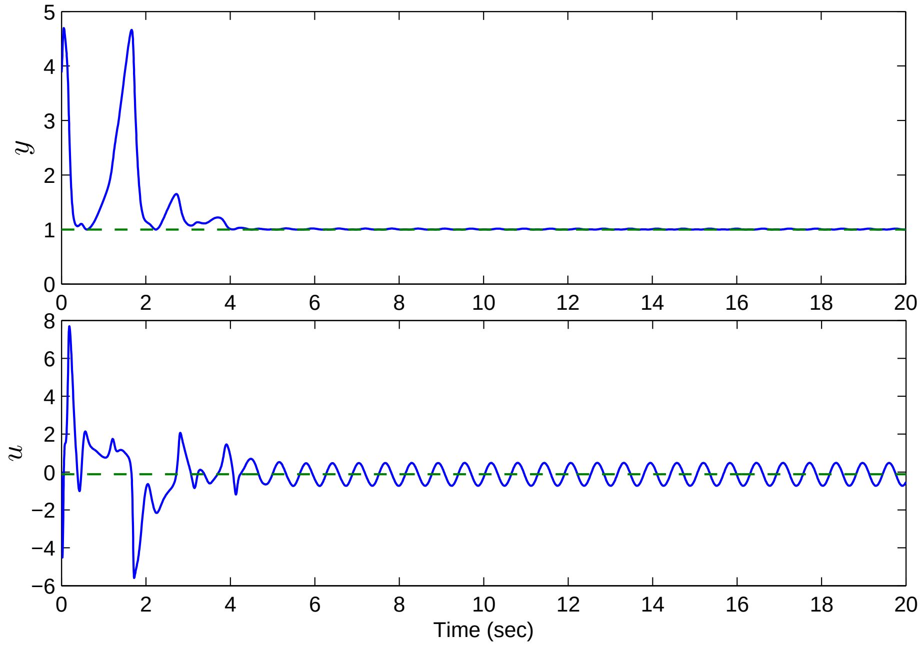

Figure 6.2: The cost function and the corresponding control input (P-ESC, $k_g = 1$): optimal value (— — —) and its estimate (——).

As expected, by comparing Figure 6.1, and Figure 6.2, the higher gain leads to the better performance and the cost function track the optimum value more effectively. The large gain value not only affects the optimization accuracy, but also result in faster convergence. However, a bias in both the output function and the corresponding control input is obtained.

In order to reduce the bias, we introduce the integral action with $k_g = 0.1$ and

$\tau_I = 1$. The corresponding simulation result is depicted in Figure 6.3. The introduction of the integral action effectively reduces the bias without the need for a larger proportional gain.

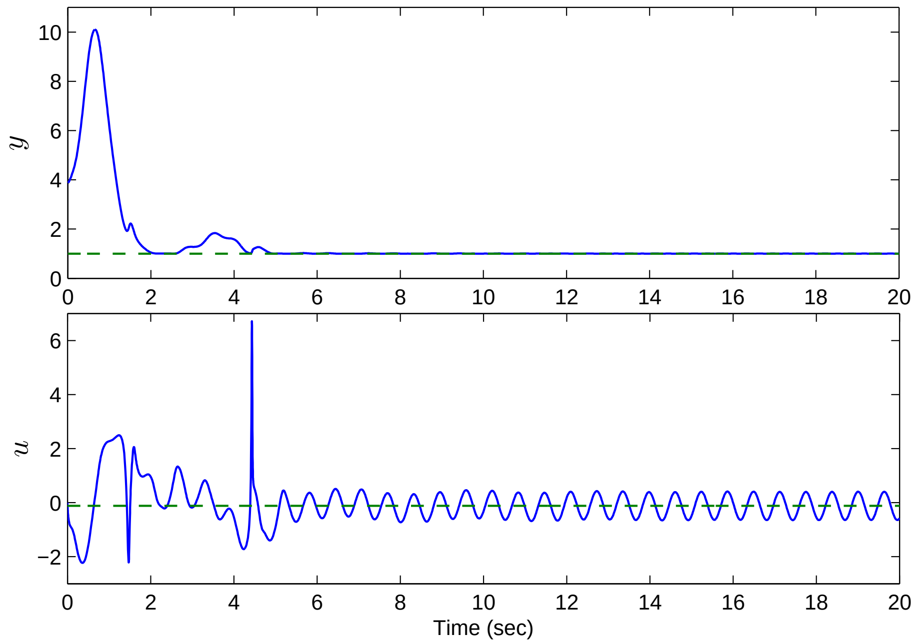

Figure 6.3: The cost function and the corresponding control input (PI-ESC, $k_g = 0.1$ and $\tau_I = 1$): optimal value ($- - -$) and its estimate (——).

For a better comparison between the performance of PI-ESC and P-ESC with $k_g = 1$, the corresponding optimization errors are shown in Figure 6.4 in smaller scale. The maximum variation of the error (at the steady-state) of the PI-ESC is about 0.005, while this value for P-ESC is approximately 0.017. Also, by comparing the control inputs in Figure 6.2 and Figure 6.3, we realize that the area under the input trajectory and the maximum peak in P-ESC is much higher than PI-ESC. The simulation results show that PI-ESC can improve the steady-state performance of the closed-loop optimization problem with acceptable control effort while improving the convergence rate to the optimal value.

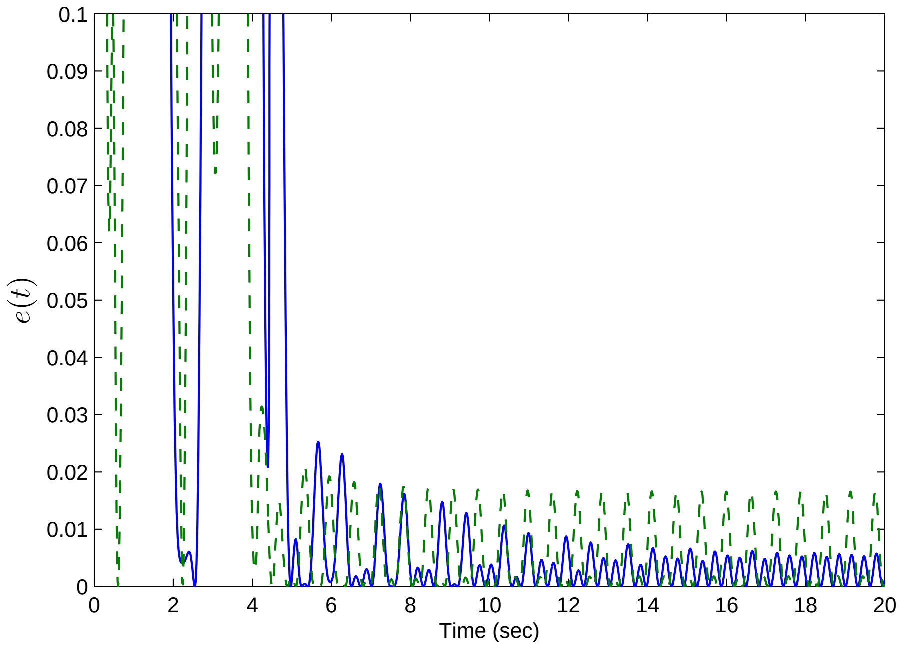

Figure 6.4: Trajectory of the optimization error $e(t) = y - y^*$: P-ESC ($- - -$) and PI-ESC (——).

6.4.2 Case 2

The uncertain dynamical system with unknown cost function are given as

$$\begin{aligned}\dot{x}_1 &= x_1 + x_2^2 + u_1 \\ \dot{x}_2 &= 0.5x_2 + u_2,\end{aligned} \tag{6.55}$$

$$y = 4 + (x_1 - 2)^2 + (x_2 - 3)^2. \tag{6.56}$$

The unforced system has one unstable equilibrium point at $(0, 0)$. All the assumptions of the problem are observed due to the convexity of the cost. The objective is to minimize the unknown output y with respect to uncertain dynamical model. We apply the proposed extremum-seeking control algorithm with $b = 2$, $k = 20$ and $k_g = 0.001$. The arbitrary dither signals are $d_1(t) = a_1 \sin(\omega_1 t)$ and $d_2(t) = a_2 \sin(\omega_2 t)$, where $a_1 = a_2 = 0.02$, $\omega_1 = 20$ and $\omega_2 = 40$. The initial conditions are $x(0) = \hat{\theta}_1(0) = [1, 1]^T$ and $\hat{\theta}_2(0) = [10, 10]^T$ where $\hat{\theta}_i$ for $i = 1, 2$ are the estimates of the unknown time-varying parameters.

The results are shown in Figures 6.5 and 6.6. The ESC state trajectories are shown in Figure 6.5. The changes of the unknown cost function y and input trajectories is depicted in Figure 6.6. The results demonstrate that the proposed extremum-seeking control algorithm provides a rapid progression to the unknown optimum $x^* = [2, 3]^T$ and optimum output $y^* = 4$. Also, a satisfactory performance of the control inputs has been achieved.

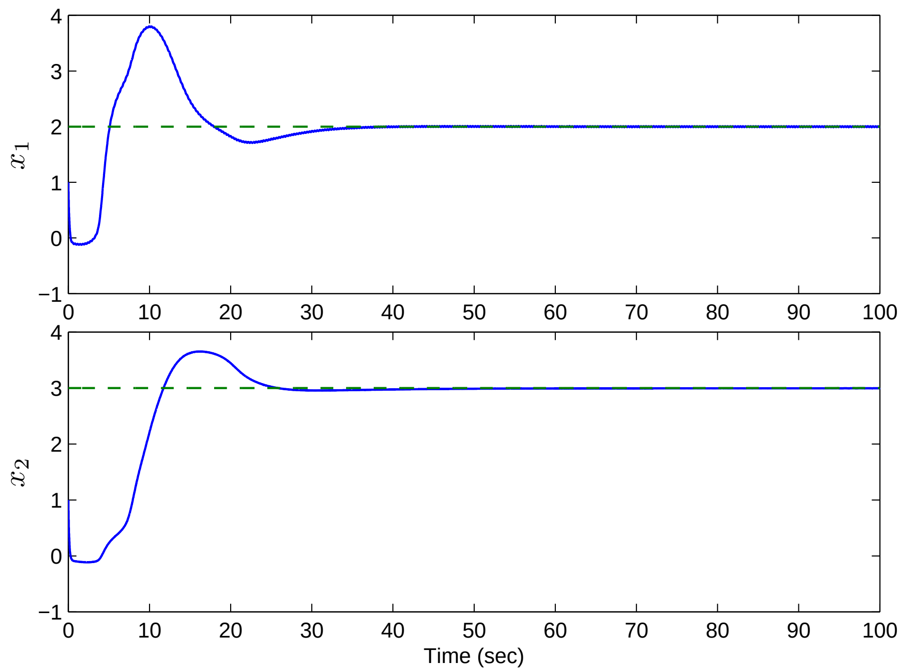

Figure 6.5: Trajectories of the states: optimal value $(- - -)$ and its estimate (——).

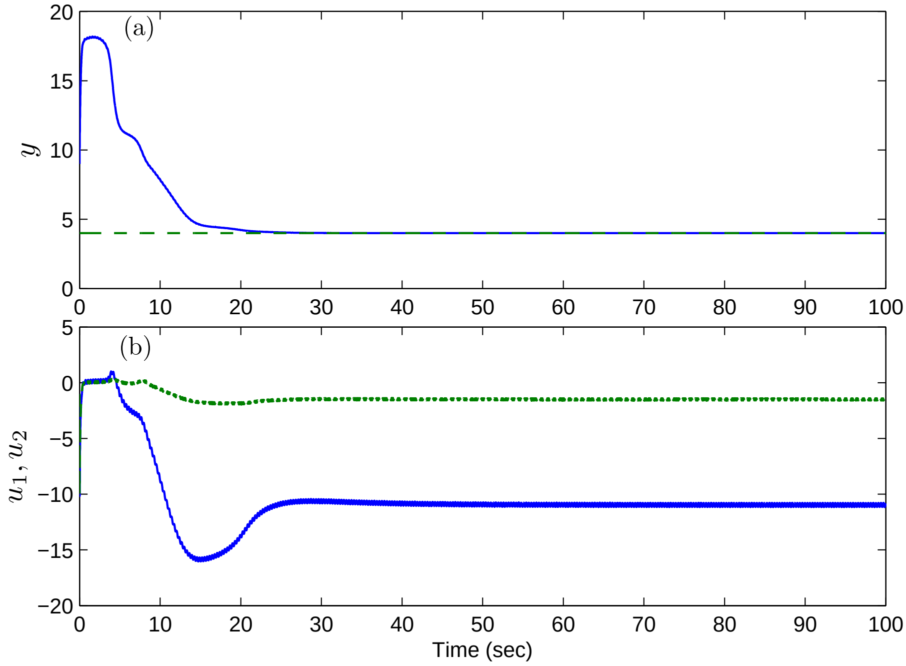

Figure 6.6: The cost function and the corresponding control inputs: (a) optimal value (− − −) and its estimate (——); (b) u_1 (——) and u_2 (− − −).

As previously mentioned, the removal of the need for slowly unstable systems has been one of the contributions of the direct adaptive ESC techniques. To evaluate the effect of fast unstable poles, an additional simulation is provided. For this purpose the following nonlinear dynamical system is considered.

$$\begin{aligned}\dot{x}_1 &= 5x_1 + x_2^2 + u_1\\ \dot{x}_2 &= 5x_2 + u_2.\end{aligned} \tag{6.57}$$

Similar to (6.55) the unforced system (6.57) has an unstable equilibrium point at $(0, 0)$. However, the qualitative behavior of system (6.57) is faster compare to the system (6.55) near the equilibrium point. The simulations are repeated with the same tuning parameters and initial conditions. The resulting ESC system trajectories for the states x_1 and x_2 are shown in Figure 6.7. The objective function trajectory with the corresponding control inputs are shown in Figure 6.8. As expected, the impact of the faster unstable poles is shown to preserve feasibility of the closed-loop trajectories while achieving the minimizer of the optimization problem.

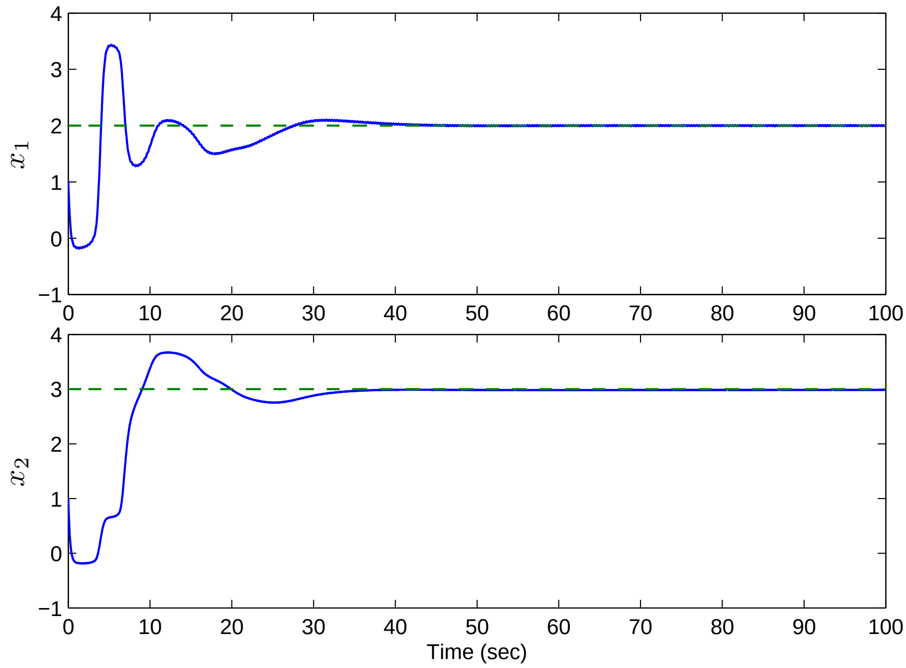

Figure 6.7: Trajectories of the states (fast unstable poles): optimal value (— — —) and its estimate (——).

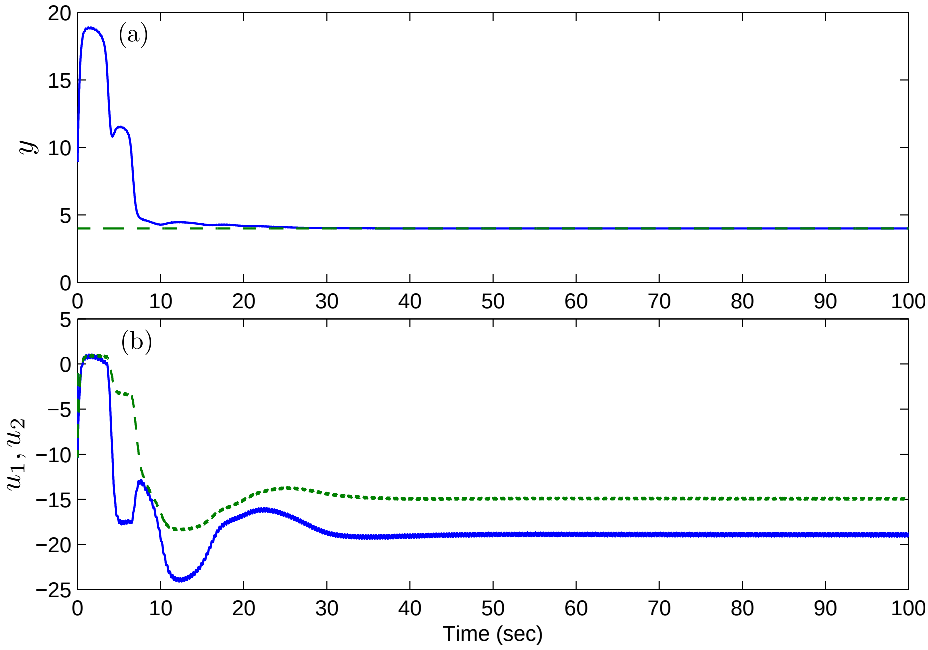

Figure 6.8: The cost function and the corresponding control inputs (fast unstable poles): (a) optimal value (– – –) and its estimate (——); (b) u_1 (——) and u_2 (– – –).

6.5 Summary

In this chapter, an alternative time-varying ESC technique is proposed to solve a class of nonlinear optimization problems subject to uncertain system dynamics. First, it is assumed that the system dynamics are unknown. A nonlinear proportional-integral ESC algorithm is proposed to solve the optimization problem. Second, it is assumed that the nonlinear dynamics are partially known, a direct gradient based ESC is proposed to achieve a rigorous performance of the real-time optimization problem. Both techniques are based on the time-varying estimation of the unknown dynamics and the cost based on the definition of almost invariant manifolds. Simulation examples are provided to demonstrate the effectiveness of the proposed algorithms.

Chapter 7

Conclusions

The results presented in this thesis suggest a new approach for estimation of time-varying parameters and its application to ESC problems. In this chapter, a brief summary and the contributions of this thesis are presented in Section 7.1. Some application problems and future directions for research are discussed in Section 7.2.

7.1 Synopsis

The control and stabilization of nonlinear systems with parametric uncertainties is intrinsically related to the ability to estimate parameters effectively and accurately. The problem is more challenging in the presence of time-varying parametric uncertainties. The development of a clear framework to estimate the time-varying uncertainties is extremely important.

In Chapter 3, we established a geometric approach for time-varying parameter estimation in a class of nonlinear dynamical systems. The concept of almost invariant manifold was defined and exploited to generate an adaptive update law for time-varying parameter estimation. The proposed estimation technique is independent of the choice of control input. Lyapunov-based techniques are used to establish the exponential convergence of the estimated parameters to their true values. A faster convergence and more accurate performance of the algorithm can be achieved by the adjustment of only one tuning gain. The proposed algorithm has the advantage to consider higher order derivatives of the uncertain parameters to improve the invariance property. This advantage has been considered in Chapter 4 for the estimation of periodic uncertainties. In this case, the generalization of almost invariant manifolds results to a perfect invariant manifold. The modification provides an estimation of periodic uncertainties and an estimate of the upper bound of the norm of their rate of change.

Chapter 5 employs the adaptive estimation technique developed in Chapter 3 to provide a formal extremum-seeking scheme for static optimization problem. The gradient of the unknown cost with respect to the inputs are considered as uncertain time-varying parameters. The proposed ESC approach consists of two main parts. An estimation of the uncertain time-varying parameters and then develop a gradient-based control law to solve the unknown optimization problem. The algorithm was extended to handle unknown constrained optimization problems. Augmented barrier functions are the key tools to provide an unconstrained optimization problem where the same time-varying ESC approach was used to find the optimal values. The main benefits of this technique are to remove the need for averaging analysis and reduce the sensitivity on the choice of dither signals on the general performance. These benefits result in good transient performance which is difficult (or impossible) to achieve with standard ESC approaches.

In Chapter 6, we considered unknown real-time optimization problems that are subject to unknown/uncertain nonlinear dynamics. For the case where cost function and system dynamics are unknown, a nonlinear proportional-integral controller was proposed to find the optimal solutions. The Lie derivatives of the cost function with respect to the drift term and the high frequency gain of the system dynamics are treated as unknown time-varying parameters. The almost invariance property was invoked to estimate the time-varying uncertainties. The estimation based PI-ESC has the ability to stabilize and optimize the unstable systems with relatively slow unstable poles.

In another approach, it is assumed that the high frequency gain of the system dynamics is available. The unknown terms of the almost invariant manifold are the drift terms and the gradient of the objective function with respect to the state variables. Suitable adaptive update laws were provided for uncertainty estimation. An inverse optimal control technique was used in a gradient descent algorithm for the design of the controller. It is shown that the proposed technique achieves stabilization and optimization for control-affine systems with unstable poles.

The proposed estimation based ESC can alleviate the limitations associated with time-scale separation in standard ESC problems. Therefore, a good transient performance is achieved with a small steady-state error between the output trajectory and its optimal value.

7.2 Future Research Problems

The vast majority of existing results on ESC have considered continuous-time systems. However, due to the recent improvement on computer and digital technology, there is a great motivation of discrete-time algorithms and sampled-data systems. Although, discrete-time systems can often be treated like their continuous counterparts, the sampling of physical systems results in some information loss. A discrete-time version of the standard ESC loop and estimation-based ESC were studied in [20] and [33], respectively. One potential subject for future studies would be to consider the approaches of Chapters 3 to 6 for discrete-time systems. In these cases, one would most surely require additional assumptions on the sampling time with respect to the adjustable gains to establish the convergence and stability of closed-loop ESC systems.

All the theoretical development of the proposed estimation-based ESC approaches, consider systems with relative degree one. The procedure in Chapter 6 can be potentially extended to find an optimal solution for nonlinear dynamics with objective functions with relative degree greater than one. This allows to consider a more robust controllers such as backstepping design and sliding mode approach. We can combine the proposed time-varying parameter estimation algorithm of this thesis with nonlinear control techniques to solve the uncertain optimization problem for systems with relative degree two or higher. Therefore, an adaptive optimization is possible without any change of coordinates. Ultimately, one would need to consider the design of ESC for nonlinear systems with unknown relative degree.

Another direction of future work would consider the proposed ESC approaches to solve more complex optimization problems that are subject to unknown/uncertain dynamics. The proposed estimation-based ESC can be used to solve minmax optimization problem that arise in several applications of robust optimization problems. One can also consider multi-objective optimization problems using estimation-based ESC. These type of problems are common in practical situations where multiple objectives must be systematically handled to achieve multiple simultaneous design goals. The on-line adaptive estimation of the unknown nonlinear parts of the dynamical system can reduce the model uncertainties and improve the overall performance.

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Appendix A

Lyapunov Stability

This section reviews elements of Lyapunov stability theory. First consider a nonlinear time-invariant dynamical system of the form

$$\dot{x} = f(x, u) \tag{A.1}$$

where $x \in \mathbb{R}^n$ and $u = \alpha(x) \in \mathbb{R}^m$ can be any state feedback control law. The following theorem is proved in Khalil, 2002 [43].

Theorem A.1. *(Lyapunov Stability Theorem). Let $x^* = 0$ be an equilibrium point of (A.1) and $\mathcal{O} \subset \mathbb{R}^n$ be a domain containing the origin. Let $V : \mathcal{O} \to \mathbb{R}$ be a continuously differentiable function such that*

- $V(0) = 0$*,*
- $V(x) > 0$ *for* $x \in \mathcal{O}\backslash\{0\}$*,*
- $\dot{V} \leq 0$ *for* $x \in \mathcal{O}$*.*

Then $x^* = 0$ is stable. Moreover if

- $\dot{V} < 0$ *for* $x \in \mathcal{O} \backslash \{0\}$,

Then $x^* = 0$ *is asymptotically stable.*

In the sequel, a function V such that the conditions of the above theorem hold will be called a Lyapunov function.

Now, consider the following time-varying system

$$\dot{x} = f(x, u, t) \tag{A.2}$$

where $u = \alpha(x, t)$ can be any smooth controller and $f(x, \alpha(x), t)$ is assumed to be smooth in x, continuous and bounded over bounded intervals I in $\mathbb{R}_+$. Assume that $f(0, 0, t) = 0, \forall t \in I$. The next theorems, proved in Khalil, 2002 [43], summarize the extension of Lyapunov Stability Theorem to the time-varying case.

Theorem A.2. *Let* $x^* = 0$ *be an equilibrium point for (A.2) with* $u = \alpha(x, t)$ *can be any smooth controller, and* $\mathcal{O} \subset \mathbb{R}^n$ *be a domain containing the origin. Let* V : $[0, \infty] \times \mathbb{R}$ *be a continuously differentiable function such that*

$$W_1(x) \leq V(x, t) \leq W_2(x) \tag{A.3}$$

$$\dot{V}(x, t) \leq 0 \tag{A.4}$$

for all $t \geq 0$ *and for all* $x \in \mathcal{O}$, *where* $W_1(x)$ *and* $W_2(x)$ *are continuous positive functions on* $\mathcal{O}$. *Then* $x^* = 0$ *is uniformly stable. If inequality (A.4) strengthened to*

$$\dot{V}(x, t) \leq -W_3(x) \tag{A.5}$$

where $W_3(x)$ *is a continuous positive definite function on* $\mathcal{O}$. *Then* $x^* = 0$ *is uniformly asymptotically stable. Finally, if* $\mathcal{O} = \mathbb{R}^n$ *and* $W_1(x)$ *is radially unbounded, then* $x^* = 0$ *is globally uniformly asymptotically stable.*

Appendix B

Sufficient Richness

This section reviews elements of sufficient richness condition on the input signal for LTI systems. First the following definitions are provided [74].

Definition B.1. *A signal $r(t) : \mathbb{R}_+ \to \mathbb{R}^m$ is said to be stationary if the autocovariance matrix*

$$R(t) = \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} r(\tau) r^T(t+\tau) d\tau, \tag{B.1}$$

exists uniformly in t_0. The spectral measure of signal r(t) is stated as

$$S(\omega) = \int_{-\infty}^{+\infty} R(\tau) e^{-j\omega\tau} d\tau \tag{B.2}$$

In particular, if $r(t)$ has a sinusoidal term at frequency ω_1, then r is said to have a spectral content at frequency ω_1.

The persistency of excitation for the stationary signal $r(t)$ is equivalent to say the autocovariance matrix $R(0) > 0$.

Definition B.2. *A stationary signal $r(t) : \mathbb{R}_+ \to \mathbb{R}$ is called sufficiently rich of order k, if the spectral density $S(\omega)$ contains at least k atoms.*

For example, consider the scalar signal $r(t) = \sin(\omega_1 t) + \sin(\omega_2 t)$. Since

$$\lim_{T\to\infty} \frac{1}{T}\left(\int_0^T \sin(2\omega_1\tau)d\tau = \int_0^T \cos(2\omega_1\tau)d\tau = \int_0^T \sin(\omega_1\tau)\cos(\omega_2\tau)d\tau\right) = 0$$

$$\lim_{T\to\infty} \frac{1}{T}\left(\int_0^T \sin^2(\omega_1\tau)d\tau = \int_0^T \sin^2(\omega_2\tau)d\tau\right) = \frac{1}{2},$$

the autocovariance matrix is achieved as $R(t) = \frac{1}{2}\cos(\omega_1 t) + \frac{1}{2}\cos(\omega_2 t)$. Based on (B.2) the spectral density function is

$$S(\omega) = \frac{\sqrt{2\pi}}{2}\left(\delta(\omega-\omega_1) + \delta(\omega+\omega_1) + \delta(\omega-\omega_2) + \delta(\omega+\omega_2)\right)$$

where $\delta(\cdot)$ is a Dirac delta function. Therefore, the signal $r(t)$ contributes four points to the spectrum at ω_1, $-\omega_1$, ω_2 and $-\omega_2$, i.e., $r(t)$ is sufficiently rich of order 4.

The following theorem is proved in Sastry and Bodson, 1994 [74].

Theorem B.1. *(PE and Sufficient Richness). Let $y(t) \in \mathbb{R}^n$ be the output of a stable LTI system with transfer function $G_0(s)$ and stationary input $r(t)$. Assume there exists frequencies $\omega_1, \omega_2, \cdots, \omega_n$ such that $G_0(j\omega_1), G_0(j\omega_2), \cdots, G_0(j\omega_n)$, are linearly independent on $\mathbb{C}^n$. Then y is PE $\iff$ r is sufficiently rich of order n.*