Distributed Control of Multi-agent systems using Extremum Seeking

by

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A thesis submitted to the
Department of Chemical Engineering
in conformity with the requirements for
the degree of Master of Applied Science

Queen's University
Kingston, Ontario, Canada
March 2016

Abstract

A model free control technique (extremum seeking) is employed to address problems of large-scale systems involving multi-agents in real-time. This thesis focuses on the use of extremum seeking control in a distributed, coordinated and a cooperative fashion to solve distributed control and optimization problems.

First, the problem of maximizing the power produced in a wind farm is considered. To tackle this problem, a distributed time-varying extremum seeking control (TVESC) technique is employed to overcome the need to provide accurate models of aerodynamic wake interactions among the wind turbines. Solutions in continuous-time and discrete-time are presented.

Secondly, precise knowledge of the structure of network connectivity has been utilized in solving cooperative optimization problems of multi-agent systems to achieve global objectives. In this thesis, a distributed proportional-integral extremum seeking control technique is designed to tackle such problems over unknown networks.

Acknowledgments

In a special way, I want to thank Dr. Martin Guay for choosing to work with me, I couldn't have asked for a better supervisor. Studying under his supervision has been a great learning experience.

I am indebted to my parents, they are the best thing that has happened to me. They have supported me with prayers and words of encouragement.

To my siblings Felicia, Edmund, Lilian and Cletus, I say thank you so much for the support.

I also want to say a big thank you to every member of G37, Samuel, Dan, Isaac, Hussain and Sebastian, you guys are awesome.

To Ehsan Moshksar, thank you so much for the wonderful discussions we had, they were really helpful.

To Emmanuel, Khadijat and Philip, thank you so much for making my stay at Queen's and in Kingston memorable.

I am so thankful to the Sobanjo, Osazuwa, Moody, Okpotse and Egbunonu families for their kindness, prayers and support.

God, it has been you all the way, you have made my life so beautiful and have crowned my efforts with success. From the bottom of my heart, I say thank you so much for all you have done for me.

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Chapter 1

Introduction

The optimization of an unknown cost function to its extremum and the stabilization of a complex dynamical system require the use of control approaches ranging from model based (e.g Optimal Control and Model predictive Control) to model free (e.g Extremum Seeking Control) design techniques. When it comes to solving large-scale real-time optimization problems involving multi-agents, control approaches employed could either be centralized or distributed (decentralized). In a centralized approach, the individual agents that make up a system are controlled by a single decision maker. This decision maker monitors, receives, and processes information from all agents and also sends processed results back to the agents. The main advantage of this control approach is that the decision maker has full knowledge of the state of the system at any given time and can make useful decision(s) that help meet global objective(s). As the number of agents increases, it becomes difficult for the decision maker to perform its task due to the increase in computational complexity. This phenomenon could result in the transmission of inaccurate information, loss of information and increase in computation time. In addition, centralized approaches are complex and most importantly lack system robustness as failure of the decision maker could mean failure of the entire system.

Most of the challenges highlighted above can be effectively avoided if one adopts a distributed approach. In a distributed environment, multiple decision makers can be utilized. Each subsystem or agent is controlled by a local decision maker. With the help of the decision maker, each agent's task can be limited to the solution of a simpler local problem. Each local problem can be solved with or without the cooperation of neighbouring agents. In a cooperative environment, agents work with other agents through local communication to find the best solution that meets global (or centralized) objective(s) of the system. Some of the advantages of distributed control includes but are not limited to effectiveness, flexibility, scalability and adaptiveness. The greatest advantage of distributed control is system robustness. This implies that the failure of a decision maker does not necessarily mean overall system failure as the system can absorb the effect of a failure and quickly recover through the help of other decision makers.

Graph theory provides the tools necessary to identify structures and the properties of distributed networks. A network can be effectively represented as a graph in which agents are identified as nodes and edges identify the existence of communication links between agents. Figures 1.1 and 1.2 provide a schematic representation of a centralized network with 20 agents and a distributed network with 30 agents, respectively. It is seen from Figure 1.1 that all the agents can communicate their information with agent 20. Agent 20 can also communicate with all the agents, it has perfect knowledge of the system. As a result, the system is controlled by a single decision maker.

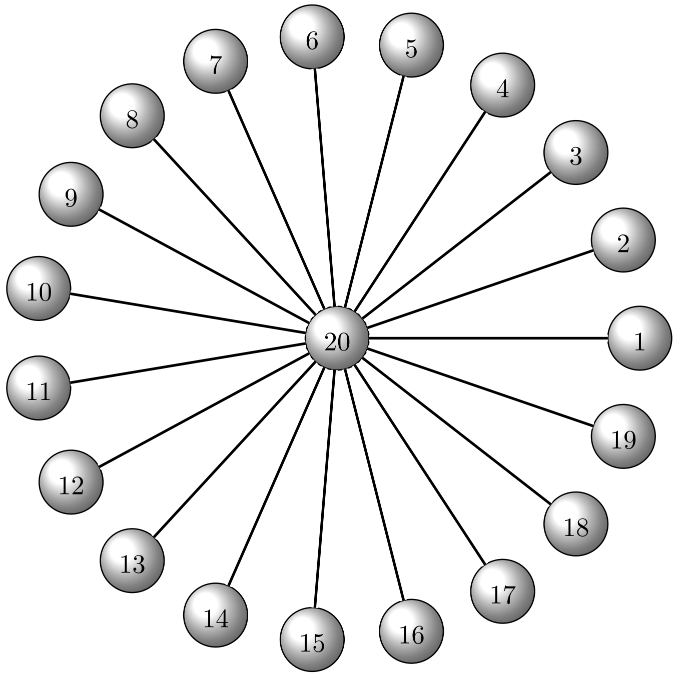

Figure 1.1: Centralized network.

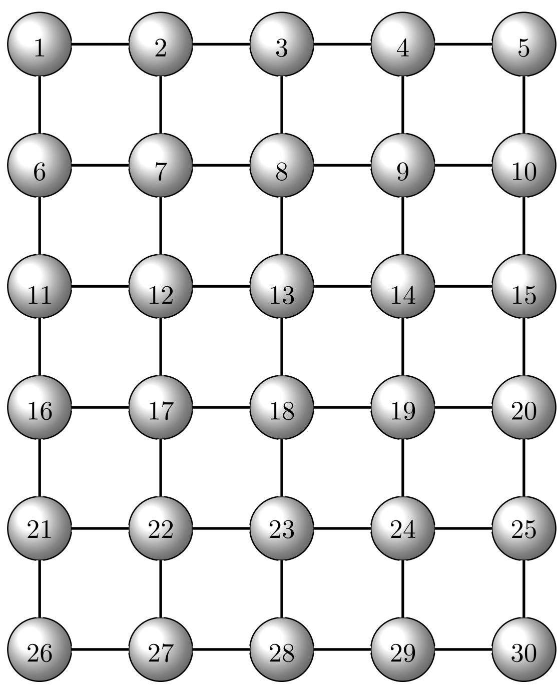

Figure 1.2: Distributed network.

From Figure 1.2, each agent is controlled by a local decision maker and can only communicate with a few other agents. There is no central decision maker in this case so the agents solve their local problem with the cooperation of neighbouring agents.

This thesis focuses on the optimization and control of large-scale multi-agent systems (MAS) in a distributed fashion using a model free real-time optimization technique called Extremum Seeking Control (ESC). Optimal Control and Model Predictive Control are model based control approaches that rely on the knowledge of the structure of the cost function to be optimized and the system dynamics. In Model Predictive Control, the model of the system is used in the computation of optimal actions that optimizes a known function. The model is assumed to be complete and perfect [4], which is unrealistic. For example, in the area of wind farm power maximization, it is known that aerodynamic interactions among wind turbines limit overall power capture. It is also known that there are no accurate models that completely describe these interactions. In this situation, the use of a model based control approach only produces sub-optimal results, optimality can be achieved when a model free control approach is considered. The use of a model-free approach such as ESC eliminates the need for knowledge of the structure of the cost function.

ESC is a gradient based optimization and an adaptive control approach used in locating and maintaining the optimum of a cost function provided that this function and its optimum exist [5]. ESC does not assume knowledge of the structure of this cost to track the optimum, it requires measurements of this unknown cost to estimate its gradient. This implies that the resulting ESC system utilizes a basic gradient descent to identify a local optimum of the measured cost. ESC has been applied in heating and cooling systems, flow control, energy conversion, agents and sensor networks, plasma control, optimization in bio-processes, brake systems, formation control, process control and internal combustion engines [4]. ESC has developed over the years leading to the propositions of several alternative design techniques. This thesis focuses on the design and implementation of ESC in a distributed manner for the control and optimization of MAS.

1.1 Objectives and Contributions

A cooperative ESC technique is used to overcome the need to provide accurate models of aerodynamic interactions among wind turbines to provide an effective technique for the maximization of power production in a wind farm. The first objective of this thesis is to address the wind-farm power maximization problem posed in [3] in a distributed fashion using the time-varying extremum seeking control (TVESC) technique proposed in [6] for continuous-time systems. It is important at this point to clearly state that this problem has been tackled using the standard perturbation based extremum seeking control (PBESC) technique [3]. Solving this problem using the time-varying technique will provide the opportunity to effectively make meaningful comparison between the two techniques based on the results obtained. Currently no solution to this problem using extremum seeking control in discrete-time is available so the second objective of this thesis is to provide a solution in discrete-time using the TVESC technique for discrete-time systems proposed in [7].

Most of the model free approaches that have been employed in solving distributed optimization problems have used the exact knowledge of network connectivity to achieve system-wide objectives. The third objective of this thesis is to design a distributed proportional-integral extremum seeking control technique to solve the problem of dynamic consensus estimation and distributed optimization of large-scale MAS over unknown networks in real-time.

1.2 Organization of Thesis

This thesis is structured as follows. A rigorous review on the development of extremum seeking control and some of the proposed extremum seeking control techniques is given in Chapter 2. Chapter 2 also provides a review of existing works on distributed control and optimization of MAS.

In Chapter 3, a wind farm power maximization problem is presented. The wind farm model, the communication network, an average dynamic consensus estimator and the distributed control algorithm are described. With all these in place, solutions of the wind farm optimization problem are presented in both continuous-time and discrete-time. The results obtained using the distributed TVESC approach are compared to the performance of the PBESC technique. More simulation examples are included to show the effectiveness of the algorithm and also demonstrate that the application to a large-scale is possible.

The control and optimization of large-scale systems involving multi-agents is addressed from a different point of view in Chapter 4. In the absence of exact knowledge of network connectivity among agents, it is shown that global objectives can still be met in real-time using a distributed proportional-integral extremum seeking control technique. First, the optimization problem is presented then the communication network and the control algorithm are described. Simulation examples showing the effectiveness of the proposed control technique are also included. Convergence of the algorithm to a small neighbourhood of the unknown minimizer of the overall cost is established.

In Chapter 5, a summary of the contributions of this thesis is presented. Areas of future research are proposed.

Chapter 2

Literature Review

The basic idea of ESC for a static map as proposed in [1] is first presented. Consider the basic ESC scheme shown in Figure 2.1 where θ is the input, k is the adaptation gain, a is the amplitude of the excitation signal and ω is the frequency of the excitation signal. Let $\tilde{\theta} = \theta^* - \hat{\theta}$, $\tilde{y} = y^* - \hat{y}$ and $k > 0$. Recall that the goal of ESC is to drive the cost function to its extremum and this can only be achieved when $\tilde{\theta} \longrightarrow 0$ and as such $\hat{\theta} \longrightarrow \theta^*$ and $\hat{y} \longrightarrow y^*$. Let the cost function be represented as

$$\begin{aligned} f(\theta) &= f^* + \frac{f''}{2}(\theta - \theta^*)^2 \\ y &= f(\theta) \end{aligned} \tag{2.1}$$

where $f \in C^2$. If the minimization of the $f(\theta)$ is considered, then the Hessian $f'' > 0$. From Figure 2.1, $\theta = \hat{\theta} + a \sin \omega t$. The excitation signal $(a \sin \omega t)$ is added to the input to ensure that the measured cost is sufficiently excited to estimate the unknown gradient.

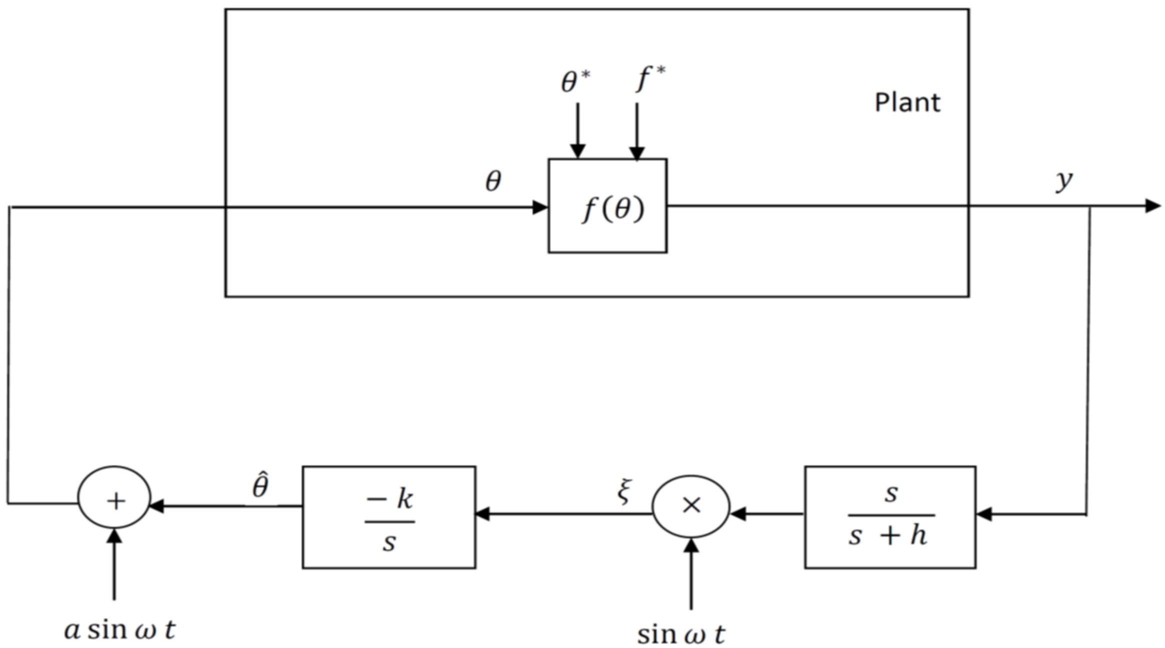

Figure 2.1: Basic extremum seeking scheme [1].

The addition of the excitation signal to the input generates a periodic response of the cost and when passed through the washout or the high pass filter, the bias component is eliminated. This signal response undergoes demodulation which results in the generation of high frequency signals that diminish when passed through an integrator. This is done to estimate the gradient of the cost and to drive $\hat{\theta}$ to θ^*. Eventually, it can be shown that the average trajectories of the ESC system are such that

$$\dot{\tilde{\theta}} \approx \frac{-kaf''}{2}\tilde{\theta}$$

Since the averaged system converges to the unknown optimum, the actual system can be shown to enter a neighbourhood of the unknown optimum for a specific choice of the ESC tuning parameters, k, a and ω.

2.1 Background

Extremum Seeking Control was first introduced in [8]. Even in Russia as early as 1943, reports have it that remarkable investigations in this area had already begun [9]. The application of ESC to the optimization of internal combustion engines was first reported in [10], where ESC was referred to as extremum control and self-optimizing control. At this time, there were no concrete analytical or even systematic schemes for ESC. As a result, it became less appealing as other optimization techniques and adaptive control methods became advantageous. ESC deals with regulation to unknown set points or reference trajectories but these adaptive control techniques depended on the knowledge of the set points to control linear [11] and nonlinear [12] systems.

In the early 2000, ESC made a strong come back after stability results based on averaging analysis and singular perturbation (standard perturbation based extremum seeking control approach) for a class of general nonlinear dynamic systems was provided [13]. This contribution ignited significant research effort to address the numerous challenges or limitations associated with ESC and the development of alternative ESC techniques with improved performance and robustness. In [13], the system dynamics and the stabilizing controller have to operate on a fast time-scale compared to the dynamics of the ESC. This limitation restricts the tuning of the excitation signal and the filters (High-pass and the low-pass) which must remain slower than the system dynamics. The design of the ESC requires that the chosen adaptation gain (k), the amplitude (a) of the excitation signal are small. The frequency (ω) of this signal must be chosen to be small but larger than the tuning parameters of the filters. Consequently, it follows that convergence of the system to a neighbourhood of the unknown optimum is required to be slow.

2.1.1 Addressing the limitations associated with the selection of ESC tuning parameters

In [14], ESC design for a class of Hammerstein/Wiener processes was proposed. A dynamic phase-lead compensator is incorporated to the ESC scheme. The addition of the compensator is such that the adaptation gain can be increased freely to improve the transient performance. Also, ω and ka^2 must be chosen such that ω is larger than ka^2, large enough to ensure time-scale separation. Improvement in adaptation transients, stability and performance of the scheme was reported. A discrete-time version of this work was presented in [15].

In [16], an analysis of the non-local properties of ESC was presented. The results established new criteria for the selection of tuning parameters for PBESC. In this work, systems of the following form are considered:

$$\dot{x} = f(x, u) \tag{2.2}$$

$$y = h(x) \tag{2.3}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the input, $y \in \mathbb{R}$ is the output. $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}$ are continuously differentiable functions. The control law is of the form:

$$u = \alpha(x, \theta) \tag{2.4}$$

where θ is the scalar parameter. θ is of the form:

$$\theta = \hat{\theta} + a\sin(\omega t)$$

so the system (2.2) becomes:

$$\dot{x} = f(x, \alpha(x, \hat{\theta} + a\sin(\omega t))) \tag{2.5}$$

and

$$\dot{\hat{\theta}} = kh(x)b\sin(\omega t).$$

Under mild assumptions concerning the dynamics of the system and the unknown cost function, semi-global practical stability of the unknown optimum equilibrium of the ESC system was demonstrated. An extension of this work to global ESC was studied in [17] where convergence to a small neighbourhood of the global optimum is shown from an arbitrarily large domain of attraction. The satisfaction of the conditions implies that the controller can be easily tuned to ensure convergence to this small neighbourhood.

Further contributions to the improvement of the performance of ESC were reported in [18] and [19] where the impact of the choice of the excitation signal was studied. A study of the effect of the shape of the excitation signal on the speed of convergence, domain of attraction and accuracy of ESC was carried out in [18]. The effect was studied using a gradient system that behaved like the true system in the presence of three different excitation signals $d(.)$. The excitation signals are sine wave, square wave and triangle wave satisfying

$$\int_0^T d(s)ds; \qquad \frac{1}{T}\int_0^T d^2(s)ds{>}0; \qquad \max_{s\in[0,T]}|d(s)| = a$$

where $a > 0$ is the amplitude of the signal. The excitation signals are periodic functions of period $T > 0$ and $\omega = \frac{2\pi}{T}$. From the results obtained in [18], the authors arrived at the following conclusions:

- the performance of the ES controller (in terms of domain of attraction and accuracy) using the different excitation signals is almost the same as the amplitude a and a controller parameter δ_c approach zero for $\omega > 0$;
- the speed of convergence of the true system is determined by the speed of convergence of the gradient system and a multiplier factor;
- this multiplier factor depends on a and w of the excitation signal, δ_c and the power of some normalized excitation signal p_d;
- the normalized excitation signal has to be of the same shape as the excitation signal but with $a = 1$ and $T = 2\pi$ and p_d differs for each signal;
- using the same a and w for all three excitation signals, the speed of convergence of the controller is fastest with the square wave, followed by the sine wave then the triangle wave.

A Newton-based extremum seeking control technique was proposed in [20]. By using averaging and singular perturbation to prove stability of the proposed scheme, the authors were able to show (as opposed to using gradient-based techniques) that convergence of the ESC can be independent of the unknown Hessian of the unknown cost function.

A class of nonlinear systems with parametric uncertainties was studied in [21] using a different ESC technique that involves the estimation of unknown parameters and the design of an adaptive extremum seeking controller to solve the required optimization problem. In this study, the structure of the cost function to be optimized was assumed to be known (so no measurement is required) and depended on the states and some unknown parameters. The successful application of this technique also required the excitation signal to be designed in a such a way that it meets a persistence of excitation (PE) condition that ensures parameter convergence. One limitation associated with this technique is the difficulty associated with designing this excitation signal such that it meets a PE condition. This challenge was addressed in [22]. The PE condition was modified and an online technique for designing the excitation signal to easily meet this condition was suggested.

A time-varying ESC technique was proposed in [6] and a discrete time version in [7]. The unknown gradient is estimated as a time-varying parameter in this technique. In contrast to [22] and [21], measurements of the cost is needed but no knowledge of the structure of the cost is required and no parametrization of the cost is needed. Flexibility in selecting the ESC tuning parameters to achieve convergence to a neighbourhood of the unknown optimum of the measured cost function and improvement in transient performance can be claimed. This ESC technique will be applied to solve the wind farm maximization problem presented in chapter 3 of this thesis.

A discrete-time ESC technique was proposed for tracking the optimum of an unknown cost function in [23]. A quadratic cost approximates the unknown cost and an estimation routine employed in estimating the gradient and the Hessian of the cost. With a linear time-varying Kalman filter as the estimator, the optimum of the cost can be located using a Newton based optimization or a steepest descent method. This technique was applied to solve the problem of formation flight for drag reduction.

A sampling based optimization technique for ESC of dynamical systems was proposed in [24]. This technique incorporates the discrete-time Shubert algorithm to extremum seeking control and knowledge of the Lipschitz constant of the plant's model is required. Provided the input (u) is initiated from a compact set, it was shown that the output (y) is guaranteed to reach a neighbourhood of the global optimum. A key tuning parameter of the ESC is the sampling period. The sampling period can be adjusted to enlarge the domain of attraction of the ESC. The sampling period also determines the speed of convergence and accuracy of this technique. The longer the sampling period, the slower the convergence rate but accuracy is improved
.

2.1.2 Addressing the limitations associated with time-scale separation

In an attempt to improve the performance of ESC, one needs to eliminate the requirement of a time-scale separation. The technique proposed in [21] does not require time-scale separation but can only be implemented when the structure of the cost function is known as discussed above. In the absence of such knowledge but in the presence of multiple identical units with different inputs, [25] has shown that the need for time-scale separation can be eliminated. The technique ensures that the optimization and the system dynamics are nearly in the same time-scale. In contrast to the perturbation way of estimating the gradient of the unknown cost function, the finite difference between the outputs measurements can be used in gradient estimation. Fast convergence to the unknown optimum was shown.

A proportional-integral extremum seeking control (PI-ESC) technique was proposed in [26] that requires no knowledge of the model of the system and the cost function to be optimized. A time-varying parameter estimation technique is used to estimate key parameters of the cost function dynamics. Using the estimated parameters, a PI controller is proposed to drive the system to its steady-state optimum. The controller is of the form:

$$\begin{aligned} u &= -k_g\hat{\theta}_1 + \hat{u} + d(t) \\ \dot{\hat{u}} &= -\frac{1}{\tau_I}\hat{\theta}_1 \end{aligned} \tag{2.6}$$

where $k_g > 0$ and $\tau_I > 0$ are the proportional gain constant and the integral gain constant respectively. $\hat{\theta}_1$ can be interpreted as an estimate of the real-time gradient of the unknown cost. It is not the steady-state gradient that is estimated in standard ESC techniques. The bounded dither signal, $d(t)$ is such that $\|d(t)\| \leq D \ \forall t \geq 0$ where $D > 0$ is a constant.

Modification of this technique for solving large-scale problems involving multi-agents in a distributed fashion was proposed in [27]. The controller for an agent, say agent

i is of the form:

$$\begin{aligned} u_i &= -k_g \hat{\theta}_{1,i} + \hat{u}_i + d_i(t) \\ \dot{\hat{u}}_i &= -\frac{1}{\tau_I} \hat{\theta}_{1,i}. \end{aligned} \tag{2.7}$$

A numerical optimization-based extremum seeking control technique was proposed in [28]. This work was an improvement of the work carried out in [29]. It involves the design of a robust state regulator that can handle disturbances and unknown plant dynamics for state feedback linearizable systems. It also avoids the need for time-scale separation but requires the availability of the measurement of the true gradient.

2.2 Distributed control of multi-agent systems

In light of the advantages of distributed control over centralized control, many researchers have focused on the development and analysis of new distributed techniques for the solution of large-scale problems of MAS. The solution of cooperative and coordinated control problems associated with MAS requires the coordination of the actions of the individual agents to optimize a given global cost function. This idea has been applied to resource allocation, formation control, source seeking, mobile communication, power maximization, *etc.* Some contributions on distributed control and optimization of MAS are reviewed below.

2.2.1 Resource Allocation

Ideas from the field of game theory have been employed in addressing resource allocation problems. One such problem was tackled in a distributed fashion in [30] where a limited set of resources $R = \{r_1, \ldots, r_n\}$ are to be shared among a set of agents $V = \{1, \ldots, p\}$. Here, p refers to the number of agents involved. Each agent $i \in V$ can only take a single action or resource $a_i = r \in \mathcal{A}_i \subseteq R$. Each resource $r \in \mathcal{A}_i$ has its welfare function (Y_r) that is unknown and separable, the numerical value of this unknown function depends on the number of agents using r. This is to say that:

$$\forall r \in \mathcal{A}_i \colon Y_r \colon \{0, 1, \ldots, p\} \longrightarrow \mathbb{R}^+.$$

The overall welfare function is of the form:

$$J(a) = \sum_{r \in \mathcal{A}_i} Y_r(|a|_r) \tag{2.8}$$

where $a = (a_1, a_2, \ldots, a_p) \in \mathcal{A}$ represents an allocation and $|a|_r = |\{i \in V \colon a_i = r\}|$ denotes the number of agents that used resource r in a. The set $\mathcal{A} = \prod_{i \in V} \mathcal{A}_i$ is the Cartesian product of sets $\mathcal{A}_i$ (the set of allocations).

Since the welfare function is separable and unknown, the unknown utility function for agent i is simply:

$$Y_i = \frac{Y_r(|a|_r)}{|a|_r} \tag{2.9}$$

The objective is to find a^* that maximizes $J(a)$, that is:

$$a^* \in \arg\max_{a \in \mathcal{A}} J(a) \tag{2.10}$$

It was demonstrated that the agents can work independently to achieve the desired objective. Results show convergence to a Nash equilibrium.

Consider a set of agents $V = \{1, \ldots, p\}$ faced with an optimization problem of the form:

$$\min_x J(x) \tag{2.11}$$

s.t.

$$J(x) = \sum_{i=1}^{p} h_i(x). \tag{2.12}$$

where $h_i : \mathbb{R}^n \longrightarrow \mathbb{R}$, $x \in \mathbb{R}^n$. For $i \in V$, h_i refers to agent i's local cost function that depends on x, the entire resource allocation vector. J is the overall cost function to be minimized. A cooperative approach for solving this problem in a distributed fashion using a subgradient technique was proposed in [31]. The agents work to minimize J by minimizing their respective local cost functions that are known, convex and nonsmooth. Each agent estimates x, by $x^i(k)$. Agent i communicates x^i to its neighbours over a directed connected time-varying network. Agent i updates x^i at every k step using x^j from its j neighbours and the subgradient information of its local cost function. This is done using:

$$x^i(k+1) = \sum_{j=1}^{p} a_j^i(k) x^j(k) - \lambda^i(k) d_i(k) \tag{2.13}$$

where a_j^i is the weight agent i gives to the information x^j that it receives from j, $\lambda^i > 0$ is the step size used by agent i and d_i is the vector of the subgradient of agent i's local cost at x^i. a_j^i is an element of a^i and $a^i \in \mathbb{R}^p$ $\forall k \geq 0$ is agent i's vector of weights, a stochastic vector, such that:

$$\sum_{j=1}^{p} a_j^i(k) = 1.$$

Through communication with their respective neighbours, [31] has shown that the agents reach consensus on their estimates of x and converge to a neighbourhood of x^*.

The combination of extremum seeking control with ideas from game theory (local replicator system proposed in [32]) was proposed in [2] to solve resource allocation problems of MAS with stable dynamics. The goal is to find the optimal resource allocation that maximizes a smooth unknown overall cost function. It requires the collaboration of each agent with its neighbours over an undirected connected communication network $G = (V_G, E_G)$. Let $V_G = \{1, \ldots, p\}$ be the vertex set of the agents and $E_G \subset V_G \times V_G$ be the edge set respectively. $\forall i,\, j \in V_G$, let $N_i = \{j \colon (1, j) \in E_G\}$ be the set of agent i's neighbours. With the simplex $\Delta_X = \{x_i \geq 0 \colon \sum_{i=1}^{p} x_i = X\}$, [2] considered the problem,

$$\begin{aligned} &\max J(\theta) \\ s.t. \qquad & x_i \in \Delta_X \quad \text{and} \quad \dot{\theta}_i = g_i(\theta_i, x_i). \end{aligned} \tag{2.14}$$

$J(\theta) = \sum_{i=1}^{p} y_i$ and $y_i = f_i(\theta_i)$. $J \in \mathbb{R}$ is the unknown overall cost to be maximized, $y_i \in \mathbb{R}$ is agent $i's$ unknown local cost. $\theta_i \in \mathbb{R}$ is agent i's internal state, X is the limited resource to be shared among the agents and x_i is agent i's share of X. $f_i \colon \mathbb{R} \longrightarrow \mathbb{R}$ and $g_i \colon \mathbb{R} \times \Delta_X \longrightarrow \mathbb{R}$. The schematic representation of the system for agent i as found in [2] is shown in Figure 2.2. $a > 0$ is the amplitude of the excitation signal, $D > 0$ is a positive constant to be assigned, $\omega > 0$ is the frequency of the excitation signal, $\omega_L > 0$ is the frequency of the low pass filter and $k > 0$ is the adaptation gain. γ_i is the output of the low pass filter for the cost y_i.

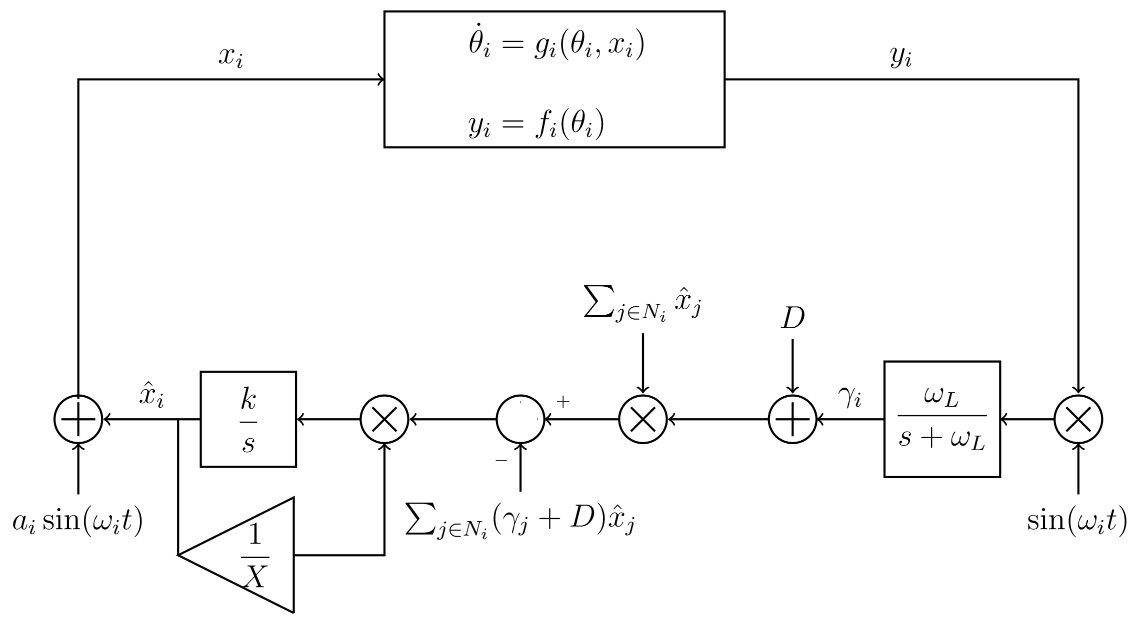

Figure 2.2: Schematic representation of the system for agent i [2].

The local replicator system applied is of the form:

$$\dot{\hat{x}}_i = k\frac{\hat{x}_i}{X}\left((\gamma_i + D)\sum_{j\in N_i}\hat{x}_j - \sum_{j\in N_i}(\gamma_j + D)\hat{x}_j\right) \tag{2.15}$$

where $\dot{\gamma}_i = -\omega_L\left(\gamma_i - f_i(\theta_i)\sin(\omega t)\right)$. Each agent i communicates its information with its neighbours and receives its neighbours' information to reach x_i^*. Under suitable assumptions, convergence of the system to a neighbourhood of the optimum was proven.

2.2.2 Source Seeking

ESC has been applied to the problem of source seeking. One such problem was considered in [33] for a single agent system where no information about the agent's position or velocity is available but the agent has access to the measurement of an unknown nonlinear function that represents the spatial distribution of a signal field. This unknown function has a local maximum that is desirable so the agent wants to maximize the unknown cost to the unknown local maximum using extremum seeking control . In the plane, the agent's position (x, y) and velocity inputs (v_x), (v_y) are of the form:

$$\dot{x} = v_x, \qquad \dot{y} = v_y. \tag{2.16}$$

The unknown function to be maximized is of the form:

$$J = f(x, y) = f^* - q_x(x - x^*)^2 - q_y(y - y^*)^2 \tag{2.17}$$

where $q_x, q_y > 0$ are constants to be specified, (x^*, y^*) is the maximizer of f and f^* the maximum. The ESC employed for the agent is represented as:

$$\begin{aligned} v_x &= a\omega\cos(\omega t) + C_x\xi\sin(\omega t) \\ v_y &= a\omega\sin(\omega t) - C_y\xi\cos(\omega t) \\ \xi &= \frac{s}{s+h}[J] \end{aligned} \tag{2.18}$$

where $a > 0$ is the amplitude of the excitation signal, ω is the frequency of the excitation signal, C_x, $C_y > 0$ are positive constants to be specified and ξ is the output signal of the high pass filter for the cost.

The multi-agent version of problem (2.16) and (2.17) was tackled in [34] using stochastic extremum seeing control in an uncooperative fashion. It was shown that the multi-agent system converge to a Nash equilibrium. The approach proposed in [34] considered random perturbations as the excitation signal in the ESC scheme. In this scheme, the effect of agent interactions was accounted for by providing each agent with a measurement of its distance to its neighbours. Let $V = \{1, \ldots, p\}$, then $\forall i \in V$, (2.16) and (2.17) becomes (2.19) and (2.20) respectively.

$$\dot{x}_i = v_{xi}, \qquad \dot{y}_i = v_{yi} \tag{2.19}$$

$$f_i(x_i, y_i) = f^* + q_x(x_i - x^*)^2 + q_y(y_i - y^*)^2 \tag{2.20}$$

Let N_i be the set of agents that interact with agent i, then the distance $d_{ij}(x, y)$ between agent i and j is represented as:

$$d_{ij}(x, y) = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}. \tag{2.21}$$

At this point, the local cost for agent i becomes:

$$J_i = f_i + \sum_{j \in N_i} w_{ij} d_{ij}^2 \tag{2.22}$$

where w_{ij} is the weight that agent i gives to the distance it measures from j.

The objective is to use stochastic extremum seeking to solve the optimization problem

$$\min_{x_i, y_i} J_i \tag{2.23}$$

This problem was tackled in [34] following a leader-follower approach. The proposed ESC for agent i is of the form:

$$\begin{aligned}
v_{xi} &= a\dot{\zeta}_{1i} + C_x \xi_i \zeta_{1i} + \nu_{xi} \\
v_{yi} &= a\dot{\zeta}_{2i} + C_y \xi_i \zeta_{2i} + \nu_{yi} \\
\xi_i &= \frac{s}{s+h}[J_i] \\
\zeta_{1i} &= \cos(B_i(\frac{t}{\epsilon})) \\
\zeta_{2i} &= \sin(B_i(\frac{t}{\epsilon}))
\end{aligned} \tag{2.24}$$

where B_i is the 1-dimensional Brownian motion for agent i. The terms ν_{xi} and ν_{yi} are added to enlarge the area where the agents are deployed. Additionally, these terms are such that ν_{xi}, $\nu_{yi} = 0$ when agent i is a follower agent and ν_{xi}, $\nu_{yi} \neq 0$ when it is an anchor agent.

2.2.3 Formation Control

Distributed extremum seeking control has been applied to formation control problems. In a dynamic or an uncertain environment, it is known that changes in the environment can affect communication quality and control of the agents so the use of non adaptive techniques will only result in sub-optimal separation distance among the agents. Problems of this form have been tackled in [35] and [36] in a cooperative fashion in real time using distributed extremum seeking control where the goal is to find the optimal separation distance between agents by maximizing a cost function.

2.2.4 Stabilization and Optimization

The use of a dynamic average consensus estimator and a proportional integral extremum seeking control technique for optimization and stabilization have been studied. In [27], the distributed optimization of MAS with unknown dynamics was studied while stabilization of unstable dynamics and distributed optimization of MAS was proposed in [37]. The distributed optimization process requires the agents to share information with their neighbours over an undirected connected communication network G. Let $G = (V_G, E_G)$ where $V_G = \{1, \ldots, p\}$ is the vertex set and $E_G \subset V_G \times V_G$ is the edge set. p is the number of agents in the network and $N_i = \{j \in V_G \colon (i, j) \in E_G\}$ is the set of agent i's neighbours $\forall i \in V_G$. In [27] and [37], the following problem is considered

$$\min J(x) = \sum_{i=1}^{p} f_i(x) \tag{2.25}$$

s.t.

$$\dot{x}_i = p_i(x) + q_i(x)u \tag{2.26}$$

$$y_i = f_i(x) \tag{2.27}$$

where $J(x)$ is the unknown overall cost function to be minimized and $y_i = f_i(x)$ is the unknown local cost function for agent i, $x \in \mathbb{R}^n$ is the vector of state variables, $u \in \mathcal{U} \subset \mathbb{R}^p$ is the vector of input variables, $p_i(x)$ and $q_i(x)$ are smooth vector valued functions of x.

As reported in [27] and [37], the solution to this problem requires:

- agent i to estimate the average of the overall cost ($\hat{J}_i$), agent i does this by communicating with its neighbours N_i over the network G and a dynamic average consensus estimator such as the one proposed in [38] ensures that the agents reach consensus on this estimate;

- the dynamics of this cost is parametrized to obtain agent i estimate of the gradient of the overall cost. This is done using a parameter estimation technique such as the one proposed in [26];

- with agent i's estimate of the gradient of the overall cost, a distributed proportional-integral extremum seeking controller is designed to reach the steady state unknown optimum of the overall cost.

The dynamic average consensus estimator for the problem is represented as:

$$\begin{bmatrix} \dot{\hat{J}} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} -I\gamma - k_P L & k_I L^T \\ -k_I L & 0 \end{bmatrix} \begin{bmatrix} \hat{J} \\ w \end{bmatrix} + \begin{bmatrix} I\gamma \\ 0 \end{bmatrix} f(x) \tag{2.28}$$

where $[\hat{J}, \omega]^T \in \mathbb{R}^{2p}$ is the internal estimator state and L is the Laplacian matrix for the network, k_P, k_I and $\gamma > 0$ are positive constant to be chosen. The distributed proportional-integral extremum seeking control is given by:

$$\begin{aligned} u_i &= -k_g \hat{\theta}_{1,i} + \hat{u}_i + d_i(t) \\ \dot{\hat{u}}_i &= -\frac{1}{\tau_I} \hat{\theta}_{1,i} \end{aligned}$$

where $k_g > 0$ is the proportional gain constant and $\tau_I > 0$ is the integral gain constant, $\hat{\theta}_{1,i}$ and $d_i(t)$ are agent i's estimate of the gradient of the unknown overall cost and bounded dither signal respectively.

Fast convergence to the unknown optimum was reported in [27] and [37]. Stabilization over short times was shown in [37].

2.3 Summary

In this chapter, the basic idea and the background of extremum seeking control was introduced. Some of the contributions made to address the challenges associated with ESC were established and a review of some developed ESC techniques for performance improvement was made. Also, some of the works on control of MAS in a distributed approach using ideas from game theory and extremum seeking control were presented.

Works on resource allocation problems of MAS were reviewed and results show that such problems can be tackled effectively in a distributed fashion to yield optimal results. In this chapter, some of contributions made in an attempt to tackle source seeking problems of MAS in an uncooperative manner using stochastic extremum seeking control was presented. Results show convergence to a Nash equilibrium. It has been established that formation control problems involving MAS have been solved using distributed extremum seeking control to find the optimal separation distance between agents while maximizing an unknown cost. Also, extremum seeking control has been shown to be effective in addressing MAS problems when it comes to optimizing systems with unknown dynamics and stabilizing systems with unstable dynamics.

Chapter 3

Distributed Extremum Seeking Control of Wind Farms

3.1 Introduction

Wind turbines extract energy from the wind for power generation and they can be located separately or grouped (wind farms) but are preferred grouped because of the reduced average cost of energy due to economy of scale [39]. In a wind farm - a group of wind turbines over an area of land (on shore) or water body (off shore), an upstream turbine generates a wake and as a result reduces the wind speed and increases the turbulence of any turbine downstream from it. Because of this wake interference the downstream turbine generates less power than is expected. Grouping brings into the picture the challenge of aerodynamic interaction leading to suboptimal power capture compared to the power that can be generated with the same number of isolated wind turbines. Control techniques employed in maximizing the power generated by a single wind turbine is inefficient for maximizing the power capture of a group of turbines. Currently, the installation of an array of wind turbines is increasing so the search for control algorithms that can help increase wind farm power capture is needed. Efforts have been made to address the effect of aerodynamic interactions among turbines. Techniques for selecting and positioning of turbines at strategic locations under fixed wind inflow conditions was proposed in [40] and under varying wind conditions in [41]. Wake models have been developed as found in [42], [43], [44] and [45]. Currently, few models exist to describe the aerodynamic wake interactions. Even the most advanced and computationally expensive models fail to accurately describe the interactions, so the use of control algorithms that are model independent is desirable. Wind farm power maximization problems have been tackled using model free control approaches in a distributed manner.

Consider a connected communication graph $G = (V_G, E_G)$. $V = \{1, \ldots, p\}$ is the vertex set representing the set of wind turbines in a wind farm, $E_G \subset V_G \times V_G$ is the edge set and p represent the number of wind turbines. Each wind turbine $i \in V$ has its input $a_i \in \mathcal{A}_i$ which is the turbine's control parameter referred to as the axial induction factor. The axial induction factor a_i is the relative decrease in wind speed from the free stream to the turbine's rotor plane. Let $\mathcal{A}_i = \{a_i \colon 0 \leq a_i \leq 0.5\}$ be turbine i's discretized set of allowable axial induction factors and $a = (a_1, a_2, \ldots, a_p) \in \mathcal{A}$ be the joint axial induction factors of the turbines. The set $\mathcal{A} = \prod_{i \in V} \mathcal{A}_i$ is the Cartesian product of sets $\mathcal{A}_i$ which is the set of allowable joint axial induction factors. Under the assumption of uniform wind where the free stream wind speed and direction are constant with respect to time, using ideas from game theory, [46] considered the problem of finding a^* that maximizes $J(a)$. That is:

$$\begin{aligned} a^* &\in \arg\max_{a \in \mathcal{A}} J(a) \\ J(a) &= \sum_{i=1}^{p} h_i(a). \end{aligned} \tag{3.1}$$

$h_i(a)$ is the unknown local cost function for the power produced by turbine i and it takes the form seen in [39]. $J(a)$ is the unknown overall cost function for all the turbines which represents the total power produced in the wind farm. $h_i(a)$ depends on the wind speed seen at turbine i with the wind speed represented using park model that can be found in [39].

The authors proposed two model-free control algorithms for solving (3.1). The first is the safe experimentation dynamics (SED) distributed algorithm and the second is the payoff-based distributed learning for Pareto optimality (PDLPO). Using SED distributed algorithm requires turbine i to have access to the power produced by all the turbines in the wind farm. This is to say that $\forall t > 0$, $a_i(t)$ relies on $a_i(\tau)$ and $J(a(\tau))$ where $\tau \leq t - 1$. Restricted communication between agents is required with the payoff-based distributed learning for Pareto optimality. Since $a_i(t)$ relies on $a_j(\tau)$ and $h_i(a(\tau))$, turbine i is only required to communicate with its neighbours in the set $N_i = \{j \in V_G \colon (i, j) \in E_G\}$. As reported in [46], these decentralized algorithms provide convergence (in the probabilistic sense since turbine i lacks the knowledge of the functional form of its local cost) to a^* that maximizes the power capture of the wind farm.

The combination of dynamic average consensus estimation and perturbation based extremum seeking control has been implemented in a distributed fashion for wind farm power maximization [3]. Cooperative control was considered where each wind turbine (agent) could exchange information with its neighbours over an undirected network with the goal of maximizing the unknown overall cost function. The ability of the agents to work together to maximize the overall cost requires each agent to first maximize its local cost. With the communication network G described above, the problem considered in [3] is as follows:

$$\max_{u \in \mathbb{R}^p} J(u) \tag{3.2}$$

where

$$J(u) = \sum_{i=1}^{p} h_i(u). \tag{3.3}$$

u is the input vector for the agents, $h_i(u)$ is the unknown local cost function for the power produced by agent i, it depends on u. $J(u)$ is the unknown overall cost function, it represents the total power produced by the agents in the wind farm.

The model of the wind farm employed is found in [46]. The solution to (3.2) requires the solution to a consensus estimation problem. Agent $i \in V_G$ through communication with its neighbours is required to estimate the average $(\hat{J}_i)$ of the overall cost. That is

$$\hat{J}_i = \frac{1}{p} \sum_{i=1}^{p} h_i(u). \tag{3.4}$$

For agent i to track $(\hat{J}_i)$, a dynamic average consensus estimator is required. The estimator proposed in [3] for solving (3.2) is similar to that shown in (2.28) and takes the form:

$$\begin{bmatrix} \dot{\hat{J}} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} -I - \rho L_P & -L_I^T \\ L_I & 0 \end{bmatrix} \begin{bmatrix} \hat{J} \\ w \end{bmatrix} + \begin{bmatrix} I \\ 0 \end{bmatrix} h(u) \tag{3.5}$$

where L_I and L_P are Laplacian matrices of G and $\rho > 0$.

Figure 3.1 shows the schematic representation of the system for agent i. Convergence of the algorithm to a neighbourhood of the maximizer of J was shown.

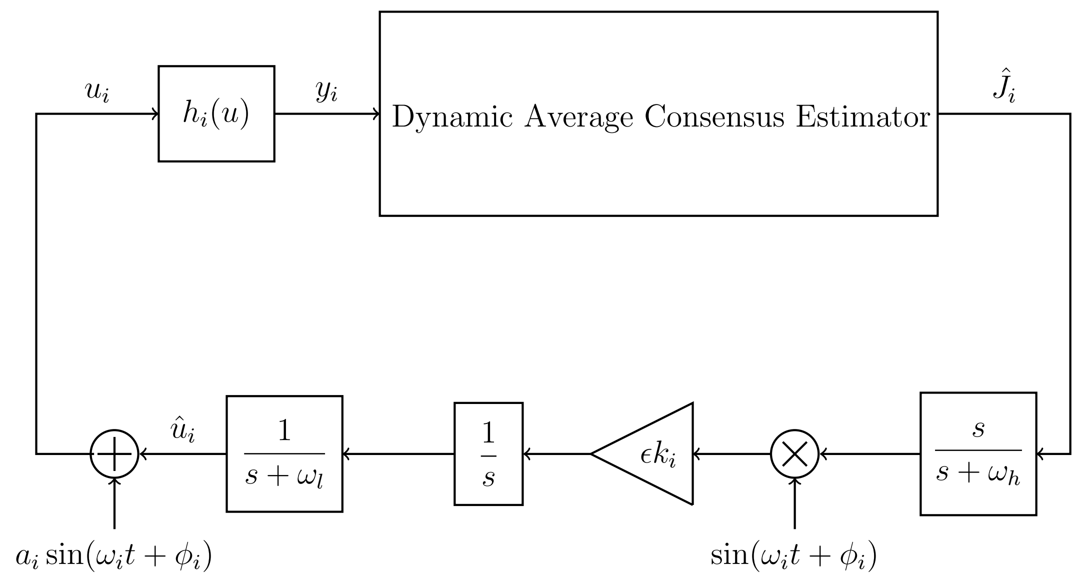

Figure 3.1: Schematic representation of the control algorithm for agent i [3].

The control algorithm is described in three steps:

- agent i takes its input u_i then measures its unknown local cost y_i, it is important to note that an excitation signal is added to the input to sufficiently excite the measured cost so that its gradient information can be obtained;
- this output signal y_i is sent into the dynamic average consensus estimator where agent i communicates with its neighbours and $\hat{J}_i$ is estimated;

- finally, $\hat{J}_i$ is fed into the extremum seeking loop where the estimate of the gradient is obtained and u_i produced.

In this chapter, a wind farm power maximization problem is addressed in a distributed fashion as mentioned in Chapter 1 using a time-varying extremum seeking control (TVESC) technique. Each wind turbine (identified as an agent) has access to the measurement of its unknown local power generation. The goal is to maximize the unknown wind farm power generation. A cooperative approach is employed where the agents work together to maximize this overall objective function. This approach requires every agent to exchange information with its neighbours over an undirected connected communication network. Each agent is also expected to estimate the mean of the overall cost. This task is achieved using a dynamic average consensus estimator to be presented in this chapter. The dynamics of each agent's cost estimate is parametrized and a parameter estimation routine is used to estimate the gradient of this cost. A distributed extremum seeking controller is designed to ensure that the overall cost is maximized. This problem is tackled via numerical simulations and results in continuous-time and discrete-time are provided.

3.2 Power maximization in continuous-time

3.2.1 Problem Description

Consider the following maximization problem,

$$\max_{u \in \mathbb{R}^p} J(u) \tag{3.6}$$

$$J(u) = \sum_{i=1}^{p} h_i(u). \tag{3.7}$$

A network of p agents is considered. Let $i = 1, \ldots, p$, u be the vector of input variables (axial induction factors for all agents) taking values in $\mathcal{U} \subset \mathbb{R}^p$, $J : \mathbb{R}^p \longrightarrow \mathbb{R}$ is the variable to be maximized referred to as the overall cost function (unknown wind farm power generation) and depends on u, $h_i : \mathbb{R}^{p_i} \longrightarrow \mathbb{R}$ is the unknown local cost (unknown local power generation) for agent i. The functions J and h_i are assumed to be smooth. To tackle this problem, the following assumptions are made.

Assumption 1: Agent i has access to the variable u_i and uses it to measure the numerical value of its unknown local cost.

Assumption 2: Let u^* be the unique maximizer of problem (3.6), the total cost function J is such that:

$$(u - u^*)^T \frac{\partial J(u)}{\partial u} \leq -\gamma_1 \|u - u^*\|^2 \tag{3.8}$$

$\forall u \in \mathcal{U}$ with $\gamma_1 > 0$.

Remark 1: Since J is smooth, then it satisfies:

$$\|J\| \leq \gamma_2 \quad \|\tfrac{\partial J}{\partial u}\| \leq \gamma_3 \quad \|\tfrac{\partial^2 J}{\partial u \partial u^T}\| \leq \gamma_4 \tag{3.9}$$

$\forall u \in \mathcal{U}$ with γ_2, γ_3, $\gamma_4 > 0$.

3.2.2 Communication Network

Assumption 3: The communication network for the agents is illustrated using a time-invariant undirected connected graph $G = (V_G, E_G)$.

$V_G = \{1, \ldots, p\}$ is the vertex set representing the set of agents in the wind farm and $E_G \subset V_G \times V_G$ is the edge set. An edge acts as the communication route between two agents. Agents i and j can exchange information means $(i, j) \in E_G \Leftrightarrow (j, i) \in E_G$ and $(i, j) \notin E_G \Leftrightarrow (j, i) \notin E_G$ means otherwise. $\forall i,\ j \in V_G\colon N_i = \{j\colon (i, j) \in E_G\}$. N_i represents the set of agent i's neighbours.

3.2.3 Wind Farm Model

The wind farm is modelled with reference to [46]. Consider an offshore wind farm with p number of agents and a matrix $C \in \mathbb{R}^p \times \mathbb{R}^p$ associated with its layout. In this wind farm, the agents are located at coordinates $\{(y_1, x_1), \ldots, (y_p, x_p)\}$ from a common point. The agents are assumed to be identical such that they have the same diameter. It is assumed that agent i is downstream from agent j. The following assumptions about the wind is made.

Assumption 4: Uniform wind with constant direction and free stream wind speed.

Assumption 5: The wind farm is oriented and wind is blowing in the positive horizontal direction.

Power Model

The power captured by agent i in the wind farm is given as:

$$h_i(u) = \frac{1}{2}\rho_{air} a_i C_p(u_i) V_i(u)^3 \tag{3.10}$$

where:

- ρ_{air} refers to the density of air $= 1.225\ kg/m^3$;
- a_i is the area of disk generated by the blades of agent i in m^2;
- $C_p(u_i)$ is the power efficiency coefficient;
- $V_i(u)$ is the aggregate wind speed seen at agent i in (m/s).

Power Efficiency Coefficient

The power efficiency coefficient $C_p(u_i)$ is of the form:

$$u_i(1-u_i)^2. \tag{3.11}$$

Wake Interaction Model

The aggregate wind speed seen at agent i is represented as:

$$V_i(u) = V_\infty\Big(1 - \delta V_i(u)\Big) \tag{3.12}$$

where:

- V_∞ is the free stream wind speed in (m/s);
- $\delta V_i(u)$ is the aggregate wind speed deficit seen at agent i and is given as

$$\delta V_i(u) = 2\sqrt{\sum_{j\in N_i:\, x_j<x_i} \Big(u_j C[j,i]\Big)^2}. \tag{3.13}$$

$$C[j,i] = \left(\frac{\bar{d}_j}{\bar{d}_j + 2k(x_i - x_j)}\right)^2 \frac{a^{\text{overlap}}_{j \longrightarrow i}}{a_i} \tag{3.14}$$

where:

- $\bar{d}_j$ is the diameter in m of the disk generated by the blades of agent j;
- $k = 0.04$ is the roughness coefficient for offshore locations, it defines the slope at which the wake expands out from the agent [46];
- x_i is agent i's distance in m from a common point in the wind direction;
- x_j is agent j's distance in m from this common point in the wind direction;
- $a^{\text{overlap}}_{j \longrightarrow i}$ is the area of overlap between the wake generated by agent j and the disk generated by the blades of agent i

Assumption 6: Agent i is in the wake of agent j such that there is no overlap. That is $a^{\text{overlap}}_{j \longrightarrow i} = a_i$.

With this assumption, (3.14) becomes:

$$C[j,i] = \left(\frac{\bar{d}_j}{\bar{d}_j + 2k(x_i - x_j)}\right)^2. \tag{3.15}$$

The wind speed seen at agent i takes into account the effect of aerodynamic wake interaction among agents. (3.12) can be written as:

$$V_i(u) = V_\infty\Big(1 - 2\sqrt{\sum_{j \in N_i:\, x_j < x_i} \big(u_j C[j,i]\big)^2}\Big) \tag{3.16}$$

3.2.4 Distributed Extremum Seeking Control

Consensus Estimation

The goal of agent i is to obtain an estimate of the overall cost in the form of the average cost, $\hat{J}_i = \frac{1}{p}\sum_{i=1}^{p} h_i$. This will be tackled using the proportional-integral dynamic average consensus estimator proposed in [38]. $\hat{J}_i$ is updated using agent i's local cost h_i and the available information from its neighbouring agents ($\hat{J}_{j\in N_i}$ from the other agents it communicates with). Consider the communication network above, let $A \in \mathbb{R}^{p\times p}$ be an adjacency matrix with elements a_{ij} equals 1 provided information can flow between i and j and 0 otherwise but a_{ii} must be zero. Let $D \in \mathbb{R}^{p\times p}$ be a degree matrix with zeros everywhere but the diagonals (the diagonals contain the number of neighbours agent i has). The Laplacian matrix $L \in \mathbb{R}^{p\times p}$ is defined as $L = D - A$. Let $\hat{J} = [\hat{J}_1, \ldots, \hat{J}_p]^T$ then the estimator takes the form:

$$\begin{bmatrix} \dot{\hat{J}} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} -I\gamma - k_P L & k_I L^T \\ -k_I L & 0 \end{bmatrix} \begin{bmatrix} \hat{J} \\ w \end{bmatrix} + \begin{bmatrix} I\gamma \\ 0 \end{bmatrix} h(u) \tag{3.17}$$

where $[\hat{J}, \omega]^T \in \mathbb{R}^{2p}$ is the internal estimator state, k_P, k_I and $\gamma > 0$ are positive constants to be assigned.

Parametrization of the local cost dynamics of agent i

The dynamics of the overall cost is given as:

$$\dot{J} = \dot{u}^T \frac{\partial J(u)}{\partial u} \tag{3.18}$$

where $\theta_1 = \frac{\partial J(u)}{\partial u} = [\theta_{1,1}, \ldots, \theta_{1,p}]^T$. The local cost dynamics for agent i is given by:

$$\frac{\dot{J}}{p} = \dot{u}_i \frac{\partial J(u)}{\partial u_i} \tag{3.19}$$

The local cost dynamics for agent i is parametrized as:

$$\frac{\dot{J}}{p} = \dot{u}_i \theta_{1,i}. \tag{3.20}$$

where $\theta_{1,i} = \frac{\partial J(u)}{\partial u_i}$. Recall that agent i's local cost depends on the inputs of other agents so this effect has to be accounted for. Let $\dot{\breve{u}}^T \breve{\theta} = \theta_{0,i} \in \mathbb{R}$ be the bias term accounting for the effect of the other agents and $\dot{J}_i = \frac{\dot{J}}{p}$ then (3.20) becomes:

$$\dot{J}_i = \theta_{0,i} + \dot{u}_i \theta_{1,i} \tag{3.21}$$

$\theta_{0,i}$ and $\theta_{1,i}$ are the time-varying parameters to be estimated. It is important to note that agent i has access to $\hat{J}_i$ not J_i so $\hat{J}_i$ is used instead of J_i in the estimation routine.

Parameter Estimation

The local parameters $\theta_{1,i}$ and $\theta_{0,i}$ for agent i are estimated using the parameter estimation routine found in [6]. Let the vector of parameter estimates be $\hat{\theta}_i = [\hat{\theta}_{0,i}, \hat{\theta}_{1,i}]^T$. The regressor vector is defined as $\phi_i = [1,\ \dot{u}_i]^T$. The predicted output for a given value of the estimate $\hat{\theta}_i$ is denoted by $\hat{z}_i$. The output prediction error is given by $e_i = \hat{J}_i - \hat{z}_i$. The prediction dynamics for (3.21) is written as:

$$\dot{\hat{z}}_i(t) = \phi_i^T \hat{\theta}_i(t) + K e_i(t) + c_i(t)^T \dot{\hat{\theta}}_i(t), \quad \hat{z}_i(0) = h_i(0) \tag{3.22}$$

where $K > 0$ is a constant to be specified. The dynamics of $c_i(t)$ is given by:

$$\dot{c}_i(t)^T = -Kc_i(t)^T + \phi_i^T(t), \quad c_i(0) = 0. \tag{3.23}$$

Let η_i be an auxiliary variable,

$$\eta_i(t) = e_i(t) - c_i^T(t)\tilde{\theta}_i(t) \tag{3.24}$$

where $\tilde{\theta}_i = \theta_i - \hat{\theta}_i$. Recall that agent i does not have access to θ_i but $\hat{\theta}_i$. This simply means that it only has an estimate of η_i. Let this estimate be $\hat{\eta}_i$, the dynamics of $\hat{\eta}_i$ is of the form:

$$\dot{\hat{\eta}}_i(t) = -K\hat{\eta}_i(t). \tag{3.25}$$

$\Sigma_i \in \mathbb{R}^{2\times 2}$ is a covariance matrix with dynamics

$$\dot{\Sigma}_i(t) = c_i(t)^T c_i(t) - K_T\Sigma_i(t) + \delta I, \quad \Sigma_i(0) = \alpha I > 0. \tag{3.26}$$

Its inverse is defined by the matrix differential equation

$$\dot{\Sigma}_i^{-1}(t) = -\Sigma_i^{-1}(t)c_i(t)c_i^T(t)\Sigma_i^{-1}(t) + K_T\Sigma_i^{-1}(t) - \delta\Sigma^{-2}(t) \tag{3.27}$$

where $\Sigma_i^{-1}(0) = \frac{1}{\alpha}I > 0$, α, δ and $K_T > 0$ are constants to be assigned. The proposed parameter update law is given as:

$$\dot{\hat{\theta}}_i = \mathrm{Proj}\Big(\Sigma_i^{-1}(t)\big(c_i(t)(e_i(t) - \hat{\eta}_i(t)) - \sigma\hat{\theta}_i(t)\big), \Theta\Big) \tag{3.28}$$

where $\sigma > 0$ and Θ is a ball of finite radius centered at 0. Proj$\{.,\Theta\}$ denotes a Lipschitz projection operator onto the set Θ [12]. The application of the projection algorithm is such that:

$$\hat{\theta}_i(0) \in \Theta \Longrightarrow \hat{\theta}_i(t) \in \Theta, \quad \forall t \geq 0. \tag{3.29}$$

The operator bounds $\hat{\theta}_i$ and ensures that:

$$\begin{aligned} &\tilde{\theta}_i^T \Sigma_i \text{Proj}\left(\Sigma_i^{-1}(t) \Big(c_i(t)(e_i(t) - \hat{\eta}_i(t)) - \sigma\hat{\theta}_i(t) \Big), \Theta \right) \\ &\qquad\qquad \leq \tilde{\theta}_i^T \Big(c_i(t)(e_i(t) - \hat{\eta}_i(t)) - \sigma\hat{\theta}_i(t) \Big) \end{aligned} \tag{3.30}$$

The signals of the closed-loop ESC system must be such that the following Persistence of Excitation condition is met.

Assumption 7: There exist constants α_1 and $T > 0$ such that

$$\int_t^{t+T} c_i(\tau) c_i(\tau)^T d\tau \geq \alpha_1 I, \quad \forall t > 0. \tag{3.31}$$

Distributed Extremum Seeking Controller Design

The distributed extremum seeking controller of the form (3.32) is proposed to solve the optimization task

$$\dot{u}_i(t) = k_g \hat{\theta}_{1,i}(t) + d_i(t) \tag{3.32}$$

where $k_g > 0$ is the optimization gain, $d_i(t)$ is the dither signal and is such that $\|d_i(t)\| \leq D_1$ $\forall t \geq 0$ and $D_1 > 0$. The schematic representation of the control algorithm for agent i is shown in Figure 3.2.

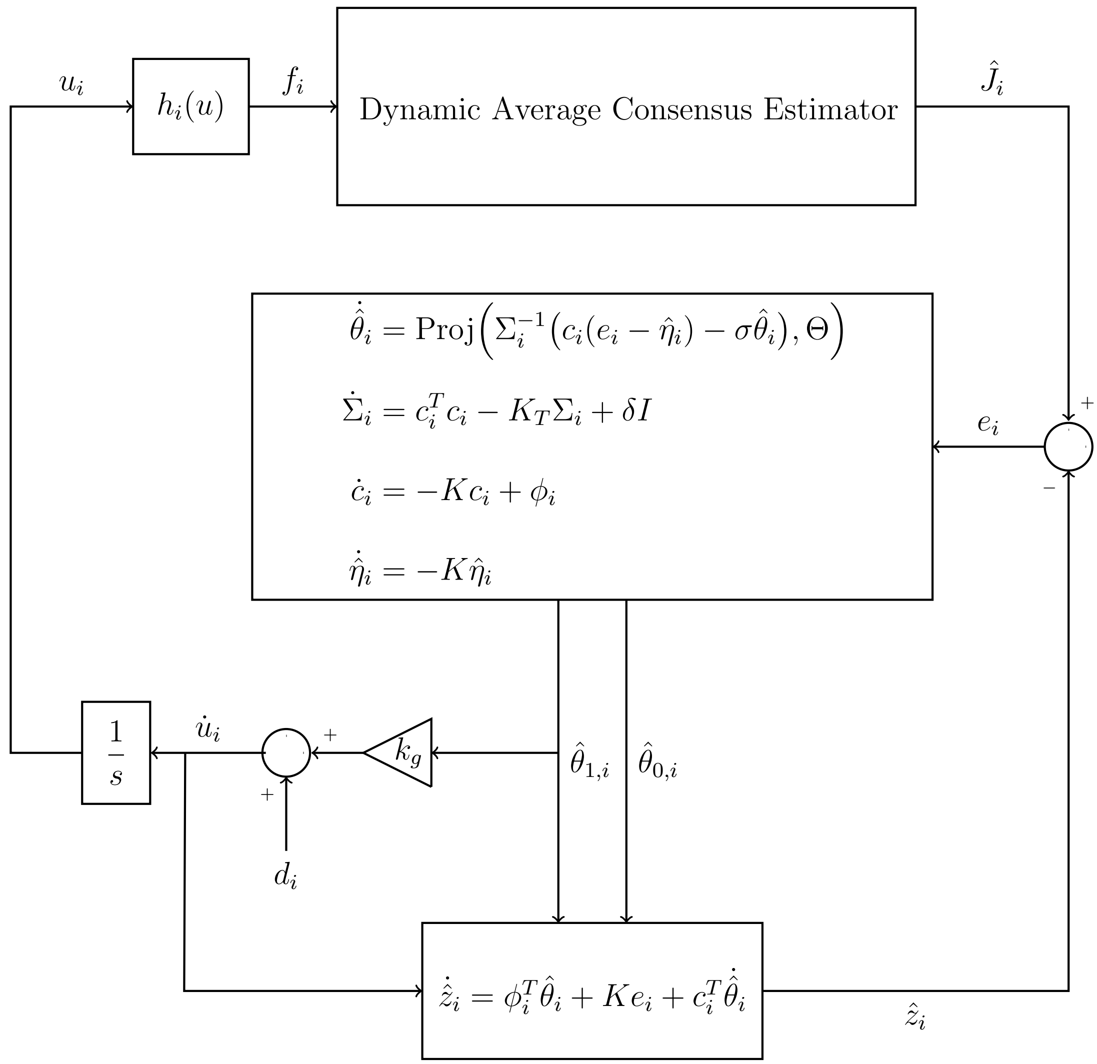

Figure 3.2: Schematic representation of the distributed control algorithm for agent i.

3.2.5 Simulation Examples

A three agent system

Consider a three agent system with agents located at coordinates $\{(0,0),(0,5\bar{d}),(0,10\bar{d})\}$ such that $x_j < x_i$ and $\bar{d} = 77$ m for all agents. The ESC tuning parameters were selected as: $\sigma = 1\times10^{-9}$, $K = K_T = 50$, $\delta = 4.5\times10^{-9}$, $\alpha = 1\times10^{-9}$, $k_g = 1$

$\times 10^{-3}$, $d(t) = 0.1\ [\sin(100t),\ \sin(130t),\ \sin(190t)]^T$. $\gamma = k_P = k_I = 10000$. The control algorithm was initiated at: $u_i(0) = 0.33$, $\hat{\theta}_i(0) = [0.01, 0]^T$, $c_i(0) = [0, 0]^T$ $\Sigma_i(0) = I_{2\times 2}$, $\hat{J}_i(0) = h_i(0)$, $w_i(0) = 0$ and $\hat{z}_i(0) = h_i(0)$.

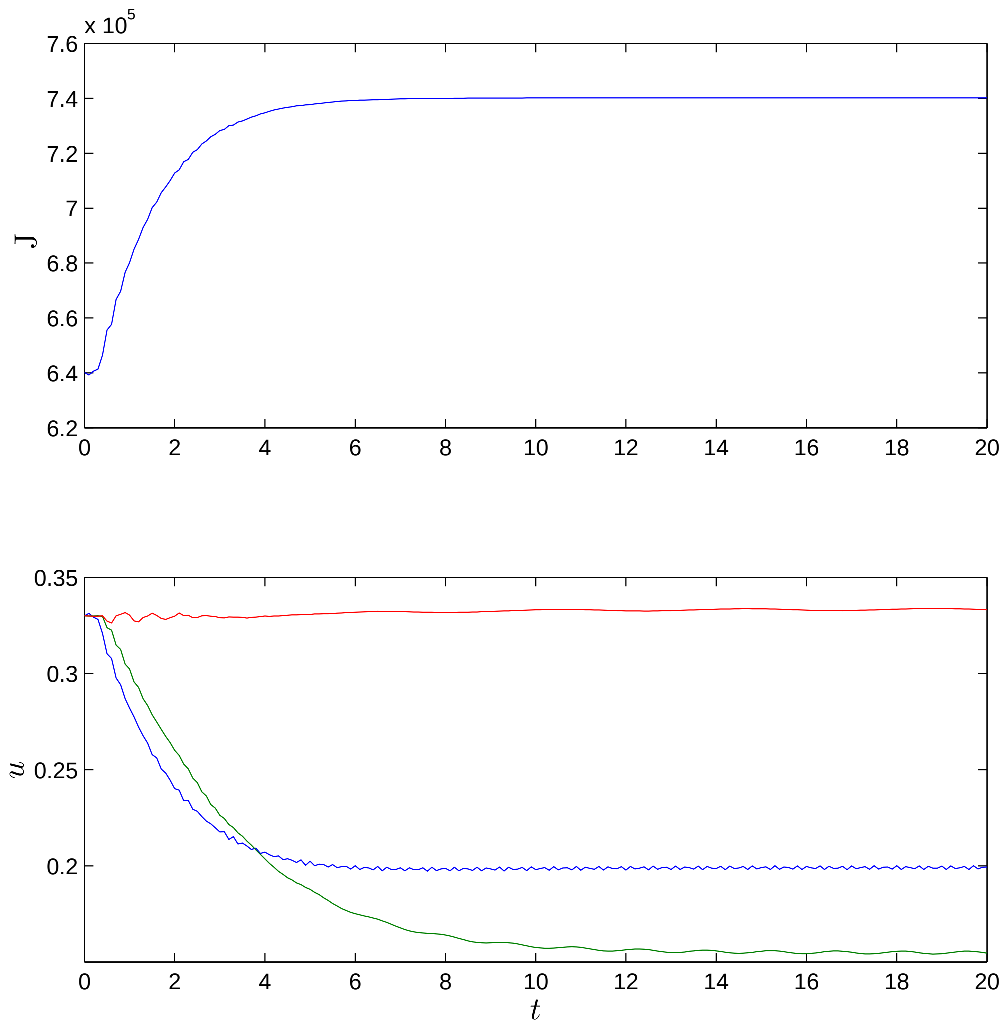

Figure 3.3: The overall cost and the inputs as a function of time for a three agent system using distributed TVESC technique.

D, A and L for this system are:

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad L = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Result using TVESC technique

The simulation result is shown in Figure 3.3. The unknown optimum power generation is $J^* = 7.4016 \times 10^5$ Watts with maximizer $u^* = [0.1990,\ 0.1548,\ 0.3333]^T$. It is known that a single isolated wind turbine is maximized at an axial induction factor of $\frac{1}{3}$. However, this may not be optimal for an array of wind turbines in a wind farm. The power captured by the third agent is maximized at $u_3^* = 0.3333$. This is expected as there are no other agents operating in its wake. From the results obtained, convergence to the unknown optimum is recorded in approximately 12 seconds.

Result using PBESC technique

In Figure 3.4, we reproduce the simulation result for this example as presented in [3] where the PBESC technique was used. The wind farm model described above is the same as the wind farm model used in [3] except for (3.16). The tuning parameters are found in [3]. From Figure 3.4, convergence to a neighbourhood of the optimum is observed in approximately 130 seconds. The transient behaviour is significantly slower than that observed for the distributed TVESC optimization technique.

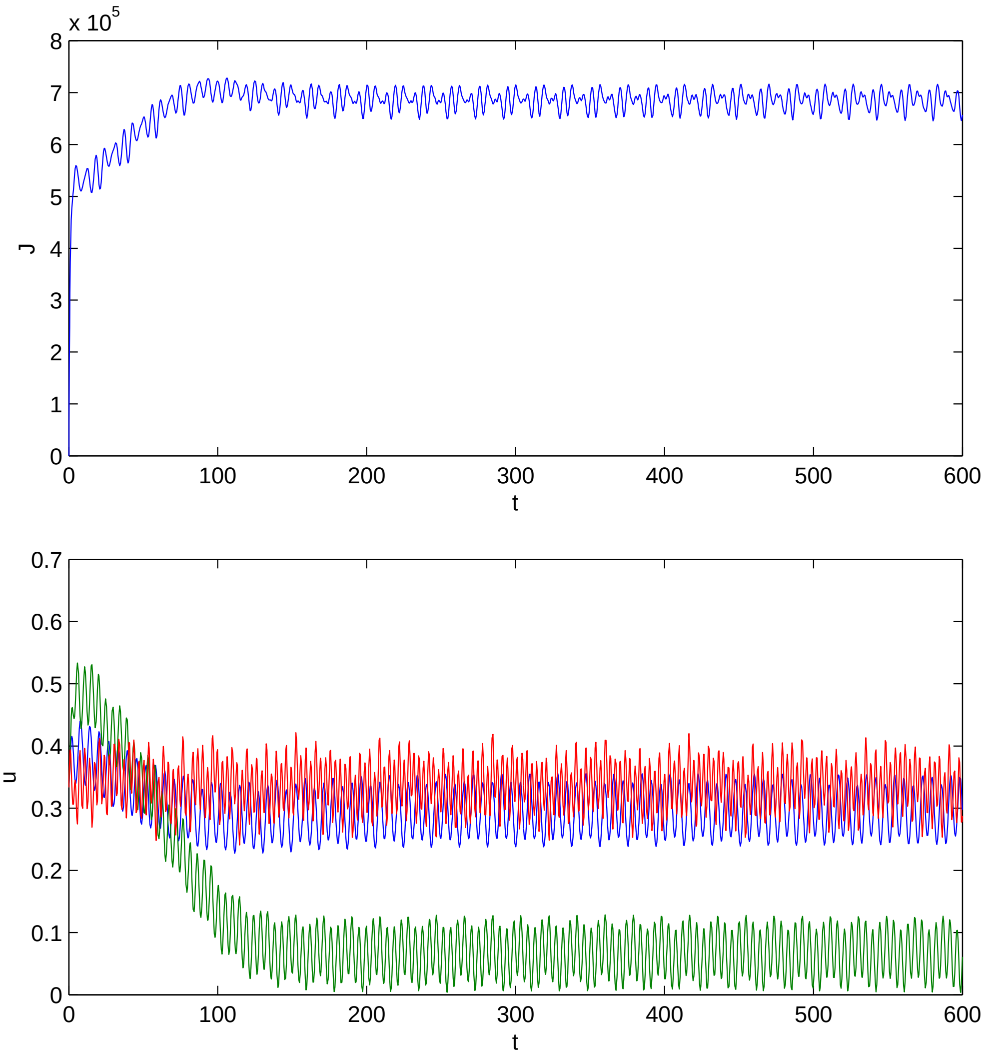

Figure 3.4: The overall cost and the inputs as a function of time for a three agent system using distributed PBESC technique.

3.3 Power maximization in discrete-time

In this section, the time-varying extremum seeking control technique proposed in [7] is generalized for the design of distributed optimization algorithms, as in the previous section, for discrete-time systems.

3.3.1 Problem Description

Consider the wind farm maximization problem below,

$$\max_{u_k \in \mathbb{R}^p} J(u_k) \tag{3.33}$$

$$J(u_k) = \sum_{i=1}^{p} h_i(u_k) \tag{3.34}$$

The same network described above is considered where $i = 1, \ldots, p$. u_k is the vector of input variables at the k^{th} time step taking values in $\mathcal{U} \subset \mathbb{R}^p$. J: $\mathbb{R}^p \longrightarrow \mathbb{R}$ is the variable to be maximized at the k^{th} time step referred to as the overall cost function and $h_i(u_k)$ is the unknown local cost function for agent i. J and h_i are assumed to be smooth functions of u_k. The objective is to find the u^* that maximizes (3.33). As in the previous section, assumptions 1, 2 and 3 are invoked. Similarly, the communication network and the wind farm model fulfill assumptions 4, 5 and 6.

3.3.2 Distributed Extremum Seeking Control

Consensus Estimation

For agent i to obtain its estimate of the overall cost, a discrete-time dynamic average consensus estimator is required. Let $\hat{J}_k = [\hat{J}_{1(k)}, \ldots, \hat{J}_{p(k)}]^T$. The discretization of the continuous-time estimator as seen in (3.17) yields an estimator of the form:

$$\begin{bmatrix} \hat{J}_{k+1} - \hat{J}_k \\ w_{k+1} - w_k \end{bmatrix} = \delta_1 \begin{bmatrix} -I\gamma - k_P L & -k_I L^T \\ -k_I L & 0 \end{bmatrix} \begin{bmatrix} \hat{J}_k \\ w_k \end{bmatrix} + \delta_1 \begin{bmatrix} I\gamma \\ 0 \end{bmatrix} h_k \tag{3.35}$$

(3.35) can be rewritten as:

$$\begin{bmatrix} \hat{J}_{k+1} \\ w_{k+1} \end{bmatrix} = \left(I + \delta_1 \begin{bmatrix} -I\gamma - k_P L & -k_I L^T \\ -k_I L & 0 \end{bmatrix} \right) \begin{bmatrix} \hat{J}_k \\ w_k \end{bmatrix} + \delta_1 \begin{bmatrix} I\gamma \\ 0 \end{bmatrix} h_k \tag{3.36}$$

where $[\hat{J}_k \ w_k]^T \in \mathbb{R}^{2p}$ represents the estimator's internal state, k_P, k_I, γ and δ_1 are positive constants. In what follows, $\Delta \hat{J}_k = \hat{J}_{k+1} - \hat{J}_k$.

Parametrization of the local cost dynamics of agent i

The overall cost dynamics is parametrized as:

$$\Delta J_k = \theta_{1(k)}^T \Delta u_k \tag{3.37}$$

where $\Delta J_k = J_{k+1} - J_k$, $\Delta u_k = u_{k+1} - u_k$ and $\theta_{1(k)} = \int_0^1 \frac{\partial J}{\partial u}(\lambda u_{k+1} + (1-\lambda)u_k)d\lambda = [\theta_{1,1}, \dots, \theta_{1,p}]^T$. Parametrization of the local cost dynamics of agent i takes the form:

$$\frac{\Delta J_k}{p} = \theta_{1,i(k)} \Delta u_{i(k)} \tag{3.38}$$

$\Delta J_{i(k)} = \frac{\Delta J_k}{p}$. Let $\Delta \breve{u}_k^T \breve{\theta}_k = \theta_{0,i(k)} \in \mathbb{R}$ be the bias term accounting for the effect of other agents on agent i, then (3.38) becomes:

$$\Delta J_{i(k)} = \theta_{0,i(k)} + \theta_{1,i(k)} \Delta u_{i(k)} = \phi_{i(k)}^T \theta_{i(k)} \tag{3.39}$$

where $\theta_{i(k)} = [\theta_{0,i(k)}, \theta_{1,i(k)}]^T$ is the vector of time-varying parameters that needs to be estimated and $\phi_{i(k)} = [1, \Delta u_{i(k)}]^T$ is the regressor vector. The estimation routine proposed in [7] is used.

Parameter Estimation

Let the vector of parameter estimates be $\hat{\theta}_{i(k)} = [\hat{\theta}_{0,i(k)},\, \hat{\theta}_{1,i(k)}]^T$, the regressor vector is $\phi_{i(k)} = [1, \Delta u_{i(k)}]^T$. The predicted output for a given value of the estimate $\hat{\theta}_{i(k)}$ is denoted by $\Delta\hat{z}_{i(k)}$. The output prediction error is given by $e_{i(k)} = \Delta\hat{J}_{i(k)} - \Delta\hat{z}_{i(k)}$. The estimator model for (3.39) becomes:

$$\Delta\hat{z}_{i(k)} = \phi^T_{i_{(k)}}\hat{\theta}_{i_{(k)}}. \tag{3.40}$$

Let $\Sigma_{i(k)} \in \mathbb{R}^{2\times 2}$ be a covariance matrix obtained from:

$$\Sigma_{i(k+1)} = \alpha_2\Sigma_{i(k)} + \phi_{i(k)}\phi^T_{i(k)}, \quad \Sigma_{i(0)} = \alpha_3 I > 0 \tag{3.41}$$

where α_2 and $\alpha_3 > 0$ are constants to be assigned. The parameter update law is as follows:

$$\Sigma^{-1}_{i(k+1)} = \frac{1}{\alpha_2}\Sigma^{-1}_{i(k)} - \frac{1}{\alpha_2^2}\Sigma^{-1}_{i(k)}\phi_{i(k)}\,(1 + \frac{1}{\alpha_2}\phi^T_{i(k)}\Sigma^{-1}_{i(k)}\phi_{i(k)})^{-1}\phi^T_{i(k)}\Sigma^{-1}_{i(k)} \tag{3.42}$$

where $\quad \Sigma^{-1}_{i(0)} = \frac{1}{\alpha_2}I,$

$$\hat{\theta}_{i(k+1)} = \mathrm{Proj}\left(\hat{\theta}_{i(k)} + \frac{1}{\alpha_2}\Sigma^{-1}_{i(k)}\phi_{i(k)}\,(1 + \frac{1}{\alpha_2}\phi^T_{i(k)}\Sigma^{-1}_{i(k)}\phi_{i(k)})^{-1}e_{i(k)}, \Theta_1\right) \tag{3.43}$$

where $\hat{\theta}_{i(0)} \in \Theta_1$ and Θ_1 is a ball centred at the origin. This projection operator is an orthogonal projection onto the surface of the uncertainty set applied to the parameter estimate. The application as seen in [47] of the projection algorithm is such that:

$$\hat{\theta}_{i(k+1)} \in \Theta_1, \quad \forall k \geq 0.$$

The signals of the closed-loop system must be such that the following Persistence of Excitation condition is met.

Assumption 8: There exist constants α_4 and $T_1 > 0$ [47] such that

$$\frac{1}{T_1}\sum_{m=k}^{k+T_1-1} \phi_{i(m)}\phi_{i(m)}^T > \alpha_4 I, \quad \forall k > T_1. \tag{3.44}$$

Distributed Extremum Seeking Controller Design

The distributed controller proposed to solve the extremum seeking task is of the form:

$$u_{i(k+1)} = u_{i(k)} + k_g\hat{\theta}_{1,i(k)} + d_{i(k)} \tag{3.45}$$

where $k_g > 0$ is the optimization gain, $d_{i(k)}$ is the dither signal for agent i and is such that $\|d_{i(k)}\| \leq D_2$ $\forall k \geq 0$ and $D_2 > 0$.

3.3.3 Simulation Examples

The three agent system

Consider the three agent system discussed earlier with agents located at the same coordinates, $x_j < x_i$ and $\bar{d} = 77$ m for all agents. The same network parameters D, A and L are considered. The tuning parameters were selected as: $\delta_1 = 1 \times 10^{-3}$, $\alpha_2 = 3 \times 10^{-6}$, $\alpha_3 = 1 \times 10^{-8}$, $k_g = 3 \times 10^{-4}$, $d_k = 1 \times 10^{-5}\,[\sin(20k),\ \sin(27k),$

$\sin(33k)]^T$. $\gamma = 50$, $k_P = k_I = 300$. The control algorithm was initiated at: $u_{i(0)} = 0.25$, $\hat{\theta}_{i(0)} = [0.001, 0]^T$, $\Sigma_{i(0)} = I_{2\times 2}$, $\hat{J}_{i(0)} = h_{i(0)}$ and $w_{i(0)} = 0$.

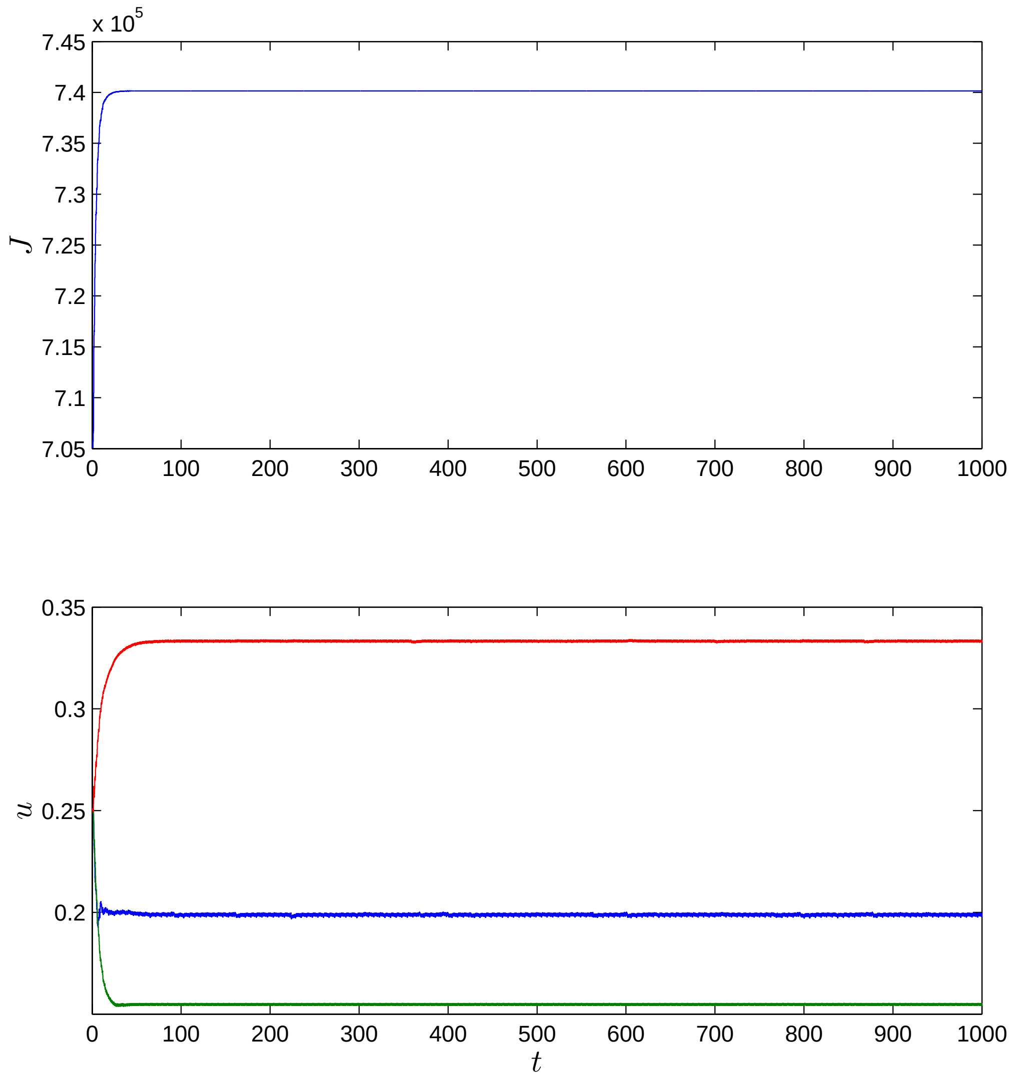

Figure 3.5: The overall cost and the inputs as a function of time for the three agent system.

Result

The simulation result for this example is seen in Figure 3.5. The following optimal conditions were obtained $J^* = 7.4016 \times 10^5$ Watts and $u^* = [0.1990,\ 0.1548,\ 0.3333]^T$. The distributed control algorithm is able to drive the system to the unknown optimum of the overall cost. Fast convergence to this optimum is observed.

A five agent system

Consider a five agent system with identical agents ($\bar{d} = 77m$). The agents are located at coordinates $\{(0,0), (0,5\bar{d}), (0,10\bar{d}), (0,15\bar{d}), (0,20\bar{d})\}$ such that $x_j < x_i$. D, A and L for this system are:

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad L = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

The tuning parameters were chosen as: $\delta_1 = 1 \times 10^{-3}$, $\alpha_2 = 3 \times 10^{-6}$, $\alpha_3 = 1 \times 10^{-8}$, $k_g = 2 \times 10^{-3}$, $d_k = 1 \times 10^{-5} [\sin(19k),\ \sin(25k),\ \sin(30k),\ \sin(34k),\ \sin(38k)]^T$. $\gamma = 30$, $k_P = k_I = 280$. The control algorithm was started off at: $u_{i(0)} = 0.1$, $\hat{\theta}_{i(0)} = [0.001, 0]^T$, $\Sigma_{i(0)} = I_{2\times 2}$, $\hat{J}_{i(0)} = h_{i(0)}$ and $w_{i(0)} = 0$.

Result

The simulation result is shown in Figure 3.6.The optimum for this problem is $u^* = [0.1893,\ 0.1390,\ 0.1479,\ 0.1606,\ 0.3333]^T$ and $J^* = 1.0603 \times 10^6$ Watts. This result confirms the effectiveness of the proposed distributed control technique.

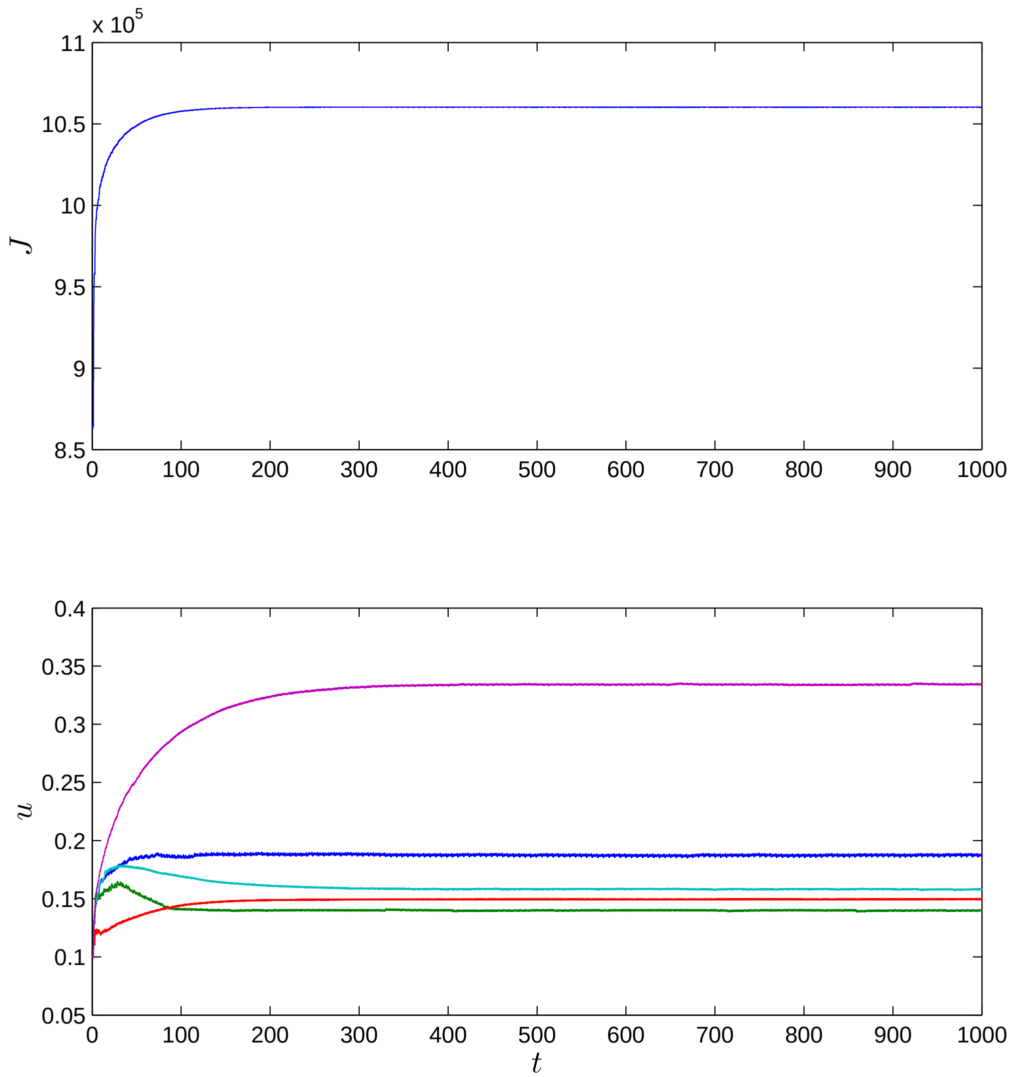

Figure 3.6: The overall cost and the inputs as a function of time for a five agent system.

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3.4 Conclusions

In this chapter, the wind farm power maximization problem was solved using a distributed time-varying extremum seeking control (TVESC) approach to be precise. This problem was tackled both in continuous and in discrete-time and useful results were obtained. Comparisons between the TVESC and the PBESC techniques demonstrated clearly the improvement in performance obtained with the TVESC. Most importantly, the results in this Chapter demonstrate the power of coordination and cooperation in the design of a distributed ESC system suitable for multi-agent systems with unknown or uncertain mathematical descriptions.

Chapter 4

Consensus Estimation and Distributed Extremum Seeking Control over Unknown Networks

4.1 Introduction

In this chapter, the solution of large-scale real-time optimization problems in the absence of precise knowledge of network connectivity is considered. In the proposed approach presented in this chapter, it is assumed that the communication network is unknown. In this environment, we consider the problem of ensuring that each agent minimizes the system's overall cost (i.e., the sum of the local cost of all the agents). Each agent has access to the measurement of two cost functions referred to as the local cost and the local disagreement cost, respectively. The objective is to optimize the system to the unknown optimum of the unknown overall cost which depends on the minimization of the local disagreement cost of all the agents. For the local disagreement cost functions to be minimized, dynamic consensus estimation is required. This ensures that the agents reach agreement on their inputs. To help tackle this challenging problem, a distributed proportional-integral extremum seeking control technique is proposed. This technique solves the distributed optimization problem by the simultaneous minimization of the local disagreement cost and the overall cost. A two-time scale approach is proposed in which the consensus can be achieved at a faster time-scale than the minimization of the overall cost. Two simulation examples are treated to show the effectiveness of this technique.

4.2 Notation

In the description of the distributed optimization approach, the following notation is adopted.

- The connected network or graph $G = (V_G, E_G)$.
- $V_G = \{1, \ldots, p\}$ is the vertex set with elements denoted as vertices or nodes. The vertices are referred to as agents.
- E_G is the edge set with elements called edges.
- p refers to the number of agents in the vertex set.
- $\forall i,\, j \in V_G, N_i = \{j \colon (i, j) \in E_G\}$ refers to the set of agent i's neighbours.
- $\mathbf{u}$ is the vector of input variables taking values in $\mathcal{U} \subset \mathbb{R}^p$.
- $\bar{u}_i$ is the vector of agent i's input and those of its neighbours. It takes values in $\mathcal{U}_i \subset \mathcal{U}$.
- $u_i \in \mathbb{R}$ is agent $i's$ local input (the decision variable).
- u is agent $i's$ local input after consensus is reached. That is u_i becomes u $\forall i$.
- d_{ij} is the weight of the shortest path from i to j considering all the possible path from i to j in the network.

4.3 Problem Description

Consider the following minimization problem

$$\min_{u} J(u) \tag{4.1}$$

s.t.

$$J(u) = \sum_{i=1}^{p} h_i(u) \tag{4.2}$$

$$u = \arg\min_{u_i} L_i(\bar{u}_i) \tag{4.3}$$

$$L_i(\bar{u}_i) = \sum_{j \in N_i} d_{ij}(u_i - u_j)^2 \quad \forall i \in V_G. \tag{4.4}$$

A network of p agents is considered. $J : \mathbb{R} \longrightarrow \mathbb{R}$ is the variable to be minimized referred to as the unknown overall cost function. It depends on u and is assumed to be an unknown smooth function of u. $h_i : \mathbb{R} \longrightarrow \mathbb{R}$ is the unknown local cost function for agent i. $L_i(\bar{u}_i) : \mathbb{R}^{p_i} \longrightarrow \mathbb{R}$ is agent i's unknown local disagreement cost function. It provides the measurement of the disagreement between agent i's input and those of its neighbours and is assumed to be an unknown smooth function of $\bar{u}_i$.

To proceed, the following assumptions are required.

Assumption 1: Agent i has access to the variable u_i and can measure h_i and L_i.

Assumption 2: $J(u)$ and $L_i(\bar{u}_i)$ are such that

$$(u - u^*)\frac{\partial J(u)}{\partial u} \geq \gamma(u - u^*)^2 \tag{4.5}$$

and

$$(\bar{u}_i - u)^T \frac{\partial L_i(\bar{u}_i)}{\partial \bar{u}_i} \geq \gamma_s \|\bar{u}_i - u\|^2 \tag{4.6}$$

$\forall \bar{u}_i \in \mathcal{U}_i$ with positive constants $\gamma > 0$ and $\gamma_s > 0$.

The total disagreement cost function $L = \sum_{i=1}^{p} L_i$ is an unknown smooth function of $\mathbf{u}$. $L : \mathbb{R}^p \longrightarrow \mathbb{R}$ and satisfies Assumption 3.

Assumption 3: $L(\mathbf{u})$ is such that

$$(\mathbf{u} - u)^T \frac{\partial L(\mathbf{u})}{\partial \mathbf{u}} \geq \gamma_d \|\mathbf{u} - u\|^2 \tag{4.7}$$

$\forall \mathbf{u} \in \mathcal{U}$ and $\gamma_d > 0$.

4.4 Distributed Extremum Seeking Control

4.4.1 Consensus Estimation

Two communication networks are considered. The first communication network for the agents is illustrated using a time-invariant undirected connected weighted graph while the second is represented as a time-invariant directed connected weighted graph. Each communication network is represented as $G = (V_G, E_G)$.

The networks are seen in Figures 4.1 and 4.2 respectively. Recall that a node represents an agent and an edge connects two agents and acts as the pathway for communication. Let $H \in \mathbb{R}^m$ be a vector of positive weights. Each edge connecting two agents is assigned a positive weight and the distance d_{ij} between agents i and j $\forall i, j \in V_G$ is computed. d_{ij} satisfies:

- $d_{ij} \geq 0 \qquad \forall i, j \in V_G$
- $d_{ij} = 0 \qquad \text{iff } i = j$
- $\forall i, j \in V_G, \quad d_{ij} = d_{ji}$ if the graph is undirected, otherwise it is directed.

$D_u \in \mathbb{R}^{p \times p}$ and $D_d \in \mathbb{R}^{p \times p}$ are distance matrices with elements d_{ij} for the undirected and the directed networks.

Recall that agent i has its input u_i, a consensus estimation approach is needed to ensure that $u_i \longrightarrow u_j \longrightarrow u$. Once all agents operate using the same value of the decision variable, it is possible to minimize the overall cost.

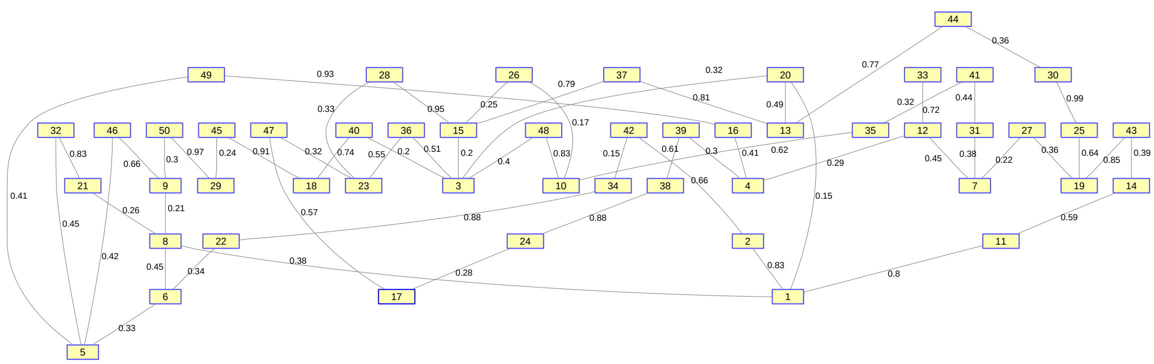

Figure 4.1: Undirected Graph G for a 50 agent system.

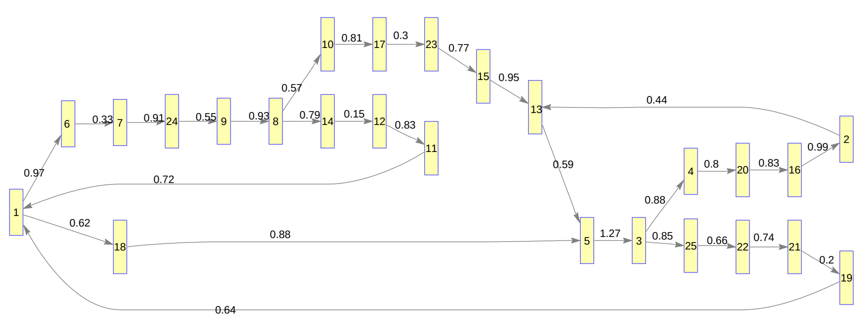

Figure 4.2: Directed Graph G for a 25 agent system.

To achieve this, a local disagreement cost function of the form (4.4) is introduced

$$L_i(\bar{u}_i) = \sum_{j \in N_i} d_{ij}(u_i - u_j)^2$$

where u_i and u_j are the inputs of agents i and j. Since agent i can communicate with its neighbours, L_i provides the measurement of the total disagreement between agent i and its neighbours. Agents i and j agree if and only if $u_i = u_j$ and they disagree otherwise. When i and j agree, no contribution is made to L_i, when they disagree, $L_i > 0$ and has to be minimized. This minimization ensures that $\forall i, j \in V_G$, the inputs achieve the consensus value u. The proposed distributed PI-ESC approach incorporates local disagreement functions and computes the minimizer u. The algorithm also ensures that u is driven to u^*, the minimizer of the overall cost, J.

4.4.2 Parametrization of agent i's local cost dynamics

The dynamics of the overall cost function J is given by:

$$\dot{J} = \frac{\partial J}{\partial u}\dot{u}. \tag{4.8}$$

Let $\theta_{J1} = \frac{\partial J}{\partial u}$, then $\dot{J} = \theta_{J1}\dot{u}$. Agent i's local cost dynamics is of the form:

$$\dot{h}_i = \dot{u}_i \frac{\partial h_i(u_i)}{\partial u_i}. \tag{4.9}$$

This local cost dynamics is parametrized as:

$$\dot{h}_i = \dot{u}_i \theta_{h1,i} \tag{4.10}$$

where $\theta_{h1,i} = \frac{\partial h_i(u_i)}{\partial u_i}$.

In a distributed environment, it is known that an agent is affected by other agents in the system. In this case, the local cost of agent i depends only on its input. We can add at term to account for the actions of other agents. Taking this into consideration, (4.10) becomes:

$$\dot{h}_i = \theta_{h0,i}(t) + \dot{u}_i\theta_{h1,i}(t) \tag{4.11}$$

The bias term $\theta_{h0,i} \in \mathbb{R}$ accounts for the effect of other agents on i.The dynamics of agent $i's$ local disagreement cost is as follows:

$$\dot{L}_i = \sum_{j\neq i} \dot{u}_j \frac{\partial L(\bar{u}_i)}{\partial u_j} + \dot{u}_i \frac{\partial L(\bar{u}_i)}{\partial u_i} \tag{4.12}$$

This cost is parametrized as follows:

$$\dot{L}_i = \theta_{L0,i}(t) + \theta_{L1,i}(t)\dot{u}_i \tag{4.13}$$

Since agent i depends on the inputs of other agents, $\theta_{L0,i}$ is the term accounting for the effect of the agents on agent i. At this point, a parameter estimation routine for estimating the local parameters $\theta_{h1,i}$, $\theta_{L1,i}$ and $\theta_{L0,i}$ is employed.

4.4.3 Parameter Estimation

The parameter estimation routine proposed in [26] is considered. Let the vectors of parameter estimates be $\hat{\theta}_{h,i} = [\hat{\theta}_{h0,i},\ \hat{\theta}_{h1,i}]^T$ and $\hat{\theta}_{L,i} = [\hat{\theta}_{L0,i},\ \hat{\theta}_{L1,i}]^T$. The regressor vectors are $\phi_{h,i} = [1,\ \dot{u}_i]^T$ and $\phi_{L,i} = [1,\ \dot{u}_i]^T$. The predicted output for a given value of the estimate $\hat{\theta}_{h,i}$ is denoted by $\hat{z}_i$. Similarly $\hat{v}_i$ is the predicted output corresponding to the estimate $\hat{\theta}_{L,i}$. The output prediction errors are given by $e_{h,i} = h_i - \hat{z}_i$ and

$e_{L,i} = L_i - \hat{v}_i$. The prediction error dynamics for (4.11) and (4.13) are written as:

$$\dot{\hat{z}}_i(t) = \phi_{h,i}^T \hat{\theta}_{h,i}(t) + K_1 e_{h,i}(t) + c_{h,i}(t)^T \dot{\hat{\theta}}_{h,i}(t), \quad \hat{z}_i(0) = h_i(0) \tag{4.14}$$

$$\dot{\hat{v}}_i(t) = \phi_{L,i}^T \hat{\theta}_{L,i}(t) + K_2 e_{L,i}(t) + c_{L,i}(t)^T \dot{\hat{\theta}}_{L,i}(t), \quad \hat{v}_i(0) = L_i(0) \tag{4.15}$$

where $K_1 > 0$ and $K_2 > 0$ are constants to be specified. The dynamics of $c_{h,i}(t)$ and $c_{L,i}(t)$ are described by

$$\dot{c}_{h,i}(t)^T = -K_1 c_{h,i}(t)^T + \phi_{h,i}^T, \quad c_{h,i}(0) = 0 \tag{4.16}$$

$$\dot{c}_{L,i}(t)^T = -K_2 c_{L,i}(t)^T + \phi_{L,i}^T, \quad c_{L,i}(0) = 0. \tag{4.17}$$

Let $\eta_{h,i}$ and $\eta_{L,i}$ be auxiliary variables

$$\eta_{h,i} = e_{h,i} - c_{h,i}^T \tilde{\theta}_{J,i}, \tag{4.18}$$

$$\eta_{L,i} = e_{L,i} - c_{L,i}^T \tilde{\theta}_{L,i}. \tag{4.19}$$

where $\tilde{\theta}_{h,i} = \theta_{h,i} - \hat{\theta}_{h,1}$ and $\tilde{\theta}_{L,i} = \theta_{L,i} - \hat{\theta}_{L,i}$. The variables $\hat{\eta}_{h,i}$ and $\hat{\eta}_{L,i}$ are filtered estimates of $\eta_{h,i}$ and $\eta_{L,i}$ with dynamics:

$$\dot{\hat{\eta}}_{h,i}(t) = -K_1 \hat{\eta}_{h,i}(t) \tag{4.20}$$

$$\dot{\hat{\eta}}_{L,i}(t) = -K_2\hat{\eta}_{L,i}(t). \tag{4.21}$$

Let $\Sigma_{h,i} \in \mathbb{R}^{2\times 2}$ and $\Sigma_{L,i} \in \mathbb{R}^{2\times 2}$ be covariance matrices with dynamics

$$\dot{\Sigma}_{h,i}(t) = c_{h,i}(t)^T c_{h,i}(t) - K_{T1}\Sigma_{h,i}(t) + \delta_2 I, \quad \Sigma_{h,i}(0) = \alpha_5 I > 0 \tag{4.22}$$

$$\dot{\Sigma}_{L,i}(t) = c_{L,i}(t)^T c_{L,i}(t) - K_{T2}\Sigma_{L,i}(t) + \delta_3 I, \quad \Sigma_{L,i}(0) = \alpha_6 I > 0 \tag{4.23}$$

where α_5, α_6, δ_2, δ_3, K_{T1} and K_{T2} > 0 are constants to be assigned. The parameter update law is:

$$\dot{\hat{\theta}}_{h,i} = \mathrm{Proj}(\Sigma_{h,i}^{-1}(c_{h,i}(e_{h,i} - \hat{\eta}_{h,i}) - \sigma_1\hat{\theta}_{h,i}), \Theta_2) \tag{4.24}$$

$$\dot{\hat{\theta}}_{L,i} = \mathrm{Proj}(\Sigma_{L,i}^{-1}(c_{L,i}(e_{L,i} - \hat{\eta}_{L,i}) - \sigma_2\hat{\theta}_{L,i}), \Theta_3) \tag{4.25}$$

where σ_1 and $\sigma_2 > 0$. Θ_2 and Θ_3 are balls of finite radii centered at 0. $\mathrm{Proj}\{.,\Theta_2\}$ and $\mathrm{Proj}\{.,\Theta_3\}$ denote Lipschitz projection operators onto the sets Θ_1 and Θ_2 [12]. The application of the projection algorithm is such that:

$$\hat{\theta}_{h,i}(0) \in \Theta_2 \Longrightarrow \hat{\theta}_{h,i}(t) \in \Theta_2, \quad \forall t \geq 0 \tag{4.26}$$

$$\hat{\theta}_{L,i}(0) \in \Theta_3 \Longrightarrow \hat{\theta}_{L,i}(t) \in \Theta_3, \quad \forall t \geq 0 \tag{4.27}$$

The operators bounds $\hat{\theta}_{h,i}$ and $\hat{\theta}_{L,i}$. They ensure that:

$$\tilde{\theta}_{h,i}^T \Sigma_{h,i} \mathrm{Proj}\left(\Sigma_{h,i}^{-1}(t) \Big(c_{h,i}(t)(e_{h,i}(t) - \hat{\eta}_{h,i}(t)) - \sigma_1 \hat{\theta}_{h,i}(t) \Big), \Theta_2 \right) \tag{4.28}$$

$$\leq \tilde{\theta}_{h,i}^T \Big(c_{h,i}(t)(e_{h,i}(t) - \hat{\eta}_{h,i}(t)) - \sigma_1 \hat{\theta}_{h,i}(t) \Big) \tag{4.29}$$

$$\tilde{\theta}_{L,i}^T \Sigma_{L,i} \mathrm{Proj}\left(\Sigma_{L,i}^{-1}(t) \Big(c_{L,i}(t)(e_{L,i}(t) - \hat{\eta}_{L,i}(t)) - \sigma_2 \hat{\theta}_{L,i}(t) \Big), \Theta_3 \right) \tag{4.30}$$

$$\leq \tilde{\theta}_{L,i}^T \Big(c_{L,i}(t)(e_{L,i}(t) - \hat{\eta}_{L,i}(t)) - \sigma_2 \hat{\theta}_{L,i}(t) \Big) \tag{4.31}$$

The signals of the closed-loop ESC system must be such that the following Persistence of Excitation condition is met.

Assumption 4: There exist constants α_7, α_8, T_1 and $T_2 > 0$ such that

$$\int_t^{t+T_1} c_{h,i}(\tau_1) c_{h,i}(\tau_1)^T d\tau_1 \geq \alpha_7 I, \quad t > 0. \tag{4.32}$$

$$\int_t^{t+T_2} c_{L,i}(\tau_2) c_{L,i}(\tau_2)^T d\tau_2 \geq \alpha_8 I, \quad t > 0. \tag{4.33}$$

4.4.4 Distributed Extremum Seeking Controller design

A distributed proportional-integral extremum seeking controller of the form (4.34) is proposed to solve the optimization task

$$\begin{aligned} \dot{\mathbf{u}} &= -k_g \hat{\theta}_{h1} - \frac{1}{\tau_I} \hat{\theta}_{L1} + \hat{\mathbf{u}} + d(t) \\ \dot{\hat{\mathbf{u}}} &= -\frac{1}{\tau_I} \hat{\theta}_{L1} \end{aligned} \tag{4.34}$$

Similarly,

$$\begin{aligned} \dot{u}_i &= -k_g\hat{\theta}_{h1,i} - \frac{1}{\tau_I}\hat{\theta}_{L1,i} + \hat{u}_i + d_i(t) \\ \dot{\hat{u}}_i &= -\frac{1}{\tau_I}\hat{\theta}_{L1,i} \end{aligned} \tag{4.35}$$

where $k_g > 0$ and $\tau_I > 0$ are the proportional gain constant and the integral gain constant respectively. $\hat{u}_i$ is the integral part of the controller that ensures that convergence is attained. The bounded dither signal, $d_i(t)$ is such that $\|d_i(t)\| \leq D_3\ \forall t \geq 0$ and $D_3 > 0$. The schematic representation of the proposed distributed PI-ESC algorithm for agent i is shown in Figure 4.3.

Theorem 1: Consider the distributed extremum seeking controller (4.35), the parameter estimation algorithm (4.22) - (4.25) and let Assumptions **1** to **4** hold, then there exist gains K_1, K_2, K_{T1}, K_{T2}, σ_1, σ_2, δ_2, δ_3, k_g and τ_I such that the system converges to a neighbourhood of the minimizer u^* of the overall cost function J.

Proof : If we have $\theta_{h1,i}$ and $\theta_{L1,i}$, at convergence, the controller can be expressed as:

$$\begin{aligned} \dot{u}_i &= -k_g\theta_{h1,i} - \frac{1}{\tau_I}\theta_{L1,i} + \hat{u}_i \\ \dot{\hat{u}}_i &= -\frac{1}{\tau_I}\theta_{L1,i}. \end{aligned}$$

Let $L = \sum_{i=1}^{p} L_i$ and $L'' = \frac{\partial^2 L}{\partial u^2}$, collecting $\dot{\hat{u}}_i$ and $\dot{u}_i$ results in:

$$\begin{aligned} \dot{\mathbf{u}} &= -k_g\theta_{h1} - \frac{1}{\tau_I}L_G + \hat{\mathbf{u}} \\ \dot{\hat{\mathbf{u}}} &= -\frac{1}{\tau_I}L_G \end{aligned} \tag{4.36}$$

where $L_G = \int_0^1 L''(\lambda u) d\lambda$ and satisfies $\mathbf{1}_p^T L_G = 0$. The multiplication of $\dot{\hat{\mathbf{u}}}$ in (4.36) by $\mathbf{1}_p^T$ results in:

$$\mathbf{1}_p^T \dot{\hat{\mathbf{u}}} = 0 \Rightarrow \mathbf{1}_p^T \hat{\mathbf{u}}(t) = \mathbf{1}_p^T \hat{\mathbf{u}}(0) = 0 \qquad \forall t \geq 0. \tag{4.37}$$

The multiplication of $\dot{\mathbf{u}}$ by $\mathbf{1}_p^T$ yields:

$$\mathbf{1}_p^T \dot{\mathbf{u}} = -k_g \mathbf{1}_p^T \theta_{h1} + \mathbf{1}_p^T \hat{\mathbf{u}}. \tag{4.38}$$

Since $\mathbf{1}_p^T \hat{\mathbf{u}} = 0$, it follows that:

$$\begin{aligned} \mathbf{1}_p^T \dot{\mathbf{u}} &= -k_g \mathbf{1}_p^T \theta_{h1} \\ \sum_{i=1}^{p} \dot{\mathbf{u}} &= -k_g \sum_{i=1}^{p} \theta_{h1} \end{aligned} \tag{4.39}$$

At $\dot{\hat{\mathbf{u}}} = 0$, this means that $\theta_{L1} = 0$ so $u_i = u_j = u$. At this point $\mathbf{u} = [u, \cdots, u]^T$ and convergence is achieved as (4.39) yields:

$$\begin{aligned} p\dot{u} &= -k_g p \theta_{h1,i} \\ \dot{u} &= -k_g \theta_{h1,i} \end{aligned} \tag{4.40}$$

Since we do not have access to $\theta_{hi,1}$ and $\theta_{Li,1}$, their estimates can be obtained using the parameter estimation routine described above. In [27] and [37], it is shown that this can be achieved. We reiterate these developments here. Recall that

$$\dot{\mathbf{u}} = -k_g \hat{\theta}_{h1} - \frac{1}{\tau_I} \theta_{L1} + \hat{\mathbf{u}} + d \tag{4.41}$$

where $\hat{\theta}_{h1} = \theta_{h1} - \tilde{\theta}_{h1}$ and $\hat{\theta}_{L1} = \theta_{L1} - \tilde{\theta}_{L1}$. To show that $\hat{\theta}_{h1} \longrightarrow \theta_{h1}$ and $\hat{\theta}_{L1} \longrightarrow \theta_{L1}$, consider the Lyapunov function:

$$W = \sum_{i=1}^{p} \left(\frac{1}{2}\tilde{\eta}_{h,i}^T \tilde{\eta}_{h,i} + \frac{1}{2}\tilde{\eta}_{L,i}^T \tilde{\eta}_{L,i} + \frac{1}{2}\tilde{\theta}_{h,i}^T \Sigma_{h,i} \tilde{\theta}_{h,i} + \frac{1}{2}\tilde{\theta}_{L,i}^T \Sigma_{L,i} \tilde{\theta}_{L,i} \right) \tag{4.42}$$

where $\tilde{\eta}_{h,i} = \eta_{h,i} - \hat{\eta}_{h,i}$, $\tilde{\eta}_{L,i} = \eta_{L,i} - \hat{\eta}_{L,i}$, $\tilde{\theta}_{h,i} = \theta_{h,i} - \hat{\theta}_{h,i}$ and $\tilde{\theta}_{L,i} = \theta_{L,i} - \hat{\theta}_{L,i}$. Solving for $\dot{\tilde{\eta}}_{h,i}$, $\dot{\tilde{\eta}}_{L,i}$, $\dot{\Sigma}_{h,i}$, $\dot{\Sigma}_{L,i}$, $\dot{\tilde{\theta}}_{h,i}$ and $\dot{\tilde{\theta}}_{L,i}$, substituting the results in $\dot{W}$ yields:

$$\begin{aligned}
\dot{W} \leq \sum_{i=1}^{p} \Big(& -K_1 \tilde{\eta}_{h,i}^T \tilde{\eta}_{h,i} - K_2 \tilde{\eta}_{L,i}^T \tilde{\eta}_{L,i} - \tilde{\eta}_{h,i}^T c_{h,i}^T \dot{\theta}_{h,i} - \tilde{\eta}_{L,i}^T c_{L,i}^T \dot{\theta}_{L,i} \\
& + \frac{1}{2}\tilde{\theta}_{h,i}^T \tilde{\theta}_{h,i} (c_{h,i}^T c_{h,i} - K_{T1}\Sigma_{h,i} + \delta_2 I) + \tilde{\theta}_{h,i}^T \Sigma_{h,i} \dot{\theta}_{h,i} \\
& + \frac{1}{2}\tilde{\theta}_{L,i}^T \tilde{\theta}_{L,i} (c_{L,i}^T c_{L,i} - K_{T2}\Sigma_{L,i} + \delta_3 I) + \tilde{\theta}_{L,i}^T \Sigma_{L,i} \dot{\theta}_{L,i} \\
& - \tilde{\theta}_{h,i}^T c_{h,i} (e_{h,i} - \hat{\eta}_{h,i}) - \tilde{\theta}_{L,i}^T c_{L,i} (e_{L,i} - \hat{\eta}_{L,i}) + \sigma_1 \tilde{\theta}_{h,i}^T \hat{\theta}_{h,i} \\
& + \sigma_2 \tilde{\theta}_{L,i}^T \hat{\theta}_{L,i} \Big).
\end{aligned} \tag{4.43}$$

Rearranging one obtains,

$$\begin{aligned}
\dot{W} \leq \sum_{i=1}^{p} \Big(& -K_1 \tilde{\eta}_{h,i}^T \tilde{\eta}_{h,i} - K_2 \tilde{\eta}_{L,i}^T \tilde{\eta}_{L,i} - \tilde{\eta}_{h,i}^T c_{h,i}^T \dot{\theta}_{h,i} - \tilde{\eta}_{L,i}^T c_{L,i}^T \dot{\theta}_{L,i} \\
& + \frac{1}{2}\tilde{\theta}_{h,i}^T c_{h,i}^T c_{h,i} \tilde{\theta}_{h,i} + \frac{1}{2}\tilde{\theta}_{L,i}^T c_{l,i}^T c_{L,i} \tilde{\theta}_{L,i} - \frac{K_{T1}}{2}\tilde{\theta}_{h,i}^T \Sigma_{h,i} \tilde{\theta}_{h,i} \\
& - \frac{K_{T2}}{2}\tilde{\theta}_{L,i}^T \Sigma_{L,i} \tilde{\theta}_{L,i} + \frac{\delta_2}{2}\tilde{\theta}_{h,i}^T \tilde{\theta}_{h,i} + \frac{\delta_3}{2}\tilde{\theta}_{L,i}^T \tilde{\theta}_{L,i} + \tilde{\theta}_{h,i}^T \Sigma_{h,i} \dot{\theta}_{h,i} \\
& + \tilde{\theta}_{L,i}^T \Sigma_{L,i} \dot{\theta}_{L,i} - \tilde{\theta}_{h,i}^T c_{h,i} (e_{h,i} - \hat{\eta}_{h,i}) - \tilde{\theta}_{L,i}^T c_{L,i} (e_{l,i} - \hat{\eta}_{L,i}) + \\
& \sigma_1 \tilde{\theta}_{h,i}^T \hat{\theta}_{h,i} + \sigma_2 \tilde{\theta}_{L,i}^T \hat{\theta}_{L,i} \Big).
\end{aligned}$$

Also recall that:

$$c_{h,i}^T\tilde{\theta}_{h,i} = e_{h,i} - \tilde{\eta}_{h,i} - \hat{\eta}_{h,i} \quad \text{and} \quad c_{L,i}^T\tilde{\theta}_{L,i} = e_{l,i} - \tilde{\eta}_{L,i} - \hat{\eta}_{L,i}.$$

Substituting the above in $\dot{W}$ yields:

$$\begin{aligned}
\dot{W} \leq \sum_{i=1}^{p} \Big(& -K_1\tilde{\eta}_{h,i}^T\tilde{\eta}_{h,i} - K_2\tilde{\eta}_{L,i}^T\tilde{\eta}_{L,i} - \tilde{\eta}_{h,i}^T c_{h,i}^T\dot{\theta}_{h,i} - \tilde{\eta}_{L,i}^T c_{L,i}^T\dot{\theta}_{L,i} \\
& + \frac{1}{2}(e_{h,i} - \tilde{\eta}_{h,i} - \hat{\eta}_{h,i})^T(e_{h,i} - \tilde{\eta}_{h,i} - \hat{\eta}_{h,i}) - \frac{K_{T1}}{2}\tilde{\theta}_{h,i}^T\Sigma_{h,i}\tilde{\theta}_{h,i} \\
& + \frac{1}{2}(e_{L,i} - \tilde{\eta}_{L,i} - \hat{\eta}_{L,i})^T(e_{L,i} - \tilde{\eta}_{L,i} - \hat{\eta}_{h,i}) - \frac{K_{T2}}{2}\tilde{\theta}_{L,i}^T\Sigma_{L,i}\tilde{\theta}_{L,i} \\
& + \frac{\delta_2}{2}\tilde{\theta}_{h,i}^T\tilde{\theta}_{h,i} + \frac{\delta_3}{2}\tilde{\theta}_{L,i}^T\tilde{\theta}_{L,i} + \tilde{\theta}_{h,i}^T\Sigma_{h,i}\dot{\theta}_{h,i} + \tilde{\theta}_{L,i}^T\Sigma_{L,i}\dot{\theta}_{L,i} \\
& - (e_{h,i} - \tilde{\eta}_{h,i} - \hat{\eta}_{h,i})^T(e_{h,i} - \hat{\eta}_{h,i}) + \sigma_1\tilde{\theta}_{h,i}^T\hat{\theta}_{h,i} \\
& - (e_{L,i} - \tilde{\eta}_{L,i} - \hat{\eta}_{L,i})^T(e_{L,i} - \hat{\eta}_{L,i}) + \sigma_2\tilde{\theta}_{L,i}^T\hat{\theta}_{L,i} \Big).
\end{aligned}$$

Simplifying results by completing the squares yields (4.44).

Since $-\frac{1}{2}(e_{h,i} - \hat{\eta}_{h,i}^T)(e_{h,i} - \hat{\eta}_{h,i}) \leq 0$, $-\frac{1}{2}(e_{L,i} - \hat{\eta}_{L,i}^T)(e_{L,i} - \hat{\eta}_{L,i}) \leq 0$, $\hat{\theta}_{h,i} = \theta_{h,i} - \tilde{\theta}_{h,i}$ and $\hat{\theta}_{L,i} = \theta_{L,i} - \tilde{\theta}_{L,i}$, (4.44) becomes (4.45)

$$\begin{aligned}
\dot{W} \leq \sum_{i=1}^{p} \Big(& -K_1\tilde{\eta}_{h,i}^T\tilde{\eta}_{h,i} - K_2\tilde{\eta}_{L,i}^T\tilde{\eta}_{L,i} - \tilde{\eta}_{h,i}^T c_{h,i}^T\dot{\theta}_{h,i} - \tilde{\eta}_{L,i}^T c_{L,i}^T\dot{\theta}_{L,i} \\
& + \frac{1}{2}\tilde{\eta}_{h,i}^T\tilde{\eta}_{h,i} - \frac{1}{2}(e_{h,i} - \hat{\eta}_{h,i})^T(e_{h,i} - \hat{\eta}_{h,i}) + \frac{1}{2}\tilde{\eta}_{L,i}^T\tilde{\eta}_{L,i} \\
& - \frac{1}{2}(e_{L,i} - \hat{\eta}_{L,i})^T(e_{L,i} - \hat{\eta}_{L,i}) - \frac{K_{T1}}{2}\tilde{\theta}_{h,i}^T\Sigma_{h,i}\tilde{\theta}_{h,i} + \frac{\delta_2}{2}\tilde{\theta}_{h,i}^T\tilde{\theta}_{h,i} \\
& - \frac{K_{T2}}{2}\tilde{\theta}_{L,i}^T\Sigma_{L,i}\tilde{\theta}_{L,i} + \frac{\delta_3}{2}\tilde{\theta}_{L,i}^T\tilde{\theta}_{L,i} + \tilde{\theta}_{h,i}^T\Sigma_{h,i}\dot{\theta}_{h,i} + \tilde{\theta}_{L,i}^T\Sigma_{L,i}\dot{\theta}_{L,i} \\
& + \sigma_1\tilde{\theta}_{h,i}^T\hat{\theta}_{h,i} + \sigma_2\tilde{\theta}_{L,i}^T\hat{\theta}_{L,i} \Big). \qquad (4.44)
\end{aligned}$$

$$\begin{aligned}\dot{W} \leq \sum_{i=1}^{p} \Big(& -K_1\tilde{\eta}_{h,i}^T\tilde{\eta}_{h,i} - K_2\tilde{\eta}_{L,i}^T\tilde{\eta}_{L,i} - \tilde{\eta}_{h,i}^T c_{h,i}^T\dot{\theta}_{h,i} - \tilde{\eta}_{L,i}^T c_{L,i}^T\dot{\theta}_{L,i} \\ & + \frac{1}{2}\tilde{\eta}_{h,i}^T\tilde{\eta}_{h,i} + \frac{1}{2}\tilde{\eta}_{L,i}^T\tilde{\eta}_{L,i} - \frac{K_{T1}}{2}\tilde{\theta}_{h,i}^T\Sigma_{h,i}\tilde{\theta}_{h,i} + \frac{\delta_2}{2}\tilde{\theta}_{h,i}^T\tilde{\theta}_{h,i} \\ & - \frac{K_{T2}}{2}\tilde{\theta}_{L,i}^T\Sigma_{L,i}\tilde{\theta}_{L,i} + \frac{\delta_3}{2}\tilde{\theta}_{L,i}^T\tilde{\theta}_{L,i} + \tilde{\theta}_{h,i}^T\Sigma_{h,i}\dot{\theta}_{h,i} + \tilde{\theta}_{L,i}^T\Sigma_{L,i}\dot{\theta}_{L,i} \\ & + \sigma_1\tilde{\theta}_{h,i}^T\theta_{h,i} - \sigma_1\tilde{\theta}_{h,i}^T\tilde{\theta}_{h,i} + \sigma_2\tilde{\theta}_{L,i}^T\theta_{L,i} - \sigma_2\tilde{\theta}_{L,i}^T\tilde{\theta}_{L,i}\Big). \end{aligned} \tag{4.45}$$

Rearranging yields:

$$\begin{aligned}\dot{W} \leq \sum_{i=1}^{p} \Bigg(& -\tilde{\eta}_{h,i}^T(K_1 - \frac{1}{2})\tilde{\eta}_{h,i} - \tilde{\eta}_{h,i}^T c_{h,i}^T\dot{\theta}_{h,i} + \tilde{\theta}_{h,i}^T\Sigma_{h,i}\dot{\theta}_{h,i} \\ & - \frac{K_{T1}}{2}\tilde{\theta}_{h,i}^T\Sigma_{h,i}\tilde{\theta}_{h,i} - (\sigma_1 - \frac{\delta_2}{2})\tilde{\theta}_{h,i}^T\tilde{\theta}_{h,i} + \sigma_1\tilde{\theta}_{h,i}^T\theta_{h,i} \\ & -\tilde{\eta}_{L,i}^T(K_2 - \frac{1}{2})\tilde{\eta}_{L,i} - \tilde{\eta}_{L,i}^T c_{L,i}^T\dot{\theta}_{L,i} + \tilde{\theta}_{L,i}^T\Sigma_{L,i}\dot{\theta}_{L,i} \\ & - \frac{K_{T2}}{2}\tilde{\theta}_{L,i}^T\Sigma_{L,i}\tilde{\theta}_{L,i} - (\sigma_2 - \frac{\delta_3}{2})\tilde{\theta}_{L,i}^T\tilde{\theta}_{L,i} + \sigma_2\tilde{\theta}_{L,i}^T\theta_{L,i}\Bigg). \end{aligned} \tag{4.46}$$

Applying Young's inequality to the indefinite terms ($\tilde{\eta}_{h,i}^T c_{h,i}^T\dot{\theta}_{h,i}$, $\tilde{\theta}_{h,i}^T\Sigma_{h,i}\dot{\theta}_{h,i}$, $\tilde{\theta}_{h,i}^T\theta_{h,i}$, $\tilde{\eta}_{L,i}^T c_{L,i}^T\dot{\theta}_{L,i}$, $\tilde{\theta}_{L,i}^T\Sigma_{L,i}\dot{\theta}_{L,i}$, $\tilde{\theta}_{L,i}^T\theta_{L,i}$), there exist constants K_3, K_4, K_5 and $K_6 > 0$ such that (4.46) becomes (4.47). Rearranging (4.47), (4.48) is obtained.

$$\begin{aligned}\dot{W} \leq \sum_{i=1}^{p} \Bigg(& -\tilde{\eta}_{h,i}^T(K_1 - \frac{1}{2})\tilde{\eta}_{h,i} + \frac{K_3}{2}\tilde{\eta}_{h,i}^T c_{h,i}^T c_{h,i}\tilde{\eta}_{h,i} + \frac{1}{2K_3}\dot{\theta}_{h,i}^T\dot{\theta}_{h,i} \\ & + \frac{K_4}{2}\tilde{\theta}_{h,i}^T\Sigma_{h,i}\tilde{\theta}_{h,i} + \frac{1}{2K_4}\dot{\theta}_{h,i}^T\Sigma_{h,i}\dot{\theta}_{h,i} - \frac{K_{T1}}{2}\tilde{\theta}_{h,i}^T\Sigma_{h,i}\tilde{\theta}_{h,i} \\ & -(\sigma_1 - \frac{\delta_2}{2})\tilde{\theta}_{h,i}^T\tilde{\theta}_{h,i} + \frac{\sigma_1}{2}\tilde{\theta}_{h,i}^T\tilde{\theta}_{h,i} + \frac{\sigma_1}{2}\theta_{h,i}^T\theta_{h,i} - \tilde{\eta}_{L,i}^T(K_2 - \frac{1}{2})\tilde{\eta}_{L,i} \\ & + \frac{K_5}{2}\tilde{\eta}_{L,i}^T c_{L,i}^T c_{L,i}\tilde{\eta}_{L,i} + \frac{1}{2K_5}\dot{\theta}_{L,i}^T\dot{\theta}_{L,i} + \frac{K_6}{2}\tilde{\theta}_{L,i}^T\Sigma_{L,i}\tilde{\theta}_{L,i} + \frac{1}{2K_6}\dot{\theta}_{L,i}^T\Sigma_{L,i}\dot{\theta}_{L,i} \\ & - \frac{K_{T2}}{2}\tilde{\theta}_{L,i}^T\Sigma_{L,i}\tilde{\theta}_{L,i} - (\sigma_2 - \frac{\delta_3}{2})\tilde{\theta}_{L,i}^T\tilde{\theta}_{L,i} + \frac{\sigma_2}{2}\tilde{\theta}_{L,i}^T\tilde{\theta}_{L,i} + \frac{\sigma_2}{2}\theta_{L,i}^T\theta_{L,i}\Bigg). \end{aligned} \tag{4.47}$$

$$\begin{aligned}\dot{W} \leq \sum_{i=1}^{p} \Bigg(& -\tilde{\eta}_{h,i}^T(K_1 - \frac{1}{2})\tilde{\eta}_{h,i} + \frac{K_3}{2}\tilde{\eta}_{h,i}^T c_{h,i}^T c_{h,i}\tilde{\eta}_{h,i} + \frac{1}{2K_3}\dot{\theta}_{h,i}^T\dot{\theta}_{h,i} \\ & -(\frac{K_{T1}}{2} - \frac{K_4}{2})\tilde{\theta}_{h,i}^T\Sigma_{h,i}\tilde{\theta}_{h,i} + \frac{1}{2K_4}\dot{\theta}_{h,i}^T\Sigma_{h,i}\dot{\theta}_{h,i} + \frac{\sigma_1}{2}\theta_{h,i}^T\theta_{h,i} - (\frac{\sigma_1}{2} - \frac{\delta_2}{2})\tilde{\theta}_{h,i}^T\tilde{\theta}_{h,i} \\ & -\tilde{\eta}_{L,i}^T(K_2 - \frac{1}{2})\tilde{\eta}_{L,i} + \frac{K_5}{2}\tilde{\eta}_{L,i}^T c_{L,i}^T c_{L,i}\tilde{\eta}_{L,i} + \frac{1}{2K_5}\dot{\theta}_{L,i}^T\dot{\theta}_{L,i} - (\frac{K_{T2}}{2} - \frac{K_6}{2})\tilde{\theta}_{L,i}^T\Sigma_{L,i}\tilde{\theta}_{L,i} \\ & +\frac{1}{2K_6}\dot{\theta}_{L,i}^T\Sigma_{L,i}\dot{\theta}_{L,i} + \frac{\sigma_2}{2}\theta_{L,i}^T\theta_{L,i} - (\frac{\sigma_2}{2} - \frac{\delta_3}{2})\tilde{\theta}_{L,i}^T\tilde{\theta}_{L,i}\Bigg) . \end{aligned} \tag{4.48}$$

Let $\tilde{K}_{T1} = K_{T1} - K_4 > 0$ and $\tilde{K}_{T2} = K_{T2} - K_6 > 0$, then the inequality becomes:

$$\begin{aligned}\dot{W} \leq \sum_{i=1}^{p} \Bigg(& -\tilde{\eta}_{h,i}^T(K_1 - \frac{1}{2})\tilde{\eta}_{h,i} + \frac{K_3}{2}\tilde{\eta}_{h,i}^T c_{h,i}^T c_{h,i}\tilde{\eta}_{h,i} + \frac{1}{2K_3}\dot{\theta}_{h,i}^T\dot{\theta}_{h,i} \\ & -\frac{\tilde{K}_{T1}}{2}\tilde{\theta}_{h,i}^T\Sigma_{h,i}\tilde{\theta}_{h,i} + \frac{1}{2K_4}\dot{\theta}_{h,i}^T\Sigma_{h,i}\dot{\theta}_{h,i} - (\frac{\sigma_1}{2} - \frac{\delta_2}{2})\tilde{\theta}_{h,i}^T\tilde{\theta}_{h,i} \\ & +\frac{\sigma_1}{2}\theta_{h,i}^T\theta_{h,i} - \tilde{\eta}_{L,i}^T(K_2 - \frac{1}{2})\tilde{\eta}_{L,i} + \frac{K_5}{2}\tilde{\eta}_{L,i}^T c_{L,i}^T c_{L,i}\tilde{\eta}_{L,i} + \frac{1}{2K_5}\dot{\theta}_{L,i}^T\dot{\theta}_{L,i} \\ & -\frac{\tilde{K}_{T2}}{2}\tilde{\theta}_{L,i}^T\Sigma_{L,i}\tilde{\theta}_{L,i} + \frac{1}{2K_6}\dot{\theta}_{L,i}^T\Sigma_{L,i}\dot{\theta}_{L,i} - (\frac{\sigma_2}{2} - \frac{\delta_3}{2})\tilde{\theta}_{L,i}^T\tilde{\theta}_{L,i} + \frac{\sigma_1}{2}\theta_{h,i}^T\theta_{h,i}\Bigg) . \end{aligned}$$

Collecting terms,

$$\begin{aligned}\dot{W} \leq \sum_{i=1}^{p} \Bigg(& -\tilde{\eta}_{h,i}^T(K_1 - \frac{1}{2} - \frac{K_3}{2}c_{h,i}^T c_{h,i})\tilde{\eta}_{h,i} + \frac{1}{2K_3}\dot{\theta}_{h,i}^T\dot{\theta}_{h,i} \\ & -\frac{\tilde{K}_{T1}}{2}\tilde{\theta}_{h,i}^T\Sigma_{h,i}\tilde{\theta}_{h,i} + \frac{1}{2K_4}\dot{\theta}_{h,i}^T\Sigma_{h,i}\dot{\theta}_{h,i} - (\frac{\sigma_1}{2} - \frac{\delta_2}{2})\tilde{\theta}_{h,i}^T\tilde{\theta}_{h,i} \\ & +\frac{\sigma_1}{2}\theta_{h,i}^T\theta_{h,i} - \tilde{\eta}_{L,i}^T(K_2 - \frac{1}{2} - \frac{K_5}{2}c_{L,i}^T c_{L,i})\tilde{\eta}_{L,i} + \frac{1}{2K_5}\dot{\theta}_{L,i}^T\dot{\theta}_{L,i} \\ & -\frac{\tilde{K}_{T2}}{2}\tilde{\theta}_{L,i}^T\Sigma_{L,i}\tilde{\theta}_{L,i} + \frac{1}{2K_6}\dot{\theta}_{L,i}^T\Sigma_{L,i}\dot{\theta}_{L,i} - (\frac{\sigma_2}{2} - \frac{\delta_3}{2})\tilde{\theta}_{L,i}^T\tilde{\theta}_{L,i} \\ & +\frac{\sigma_2}{2}\theta_{L,i}^T\theta_{L,i}\Bigg) . \end{aligned} \tag{4.49}$$

We proceed to establish the boundedness of the positive definite matrices $\Sigma_{h,i}$ and $\Sigma_{L,i}$. Integrating (4.22) and (4.23) and using Assumption 4 for $t \geq T_1$ and $t \geq T_2$ respectively, then

$$
\begin{aligned}
\Sigma_{h,i} &= e^{-K_{T1}t}\Sigma_{h,i}(0) + \int_0^t e^{-K_{T1}(t-\tau_1)} \left(c_{h,i}(\tau_1)c_{h,i}^T(\tau_1) + \delta_2 I\right) d\tau_1 \\
&\geq \int_{t-T_1}^t e^{-K_{T1}(t-\tau_1)} \left(c_{h,i}(\tau_1)c_{h,i}^T(\tau_1) + \delta_2 I\right) d\tau_1 \\
&\geq e^{-K_{T1}T_1}(\alpha_7 + \delta_2 T_1)I \\
\Sigma_{L,i} &= e^{-K_{T2}t}\Sigma_{L,i}(0) + \int_0^t e^{-K_{T2}(t-\tau_2)} \left(c_{L,i}(\tau_2)c_{L,i}^T(\tau_2) + \delta_3 I\right) d\tau_2 \\
&\geq \int_{t-T_2}^t e^{-K_{T2}(t-\tau_2)} \left(c_{L,i}(\tau_2)c_{L,i}^T(\tau_2) + \delta_3 I\right) d\tau_2 \\
&\geq e^{-K_{T2}T_2}(\alpha_8 + \delta_3 T_2)I.
\end{aligned}
$$

When $t < T_1$ and $t < T_2$,

$$\Sigma_{h,i} \geq e^{-K_{T1}T_1}\alpha_5 I \quad \text{and} \quad \Sigma_{L,i} \geq e^{-K_{T2}T_2}\alpha_6 I.$$

Consequently,

$$\Sigma_{h,i} \geq e^{-K_{T1}T_1} \min[(\alpha_7 + \delta_2 T_1), \alpha_5]I \qquad \forall t > 0$$

and

$$\Sigma_{L,i} \geq e^{-K_{T2}T_2} \min[(\alpha_8 + \delta_3 T_2), \alpha_6]I \qquad \forall t > 0.$$

The Lipschitz projection operator employed in the parameter estimation routine ensures that $\hat{\theta}_{h,i}$ and $\hat{\theta}_{L,i}$ are bounded. As a result of the boundedness of $\hat{\theta}_{h,i}$ and $\hat{\theta}_{L,i}$, $\dot{u}_i$, $c_{h,i}$ and $c_{L,i}$ are bounded. This implies that there exist positive constants g_1, and

g_2 such that:

$$c_{h,i}c_{h,i}^T < g_1 I \quad \text{and} \quad c_{L,i}c_{L,i}^T < g_2 I \qquad \forall t > 0.$$

Consequently,

$$\begin{aligned}
\Sigma_{h,i} &= e^{-K_{T1}t}\Sigma_{h,i}(0) + \int_0^t e^{-K_{T1}(t-\tau_1)}\left(c_{h,i}(\tau_1)c_{h,i}^T(\tau_1) + \delta_2 I\right)d\tau_1 \\
&\leq (\alpha_5 + \frac{g_1+\delta_2}{K_{T1}})I \\
\Sigma_{l,i} &= e^{-K_{T2}t}\Sigma_{L,i}(0) + \int_0^t e^{-K_{T2}(t-\tau_2)}\left(c_{L,i}(\tau_2)c_{L,i}^T(\tau_2) + \delta_3 I\right)d\tau_2 \\
&\leq (\alpha_6 + \frac{g_2+\delta_3}{K_{T2}})I.
\end{aligned}$$

Now, let

$$\begin{aligned}
\zeta_1 &= e^{-K_{T1}T_1}\min[(\alpha_7 + \delta_2 T_1), \alpha_5] \quad \text{and} \quad \zeta_2 = (\alpha_5 + \tfrac{g_1+\delta_2}{K_{T1}}) \\
\zeta_3 &= e^{-K_{T2}T_2}\min[(\alpha_8 + \delta_3 T_2), \alpha_6] \quad \text{and} \quad \zeta_4 = (\alpha_6 + \tfrac{g_2+\delta_3}{K_{T2}}),
\end{aligned}$$

then

$$\zeta_1 I \leq \Sigma_{h,i} \leq \zeta_2 I \quad \text{and} \quad \zeta_3 I \leq \Sigma_{L,i} \leq \zeta_4 I.$$

Recall that $c_{h,i}^T c_{h,i} < g_1$ and $c_{L,i}^T c_{L,i} < g_2$. Substituting $\delta_2 = \sigma_1$, $\delta_3 = \sigma_2$, $K_1 = k_1 + \frac{1}{2}$, $K_2 = k_2 + \frac{1}{2}$ and the bounds for $c_{h,i}^T c_{h,i}$, $c_{L,i}^T c_{L,i}$, $\Sigma_{h,i}$ and $\Sigma_{L,i}$, in (4.49), then

$$\begin{aligned}
\dot{W} \leq \sum_{i=1}^{p} \Bigg(& -\tilde{\eta}_{h,i}^T(k_1 - \frac{K_3}{2}g_1)\tilde{\eta}_{h,i} + \frac{1}{2K_3}\dot{\theta}_{h,i}^T\dot{\theta}_{h,i} \\
& -\frac{\tilde{K}_{T1}}{2}\zeta_1\tilde{\theta}_{h,i}^T\tilde{\theta}_{h,i} + \frac{1}{2K_4}\zeta_2\dot{\theta}_{h,i}^T\dot{\theta}_{h,i} + \frac{\sigma_1}{2}\theta_{h,i}^T\theta_{h,i} \\
& -\tilde{\eta}_{L,i}^T(k_2 - \frac{K_5}{2}g_2)\tilde{\eta}_{L,i} + \frac{1}{2K_5}\dot{\theta}_{L,i}^T\dot{\theta}_{L,i} \\
& -\frac{\tilde{K}_{T2}}{2}\zeta_3\tilde{\theta}_{L,i}^T\tilde{\theta}_{L,i} + \frac{1}{2K_6}\zeta_4\dot{\theta}_{L,i}^T\dot{\theta}_{L,i} + \frac{\sigma_2}{2}\theta_{L,i}^T\theta_{L,i}\Bigg).
\end{aligned} \tag{4.50}$$

Let

$$G_1 = k_1 - \tfrac{K_3}{2}g_1, \quad G_2 = \tfrac{\tilde{K}_{T1}}{2}\zeta_1, \quad G_3 = \tfrac{1}{2K_3} + \tfrac{1}{2K_4}\zeta_2,$$
$$G_4 = k_2 - \tfrac{K_5}{2}g_2, \quad G_5 = \tfrac{\tilde{K}_{T2}}{2}\zeta_3, \quad G_6 = \tfrac{1}{2K_5} + \tfrac{1}{2K_6}\zeta_4.$$

Recall that $\tilde{K}_{T1} > 0$ and $\tilde{K}_{T2} > 0$, let $k_1 > \frac{K_3}{2}g_1$ and $k_2 > \frac{K_5}{2}g_2$, then

$$\begin{aligned}\dot{W} \leq \sum_{i=1}^{p} \Big(& -G_1\|\tilde{\eta}_{h,i}\|^2 - G_2\|\tilde{\theta}_{h,i}\|^2 + G_3\|\dot{\theta}_{h,i}\|^2 + \frac{\sigma_1}{2}\|\theta_{h,i}\|^2 \\ & -G_4\|\tilde{\eta}_{L,i}\|^2 - G_5\|\tilde{\theta}_{L,i}\|^2 + G_6\|\dot{\theta}_{L,i}\|^2 + \frac{\sigma_2}{2}\|\theta_{L,i}\|^2 \Big). \end{aligned} \tag{4.51}$$

From (4.51), it is seen that provided $\theta_{h,i}$, $\dot{\theta}_{h,i}$. $\theta_{L,i}$ and $\dot{\theta}_{L,i}$ are bounded, $\tilde{\eta}_{h,i}$, $\tilde{\theta}_{h,i}$, $\tilde{\eta}_{L,i}$ and $\tilde{\theta}_{L,i}$ will be bounded and approach a neighbourhood of the origin. Therefore, the parameter estimates will reach a neighbourhood of their true values.

Since the parameter estimates converge to a neighbourhood of their true values, the next step is to show that the bias term $\hat{u}_i$ converges to a neighbourhood of u. Let $\tilde{u}_i = \hat{u}_i - u$ then consider the Lyapunov function:

$$Y = W + M \tag{4.52}$$

where $M = \frac{1}{2}\tilde{\mathbf{u}}^T\tilde{\mathbf{u}}$. The differentiation of (4.52) gives $\dot{Y} = \dot{W} + \tilde{\mathbf{u}}^T\dot{\tilde{\mathbf{u}}}$. Recalling that $\tilde{\mathbf{u}} = \hat{\mathbf{u}} - u$, $\dot{\tilde{\mathbf{u}}} = \dot{\hat{\mathbf{u}}}$, $\dot{\hat{\mathbf{u}}} = -\frac{1}{\tau_I}\hat{\theta}_{L1}$ and $\hat{\theta}_{L1} = \theta_{L1} - \tilde{\theta}_{L1}$, then:

$$\dot{Y} = \dot{W} + \tilde{\mathbf{u}}^T\Big(\frac{1}{\tau_I}\tilde{\theta}_{L1} - \frac{1}{\tau_I}\theta_{L1}\Big). \tag{4.53}$$

(4.53) can be written as:

$$\dot{Y} = \dot{W} + \sum_{i=1}^{p}\Big(\frac{1}{\tau_I}\tilde{u}_i\tilde{\theta}_{L1,i} - \frac{1}{\tau_I}\tilde{u}_i\theta_{L1,i}\Big). \tag{4.54}$$

The substitution of $\dot{W}$ in (4.54) results in:

$$\dot{Y} \leq \sum_{i=1}^{p} \Big(-G_1\|\tilde{\eta}_{h,i}\|^2 - G_2|\tilde{\theta}_{h,i}\|^2 + G_3\|\dot{\theta}_{h,i}\|^2 + \frac{\sigma_1}{2}\|\theta_{h,i}\|^2 - G_4\|\tilde{\eta}_{L,i}\|^2$$
$$-G_5|\tilde{\theta}_{L,i}\|^2 + G_6\|\dot{\theta}_{L,i}\|^2 + \frac{\sigma_2}{2}\|\theta_{L,i}\|^2 + \frac{1}{\tau_I}\tilde{u}_i\tilde{\theta}_{L1,i} - \frac{1}{\tau_I}\tilde{u}_i\theta_{L1,i} \Big).$$

Collecting the $\tilde{u}_i\theta_{L1,i}$ terms and using Assumption 3, it follows that:

$$\dot{Y} \leq \sum_{i=1}^{p} \Big(-G_1\|\tilde{\eta}_{h,i}\|^2 - G_2|\tilde{\theta}_{h,i}\|^2 + G_3\|\dot{\theta}_{h,i}\|^2 + \frac{\sigma_1}{2}\|\theta_{h,i}\|^2 - G_4\|\tilde{\eta}_{L,i}\|^2$$
$$-G_5|\tilde{\theta}_{L,i}\|^2 + G_6\|\dot{\theta}_{L,i}\|^2 + \frac{\sigma_2}{2}\|\theta_{L,i}\|^2 + \frac{1}{\tau_I}\tilde{u}_i\tilde{\theta}_{L1,i} - \frac{1}{\tau_I}\gamma_d\|\tilde{u}_i\|^2 \Big). \tag{4.55}$$

Applying Young's inequality to the indefinite terms in (4.55) then there exist a constant $\alpha_9 > 0$ such that:

$$\dot{Y} \leq \sum_{i=1}^{p} \Big(-G_1\|\tilde{\eta}_{h,i}\|^2 - G_2|\tilde{\theta}_{h,i}\|^2 + G_3\|\dot{\theta}_{h,i}\|^2 + \frac{\sigma_1}{2}\|\theta_{h,i}\|^2 - G_4\|\tilde{\eta}_{L,i}\|^2$$
$$-G_5|\tilde{\theta}_{L,i}\|^2 + G_6\|\dot{\theta}_{L,i}\|^2 + \frac{\sigma_2}{2}\|\theta_{L,i}\|^2 - \frac{1}{\tau_I}\gamma_d\|\tilde{u}_i\|^2 + \frac{\alpha_9}{2\tau_I}\|\tilde{u}_i\|^2$$
$$+\frac{1}{2\alpha_9\tau_I}\|\tilde{\theta}_{L1,i}\|^2 \Big).$$

Rearranging the last inequality gives:

$$\dot{Y} \leq \sum_{i=1}^{p} \Big(-G_1\|\tilde{\eta}_{h,i}\|^2 - G_2|\tilde{\theta}_{h,i}\|^2 + G_3\|\dot{\theta}_{h,i}\|^2 + \frac{\sigma_1}{2}\|\theta_{h,i}\|^2 - G_4\|\tilde{\eta}_{L,i}\|^2$$
$$-G_5|\tilde{\theta}_{L0,i}\|^2 + G_6\|\dot{\theta}_{L,i}\|^2 + \frac{\sigma_2}{2}\|\theta_{L,i}\|^2 - (\frac{1}{\tau_I}\gamma_d - \frac{\alpha_9}{2\tau_I})\|\tilde{u}_i\|^2$$
$$-(G_5 - \frac{1}{2\alpha_9\tau_I})\|\tilde{\theta}_{L1,i}\|^2 \Big). \tag{4.56}$$

Let $G_7 = \left(\frac{\gamma_d}{\tau_I} - \frac{\alpha_9}{2\tau_I}\right)$ and $G_8 = \left(G_5 - \frac{1}{2\alpha_9\tau_I}\right)$, to make $G_7 > 0$, we pick $\alpha_9 < 2\gamma_d$. To make $G_8 > 0$, we pick $\tau_I < \frac{1}{2\alpha_9 G_5}$. Making τ_I large enough but less than $\frac{1}{2\alpha_9 G_5}$ ensures that $\tilde{u}_i$ is bounded and enters a neighbourhood of the origin as $\hat{u}_i$ approaches a neighbourhood of u. Therefore:

$$\begin{aligned}\dot{Y} \leq \sum_{i=1}^{p} \Big(& -G_1\|\tilde{\eta}_{h,i}\|^2 - G_2|\tilde{\theta}_{h,i}\|^2 + G_3\|\dot{\theta}_{h,i}\|^2 + \frac{\sigma_1}{2}\|\theta_{h,i}\|^2 - G_4\|\tilde{\eta}_{L,i}\|^2 \\ & -G_5|\tilde{\theta}_{L0,i}\|^2 + G_6\|\dot{\theta}_{L,i}\|^2 + \frac{\sigma_2}{2}\|\theta_{L,i}\|^2 - G_7\|\tilde{u}_i\|^2 - G_8\|\tilde{\theta}_{L1,i}\|^2\Big). \end{aligned} \tag{4.57}$$

Since $\hat{u}_i$ approaches a neighbourhood of u, this implies that $\dot{\hat{u}}_i = 0$ therefore, the ESC algorithm reduces to $\dot{\mathbf{u}} = -k_g\hat{\theta}_{h1} + \hat{\mathbf{u}} + d$, where $\mathbf{u} = [u, \cdots, u]^T$. The multiplication of $\dot{\mathbf{u}}$ by $\mathbf{1}_p^T$ yields:

$$\mathbf{1}_p^T\dot{\mathbf{u}} = -k_g\mathbf{1}_p^T\hat{\theta}_{h1} + \mathbf{1}_p^T\hat{\mathbf{u}} + \mathbf{1}_p^T d \tag{4.58}$$

But $\mathbf{1}_p^T\hat{\mathbf{u}} = 0$, (4.58) simplifies to:

$$\begin{aligned}\sum_{i=1}^{p}\dot{u} &= -k_g\sum_{i=1}^{p}\hat{\theta}_{h1,i} + \sum_{i=1}^{p} d_i \\ p\dot{u} &= -k_g p\hat{\theta}_{h1,i} + pd_i \\ \dot{u} &= -k_g\hat{\theta}_{h1,i} + d_i\end{aligned} \tag{4.59}$$

Finally, we focus on showing that u approaches a neighbourhood of the u^*, the minimizer of the overall cost. Let $\tilde{u} = u - u^*$, $\tilde{\mathbf{u}} = \mathbf{u} - u^*$ and $V = \frac{1}{2}\tilde{\mathbf{u}}^T\tilde{\mathbf{u}}$, consider the

Lyapunov function:

$$\mathcal{W} = Y + V \tag{4.60}$$

The differentiation of (4.60) w.r.t. time yields:

$$\dot{\mathcal{W}} = \dot{Y} + \tilde{\mathbf{u}}^T \dot{\tilde{\mathbf{u}}}. \tag{4.61}$$

Since $\tilde{\mathbf{u}} = \mathbf{u} - u^*$ then $\dot{\tilde{\mathbf{u}}} = \dot{\mathbf{u}}$. Recalling that $\hat{\theta}_{h1} = \theta_{h1} - \tilde{\theta}_{h1}$, the substitution of $\dot{\mathbf{u}}$ in (4.61) gives:

$$\dot{\mathcal{W}} = \dot{Y} - k_g \tilde{\mathbf{u}}^T \theta_{h1} + k_g \tilde{\mathbf{u}}^T \tilde{\theta}_{h1} + \tilde{\mathbf{u}}^T d. \tag{4.62}$$

(4.62) can be written as:

$$\dot{\mathcal{W}} = \dot{Y} + \sum_{i=1}^{p} \Big(k_g \tilde{u} \tilde{\theta}_{h1,i} - k_g \tilde{u} \theta_{h1,i} + \tilde{u} d_i \Big) \tag{4.63}$$

Substituting $\dot{Y}$ yields:

$$\begin{aligned}\dot{\mathcal{W}} \leq \sum_{i=1}^{p} \Big(& -G_1 \|\tilde{\eta}_{h,i}\|^2 - G_2 |\tilde{\theta}_{h,i}\|^2 + G_3 \|\dot{\theta}_{h,i}\|^2 + \frac{\sigma_1}{2} \|\theta_{h,i}\|^2 - G_4 \|\tilde{\eta}_{L,i}\|^2 \\ & -G_5 |\tilde{\theta}_{L0,i}\|^2 + G_6 \|\dot{\theta}_{L,i}\|^2 + \frac{\sigma_2}{2} \|\theta_{L,i}\|^2 - G_7 \|\tilde{u}_i\|^2 - G_8 \|\tilde{\theta}_{L1,i}\|^2 \, . \\ & -k_g \tilde{u} \theta_{h1,i} + k_g \tilde{u} \tilde{\theta}_{h1,i} + \tilde{u} d_i \Big) \end{aligned} \tag{4.64}$$

It is important to state that at consensus, $\tilde{u}$ and $\theta_{h1,i}$ are same for all the agents so it follows that $p\tilde{u}\theta_{h1,i} = \tilde{u}\theta_{J1}$. By the convexity of the overall cost we have:

$$\begin{aligned} p\tilde{u}\theta_{h1,i} &\geq \gamma\tilde{u}^2 \\ \tilde{u}\theta_{h1,i} &\geq \frac{\gamma}{p}\tilde{u}^2 \\ -k_g\tilde{u}\theta_{h1,i} &\leq -\frac{k_g}{p}\gamma\tilde{u}^2 \end{aligned} \tag{4.65}$$

Substituting (4.65) in (4.64) results in:

$$\begin{aligned} \dot{\mathcal{W}} \leq \sum_{i=1}^{p} \Big(& -G_1\|\tilde{\eta}_{h,i}\|^2 - G_2|\tilde{\theta}_{h,i}\|^2 + G_3\|\dot{\theta}_{h,i}\|^2 + \frac{\sigma_1}{2}\|\theta_{h,i}\|^2 \\ & -G_4\|\tilde{\eta}_{L,i}\|^2 - G_5|\tilde{\theta}_{L0,i}\|^2 + G_6\|\dot{\theta}_{L,i}\|^2 + \frac{\sigma_2}{2}\|\theta_{L,i}\|^2 \\ & -G_7\|\tilde{u}_i\|^2 - G_8\|\tilde{\theta}_{L1,i}\|^2 - \frac{k_g}{p}\gamma\tilde{u}^2 + k_g\tilde{u}\tilde{\theta}_{h1,i} + \tilde{u}d_i \Big). \end{aligned} \tag{4.66}$$

Applying Young's inequality to the indefinite terms, there exist constants $\alpha_{10} > 0$ and $\alpha_{11} > 0$ such that:

$$\begin{aligned} \dot{\mathcal{W}} \leq \sum_{i=1}^{p} \Big(& -G_1\|\tilde{\eta}_{h,i}\|^2 - G_2|\tilde{\theta}_{h,i}\|^2 + G_3\|\dot{\theta}_{h,i}\|^2 + \frac{\sigma_1}{2}\|\theta_{h,i}\|^2 \\ & -G_4\|\tilde{\eta}_{L,i}\|^2 - G_5|\tilde{\theta}_{L0,i}\|^2 + G_6\|\dot{\theta}_{L,i}\|^2 + \frac{\sigma_2}{2}\|\theta_{L,i}\|^2 \\ & -G_7\|\tilde{u}_i\|^2 - G_8\|\tilde{\theta}_{L1,i}\|^2 - \frac{k_g}{p}\gamma\|\tilde{u}\|^2 + \frac{\alpha_{10}}{2}k_g\|\tilde{u}\|^2 \\ & + \frac{1}{2\alpha_{10}}k_g\|\tilde{\theta}_{h1,i}\|^2 + \frac{\alpha_{11}}{2}\|\tilde{u}\|^2 + \frac{1}{2\alpha_{11}}\|d_i\|^2 \Big). \end{aligned} \tag{4.67}$$

We can rewrite (4.67) as:

$$\begin{aligned}\dot{\mathcal{W}} \leq \sum_{i=1}^{p} \Big(& -G_1\|\tilde{\eta}_{h,i}\|^2 - G_2|\tilde{\theta}_{h0,i}\|^2 + G_3\|\dot{\theta}_{h,i}\|^2 + \frac{\sigma_1}{2}\|\theta_{h,i}\|^2 \\ & -G_4\|\tilde{\eta}_{L,i}\|^2 - G_5|\tilde{\theta}_{L0,i}\|^2 + G_6\|\dot{\theta}_{L,i}\|^2 + \frac{\sigma_2}{2}\|\theta_{L,i}\|^2 \\ & -G_7\|\tilde{u}_i\|^2 - G_8\|\tilde{\theta}_{L1,i}\|^2 - (\frac{k_g}{p}\gamma - \frac{\alpha_{10}}{2}k_g - \frac{\alpha_{11}}{2})\|\tilde{u}\|^2 \\ & -(G_2 - \frac{1}{2\alpha_{10}}k_g)\|\tilde{\theta}_{h1,i}\|^2 + \frac{1}{2\alpha_{11}}\|d_i\|^2 \Big). \end{aligned} \tag{4.68}$$

Let $G_9 = (\frac{k_g}{p}\gamma - \frac{\alpha_{10}}{2}k_g - \frac{\alpha_{11}}{2})$, by choosing $\alpha_{10} = \frac{\gamma}{p}$ then k_g can be chosen such that $k_g > p\frac{\alpha_{11}}{\gamma}$ and this ensures that $G_9 > 0$. Also let $G_{10} = (G_2 - \frac{1}{2\alpha_{10}}k_g)$, $G_{10} > 0$ by choosing k_g such that $k_g < 2\alpha_{10}G_2$. Therefore,

$$\begin{aligned}\dot{\mathcal{W}} \leq \sum_{i=1}^{p} \Big(& -G_1\|\tilde{\eta}_{h,i}\|^2 - G_2|\tilde{\theta}_{h0,i}\|^2 + G_3\|\dot{\theta}_{h,i}\|^2 + \frac{\sigma_1}{2}\|\theta_{h,i}\|^2 - G_4\|\tilde{\eta}_{L,i}\|^2 \\ & -G_5|\tilde{\theta}_{L0,i}\|^2 + G_6\|\dot{\theta}_{L,i}\|^2 + \frac{\sigma_2}{2}\|\theta_{L,i}\|^2 - G_7\|\tilde{u}_i\|^2 - G_8\|\tilde{\theta}_{L1,i}\|^2 \\ & -G_9\|\tilde{u}\|^2 - G_{10}\|\tilde{\theta}_{h1,i}\|^2 + \frac{1}{2\alpha_{11}}\|d_i\|^2 \Big). \end{aligned} \tag{4.69}$$

We know that the dither signal is bounded. Since $\dot{u}$ is bounded, $\dot{\theta}_{h1,i}$ and $\dot{\theta}_{L1,i}$ are bounded. Likewise $\dot{\theta}_{h0,i}$ and $\dot{\theta}_{L0,i}$ are bounded since $\theta_{h0,i}$ and $\theta_{L0,i}$ are bias terms unaffected by the controller and can be bounded by positive constants. Therefore $\dot{\theta}_{h,i}$, $\theta_{h,i}$, $\dot{\theta}_{L,i}$ and $\theta_{L,i}$ are bounded. This Lyapunov derivative shows that $\tilde{\eta}_{h,i}$, $\tilde{\eta}_{L,i}$, $\tilde{\theta}_{h,i}$, $\tilde{\theta}_{L,i}$, $\tilde{u}_i$ and $\tilde{u}$ are bounded. $\tilde{\eta}_{L,i}$, $\tilde{\theta}_{L,i}$, and $\tilde{u}_i$ enter a neighbourhood of the origin as $\hat{u}_i$ approaches a neighbourhood of u. $\tilde{\eta}_{h,i}$, $\tilde{\theta}_{h,i}$ and $\tilde{u}$ also approach a neighbourhood of the origin as u converges to a neighbourhood of u^*. The size of this neighbourhood depends on the positive terms in (4.69). This completes the proof.

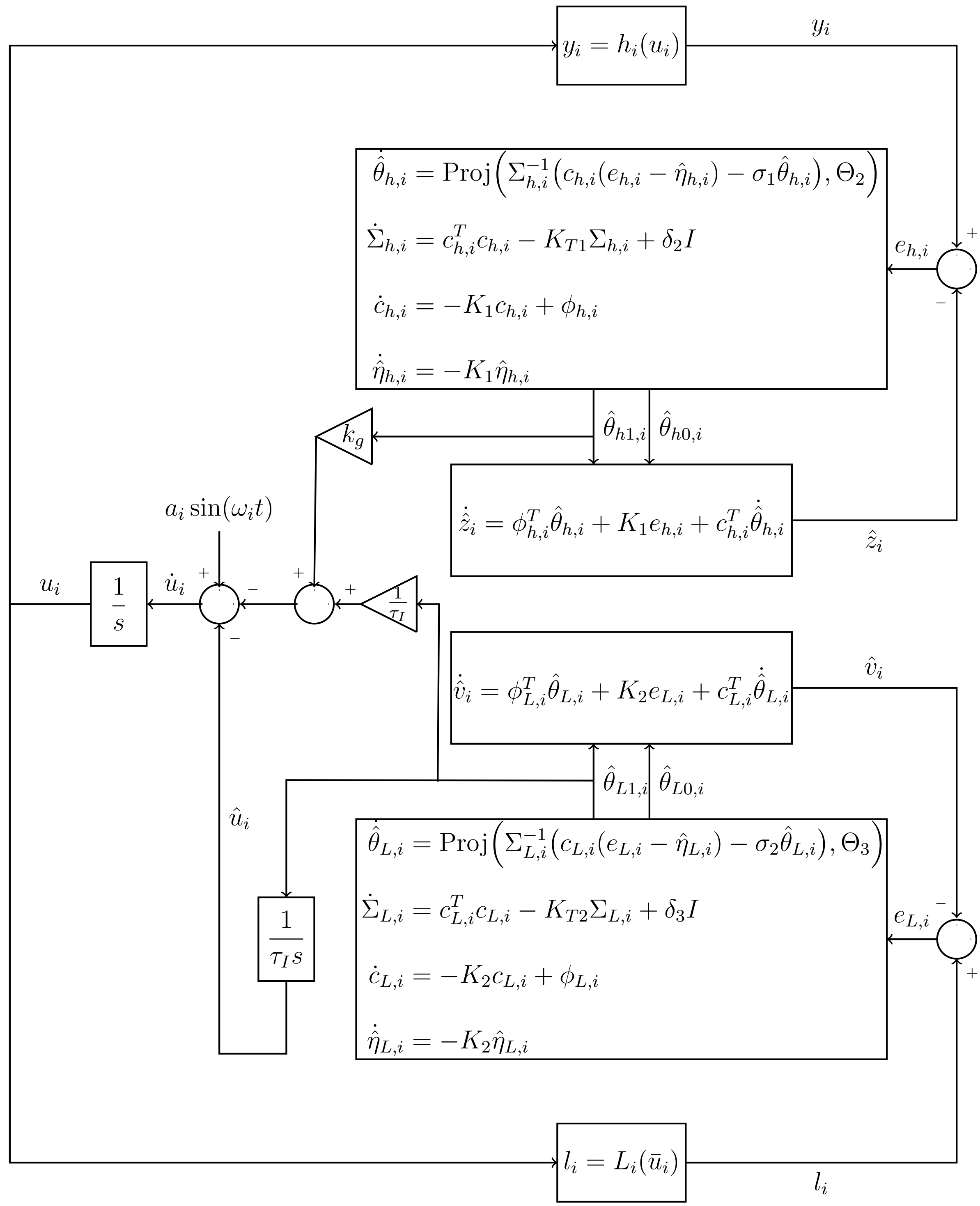

Figure 4.3: Distributed control algorithm for agent i.

4.5 Simulation Examples

4.5.1 Example 1

Consider a 50 agent system where the agents communicate over the undirected network shown in Figure 4.1. Weight were assigned to the edges from the vector $H \in \mathbb{R}^m$. Using the edge weights, $d_{i,j}$ was computed. The distance matrix D_u contains the elements d_{ij} from agent i to j.

$$H = \begin{bmatrix} 0.99 \\ 0.97 \\ 0.95 \\ \vdots \\ 0.17 \end{bmatrix} \qquad D_u = \begin{bmatrix} 0 & 0.83 & 0.47 & \dots & 0.89 \\ 0.83 & 0 & 1.30 & \dots & 1.72 \\ 0.47 & 1.30 & 0 & \dots & 1.36 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0.89 & 1.72 & 1.36 & \dots & 0 \end{bmatrix}$$

The objective is to minimize the sum of the local cost of all the agents. The local cost are:

$$h_1 = 0.5e^{-0.5u_1} + 0.4e^{0.3u_1} \qquad h_2 = (u_2 - 4)^2$$

$$h_3 = 0.5u_3^2 \ln(1 + u_3^2) + u_3^2 \qquad h_4 = u_4^2 + e^{0.1u_4}$$

$$h_5 = \ln(e^{-0.1u_5} + e^{0.3u_5}) + 0.1u_5^2 \qquad h_6 = \frac{u_6^2}{\ln(2+u_6^2)}$$

$$h_7 = 0.2e^{-0.2u_7} + 0.4e^{0.4u_7} \qquad h_8 = u_8^4 + 2u_8^2 + 2$$

$$h_9 = \frac{u_9^2}{\sqrt{1+u_9^2}} + 0.1u_9^2 \qquad h_{10} = (u_{10} + 2)^2$$

$$h_{11} = 5u_{11}^2 + e^{3u_{11}} \qquad h_{12} = (u_{12} + 5)^2 + (u_{12} + 5)^2$$

$$h_{13} = 4u_{13}^2 + \frac{0.3u_{13}^2}{\sqrt{u_{13}^2+1}} \qquad h_{14} = (u_{14} - 10)^2 + (u_{14} + 2)^2$$

$$h_{15} = e^{-10u_{15}} + e^{-0.5u_{15}} \qquad h_{16} = (u_{16}^2 + 0.5)^2 + 8u_{16}^2$$

$$h_{17} = e^{-0.3u_{17}} + e^{0.2u_{17}} \qquad h_{18} = e^{-10u_{18}} + (4 + u_{18})^2$$

$$h_{19} = 0.6u_{19}^2 e^{0.6u_{19}} + 0.5u_{19}^2 \qquad h_{20} = 0.7u_{20}^2 + (0.5 - u_{20})^2$$

$$h_{21} = u_{21}^2 e^{2u_{21}} \qquad h_{22} = u_{22}^2 + e^{2u_{22}}$$
$$h_{23} = (0.7u_{23})^2 \qquad h_{24} = (4 - u_{24})^2$$
$$h_{25} = u_{25}^2 \qquad h_{26} = (u_{26} - 1)^2 + (u_{26} - 2)^2$$
$$h_{27} = (u_{27} - 2)^2 + (u_{27} - 3)^2 \qquad h_{28} = (u_{28} - 3)^2 + (u_{28} - 4)^2$$
$$h_{29} = (u_{29} - 4)^2 + (u_{29} - 5)^2 \qquad h_{30} = (u_{30} - 5)^2 + (u_{30} - 6)^2$$
$$h_{31} = (u_{31} - 6)^2 + (u_{31} - 7)^2 \qquad h_{32} = (u_{32} - 7)^2 + (u_{32} - 8)^2$$
$$h_{33} = (u_{33} - 8)^2 + (u_{33} - 9)^2 \qquad h_{34} = (u_{34} - 9)^2 + (u_{34} - 1)^2$$
$$h_{35} = (u_{35} - 2)^2 + (u_{35} - 9)^2 \qquad h_{36} = (u_{36} - 3)^2 + (u_{36} - 9)^2$$
$$h_{37} = (u_{37} - 4)^2 + (u_{37} - 9)^2 \qquad h_{38} = (u_{38} - 5)^2 + (u_{38} - 9)^2$$
$$h_{39} = (u_{39} - 6)^2 + (u_{39} - 9)^2 \qquad h_{40} = (u_{40} - 7)^2 + (u_{40} - 9)^2$$
$$h_{41} = \ln(e^{0.5u_{41}} + e^{0.3u_{41}}) + u_{41}^2 \qquad h_{42} = \frac{10u_{42}^2}{\ln(10+u_{42}^2)}$$
$$h_{43} = u_{43}^2 + 0.5u_{43}^2 \qquad h_{44} = e^{-u_{44}} + u_{44}^2$$
$$h_{45} = 0.1u_{45}^2 e^{0.1u_{45}} + 2u_{45}^2 \qquad h_{46} = 7u_{46}^2 + (7 - u_{46})^2$$
$$h_{47} = 4u_{47}^2 e^{4u_{47}} \qquad h_{48} = 0.4u_{48}^2 + e^{0.4u_{48}}$$
$$h_{49} = (u_{49} - 10)^2 + (u_{49} - 10)^2 \qquad h_{50} = (u_{50} - 8)^2 + (u_{50} - 8)^2$$

The tuning parameters were selected as: $\sigma_1 = \sigma_2 = 1 \times 10^{-8}$, $K_1 = K_2 = K_{T_1} = K_{T_2} = 100$, $\delta_2 = \delta_3 = 5 \times 10^{-9}$, $\alpha_5 = \alpha_6 = 1$, $k_g = 6 \times 10^{-3}$, $\tau_I = 16.66$, $d(t) = 0.3\,[\sin(500t), \sin(494t), \ldots, \sin(361t), \sin(358t)]^T$. The initial conditions were chosen as: $\mathbf{u}(0) = [0.4, \ldots, 0.6]^T$, $\hat{\theta}_{h,i}(0) = \hat{\theta}_{L,i}(0) = [0.01, 0]^T$, $c_{h,i}(0) = c_{L,i}(0) = [0, 0]^T$, $\Sigma_{h,i}(0) = \Sigma_{L,i}(0) = I_{2\times 2}$, $\hat{z}_i(0) = h_i(0)$ and $\hat{v}_i(0) = L_i(0)$.

Result 1

Figure 4.4 shows the result obtained for example 1, $J^* = 1580$ and $u^* = 0.7201$. The overall cost and the inputs are shown to converge very fast to J^* and u^* respectively.

This result shows the effectiveness and robustness of the proposed technique and validates the role of cooperation and coordinated control in multi-agent systems.

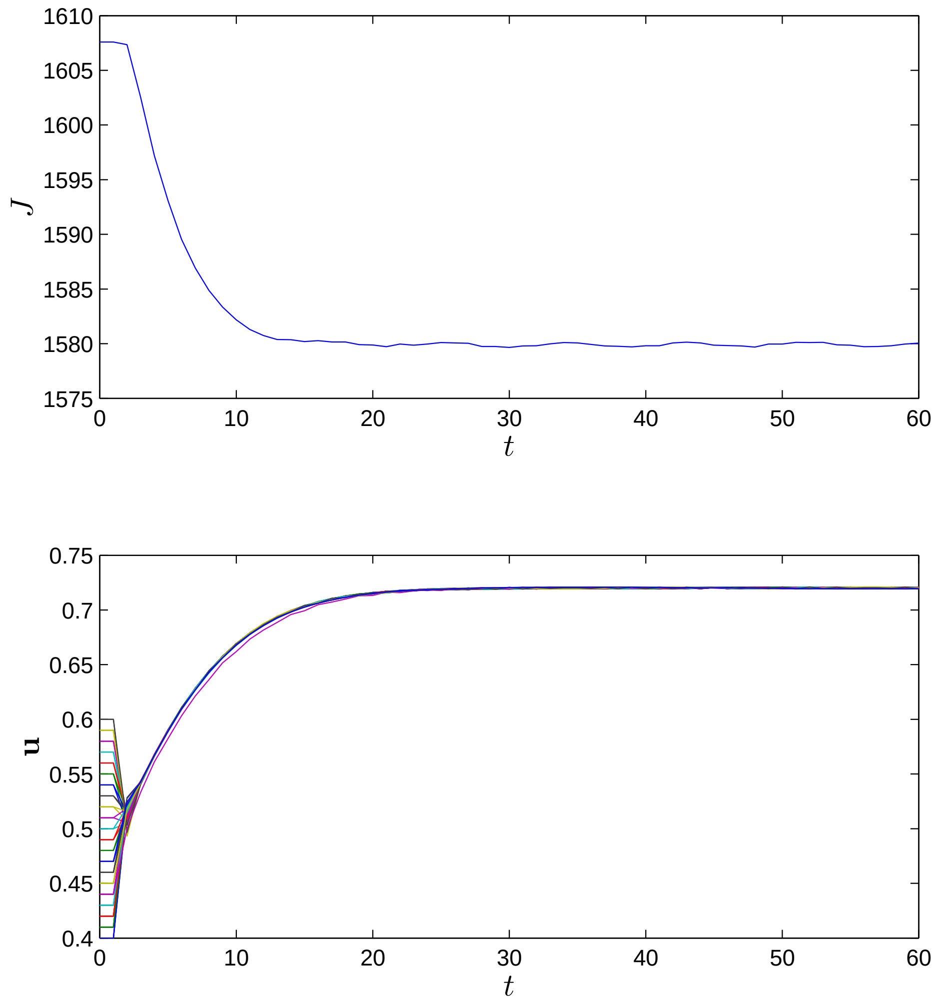

Figure 4.4: Plot of the overall cost and the inputs as a function of time for a 50 agent system when the communication network is undirected.

4.5.2 Example 2

Consider a system with 25 agents where the communication network is directed as seen in Figure 4.2. The local cost for the agents are the last 25 cost functions used in example 1.

$$D_d = \begin{bmatrix} 0 & 6.27 & 2.77 & \dots & 3.62 \\ 5.39 & 0 & 2.30 & \dots & 3.15 \\ 3.09 & 3.50 & 0 & \dots & 0.85 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2.29 & 8.51 & 5.01 & \dots & 0 \end{bmatrix}$$

The same initial conditions and tuning parameters as seen above were used except for ω and $\mathbf{u}(0)$. $d(t) = 0.3\,[\sin(500t), \sin(497t), \ldots, \sin(432t), \sin(429t)]^T$ and $\mathbf{u}(0) = [0, \ldots, 0.24]^T$.

Result 2

The result for example 2 is presented in Figure 4.5, $J^* = 1328$ and $u^* = 0.7731$. The overall cost is minimized and fast convergence (in approximately 22 seconds) to the optimum is recorded. This result confirms the effectiveness of the proposed technique.

4.6 Conclusion

A distributed proportional-integral extremum seeking control technique is proposed to solve a distributed optimization problem for a system with unknown network and unknown cost functions. The results show that in the absence of precise knowledge of network connectivity, global objectives can still be met. By taking measurements of the local cost and the local disagreement cost functions, the proposed technique ensures that the disagreement among the agents is minimized so that consensus on their inputs is reached and ultimately, the overall cost is minimized. The results adds to the literature on distributed extremum seeking control.

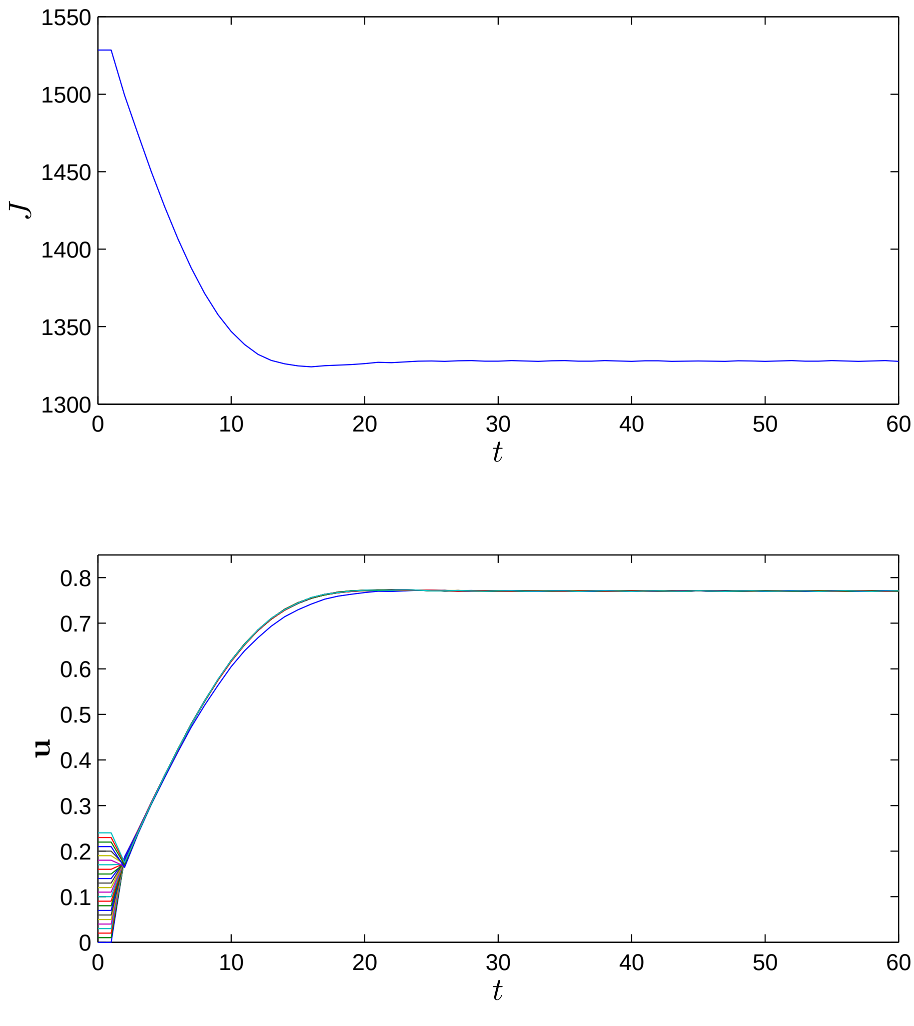

Figure 4.5: Plot of the overall cost and the input as a function of time for a 25 agent system when the communication network is directed.

Chapter 5

Conclusions

In Chapter 1 of this thesis, the basic idea of distributed control for use in the optimization of large scale systems involving multi-agents was introduced. The advantages of distributed control over centralized control were enumerated. Extremum seeking control (ESC) was also introduced as the real-time optimization technique to be employed since it requires no knowledge of the mathematical model(s) describing a complex nonlinear system to drive the system to its unknown optimum. In Chapter 2, a review of some of the useful contributions made in the area of extremum seeking control (which includes some of the proposed ESC techniques) was presented. Research work involving the application of extremum seeking control and ideas from the field of game theory in addressing problems of MAS were also reviewed. The focus of Chapter 3 was on the control and optimization of the power produced by wind farms using a distributed TVESC technique. Since aerodynamic interactions among wind turbines limit overall power capture in a wind farm and at the moment, there are no accurate models that perfectly describe these interaction. The use of extremum seeking control provides an effective mechanism to maximize the power produced. Useful results were obtained both in continuous-time and discrete-time. Comparisons were made between these results and those obtained in [3] were the PBESC technique was employed. In Chapter 4, the problem of distributed optimization over unknown networks was considered since most of the work done on distributed control of MAS have used the knowledge of network connectivity to achieve global objectives [3],[27] and [37]. It was shown that if an agent can have access to the measurements of the disagreement between its input and those of its neighbours, a PI-ESC technique can be designed such that global objectives are met. From the results obtained, it can be concluded that extremum seeking control is a powerful and an effective technique for steady-state real-time optimization of MAS in a distributed fashion. The distributed TVESC and the distributed PI-ESC techniques employed in this thesis are robust and show fast convergence to the unknown optimum. These techniques can be utilized in addressing other MAS problems as their effectiveness cannot be overemphasized.

5.1 Future Work

The assumptions made about the wind in Chapter 3 are not necessarily realistic since the wind is chaotic in nature. The next step will be to relax the assumptions and improve upon the control algorithm to accommodate the varying nature of the wind. Another future work will be to incorporate dynamics (unstable dynamics) to the problem addressed in Chapter 4, so that the problem of stabilization and optimization can be addressed simultaneously. Finally the two problems tackled in this thesis will be addressed using a time-varying communication network.

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