Adaptive Identification of Nonlinear Systems

by

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Abstract

This work presents three techniques for parameter identification for nonlinear systems. The methods presented are expanded from those presented in Adetola and Guay [3, 4, 5] and are intended to improve the performance of existing adaptive control systems. The first two methods exactly recover open-loop system parameters once a defined convergence condition is met. In either case, the true parameters are identified when the regressor matrix is of full rank and can be inverted. The third case uses a novel method developed in Adetola and Guay [5] to define a parameter uncertainty set. The uncertainty set is periodically updated to shrink around the true value of the parameters. Each method is shown to be applicable to a large class of linearly parameterized nonlinear discrete-time system. In each case, parameter convergence is guaranteed subject to an appropriate convergence condition, which has been related to a classical persistence of excitation condition. The effectiveness of the methods is demonstrated using a simulation example. The application of the uncertainty set technique to nonlinearly parameterized systems constitutes the main contribution of the thesis. The parameter uncertainty set method is generalized to the problem of adaptive estimation in nonlinearly parameterized systems, for both continuous-time and discrete-time cases. The method is demonstrated to perform well in simulation for a simplified model of a bioreactor operating under Monod kinetics.

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Chapter 1

Introduction

1.1 Motivation

Parameter identification is an important feature in many control situations. Many adaptive control systems are centered on an unknown reference trajectory. System dynamics often rely on a set of unknown system parameters. The ability to estimate these unknown model parameters is desirable in order reduce the impact of uncertainties and improve the robustness of closed-loop systems. It is generally known that efficient parameter convergence increases the robustness properties of closed-loop adaptive systems (Lin and Kanellakopoulos [22]). One such class of control applications are known as adaptive extremum-seeking control systems. The objective of an adaptive extremum-seeking control system is the optimization of a user defined cost function, that may depend on unknown system parameters. In applications of these control systems such as Guay et al. [15] and Wang et al. [27], knowledge of the system parameters is crucial. The performance of the parameter identification method has a great impact on the performance of the system.

1.2 Literature Survey

1.2.1 Adaptive Control

Goodwin and Sin [13] categorize control problems in three classes according to difficulty. Deterministic control problems deal with control systems that are perfectly known with no disturbances. Stochastic control problems deal with systems with known dynamics, subject to stochastic disturbances. Adaptive control refers to a class of problems for uncertain systems subject to parametric uncertainties and disturbances. Adaptive control systems are considered when the system's dynamics contain unknown parameters whose values are required for the design of a well-posed control system. The goal of adaptive control is to implement a control algorithm without complete knowledge of the system. Adaptive control has found many interesting applications in air flight control, bioprocess control (Bastin and Dochain [7]), and many other fields. It has recently been considered in the development of new techniques for real-time optimization of dynamic processes.

Incomplete knowledge of system dynamics can be a result of a number of uncertainties. Uncertainty can be introduced by exogenous disturbances, unmodelled dynamics or parametric uncertainty. This work concerns uncertainty as a result of unknown parametric. Ioannou and Sun [18] describe classical adaptive control as algorithms where the control system consists of a state or output feedback law that depends on uncertain parameters. Some identification scheme is utilized to provide estimates of the unknown parameters on-line, using the available information. States or outputs are estimated using estimates of the unknown parameters, a controller is designed to execute some control task on this predicted feedback.

Adaptive extremum-seeking control solves a class of adaptive control problems that deal with applying control to a system in order to track an optimum on a defined objective function that is dependent on unknown parameters. Parameter estimation or identification

is of great importance as parameter convergence is necessary to ensure that the true optimum of the unmeasured objective function is found. Conditions under which identification is guaranteed can be difficult to meet, and deal primarily with system state trajectories in closed-loop. Nevertheless, it is necessary to rely on such conditions to be able to guarantee that a given estimation scheme converges to the true parametric values. A common approach to solving these problems is Lyapunov-based adaptive extremum-seeking control. Examples for these applications are presented in Guay et al. [15] and Guay and Zhang [14].

Persistence of excitation

The process of online estimation is integral to adaptive control algorithms. In order to gather information regarding the system, there must be some sort of system excitation. In open loop system, input variables may be used to provide the necessary excitation. In closed-loop system, it is customary to consider injection of a perturbation signal in the system by introducing a dither signal about the desired setpoint.

Persistence of excitation (PE) conditions are defined to guarantee convergence of the parameter values to their true values (or to a neighbourhood of the true values). It is akin to the full rank requirement of the regressor matrix in standard regression analysis. They are sufficient conditions that must be imposed on the system's trajectories to ensure there is enough information about the parameters to claim parameter convergence. For systems subject to external excitation signals, it is said that the signals are sufficiently rich if they yield system trajectories that are persistently exciting. The book by Khalil [19] provides the following formal definition of persistence of excitation:

Definition 1.2.1. *A matrix of signals $\omega \in \mathbb{R}^{n \times p}$ is said to be persistently exciting if, at time t, there exists constants, c_1, c_2 and T_0 such that*

$$c_2 I_{p \times p} \geq \int_t^{t+T_0} \omega^T(\tau)\omega(\tau)\mathrm{d}\tau \geq c_1 I_{p \times p}. \tag{1.1}$$

From Goodwin and Sin [13], it follows that for discrete time signals,

Definition 1.2.2. *A matrix of signals $\omega_k \in \mathbb{R}^{n \times p}$ can be said to be persistently exciting, at time step k, if there exist constants, c_1, c_2 and K_0 such that*

$$c_2 I_{p \times p} \geq \sum_{i=k}^{k+K_0} \omega_i^T \omega_i \geq c_1 I_{p \times p}. \tag{1.2}$$

The implication of Definitions 1.2.1 and 1.2.2 is that at any time or time step, there exists a subsequent time or time step where, over the defined time period, the sum of the magnitudes of the signal in question (if a vector of signals or a single signal is concerned) is non-zero, or a matrix of signals is positive definite. Adetola and Guay [1] note that in many cases, including the ones presented in this work, the persistence of excitation condition is dependant on sufficient signal richness of an injected perturbation.

Parameter convergence in adaptive control may require external excitation that is sufficiently rich to ensure that some suitable PE condition is met. The main problem associated the addition of an external excitation signal (or dither signal) is that this additional excitation acts in a manner similar to an injected disturbance. This feature of adaptive control systems can, in fact, reduce transient performance. As stated above, PE conditions imply that the excitation requirements hold for all times. Recently, several results on intelligent excitation signals, where the magnitude of the excitation signal is adjusted as needed to ensure convergence, have become available from Adetola and Guay [2] and Cao and Wang [10]. Such results provide conditions under which one can reduce the impact of the external signals. However, they do no provide mechanisms to remove the external dither signal when parameter convergence can be confirmed.

Lin and Kanellakopoulos [22] and Adetola and Guay [1] identify a common issue in the study of persistence of excitation conditions with respect to convergence in adaptive control algorithms. In many cases, the relation between the system trajectories, the external dither signal and the persistence of excitation is unknown because the persistence of excitation condition may depend on the system's closed-loop trajectories which are dependent on the parameter estimates. As a result, it is very difficult to define conditions under which a dither signal can be guaranteed to be sufficiently rich a priori.

In this thesis, we focus on the approaches presented in Adetola and Guay [3, 4, 5]. These recent results provide effective techniques to guarantee parameter convergence that yield a number of alternatives to monitor external excitation. The techniques rely on alternative persistence of excitation conditions (referred to as convergence conditions) that can be monitored online to confirm that parameter convergence can be achieved in finite-time. As a result, it is possible to introduce mechanisms for dither signal removal when sufficient information about the unknown parameters has been gathered.

Lyapunov Stability

The concept of Lyapunov stability is central to several of the conclusions that are made in this work. Khalil [19] demonstrates that the Lyapunov stability theorem for continuous-time systems is

Theorem 1.2.1. *Consider the system:*

$$\dot{x} = f(x). \tag{1.3}$$

Let $x = 0$ be an equilibrium point for (1.3), and $D \subset R^n$ be a domain containing $x = 0$. Let $V : D \to R$ be a continuously differentiable function such that

$$V(0) = 0 \quad and \quad V(x) > 0 \quad in \quad D - \{0\}, \tag{1.4}$$

if

$$\dot{V}(x) \leq 0 \quad in \quad D - \{0\}, \tag{1.5}$$

then $x = 0$ is stable. Further, if

$$\dot{V}(x) < 0 \quad in \quad D - \{0\}, \tag{1.6}$$

then $x = 0$ is asymptotically stable.

Lin [23] shows that in the case of discrete-time systems the Lyapunov stability theorem becomes:

Theorem 1.2.2. *For the system*

$$x_{k+1} = x_k + F(x_k). \tag{1.7}$$

Let x_e be an equilibrium point for (1.7) where $V(x_e) = 0$, where $V(x_k)$ is a continuous, positive definite function $V : R^n \to R$, locally defined on a neighborhood U of the equilibrium point $x_e \in R^n$. Then if

$$V(x_{k+1}) - V(x_k) \leq 0 \quad \forall x_k \in U, \tag{1.8}$$

then x_e is stable. Further, if

$$V(x_{k+1}) - V(x_k) < 0 \quad \forall x_k \in U, \tag{1.9}$$

then x_e is asymptotically stable.

Theorems 1.2.1 and 1.2.2 will be used throughout this work to analyze the asymptotic behaviour of the proposed estimation techniques.

1.2.2 Identification Methods

In adaptive control, there exist two large classes of applications, known as identifier-based and non-identifier-based adaptive control algorithms. A non-identifier-based algorithm applies a control law which is defined for the appropriate class of systems, the control law does not require any knowledge of the open-loop system parameters. Non-identifier-based control algorithms use the relation between inputs and states or outputs to design control laws that will solve stabilization or tracking problems. Identifier-based methods differ in that they attempt to provide estimates for unknown system parameters, and then to achieve the desired control task based on these estimates. The methods presented in this thesis are directly applicable to identifier-based control algorithms. The identification methods used in these algorithms will be discussed more thoroughly below.

Ilchmann [17] states that the primary purpose of most non-identifier-based control algorithms is to develop a feedback control law that solves a stabilization or tracking problem without the complexity of an identification step. Though they appear starkly different from the identifier-based methods developed in this thesis, the results presented herein suggest that they could be coupled to methods we discuss. As a result of the decoupled control and identification tasks developed by Adetola and Guay [3, 4, 5] and expanded in this work, a control law developed for a non-identifier-based control algorithm, could potentially be utilized in parallel with the identification methods presented here.

Least Squares Methods

The application of least squares methods to on-line parameter estimation is well discussed in the literature. Goodwin and Sin [13] describe a least squares method as an optimization problem where a cost function, a sum of squared residuals, is minimized. The cost function or objective function is constructed based on the magnitude of the difference between the

true system state or output, and a system state or output predicted from estimated parameters. The solution to this problem is a set of parameter that minimizes the cost function. This corresponds to a set of parameters at which the gradient of the cost function is equal to zero. The application of a least squares algorithm to a linear system is rather simple as analytical solutions to the linear least squares problem are well known. When applied to a nonlinearly parameterized system it is often necessary to apply a numerical technique to find a solution.

A recursive least squares method may be applied to the type of identification problems addressed in this thesis. This method involves defining a function of the true state and the predicted state at some time step k and all time steps previous to k. The gradient of the cost function with respect to the unknown parameters is taken at each time step. The solution to the recursive least squares problem is a set of parameters that minimizes the cost function, or the set of parameters that produces the smallest sum of residuals. Billings and Voon [8] apply a common variation on the least squares method for nonlinear systems by introducing a forgetting factor. This factor applies a weight to the residuals in the cost function that emphasizes the influence of recent measurements.

Other Identification Techniques

Few recent studies are available regarding parameter identification for discrete-time systems in the context of adaptive control. Zhao and Kanellakopoulos [31, 32] propose a control method for both output-feedback systems and strict-feedback systems using a two-step approach that separates the identification algorithm from the control task. In the first step of this approach, the discrete-time system is driven to a state where an orthogonalized projection scheme is known to converge in finite-time. It is guaranteed that a sufficient amount of parameter information is gathered in a finite number of time steps. The second step uses the identified parameters to apply a control law as if the parameters were known. This approach assumes that the system is free from noise, though robustness is shown for

a small additive random noise.

1.3 Adaptive Identification

The primary contributions of this work are the adaptation and extension of parameter identification methods developed in Adetola and Guay [3, 4, 5], and the application of one of the methods to a broader class of systems.

The first method, the finite-time identification method, allows the direct and exact recovery of parameters immediately once a convergence condition is met. This method requires the online inversion and computation of the rank of a regressor matrix. Because the parameters are identified immediately, it is possible to remove any excitation signals at the moment the parameters are recovered. To apply this method to discrete time systems it was necessary to apply a modified state predictor that ensured that the auxiliary variable remained finitely-summable, and could be tracked by a simple recursion.

The second method is a refinement of the first, it uses an adaptive compensator to eliminate the need for online inversion and rank computation of a matrix. The parameter estimation error can be shown to be non-increasing, and convergence is guaranteed once a convergence condition is met. Minimal modification was required to adapt this method for use with discrete-time systems. The proof of convergence of the parameter estimation error is presented in a simpler manner than in the literature while demonstrating the guarantee that performance is recovered from the finite-time algorithm.

The third method defines a parameter uncertainty set, similar to the one developed in Adetola and Guay [5], that evolves based on a worst-case estimate in order to shrink around the true parameter values. Further, the parameter estimates are not allowed to fall outside the uncertainty set. This method ensures convergence of the parameter estimates to the true parameters, provided the true parameters fall within the initial uncertainty set, as the update algorithm ensures non-exclusion of the true parameter values. In a practical sense

this algorithm guarantees parameter convergence as long as a nominally good estimate of the parameters can be made to initiate the algorithm.

The parameter uncertainty set method presented in Adetola and Guay [5] is modified significantly in its application to discrete-time systems. The new state predictor developed for the finite-time identification method is implemented to ensure that the auxiliary variable estimation error is a finitely-summable signal. A projection algorithm is developed based on the principles presented in Goodwin and Sin [13]. The project algorithm guarantees that the parameter estimates remain within a known compact set while preserving the monotonous decrease of the candidate Lyapunov function for the parameter estimation error. The convergence properties of the system, operating under the parameter uncertainty set are guaranteed, provided a defined convergence criteria is met. This convergence criteria has been related to a classical definition of persistence of excitation.

The main contribution of this work is the generalization of an uncertainty set-update method to nonlinearly parameterized systems. Work in this area has been limited in the literature to specific classes of systems. The most significant approach to solve this problem can be found in several works by Cao et al. [11], Annaswamy et al. [6], Kojic et al. [20] and Netto et al. [26]. Their approach exploits convexity of the system dynamics with respect to the parameters to develop a class of min-max adaptive estimation routines. A gradient-based approach is proposed subject to a worst-case parameter set. Several authors have also studied this class of problems for specific applications. One such application presented by Boskovic [9] and Zhang and Guay [29], is the study of microbial growth kinetics where most models, due to the importance of classical enzyme kinetics models, are nonlinearly parameterized. The nonlinearity of these models precludes the use of normal techniques to establish parameter convergence. For Monod models, it can be demonstrated that parameter convergence can be achieved subject to a conservative persistence of excitation condition that can only be derived using highly tailored Lyapunov-based arguments. Another leading approach presented by Zhang et al. [30],Wang [28],Ge [12] consists of approximating the nonlinearity

using neural networks. The main drawbacks of these techniques is that such approximations cannot be used to uniquely reconstruct the unknown parameter vector. In this work, a different approach is undertaken and it is demonstrated that nonlinearly parameterized systems may be treated as linearly parameterized systems subject to bounded uncertainties, where the nonlinearity is treated as a bounded disturbance.

1.4   Overview of the Thesis

The primary focus of this thesis is a set of solutions to a parametric identification problem. Solutions are presented for a series of different incarnations of the system in question. In Chapter 2 two simple methods to recover exact parameter values in finite-time are applied to systems without uncertainty, other than the unknown parameters. Furthermore, this chapter demonstrates the application of the parameter uncertainty set method to linearly parameterized systems with bounded disturbances. In Chapters 3 and 4 it is demonstrated that the method in question can be applied to nonlinearly parameterized systems, where the nonlinearity is treated as a bounded disturbance. In total this work comprises three methods in five applications, each method is proven to guarantee convergence of the parameter estimates to the true values of the parameters given a defined convergence condition. The convergence condition have been related to a classical definition of persistence of excitation, and it is concluded that persistently exciting state trajectories are a sufficient condition in all cases to guarantee convergence.

Chapter 2

Linearly Parameterized Discrete-time Systems

The methods presented in this section are adapted from those presented in Adetola and Guay [3, 4, 5]. They are modified to guarantee convergence of parameter error for discrete-time systems of the form (2.1), in which the parameters appear linearly. The first section in this chapter applies a method from literature, that guarantees parameter convergence in finite-time for continuous-time systems, to discrete-time systems. The second method solves a computational complexity problem presented in the first method. The third section demonstrates a method that provides guaranteed convergence (provided a convergence condition is met) of the parameter estimation error in the presence of certain types of exogenous disturbances.

2.1 Finite-time Parameter Identification

2.1.1 Overview

This section demonstrates the simplest, most straight forward identification technique presented in this work. The finite-time identification method is most applicable to simple systems, where problems of computational complexity will not arise. Because this method involves the inversion of the parameter covariance matrix (of size $p \times p$ where p is the number of parameters) at each time step, for systems with a large number of parameters, the computational complexity can become quite high. However, for systems with few parameters this method guarantees, exact and immediate convergence of the parameter values to their true values, once the appropriate convergence condition is met.

2.1.2 Problem Description and Assumptions

Consider the system:

$$x_{k+1} = x_k + F(x_k, u_k) + G(x_k, u_k)\theta, \tag{2.1}$$

where $x_k \in \mathbb{R}^n$ is a state at some time step k, $u_k \in \mathbb{R}^m$ is the control input at some time step k, and $\theta \in \mathbb{R}^p$ is a column vector of system parameters.

Assumption 2.1.1. *The state of the system, x_k, is known at all time steps k.*

Assumption 2.1.2. *There is some known, bounded control law, u_k, that achieves some control objective.*

Assumption 2.1.3. *The state and input variables evolve on a compact set, $x_k \in \mathbb{X} \subset \mathbb{R}^n$, $u_k \in \mathbb{U} \subset \mathbb{R}^m$.*

2.1.3 Theory

Consider the following state predictor:

$$\hat{x}_{k+1} = \hat{x}_k + F(x_k, u_k) + G(x_k, u_k)\hat{\theta}_{k+1} + K_k e_k - \omega_k(\hat{\theta}_k - \hat{\theta}_{k+1}) + K_k \omega_k(\hat{\theta}_k - \hat{\theta}_{k+1}), \quad (2.2)$$

where $\hat{\theta}_k$ is the vector of parameter estimates at time step k given by any update law, K_k is a correction factor at time step k, $e_k = x_k - \hat{x}_k$ is the state estimation error at time step k. The variable ω_k is an output filter generated by

$$\begin{aligned} \omega_{k+1} &= \omega_k + G(x_k, u_k) - K_k \omega_k \\ \omega_0 &= 0. \end{aligned} \quad (2.3)$$

Let the parameter estimation error at some time step k be $\tilde{\theta}_k = \theta - \hat{\theta}_k$. Now from (2.2) and (2.1) the state estimation error at time step $k + 1$ is given by

$$e_{k+1} = e_k + G(x_k, u_k)\tilde{\theta}_{k+1} - K_k e_k + \omega_k(\hat{\theta}_k - \hat{\theta}_{k+1}) - K_k \omega_k(\hat{\theta}_k - \hat{\theta}_{k+1}). \quad (2.4)$$

Define the auxiliary variable:

$$\eta_k = e_k - \omega_k \tilde{\theta}_k. \quad (2.5)$$

From (2.3),(2.4) and (2.5) it follows that

$$\begin{aligned} \eta_{k+1} &= \eta_k - K_k \eta_k \\ \eta_0 &= e_0. \end{aligned} \quad (2.6)$$

Let $Q_k \in \mathbb{R}^{p \times p}$ and $C_k \in \mathbb{R}^p$ be defined as

$$
\begin{aligned}
Q_{k+1} &= Q_k + \omega_k^T \omega_k \\
Q_0 &= 0 \\
C_{k+1} &= C_k + \omega_k^T(\omega_k \hat{\theta}_k + e_k - \eta_k) \\
C_0 &= 0.
\end{aligned}
$$

(2.7)

(2.8)

Lemma 2.1.1. *If there exists some time step k_c such that Q_{k_c} is invertible, that is,*

$$
Q_{k_c} = \sum_{i=0}^{k_c} \omega_i^T \omega_i \succ 0,
$$

(2.9)

then the parameters are given by

$$
\theta = Q_k^{-1} C_k \qquad \forall k \geq k_c.
$$

(2.10)

Proof. This result can be shown from

$$
Q_k \theta = \sum_{i=0}^{k} \omega_i^T \omega_i [\hat{\theta}_i + \tilde{\theta}_i].
$$

(2.11)

Upon substitution with (2.5), it follows that

$$
\begin{aligned}
\theta &= Q_k^{-1} \sum_{i=0}^{k} \omega_i^T (\omega_i \hat{\theta}_i + e_i - \eta_i) \\
&= Q_k^{-1} C_k \quad \forall k \geq k_c,
\end{aligned}
$$

(2.12)

which proves the result. $\qquad \square$

Persistence of Excitation

For the purposes of the result presented in this section, persistence of excitation conditions are imposed in an attempt to guarantee the convergence of the identification task. The persistence of excitation conditions are sufficient to guarantee that will occur. In this section a necessary condition for parameter convergence is

$$rank(Q_{k_c}) = p, \tag{2.13}$$

or

$$rank(\sum_{i=0}^{k_c} \omega_i^T \omega_i) = p. \tag{2.14}$$

The classical persistence of excitation condition from (1.2.2) being met at any time step by the output filter, is sufficient to guarantee that the parameter convergence condition will be met at some time step k_c.

2.1.4 Simulation Example

Consider the following nonlinear system:

$$
\begin{aligned}
x_1(k+1) &= x_1(k) + 0.01[x_2(k) + u_3(k) + x_3(k)\theta_1 - x_1(k)\theta_3] \\
x_2(k+1) &= x_2(k) + 0.01[(1 + x_3(k))u_1(k) + x_1(k)\theta_2 - x_2(k)\theta_3] \\
x_3(k+1) &= x_3(k) + 0.01[x_1(k) + u_2(k) + x_2(k)\theta_3 - x_3(k)\theta_3], \tag{2.15}
\end{aligned}
$$

where $\theta = [\theta_1, \theta_2, \theta_3]^T$. The input is taken as constant, $u_k = [-0.1\ 0.1\ 0.2]^T$. The true parameter values are $\theta = [34\ 3\ 0.72]^T$. Fig. 2.1 shows the parameter estimates converging to the true values immediately at the time step when the regressor matrix Q_k has full rank. It is known from the simulation results and may be observed in the time course plot

of the parameter estimates that the convergence condition is met at time step $k_c = 95$. As expected, the convergence of the parameter estimates to their true unknown values is immediate once the persistence of excitation condition is met. The main drawback with the technique is the requirement to check the non-singularity of Q_k at each step. This requirement can add, in some cases, some computational complexity that may reduce the applicability of this technique to more complex problems. This problem will be addressed in subsequent sections.
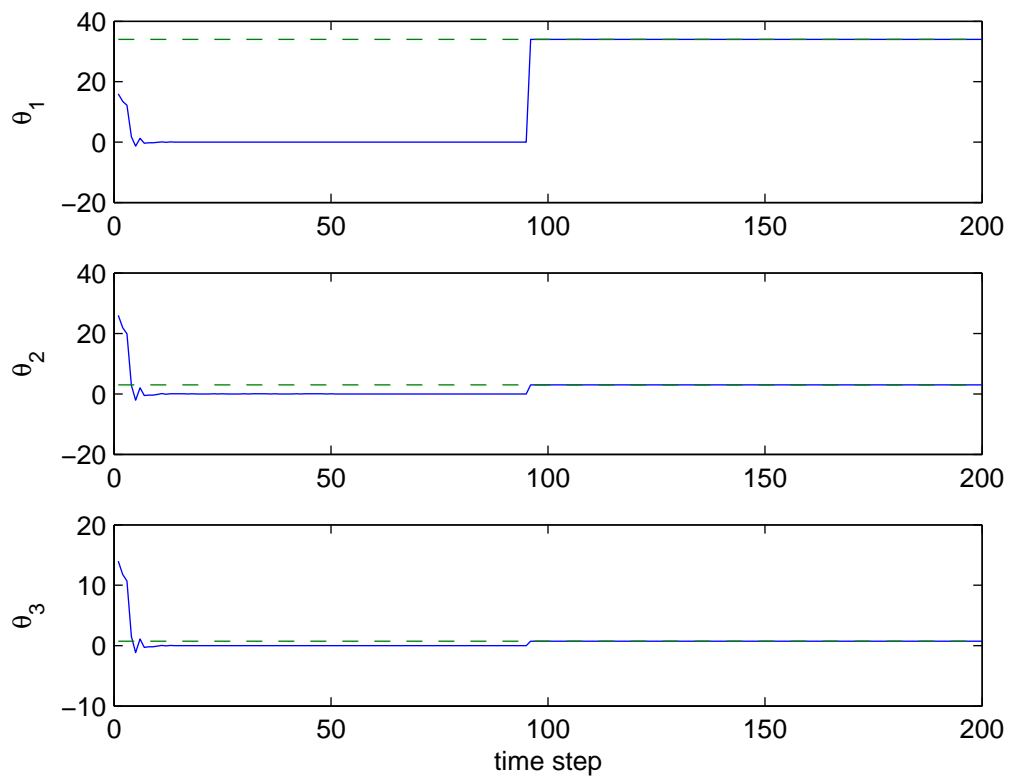
Figure 2.1: Time course plot of the parameter estimates and true values, under the finite time estimation algorithm, the dashed lines (- -) represent the true parameter values, the solid lines (–) represent the parameter estimates.

2.2 Adaptive Compensation Design

2.2.1 Overview

This section presents an identification method that provides a refinement of the finite-time identification method. Application of the finite-time identifier is problematic since it requires that the non-singularity of Q_k be checked at all time steps. In this section, an adaptive compensation design proposed by Adetola and Guay [4], is implemented. The technique recovers exponential stability of the parameter estimation error in finite-time without the need to test the matrix Q_k.

2.2.2 Adaptive Compensation

We consider the system (2.1) subject to Assumptions 2.1.1, 2.1.2 and 2.1.3.

Consider the state predictor for system (2.1):

$$\hat{x}_{k+1} \quad = \quad \hat{x}_k + F(x_k, u_k) + G(x_k, u_k)\theta^o + K_k(x_k - \hat{x}_k), \tag{2.16}$$

where $K_k > 0$ and θ^o is the vector of initial parameter estimates.

As in the preceding section, define the auxiliary variable and the filter dynamic as

$$\eta_k \quad = \quad x_k - \hat{x}_k - \omega_k(\theta - \theta^o) \tag{2.17}$$

$$\omega_{k+1} \quad = \quad \omega_k + G(x_k, u_k) - K_k\omega_k \tag{2.18}$$

$$\omega_0 \quad = \quad 0.$$

The auxiliary variable η_k can be generated from

$$\eta_{k+1} \quad = \quad \eta_k - K_k\eta_k$$

$$\eta_0 \quad = \quad e_0. \tag{2.19}$$

Let Q and C be generated by

$$
\begin{aligned}
Q_{k+1} &= Q_k + \omega_k^T \omega_k \\
Q_0 &= 0 \\
C_{k+1} &= C_k + \omega_k^T(\omega_k \theta^o + e_k - \eta_k) \\
C_0 &= 0,
\end{aligned}
$$
(2.20)

(2.21)

and let k_c be a time step at which $Q_{k_c} \succ 0$.

The parameter update law proposed in Adetola and Guay [4] is modified to become

$$
\hat{\theta}_{k+1} = \hat{\theta}_k + \Gamma_k(C_k - Q_k \hat{\theta}_k).
$$
(2.22)

It follows from (2.22) that the dynamics of the parameter estimation error are

$$
\tilde{\theta}_{k+1} = \tilde{\theta}_k - \Gamma_k(C_k - Q_k \hat{\theta}_k).
$$
(2.23)

Since, $Q_k \theta = C_k$, it follows that

$$
\begin{aligned}
\tilde{\theta}_{k+1} &= \tilde{\theta}_k - \Gamma_k(Q_k \theta - Q_k \hat{\theta}_k) \\
\tilde{\theta}_{k+1} &= (I - \Gamma_k Q_k)\tilde{\theta}_k.
\end{aligned}
$$
(2.24)

(2.25)

Define the variable: $\Gamma_k = \frac{1}{\|Q_k\|+\epsilon}$, where ϵ is some small positive number. It follows that

$$
\tilde{\theta}_{k+1} = (I - \frac{Q_k}{\|Q_k\| + \epsilon})\tilde{\theta}_k.
$$
(2.26)

It follows from (2.26) that $\tilde{\theta}$ is non increasing, and for all time steps $k \geq k_c$,

$$
\lim_{k\to\infty} \tilde{\theta}_k = 0.
$$
(2.27)

Persistence of Excitation

In this section the condition for parameter convergence, that the matrix Q_k has full rank, becomes a sufficient, not a necessary condition. It is possible that identification could occur without (2.13) being met, as it has been proven that the parameter estimation is non-increasing at all time steps, regardless of whether the convergence condition is met. It is possible to conclude however, that if the classical persistence of excitation condition is met by the output filter at any time step, the convergence condition is guaranteed.

2.2.3 Simulation Example

Consider the following nonlinear system:

$$
\begin{aligned}
x_1(k+1) &= x_1(k) + 0.01[x_2(k) + u_3(k) + x_3(k)\theta_1 - x_1(k)\theta_3] \\
x_2(k+1) &= x_2(k) + 0.01[(1 + x_3(k))u_1(k) + x_1(k)\theta_2 - x_2(k)\theta_3] \\
x_3(k+1) &= x_3(k) + 0.01[x_1(k) + u_2(k) + x_2(k)\theta_3 - x_3(k)\theta_3],
\end{aligned} \tag{2.28}
$$

where $\theta^T = [\theta_1, \theta_2, \theta_3]$. The input is taken as constant, $u_k = [-0.1 \ 0.1 \ 0.2]^T$. The true parameter values are $\theta = [3.4 \ 3 \ 0.72]^T$.

It is known from the simulation data that the parameter convergence condition is met at time step $k_c = 95$, demonstrating that performance of the finite-time identification algorithm is recovered exactly. It can be seen in Fig. 2.2 that the parameter estimates converge to the true parameter values asymptotically beyond time step k_c. While convergence is slower in this case than in the method presented in Section 2.1, eliminating the need for online matrix rank calculation and inversion decreases the computational complexity of the algorithm. Both methods solve an identification problem for the same type of system, and it is demonstrated that the parameter convergence condition is met in the same number of time steps. This method represents an improvement over the finite-time method if

the computational complexity limitation of the previous method outweighs the additional convergence time required by the adaptive compensator method. It is of note that parameter scaling becomes important in this case. An increase of an order of magnitude in one of the parameters can increase the time to identification for that parameter by 20-30 times, even though the convergence condition will be met at the same time step.

One obvious disadvantage of the finite-time and adaptive compensator techniques is that they are susceptible to the effect external disturbances and unmodelled dynamics. This is especially true in the finite-time technique since the parameter convergence condition requires the exact knowledge of the rank of Q_k. Since the knowledge of this rank can be subject to effect of unmodelled uncertainties, the finite-time technique may prove to be unreliable in application. It is important to note that some degree of nominal robustness may be achieved by an appropriate choice of the tuning parameter K_k. In this next section, a technique is presented that takes into account the effect of unmodelled uncertainties.
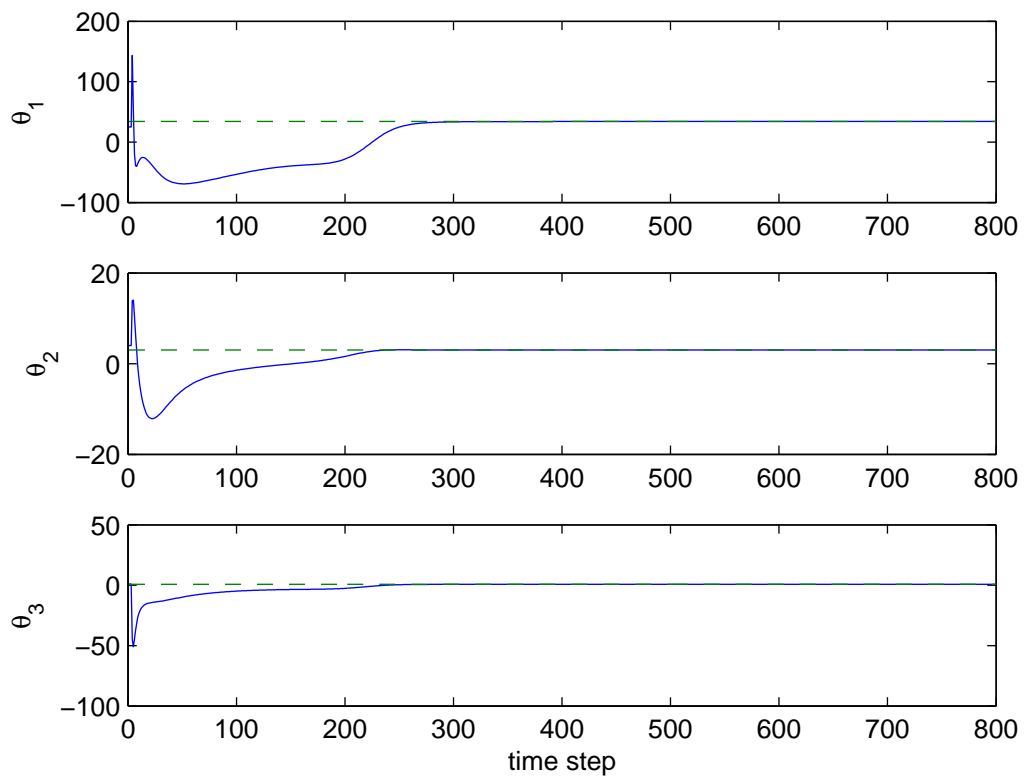
Figure 2.2: Time course plot of the parameter estimates and true values, under the adaptive compensator algorithm, the dashed lines (- -) represent the true parameter values, the solid lines (–) represent the parameter estimates.

2.3 Parameter Uncertainty Set Estimation

2.3.1 Overview

The identification method presented in this section addresses a slightly different problem than the methods presented above. The adaptive compensator and finite-time identification methods provide effective mechanisms to recover the exact system parameters under certain circumstances. However, the properties of these designs can be lost in the presence of exogenous disturbance variables and model mismatch. In this section, a parameter estimation technique is proposed to handle nonlinear systems subject to exogenous disturbance variables. The technique generalizes a novel uncertainty set-update formulation devised in Adetola and Guay [5] that provides robust performance when coupled to a parameter estimation technique similar to the adaptive compensator Design.

2.3.2 Problem Description and Assumptions

In this section, we consider a class of uncertain linearly parameterized systems defined as follows:

$$x_{k+1} = x_k + F(x_k, u_k) + G(x_k, u_k)\theta + \vartheta_k, \tag{2.29}$$

where $x_k \in \mathbb{R}^n$ is a state at some time step k, $u_k \in \mathbb{R}^m$ is the control input at some time step k, and $\theta \in \mathbb{R}^p$ is a column vector of system parameters.

Assumption 2.3.1. *The state of the system, x_k, is known at all time steps k.*

Assumption 2.3.2. *There is some known, bounded control law, u_k ,that achieves some control objective.*

Assumption 2.3.3. *The state and input variables evolve on a compact set, $x_k \in \mathbb{X} \subset \mathbb{R}^n$, $u_k \in \mathbb{U} \subset \mathbb{R}^m$.*

Assumption 2.3.4. *$\vartheta_k \in \mathbb{R}^n$ is a vector of bounded disturbances that satisfies $\|\vartheta_k\| \leq M_\vartheta < \infty \quad \forall k$, where M_ϑ is a positive constant.*

Assumption 2.3.5. *It is assumed that θ is uniquely identifiable and lies within an initially known compact set defined by the ball function $\Theta^0 = B(\theta_0, z_0)$, where θ_0 is an initial estimate of the unknown parameters and z_0 is an initial estimate of the radius of the parameter uncertainty set.*

2.3.3 Adaptive Estimation of Uncertain Systems

Parameter Update

Consider the uncertain nonlinear system (2.29). Using the state predictor defined in (2.2) and the output filter defined in (2.3), the prediction error $e_k = x_k - \hat{x}_k$ is given by

$$
\begin{aligned}
e_{k+1} &= e_k + G(x_k, u_k)\tilde{\theta}_{k+1} - K_k e_k + \omega_k(\hat{\theta}_k - \hat{\theta}_{k+1}) - K_k \omega_k(\hat{\theta}_k - \hat{\theta}_{k+1}) + \vartheta_k \\
e_0 &= x_0 - \hat{x}_0.
\end{aligned}
\tag{2.30}
$$

The auxiliary variable η_k dynamics are as follows

$$
\begin{aligned}
\eta_{k+1} &= e_{k+1} - \omega_{k+1}\tilde{\theta}_{k+1} + \vartheta_k, \\
\eta_0 &= e_0.
\end{aligned}
\tag{2.31}
$$

Since ϑ_k is unknown, it is necessary to use an estimate, $\hat{\eta}$, of η . The estimate is generated by the recursion

$$
\begin{aligned}
\hat{\eta}_{k+1} &= \hat{\eta}_k - K_k \hat{\eta}_k, \\
\hat{\eta}_0 &= \eta_0.
\end{aligned}
\tag{2.32}
$$

The resulting dynamics of the η estimation error are

$$\tilde{\eta}_{k+1} = \tilde{\eta}_k - K_k \tilde{\eta}_k + \vartheta. \tag{2.33}$$

Let the identifier matrix Σ_k be defined as

$$\begin{aligned}
\Sigma_{k+1} &= \Sigma_k + \omega_k^T \omega_k \tag{2.34}\\
\Sigma_0 &= \alpha I \succ 0,
\end{aligned}$$

where $\alpha \in \mathbb{R} > 0$. The inverse of the identifier matrix is generated by the recursion

$$\begin{aligned}
\Sigma_{k+1}^{-1} &= \Sigma_k^{-1} - \Sigma_k^{-1^T} \omega_k^T \left(I + w_k \Sigma_k^{-1} w_k^T \right)^{-1} \omega_k \Sigma_k^{-1} \tag{2.35}\\
\Sigma_0^{-1} &= \frac{1}{\alpha} I \succ 0.
\end{aligned}$$

From equations (2.2), (2.3), and (2.33) the preferred parameter update law is

$$\hat{\theta}_{k+1} = \hat{\theta}_k + \Sigma_k^{-1} \omega_k^T \left(I + w_k \Sigma_k^{-1} w_k^T \right)^{-1} (e_k - \hat{\eta}_k). \tag{2.36}$$

To ensure that the parameter estimates remain within the constraint set Θ_k, we propose to use a projection operator of the form

$$\bar{\hat{\theta}}_{k+1} = \mathrm{Proj}\{\hat{\theta}_k + \Sigma_k^{-1} \omega_k^T \left(I + w_k \Sigma_k^{-1} w_k^T \right)^{-1} (e_k - \hat{\eta}_k), \Theta_k\}. \tag{2.37}$$

The operator *Proj* represents an orthogonal projection onto the boundary of the uncertainty set applied to the parameter estimate. The parameter uncertainty set is defined by the ball function $B(\theta_c, z_{\theta_c})$, where θ_c and z_{θ_c} are the parameter estimate and set radius found at the latest set update.

Following Goodwin and Sin [13], the projection operator is designed such that:

- $\bar{\bar{\theta}}_{k+1} \in \Theta_k$

- $\bar{\bar{\theta}}_{k+1}^T \Sigma_{k+1} \bar{\bar{\theta}}_{k+1} \leq \tilde{\theta}_{k+1}^T \Sigma_{k+1} \tilde{\theta}_{k+1}.$

It will be shown that the parameter update law defined in (3.18) guarantees convergence of parameter estimates to the true values.

We employ the following lemma, as described by Haddad [16].

Lemma 2.3.1. *Consider the system:*

$$x_{k+1} = Ax_k + Bu_k, \tag{2.38}$$

where A is a stable matrix with eigenvalues inside the unit circle and B is a matrix of appropriate dimension. Then it can be shown that

$$\sum_{k=0}^{k-1} x_{k+1}^T x_{k+1} \leq \delta^2 \sum_{k=0}^{K-1} u_k^T u_k \tag{2.39}$$

for some $\delta > 0$ and $K - 1 > 0$.

Let l_2 denote the space of square finitely-summable signals and consider the following lemma.

Lemma 2.3.2. *The identifier (2.35) and parameter update law (3.18) are such that $\tilde{\theta}_k = \theta_k - \hat{\theta}_k$ is bounded. Furthermore, if*

$$\vartheta_k \in l_2 \quad or \quad \sum_{k=0}^{\infty}[\|\tilde{\eta}_k\|^2 - \gamma\|e_k - \hat{\eta}_k\|^2] < +\infty \tag{2.40}$$

and

$$\lim_{k\to\infty} \Sigma_k = \infty \tag{2.41}$$

are satisfied, then $\tilde{\theta}_k$ converges to 0 asymptotically.

Proof. Let $V_{\tilde{\theta}_k} = \tilde{\theta}_k^T \Sigma_k \tilde{\theta}_k$, it follows from the properties of the projection operator that

$$V_{\tilde{\theta}_{k+1}} - V_{\tilde{\theta}_k} = \bar{\bar{\theta}}_{k+1}^T \Sigma_{k+1} \bar{\bar{\theta}}_{k+1} - \tilde{\theta}_k^T \Sigma_k \tilde{\theta}_k \leq \tilde{\theta}_{k+1}^T \Sigma_{k+1} \tilde{\theta}_{k+1} - \tilde{\theta}_k^T \Sigma_k \tilde{\theta}_k.$$

Using the parameter update law, $\tilde{\theta}_{k+1}$ can be written as

$$\tilde{\theta}_{k+1} = \tilde{\theta}_k - \Sigma_k^{-1} \omega_k^T \left(I + w_k \Sigma_k^{-1} w_k^T\right)^{-1} (e_k - \hat{\eta}_k)$$
$$= \tilde{\theta}_k - \Sigma_k^{-1} \omega_k^T \left(I + w_k \Sigma_k^{-1} w_k^T\right)^{-1} (w_k \tilde{\theta}_k + \tilde{\eta}_k)$$

or,

$$\tilde{\theta}_{k+1} = \Sigma_{k+1}^{-1} \Sigma_k \tilde{\theta}_k - \Sigma_k^{-1} \omega_k^T \left(I + w_k \Sigma_k^{-1} w_k^T\right)^{-1} \tilde{\eta}_k. \tag{2.42}$$

Upon substitution of the parameter update law, the identifier matrix dynamics, the filter dynamics and the auxiliary variable dynamics, the rate change of the $V_{\tilde{\theta}_k}$ is given by

$$\begin{aligned} V_{\tilde{\theta}_{k+1}} - V_{\tilde{\theta}_k} &\leq -(e_k - \hat{\eta}_k)^T \left(I + w_k \Sigma_k^{-1} w_k^T\right)^{-1} (e_k - \hat{\eta}_k) \\ &+ \tilde{\eta}_k^T \left(I + w_k \Sigma_k^{-1} w_k^T\right)^{-1} \tilde{\eta}_k. \end{aligned} \tag{2.43}$$

From the $\tilde{\eta}_k$ dynamics given in (2.33), it follows from Lemma 2.3.1 if $\vartheta_k \in l_2$ then $\tilde{\eta}_k \in l_2$. Taking the limit as $k \to \infty$, the inequality becomes

$$\begin{aligned} \lim_{k \to \infty} V_{\tilde{\theta}_k} &= V_{\tilde{\theta}_0} + \sum_{k=0}^{\infty} V_{\tilde{\theta}_{k+1}} - V_{\tilde{\theta}_k} \tag{2.44} \\ &\leq V_{\tilde{\theta}_0} - \sum_{k=0}^{\infty} \left[(e_k - \hat{\eta}_k)^T \left(I + w_k \Sigma_k^{-1} w_k^T\right)^{-1} (e_k - \hat{\eta}_k) \right] \tag{2.45} \\ &+ \sum_{k=0}^{\infty} \left[\tilde{\eta}_k^T \left(I + w_k \Sigma_k^{-1} w_k^T\right)^{-1} \tilde{\eta}_k \right]. \tag{2.46} \end{aligned}$$

By the boundedness of the trajectories of the system, it follows that there exists a number

$\Gamma > 0$ such that

$$1 \geq \| \left(I + w_k \Sigma_k^{-1} w_k^T \right)^{-1} \| \geq \Gamma.$$

As a result, the following inequality is obtained

$$\lim_{k \to \infty} V_{\tilde{\theta}_k} \leq V_{\tilde{\theta}_0} - \gamma \sum_{k=0}^{\infty} \left[(e_k - \hat{\eta}_k)^T (e_k - \hat{\eta}_k) \right] + \sum_{k=0}^{\infty} \left[\tilde{\eta}_k^T \tilde{\eta}_k \right]. \tag{2.47}$$

Therefore if the conditions (2.40) are met then the right hand side of (2.47) is finite. As a result, it is possible to conclude that, if the parameter convergence condition, (2.41), is met then

$$\lim_{k \to \infty} \tilde{\theta}_k = 0, \tag{2.48}$$

as required. □

Set Update

An update law that measures the worst-case progress of the parameter update law is adapted from the one proposed in Adetola and Guay [5], and is defined as follows:

$$z_{\theta_k} = \sqrt{\frac{V_{z\theta_k}}{4\lambda_{min}[\Sigma_k]}} \tag{2.49}$$

$$V_{z\theta_{k+1}} = V_{z\theta_k} - (e_k - \hat{\eta}_k)^T \left(I + w_k \Sigma_k^{-1} w_k^T \right)^{-1} (e_k - \hat{\eta}_k) + (\frac{M_\vartheta}{K_k})^2 \tag{2.50}$$

$$V_{z\theta_0} = 4\lambda_{max}[\Sigma_0](z_{\hat{\theta}_0})^2. \tag{2.51}$$

The parameter uncertainty set, defined by the ball function $B(\theta_c, z_{\theta_c})$ is updated using the parameter update law (2.37) and the error bound (2.49) according to the following algorithm:

Algorithm 2.3.1. *Beginning at time step $k = 0$, the set is adapted according to the following iterative process*

1. **Initialize** $z_{\theta_c} = z_{\theta_0}, \theta_c = \hat{\theta}_0$

2. *At time step k, using equations (2.37) and (2.49) perform the update*

$$(\theta_c, z_{\theta_c}) = \begin{cases} (\hat{\theta}_k, z_{\theta_k}) & \text{if } z_{\theta_k} \leq z_{\theta_c} - \|\hat{\theta}_k - \theta_c\| \\ (\theta_c, z_{\theta_c}) & \text{otherwise} \end{cases} \tag{2.52}$$

3. *Return to step two and iterate, incrementing to time step $k + 1$*

Lemma 2.3.3. *The algorithm ensures that*

1. *the set is only updated when updating will yield a contraction,*

2. *the dynamics of the set error bound described in (2.49) are such that they ensure the non-exclusion of the true value $\theta \in \Theta_k$, $\forall k$ if $\theta_0 \in \Theta_0$.*

Proof. 1. If $\Theta_{k+1} \not\subseteq \Theta_k$ then

$$\sup_{s \in \Theta_{k+1}} \|s - \hat{\theta}_k\| \geq z_{\theta_k} \tag{2.53}$$

However, it is guaranteed by the set update algorithm presented, that Θ, at update times, obeys the following

$$\begin{aligned} &\sup_{s \in \Theta_{k+1}} \|s - \hat{\theta}_k\| \\ \leq\ &\sup_{s \in \Theta_{k+1}} \|s - \hat{\theta}_{k+1}\| + \|\hat{\theta}_{k+1} - \hat{\theta}_k\| \\ \leq\ &z_{\theta_{k+1}} + \|\hat{\theta}_{k+1} - \hat{\theta}_k\| \leq z_{\theta_k}. \end{aligned} \tag{2.54}$$

This contradicts (2.53). Therefore, $\Theta_{k+1} \subseteq \Theta_k$ at time steps where Θ is updated.

2. It is known, by definition, that

$$V_{\tilde{\theta}_k} \leq V_{z\theta_k}, \qquad \forall k \geq 0. \tag{2.55}$$

Since, $V_{\tilde{\theta}_k} = \tilde{\theta}_k^T \Sigma_k \tilde{\theta}_k$,

$$\|\tilde{\theta}_k\|^2 \leq \frac{V_{z\theta_k}}{\lambda_{min}[\Sigma_k]} = 4z_{\theta_k}^2, \qquad \forall k \geq 0. \tag{2.56}$$

Therefore, if $\theta \ \epsilon \ \Theta_0$, then $\theta \ \epsilon \ \Theta_k \ \ \forall \ k \ \geq \ 0$.

$\square$

Persistence of Excitation

In this section we have defined a different convergence condition. Successful identification will occur if

$$\lim_{k \to \infty} \Sigma_k = \infty. \tag{2.57}$$

It follows from the classical definition of persistence of excitation in (1.2.2), and from the dynamics of the identifier matrix (2.35), that if the output filter meets the classical persistence of excitation condition for all time steps, then the identifier matrix will grow unbounded. This is a sufficient condition for successful identification. It is possible that identification can occur without a persistently exciting output filter signal.

2.3.4 Simulation Example

Consider the following nonlinear system:

$$
\begin{aligned}
x_1(k+1) &= x_1(k) + 0.01[x_2(k) + u_3(k) + x_3(k)\theta_1 - x_1(k)\theta_3 + \vartheta_1(k)] \\
x_2(k+1) &= x_2(k) + 0.01[(1 + x_3(k))u_1(k) + x_1(k)\theta_2 - x_2(k)\theta_3 + \vartheta_2(k)] \\
x_3(k+1) &= x_3(k) + 0.01[x_1(k) + u_2(k) + x_2(k)\theta_3 - x_3(k)\theta_3 + \vartheta_3(k)], \qquad (2.58)
\end{aligned}
$$

where $\theta = [\theta_1 \ \theta_2 \ \theta_3]^T$. The input is taken as constant, $u_k = [-0.1 \ 0.1 \ 0.2]^T$. The true parameter values are $\theta = [34 \ 3 \ 0.72]^T$. The bounded noise term is $\vartheta = [sin(k) \ sin(k) \ sin(k)]^T$. The system state trajectories are shown in Fig. 2.3. The parameter estimates are shown to converge to a neighborhood around their true values in Fig. 2.4, however convergence is limited by the injected noise, and cannot converge to the true values as the prediction algorithm can not account for the unknown noise. The set is shown to contract at all update instances in Fig. 2.5. Parameter convergence is achieved at a rate similar to that achieved in Section 2.1. It is reasonable to compare the performance of this algorithm with the two previously presented methods since the system in question is effectively the same, with the addition of an injected noise. The convergence of this algorithm on a time frame similar to the finite-time identification algorithm, while sharing the advantages in computational complexity of the adaptive compensator algorithm demonstrates its effectiveness in application to the types of systems presented in Sections 2.1.2, 2.2.2 and 2.3.2.

2.4 Conclusion

In this chapter, we have demonstrated the guaranteed convergence results for all three methods discussed in this work. For each method parameter convergence has been proven, given a parameter convergence condition. The parameter convergence has been related to
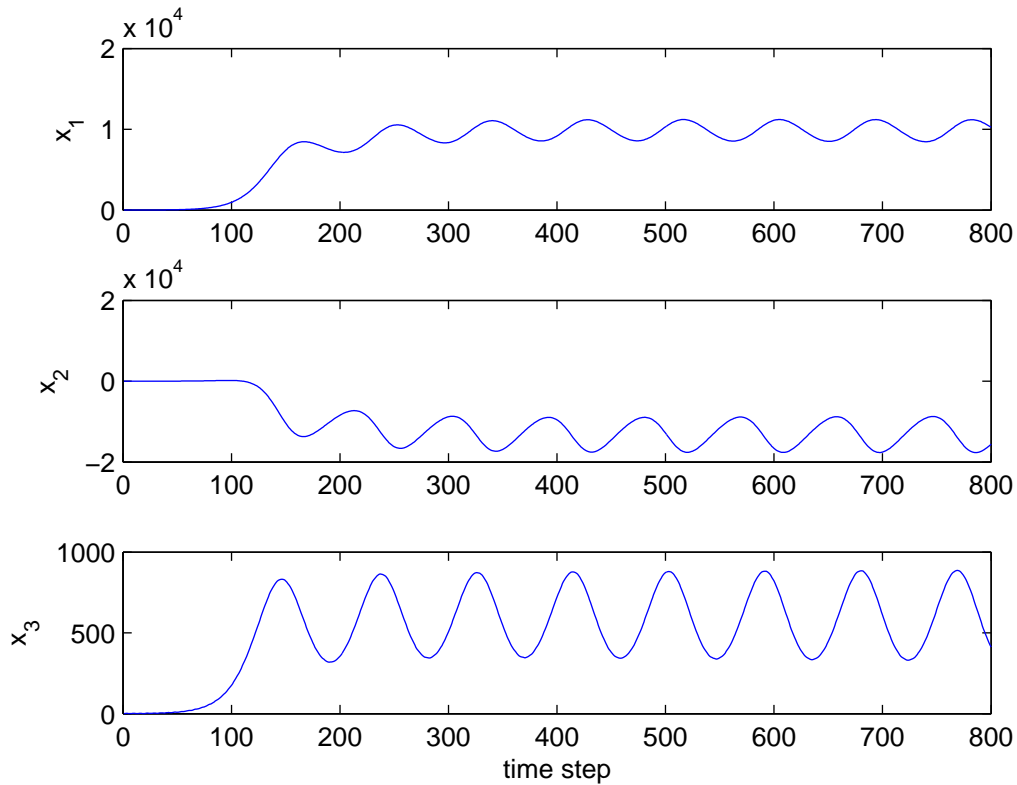
Figure 2.3: Time course plot of the system state.

the classical definition for persistently exciting signals. It can be concluded that a persis-
tently exciting output filter is, at least, a sufficient condition for parameter identification.
Each method has been illustrated using a simple simulation example, that demonstrates
successful identification of the unknown system parameters. These results present solutions
to the identification problems posed in Sections 2.1.2, 2.2.2 and 2.3.2. These identification
tasks were completed without additional excitation of the system state (except in the case
of the unmeasured exogenous disturbance in Section 2.3). The system states were, how-
ever, transient at all time steps, suggesting that the trajectories may have already been
sufficiently rich.

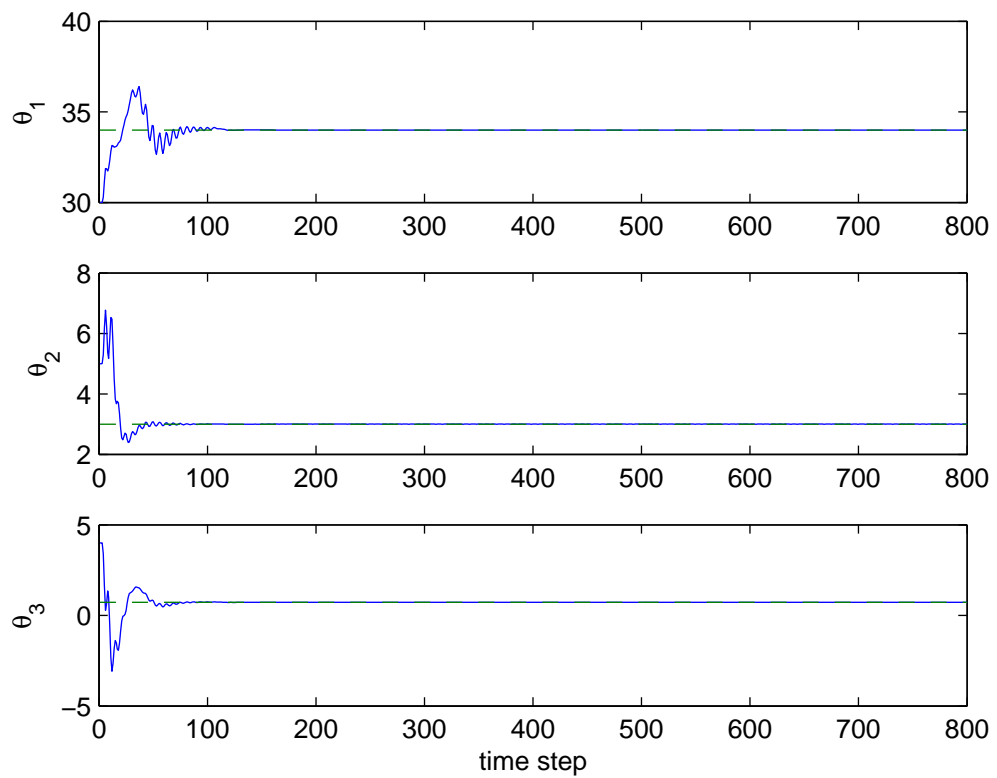Figure 2.4: Time course plot of the parameter estimates and true values under the parameter uncertainty set algorithm, the dashed lines (- -) represent the true parameter values, the solid lines (−) represent the parameter estimates.
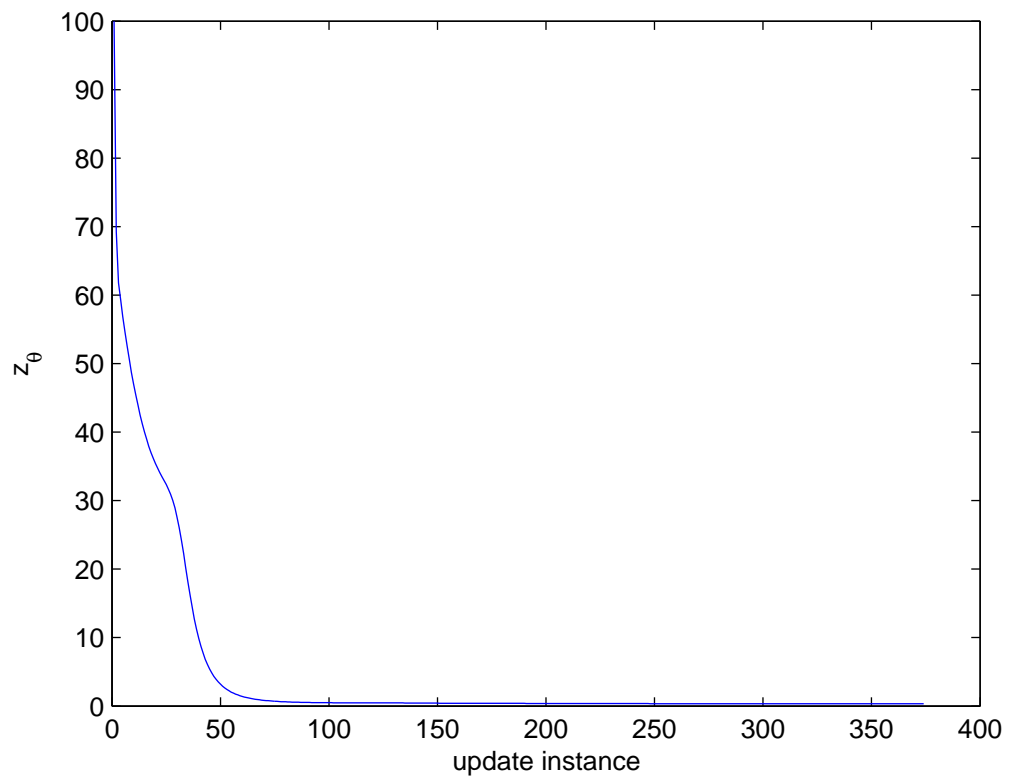
Figure 2.5: The progression of the radius of the parameter uncertainty set at time steps when the set is updated.

Chapter 3

Nonlinearly Parameterized
Discrete-time Systems

3.1 Overview

In this chapter, the parameter uncertainty set method is adapted so that it may be applied to a more general identification problem. The application in this chapter considers nonlinearly parameterized nonlinear systems. The nonlinearity in the parameters means that the algorithm cannot be applied directly as it was in Section 2.3. We apply a method of to restate the system in terms of a variable δ that represents the distance between the parameter estimate and the current center of the uncertainty set. Though the restated system remains nonlinearly parameterized, it is demonstrated in this chapter how it can be treated similarly to a linearly parameterized system.

3.2 Problem description

Consider the system:

$$x_{k+1} = x_k + F(x_k, u_k, \theta), \tag{3.1}$$

where $x_k \in \mathbb{R}^n$ is a state at some time step k, $u_k \in \mathbb{R}^m$ is the control input at some time step k, and $\theta \in \mathbb{R}^p$ is a column vector of system parameters.

Assumption 3.2.1. *The state of the system, x_k, is known at all time steps k.*

Assumption 3.2.2. *There is some known, bounded control law, u_k that achieves some control objective.*

Assumption 3.2.3. *The state and input variables evolve on a compact set, $x_k \in \mathbb{X} \subset \mathbb{R}^n$, $u_k \in \mathbb{U} \subset \mathbb{R}^m$.*

Assumption 3.2.4. *θ is uniquely identifiable and lies within an initially known compact set defined by the ball function $\Theta^0 = B(\theta_0, z_0)$, where θ_0 is some initial estimate of the parameters and z_0 is the initial radius of the uncertainty set.*

3.3 Parameter Update

The problem is first restated by describing the true parameter values as follows:

$$\theta = \theta_c + \delta, \tag{3.2}$$

and the parameter estimates by

$$\hat{\theta}_k = \theta_c + \hat{\delta}_k, \tag{3.3}$$

where θ_c represents the center of the parameter uncertainty set defined by the ball Θ. It is known from the mean-value theorem and (3.2) that

$$F(x_k, \theta_c + \delta) - F(x, \theta_c) = \left(\int_0^1 \frac{\partial F}{\partial \theta}(x_k, \theta_c + \lambda \delta) d\lambda \right) \delta. \tag{3.4}$$

Now, let

$$\Psi(x_k, \theta_c, \delta) = \int_0^1 \frac{\partial F}{\partial \theta}(x_k, \theta_c + \lambda \delta) d\lambda, \tag{3.5}$$

and

$$\Delta \Psi(x_k, \theta_c, \delta) = \Psi(x_k, \theta_c, \delta) - \Psi(x_k, \theta_c, 0). \tag{3.6}$$

The following state predictor is used:

$$\hat{x}_{k+1} = \hat{x}_k + F(x_k, \theta_c + \hat{\delta}_k) + K_k e_k + (c_k^T - Kc_k^T + \Psi(x_k, \theta_c, 0))(\hat{\delta}_{k+1} - \hat{\delta}_k). \tag{3.7}$$

Applying the state predictor (3.7), the error dynamics are given by

$$
\begin{aligned}
e_{k+1} =\ & e_k + \Delta \Psi(x_k, \theta_c, \delta)\delta - \Psi(x_k, \theta_c, 0)\tilde{\delta}_k - K_k e_k \\
& - (c_k^T - Kc_k^T + \Psi(x_k, \theta_c, 0))(\hat{\delta}_{k+1} - \hat{\delta}_k) - \Delta \Psi(x_k, \theta_c, \hat{\delta}_k)\hat{\delta}_k.
\end{aligned}
\tag{3.8}
$$

Since the term $\Delta \Psi(x_k, \theta_c, \hat{\delta}_k)\hat{\delta}_k$ is known at all time steps, the state predictor is modified to be

$$
\begin{aligned}
\hat{x}_{k+1} =\ & \hat{x}_k + F(x_k, \theta_c + \hat{\delta}_k) + K_k e_k \\
& + (c_k^T - Kc_k^T + \Psi(x_k, \theta_c, 0))(\hat{\delta}_{k+1} - \hat{\delta}_k) - \Delta \Psi(x_k, \theta_c, \hat{\delta}_k)\hat{\delta}_k,
\end{aligned}
\tag{3.9}
$$

and the error dynamics become

$$
\begin{aligned}
e_{k+1} =\ & e_k + \Delta \Psi(x_k, \theta_c, \delta)\delta + \Psi(x_k, \theta_c, 0)\tilde{\delta}_k - K_k e_k \\
& - (c_k^T - Kc_k^T + \Psi(x_k, \theta_c, 0))(\hat{\delta}_{k+1} - \hat{\delta}_k).
\end{aligned}
\tag{3.10}
$$

In this case, the auxiliary variable η_k is given by

$$\eta_k = e_k - c_k^T \tilde{\delta}_k. \tag{3.11}$$

The output filter is defined as

$$
\begin{aligned}
c_{k+1}^T &= c_k^T + \Psi(x_k, \theta_c, 0) - K_k c_k^T, \tag{3.12} \\
c_0 &= 0.
\end{aligned}
$$

The auxiliary variable dynamics are given by:

$$\eta_{k+1} = \Delta\Psi(x_k, \theta_c, \delta)\delta + \eta_k - K_k \eta_k. \tag{3.13}$$

It is necessary to employ an estimate of the true value of η_k, this estimate is given by the recursion

$$
\begin{aligned}
\hat{\eta}_{k+1} &= \hat{\eta}_k - K\hat{\eta}_k, \tag{3.14} \\
\hat{\eta}_0 &= e_0.
\end{aligned}
$$

Let the identifier matrix Σ_k be defined as

$$
\begin{aligned}
\Sigma_{k+1} &= \Sigma_k + c_k c_k^T \tag{3.15} \\
\Sigma_0 &= \alpha I \succ 0,
\end{aligned}
$$

where $\alpha \in \mathbb{R} > 0$. The inverse of the identifier matrix is generated by the recursion

$$
\begin{aligned}
\Sigma_{k+1}^{-1} &= \Sigma_k^{-1} - \Sigma_k^{-1} c_k (I + c_k \Sigma_k^{-1} c_k^T)^{-1} c_k^T \Sigma_k^{-1} \tag{3.16} \\
\Sigma_0^{-1} &= \frac{1}{\alpha} I \succ 0.
\end{aligned}
$$

The parameter update is performed indirectly by updating the variable $\hat{\delta}$,

$$\hat{\delta}_{k+1} = \hat{\delta}_k + \Sigma_k^{-1} c_k \left(I + c_k^T \Sigma_k^{-1} c_k\right)^{-1} (e_k - \hat{\eta}_k). \tag{3.17}$$

To ensure that the parameter estimates remain within the constraint set Θ_k, we propose to use a projection operator of the form:

$$\bar{\hat{\delta}}_{k+1} = \text{Proj}\{\hat{\delta}_k + \Sigma_k^{-1} c_k \left(I + c_k^T \Sigma_k^{-1} c_k\right)^{-1} (e_k - \hat{\eta}_k), \Theta_k\}. \tag{3.18}$$

The operator $Proj$ represents an orthogonal projection onto the boundary of the uncertainty set applied to the parameter estimate. The parameter uncertainty set is defined by the ball function $B(\theta_c, z_{\theta_c})$, where θ_c and z_{θ_c} are the parameter estimate and set radius found at the latest set update.

Following Goodwin and Sin [13], the projection operator is designed such that:

- $\bar{\hat{\theta}}_{k+1} \in \Theta_k$

- $\bar{\hat{\delta}}_{k+1}^T \Sigma_{k+1} \bar{\hat{\delta}}_{k+1} \le \tilde{\delta}_{k+1}^T \Sigma_{k+1} \tilde{\delta}_{k+1}$.

It will be shown that the parameter update law (3.18) guarantees convergence of the parameter estimation error $\tilde{\theta}_k = (\theta_c + \delta) - (\theta_c + \hat{\delta}_k)$ to zero.

3.4   Uncertainty Set Update

The parameter uncertainty set, defined by the ball function $B(\theta_c, z_{\theta_c})$ is updated using the parameter update law (3.18) and the error bound (3.20) according to the following algorithm:

Algorithm 3.4.1. *Beginning at time step $k = 0$, the set is adapted according to the following iterative process*

1. **Initialize** $z_{\theta_c} = z_{\theta_0}, \theta_c = \hat{\theta}_0$

2. *at time step k, using equations (3.18) and (3.20) perform the update*

$$(\theta_c, z_{\theta_c}) = \begin{cases} (\hat{\theta}_k, z_{\theta_k}) & \text{if } z_{\theta_k} \leq z_{\theta_c} - \|\hat{\theta}_k - \theta_c\| \\[2mm] (\theta_c, z_{\theta_c}) & \text{otherwise} \end{cases} \tag{3.19}$$

3. *in the case when the uncertainty set is updated, the following values are reset as follows: $c_k^T = 0$ and $\hat{\eta}_k = e_k$*

4. *Return to step two and iterate, incrementing to time step $k + 1$*

An update law that measures the worst-case progress of the parameter update law is adapted from the one proposed in Adetola and Guay [5]

$$z_{\theta_k} = \sqrt{\frac{V_{z\theta_k}}{4\lambda_{min}[\Sigma_k]}} \tag{3.20}$$

$$\bar{V}_{z\theta_{k+1}} = V_{z\theta_k} - (e_k - \hat{\eta}_k)^T \left(I + c_k^T \Sigma_k^{-1} c_k\right)^{-1} (e_k - \hat{\eta}_k) + (\frac{Lz_{\theta_k}^2}{K_k})^2 \tag{3.21}$$

$$V_{z\theta_{k+1}} = \begin{cases} V_{z\theta_k} & \text{if } \bar{V}_{z\theta_{k+1}} \geq V_{z\theta_k} \\[2mm] \bar{V}_{z\theta_{k+1}} & \text{otherwise} \end{cases} \tag{3.22}$$

$$V_{z\theta_0} = 4\lambda_{max}[\Sigma_0](z_{\theta_0})^2. \tag{3.23}$$

Since the set is updated using the same update condition as (2.3.1) it follows that the proof of contraction from (2.53) applies, hence, $\Theta_{k+1} \subset \Theta_k$. The main difference arises from the choice of update of $V_{z\theta_{k+1}}$ described by (3.22). In this case, any increase in $V_{z\theta_k}$ is disregarded to remove some conservativeness associated with the nominal uncertain set

update based on (3.21).

In addition to the non-exclusion property, one needs to ensure that the true value of the parameters, $\theta = \theta_c + \delta$ remain within the uncertainty set. In previous sections, we considered the application of the comparison principle as follows.

First, one assumes that, at step k,

$$V_{z\theta_k} \geq V_{\tilde{\delta}_k}. \tag{3.24}$$

Then if,

$$V_{z\theta_{k+1}} - V_{z\theta_k} \geq V_{\tilde{\delta}_{k+1}} - V_{\tilde{\delta}_k}, \tag{3.25}$$

it follows that

$$V_{z\theta_{k+1}} \geq V_{\tilde{\delta}_{k+1}}. \tag{3.26}$$

Hence, one concludes that

$$\|\tilde{\delta}_{k+1}\|^2 \leq \frac{V_{z\theta_{k+1}}}{\lambda_{min}\left[\Sigma_{k+1}\right]} = 4z^2_{\theta_{k+1}}. \tag{3.27}$$

For the set update of $V_{z\theta_k}$ given by (3.22), it is important to guarantee that the non-exclusion property is still preserved. The projection algorithm always guarantees that the estimation errors are as follows:

$$\|\tilde{\delta}_k\|^2 \leq 4z^2_{\theta_c}$$

Therefore, one can always assume that

$$\|\tilde{\delta}_k\|^2 \leq \frac{V_{z\theta_k}}{\lambda_{min}\left[\Sigma_k\right]} \leq 4z_{\theta_c}^2.$$

The rate of change of $V_{\tilde{\delta}}$ is given by

$$
\begin{aligned}
V_{\tilde{\delta}_{k+1}} \quad \leq \quad & V_{\tilde{\delta}_k} - (e_k - \hat{\eta}_k)^T \left(I + c_k^T \Sigma_k^{-1} c_k\right)^{-1} (e_k - \hat{\eta}_k) & (3.28) \\
& + \tilde{\eta}_k^T \left(I + c_k^T \Sigma_k^{-1} c_k\right)^{-1} \tilde{\eta}_k.
\end{aligned}
$$

One can also write,

$$V_{\tilde{\delta}_{k+1}} \quad \leq \quad V_{z\theta_k} - (e_k - \hat{\eta}_k)^T \left(I + c_k^T \Sigma_k^{-1} c_k\right)^{-1} (e_k - \hat{\eta}_k) + \tilde{\eta}_k^T \tilde{\eta}_k.$$

By construction,

$$\lambda_{\min}\left[\Sigma_{k+1}\right]\|\tilde{\delta}_{k+1}\|^2 \quad \leq \quad V_{z\theta_k} - (e_k - \hat{\eta}_k)^T \left(I + c_k^T \Sigma_k^{-1} c_k\right)^{-1} (e_k - \hat{\eta}_k) + (\frac{Lz_{\theta_c}^2}{K_k})^2.$$

Hence,

$$\|\tilde{\delta}_{k+1}\|^2 \quad \leq \quad \frac{1}{\lambda_{\min}\left[\Sigma_{k+1}\right]}\left(V_{z\theta_k} - (e_k - \hat{\eta}_k)^T \left(I + c_k^T \Sigma_k^{-1} c_k\right)^{-1} (e_k - \hat{\eta}_k) + (\frac{Lz_{\theta_c}^2}{K_k})^2\right).$$

Since,

$$\lambda_{\min}\left[\Sigma_{k+1}\right] \geq \lambda_{\min}\left[\Sigma_k\right],$$

it follows that

$$
\begin{aligned}
\|\tilde{\delta}_{k+1}\|^2 \;\; &\leq \;\; \frac{1}{\lambda_{\min}[\Sigma_k]} \left(V_{z\theta_k} - (e_k - \hat{\eta}_k)^T \left(I + c_k^T \Sigma_k^{-1} c_k\right)^{-1} (e_k - \hat{\eta}_k) + (\frac{Lz_{\theta_c}^2}{K_k})^2 \right) \\
&= \;\; \frac{V_{z\theta_k}}{\lambda_{\min}[\Sigma_k]} - \frac{\left((e_k - \hat{\eta}_k)^T \left(I + c_k^T \Sigma_k^{-1} c_k\right)^{-1} (e_k - \hat{\eta}_k) - (\frac{Lz_{\theta_c}^2}{K_k})^2\right)}{\lambda_{\min}[\Sigma_k]} \\
&\leq \;\; 4z_{\theta_c}^2 - \frac{\left((e_k - \hat{\eta}_k)^T \left(I + c_k^T \Sigma_k^{-1} c_k\right)^{-1} (e_k - \hat{\eta}_k) - (\frac{Lz_{\theta_c}^2}{K_k})^2\right)}{\lambda_{\min}[\Sigma_k]} \qquad (3.29)
\end{aligned}
$$

From the last inequality we see that by the action of the projection algorithm the set update will only lead to potential exclusion of the true value of the parameters if

$$
\left((e_k - \hat{\eta}_k)^T \left(I + c_k^T \Sigma_k^{-1} c_k\right)^{-1} (e_k - \hat{\eta}_k) - (\frac{Lz_{\theta_c}^2}{K_k})^2 \right) > 0. \qquad (3.30)
$$

Therefore, one only needs to update the value of $V_{z\theta_{k+1}}$ when it leads to a decrease, or alternatively, when the inequality (3.30) is met. We can write this in the form of a lemma as follows.

Lemma 3.4.1. *The set-update procedure 3.4.1 is such that*

1. *the set is only updated when updating will yield a contraction,*

2. *the dynamics of the set error bound described in (3.20) are such that they ensure the non-exclusion of the true value, that is, $\theta \in \Theta_k$, $\forall k$ if $\theta_0 \in \Theta_0$.*

We are now ready to state the main result of this chapter.

Theorem 3.4.1. *The identifier (3.16), the parameter update law (3.18), and the set update procedure algorithm 3.4.1 are such that $\tilde{\delta} = \delta - \hat{\delta}$ and therefore $\tilde{\theta}_k = \theta_k - \hat{\theta}_k$ are bounded. Furthermore, if*

$$
\lim_{k \to \infty} \lambda_{\min}[\Sigma_k] = \infty \qquad (3.31)
$$

is satisfied, then $\tilde{\theta}_k$ converges to 0 asymptotically.

Proof. Since, $V_{\tilde{\delta}_k} = \tilde{\delta}_k^T \Sigma_k \tilde{\delta}_k$, then it follows by the non exclusion property of the set update algorithm that,

$$\|\tilde{\delta}_k\| \leq \frac{V_{z\theta_k}}{\lambda_{min}(\Sigma_k)} = 4z_{\theta k}^2 \leq 4z_{\theta_c}^2, \qquad \forall k \geq 0. \tag{3.32}$$

Since $V_{z\theta_k}$ is non-increasing, by construction, and since $\lambda_{\min}[\Sigma_k]$ is increasing, there exists a finite integer N such that

$$z_{\theta_k} \leq z_{\theta_c} - \|\hat{\delta}_N\|,$$

leading to a shrinking of the set. Applying the set update yields to a re-centered uncertainty set with a smaller uncertainty radius containing the unknown value of the parameters. Repeating sequentially, it follows that, as $k \to \infty$, the properties of the set update are such that the uncertainty radius z_{θ_c} will tend asymptotically to zero. This is, in turn, guarantees that $\lim_{k\to\infty} \|\tilde{\delta}_k\| = 0$ and, by the non-exclusion property, that the center of the uncertainty set θ_c converges to the true value of the parameters θ. $\qquad\square$

3.5 Simulation Example

Consider the following system representing a chemostat operating under Monod kinetics, similar to those described by Monod [24] and Monod [25], discretized using a finite differences method.

$$\begin{aligned}
x_1(k+1) &= x_1(k) + \Delta t \left\{ \frac{\theta_1 x_1(k) x_2(k)}{\theta_2 + x_2(k)} - D x_1(k) \right\} \\
x_2(k+1) &= x_2(k) + \Delta t \left\{ \frac{-\theta_3 x_1(k) x_2(k)}{\theta_2 + x_2(k)} + S_0 - D x_2(k), \right\}
\end{aligned}$$

where the system states have the following units $x(k)[=][\frac{g_{cells}}{L} \quad \frac{g_{substrate}}{L}]$. The parameter vector represents $\theta = [\mu_{max} \quad K_s \quad \frac{1}{Y_{x/s}}]^T$, with units $\theta[=][h^{-1} \quad \frac{g_{substrate}}{L} \quad \frac{g_{substrate}}{g_{cells}}]^T$ and values

$\theta = [0.33 \ 0.5 \ 0.66]^T$. The control input, the substrate feed rate $S_0 = 5\frac{g_{substrate}}{L \cdot h}$ is kept constant and the input dilution rate, D is oscillated sinusoidally such that $D = 0.1 + 0.05sign(\sin(2(k+1))\frac{2}{\pi \cdot 72}) + 0.04sign(\cos\frac{2(k+1)}{\pi \cdot 36})h^{-1}$. The correction factor is $K = 0.05$, and the upper bound is $L = 2$. Δt is the size of the time step defined as $\Delta t = \frac{1}{3600}h$.

It is shown in Fig. 3.1 that the parameter estimates converge asymptotically to their true values. It is possible to observe the effect of the uncertainty set and the projection algorithm in Fig. 3.1. The magnitude of oscillations in the parameter estimates shrinks as the identification task progresses, this is as a result of the constraint imposed by the projection algorithm. The radius of the parameter uncertainty set decreases at each instance the set is updated (Fig. 3.3). Furthermore, the results shown in Fig. 3.3 also demonstrate that the magnitude of the δ variable error is always smaller than the radius of the set. Since the δ variable error is equal to the parameter estimation error, it is possible to verify that the true parameters remain within the uncertainty set throughout the simulation. It is important to note that we can confirm that the construction of $V_{z\theta}$ does not lead to an exclusion of the true parameters. The construction of the worst case Lyapunov function leads to improved convergence performance since only changes that can progress the identification task are considered. Fig. 3.2 shows the state prediction error. Convergence of the state prediction error to zero is expected and consistent with successful identification. It follows from (3.10) that the state prediction will approach the true state when $\tilde{\delta}$ approaches zero, which is a consequence of the shrinking of the uncertainty set. It follows logically that when the parameters are identified, as a result of the construction of the state predictor, the prediction will approach the true state.

3.6 Conclusion

In this chapter we have demonstrated, that the parameter uncertainty set method developed by Adetola and Guay [5] and generalized to discrete-time systems in Chapter 2, can

be applied to nonlinearly parameterized systems. The approach from Chapter 2 is applied in a sequential manner such that nonlinearly parameterized systems can be treated as linearly parameterized uncertain systems at each step. It can be concluded from the classical persistence of excitation condition, that an output filter that meets the persistence of excitation condition at all time steps is a sufficient condition under which convergence of the parameter estimates to the true parameter values is guaranteed. This conclusion follows from the convergence condition defined by (3.31) and the argument made in Section 2.3.3. The application of this method has been applied to highly nonlinear system of a bioreactor operating under Monod kinetics.
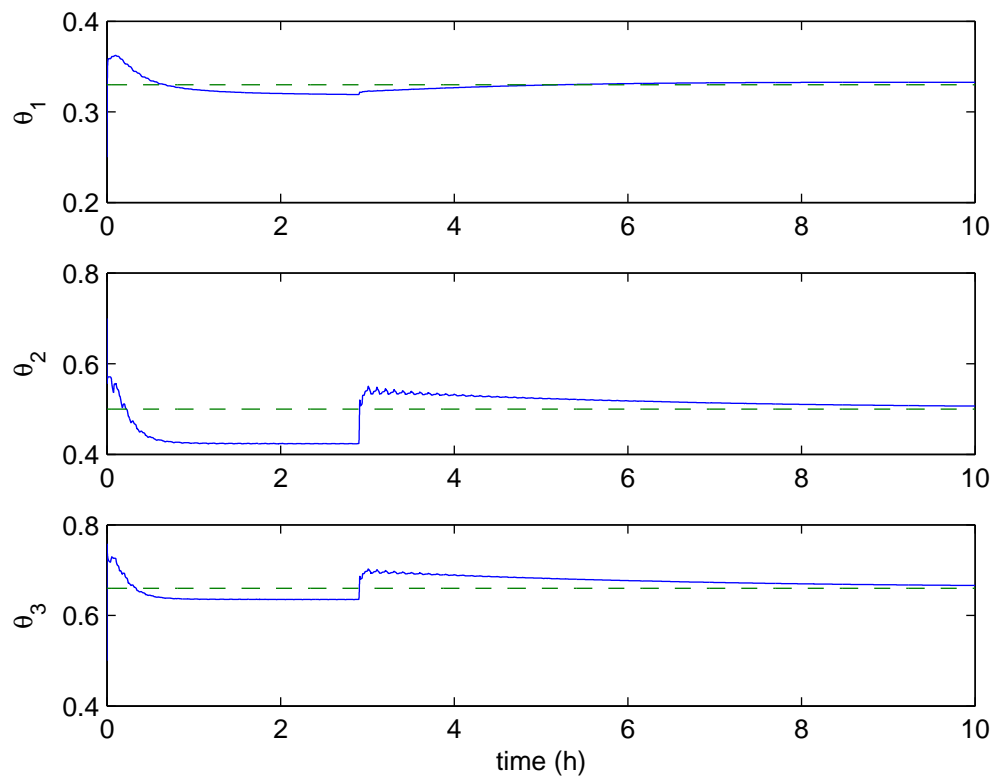
Figure 3.1: Time course plot of the parameter estimates and true values under the parameter uncertainty set algorithm, the dashed lines (- -) represent the true parameter values, the solid lines (−) represent the parameter estimates.
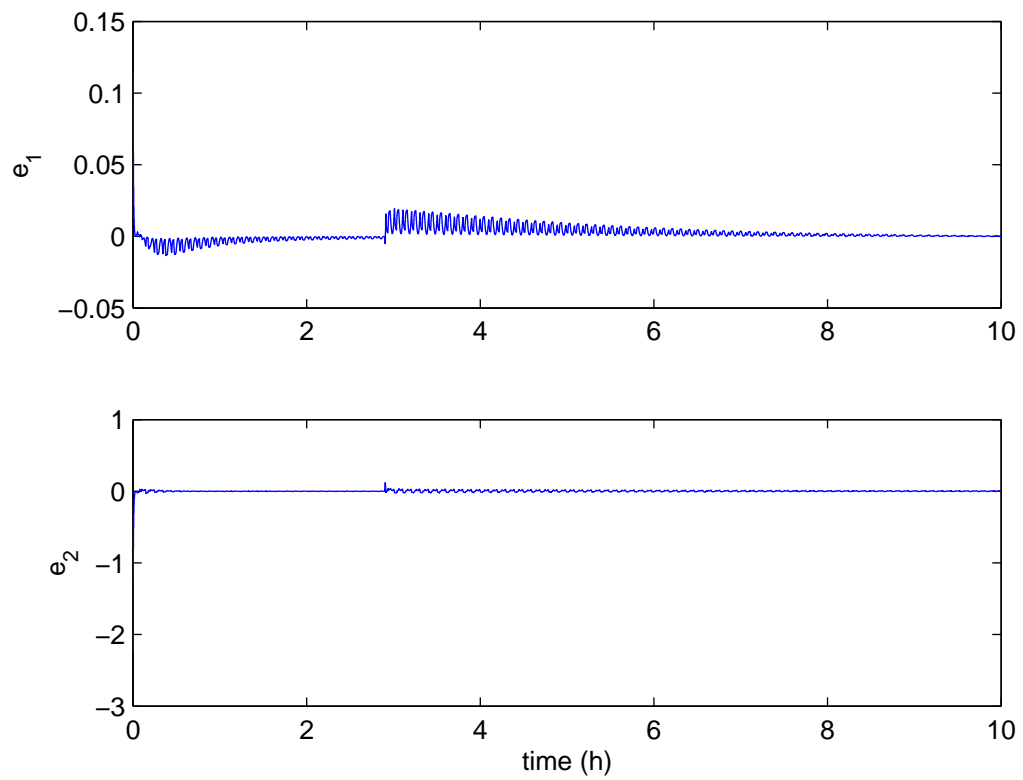
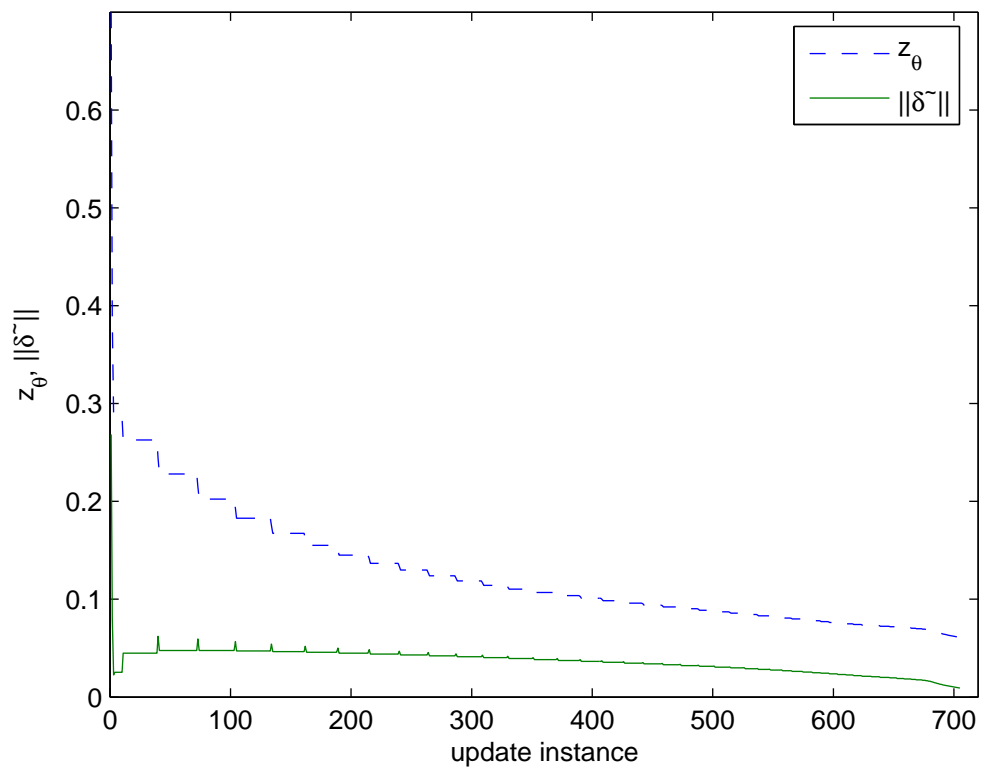Figure 3.2: Time course plot of the state prediction error $e_k = x_k - \hat{x}_k$.

Figure 3.3: The progression of the radius of the parameter uncertainty set and the magnitude of the variable $\tilde{\delta}$ at time steps when the set is updated.

Chapter 4

Nonlinearly Parameterized Continuous-time Systems

4.1 Overview

The method developed in Section 2.3 and modified in Chapter 3 is adapted further in this chapter to be applied to nonlinearly parameterized continuous-time systems.

4.2 Problem Statement and Assumptions

Consider the nonlinear system:

$$\dot{x}(t) = F(x(t), u(t), \theta) \tag{4.1}$$

where $x(t) \in \mathbb{R}^n$ is a state at some time t, $u(t) \in \mathbb{R}^m$ is a the control input at some time $,t$, and $\theta \in \mathbb{R}^p$ is a column vector of system parameters.

Assumption 4.2.1. *The state of the system, $x(t)$, is known at all times $t \geq 0$.*

Assumption 4.2.2. *There is some known, bounded control law, $u(t)$, that achieves some control objective.*

Assumption 4.2.3. *The state and input variables evolve on a compact set, $x(t) \in \mathbb{X} \subset \mathbb{R}^n$, $u(t) \in \mathbb{U} \subset \mathbb{R}^m$.*

Assumption 4.2.4. *θ is uniquely identifiable and lies within an initially known compact set defined by the ball function $\Theta^0 = B(\theta_0, z_0)$, where θ_0 is some initial estimate of the parameters and z_0 is the initial radius of the uncertainty set.*

4.3   Parameter Update

As above, the problem is restated with respect to a new variable δ. Let

$$\theta = \theta_c + \delta, \tag{4.2}$$

where θ_c is the center of the parameter uncertainty set, and δ is a vector between the center of the set and the true parameter values. In this case, the parameter estimate to be used in the state predictor (4.4) is given by

$$\hat{\theta} = \theta_c + \hat{\delta}. \tag{4.3}$$

The following state predictor, similar to the one given by (2.2), is proposed:

$$\dot{\hat{x}} = F(x, \theta_c + \hat{\delta}) + Ke + c^T \dot{\hat{\delta}}. \tag{4.4}$$

As shown in Chapter 3, let

$$\Psi(x, \theta_c, \delta) = \int_0^1 \frac{\partial F}{\partial \theta}(x, \theta_c + \lambda\delta)d\lambda, \tag{4.5}$$

and,

$$\Delta\Psi(x, \theta_c, \delta) = \Psi(x, \theta_c, \delta) - \Psi(x, \theta_c, 0). \tag{4.6}$$

From (4.4) and (4.5) and (4.6), the error dynamics can then be described by

$$\dot{e} = \Delta\Psi(x, \theta_c, \delta)\delta + \Psi(x, \theta_c, 0)\tilde{\delta} - \Delta\Psi(x, \theta_c, \hat{\delta})\hat{\delta} - Ke - c^T\dot{\hat{\delta}}. \tag{4.7}$$

From (4.7), the state predictor is modified to

$$\dot{\hat{x}} = F(x, \theta_c + \hat{\delta}) + Ke + c^T\dot{\hat{\delta}} - \Delta\Psi(x, \theta_c, \hat{\delta})\hat{\delta}, \tag{4.8}$$

and the error dynamics become

$$\dot{e} = \Delta\Psi(x, \theta_c, \delta)\delta + \Psi(x, \theta_c, 0)\tilde{\delta} - Ke - c^T\dot{\tilde{\delta}} \tag{4.9}$$

$$e(0) = x(0) - \hat{x}(0).$$

Similar to the application of the parameter uncertainty set to linearly parameterized systems, the output filter is designed as follows

$$\dot{c}^T = \Psi(x, \theta_c, 0) - Kc^T \tag{4.10}$$

$$c^T(0) = 0.$$

An auxiliary variable η is given by

$$\eta(t) = e(t) - c^T(t)\tilde{\delta}(t). \tag{4.11}$$

The auxiliary variable dynamics are then given by

$$\dot{\eta} = \Delta\Psi(x, \theta_c, \delta)\delta - K(e - c^T\tilde{\delta}). \tag{4.12}$$

Since the true value of δ is unknown, an estimate for the auxiliary variable is tracked by

$$\dot{\hat{\eta}} = -K\hat{\eta}. \tag{4.13}$$

Now, let the identifier matrix Σ be defined by

$$
\begin{aligned}
\dot{\Sigma} &= cc^T \\
\Sigma(0) &= \alpha I,
\end{aligned}
\tag{4.14}
$$

where $\alpha I \succ 0$, and $\alpha \in \mathbb{R} > 0$.

The inverse of the identifier matrix is given by

$$
\begin{aligned}
\dot{\Sigma}^{-1} &= \Sigma^{-1}cc^T\Sigma^{-1} \\
\Sigma^{-1}(0) &= \frac{1}{\alpha}I.
\end{aligned}
\tag{4.15}
$$

As above, the parameter estimate is not updated directly, rather, the $\hat{\delta}$ variable is updated with the following update law

$$\dot{\hat{\delta}} = Proj\{\Sigma^{-1}c(e - \hat{\eta})\}, \tag{4.16}$$

where $Proj$ represents an orthogonal projection of the gradient of the δ variable onto the surface of the ball Θ. More information on the project algorithm can be found in Krstic et al. [21].

Assumption 4.3.1. *For all $x \in \mathbb{X}$, $u \in \mathbb{U}$ and $\theta_0 \in \Theta_0$, there exists a constant $L > 0$ such that*

$$\|\Delta\Psi(x, \theta_c, \delta)\delta\| \leq L\|\delta\|^2 \tag{4.17}$$

As a consequence of Assumption 4.3.1, it follows that:

$$\|\Delta\Psi(x, \theta_c, \delta)\| \leq Lz_{\theta_c}^2 = \vartheta. \tag{4.18}$$

Lemma 4.3.1. *The identifier (4.15) is such that for every $\theta_0 \in \Theta_0$, the estimation error, $\tilde{\delta} = \delta - \hat{\delta}$, and the auxiliary variable estimation error, $\tilde{\eta} = \eta - \hat{\eta}$, are bounded.*

Proof. Consider the $\tilde{\eta}$ dynamics and the Lyapunov function $V_{\tilde{\eta}} = \frac{1}{2}\tilde{\eta}^T\tilde{\eta}$. The rate of change of $V_{\tilde{\eta}}$ is given by

$$\dot{V}_{\tilde{\eta}} = -K\tilde{\eta}^T\tilde{\eta} + \tilde{\eta}^T\Delta\Psi(x, \theta_c, \delta)\delta, \tag{4.19}$$

it follows that

$$\dot{V}_{\tilde{\eta}} \leq -K\|\tilde{\eta}\|^2 + \|\tilde{\eta}\|\vartheta. \tag{4.20}$$

As a result it is guaranteed that $\dot{V}_{\tilde{\eta}} \leq 0 \ \forall \tilde{\eta} \in \mathbb{R}^n$, if $\|\tilde{\eta}\| > \frac{\vartheta}{K}$. Since $\tilde{\eta}(0) = 0$, it can be concluded that $\|\tilde{\eta}(t)\| \leq \frac{\vartheta}{K}$.

Let

$$V_{\tilde{\delta}} = \tilde{\delta}^T\Sigma\tilde{\delta}. \tag{4.21}$$

It follows (4.16) that

$$
\begin{aligned}
\dot{V}_{\tilde{\delta}}(t) &= -(e - \hat{\eta})^T (e - \hat{\eta}) + \|\tilde{\eta}\|^2 \\
&\leq -(e - \hat{\eta})^T (e - \hat{\eta}) + (\frac{Lz_{\theta_c}^2}{K})^2.
\end{aligned}
\tag{4.22}
$$

Since $\|\tilde{\eta}\|^2$ is bounded, it follows that $\|\tilde{\delta}\|^2$ is also bounded. The projection algorithm guarantees that

$$
\|\tilde{\delta}\|^2 \leq \|\delta\|^2 + \|\hat{\delta}\|^2 \leq 4z_{\theta_c}^2.
\tag{4.23}
$$

□

The following persistence of excitation condition will be required to guarantee convergence of $\tilde{\delta}$ to a neighborhood of the origin.

Assumption 4.3.2. *There exists positive constants T and k_N*

$$
\int_t^{t+T} c(\tau, \theta_c)^T c(\tau, \theta_c) d\tau \geq k_N(\theta_c), \quad \forall t > 0, \quad \forall \theta_0 \in \Theta_0.
\tag{4.24}
$$

The persistence of excitation condition (4.24) considers the dependence of the filter dynamics on the current value of the center of the uncertainty set. When the set is updated, a change will occur in the regressor matrix, $\Psi(x, \theta_c, 0)$, and subsequently in the filter dynamics. Now, consider the convergence properties of the estimates $\hat{\delta}$ at a fixed value θ_c.

Lemma 4.3.2. *Assume that the persistency of excitation condition (4.24) is met. Then the parameter estimation scheme given by (4.16) and Algorithm 4.4.1 is such that the parameter estimation error converges exponentially to a neighborhood of the origin.*

Proof. It is clearly observable that $\dot{\tilde{\delta}} = -\dot{\hat{\delta}}$. It follows from the projection algorithm that the rate of change of $V_{\tilde{\delta}}$ along the trajectories of the closed-loop system about the center of

the uncertainty set is

$$
\begin{aligned}
\dot{V}_{\tilde{\delta}} &\leq -2\tilde{\delta}^T cc^T \tilde{\delta} - 2\tilde{\delta}^T c\tilde{\eta} + \tilde{\delta}^T cc^T \tilde{\delta} & (4.25) \\
&\leq -\tilde{\delta}^T cc^T \tilde{\delta} - 2\tilde{\delta}^T c\tilde{\eta}.
\end{aligned}
$$

Consider the Lyapunov function $W = V_{\tilde{\delta}} + \tilde{\eta}^T \tilde{\eta}$.

$$
\dot{W} \leq -\tilde{\delta}^T cc^T \tilde{\delta} - 2\tilde{\delta}^T c\tilde{\eta} - K\tilde{\eta}^T \tilde{\eta} + \tilde{\eta}^T \Delta\Psi(x, \theta_c, \delta)\delta. \tag{4.26}
$$

There exist positive constants k_z and K_Z such that $K_Z > k_z > 1$ such that

$$
\begin{aligned}
\dot{W} &\leq -(1 - \frac{1}{k_z})\tilde{\delta}^T cc^T \tilde{\delta} - (K_Z - k_z)\tilde{\eta}^T \tilde{\eta} + \tilde{\eta}^T \Delta\Psi(x, \theta_c, \delta)\delta & (4.27) \\
&\leq -k_1 \tilde{\delta}^T cc^T \tilde{\delta} - k_2 \tilde{\eta}^T \tilde{\eta} + k_3 \tilde{\eta}^T \tilde{\eta} + \frac{1}{k_3}(Lz_{\theta_c}^2)^2.
\end{aligned}
$$

For $k_2 > k_3$

$$
\dot{W} \leq -k_1 \tilde{\delta}^T cc^T \tilde{\delta} - k_4 \tilde{\eta}^T \tilde{\eta} + \frac{1}{k_3}(Lz_{\theta_c}^2)^2 \tag{4.28}
$$

If the persistence of excitation condition (4.24) is met, it follows that

$$
\begin{aligned}
\dot{W} &\leq -\gamma_1 c_1 V_{\tilde{\delta}} - k_4 \tilde{\eta}^T \tilde{\eta} + \frac{1}{k_3}(Lz_{\theta_c}^2)^2 & (4.29) \\
&\leq -k_5 W + \frac{1}{k_3}(Lz_{\theta_c}^2)^2,
\end{aligned}
$$

which confirms that the parameter estimation error $\tilde{\delta}$ and the η estimation error, $\tilde{\eta}$, converge exponentially to a neighborhood of the origin for any value $\theta_0 \in \Theta_0$. $\qquad \square$

4.4   Set Update

Similar to the method applied to the linearly parameterized system, an update law that measures the worst-case progress of the parameter update law is proposed based on the one utilized by Adetola and Guay [5].

$$z_\theta(t) \quad = \quad \sqrt{\frac{V_{z\theta}(t)}{4\lambda_{min}(\Sigma(t))}} \tag{4.30}$$

$$V_{z_\theta}(0) \quad = \quad 4\lambda_{max}(\Sigma(0))z_{\theta_0}^2 \tag{4.31}$$

$$V_\eta(0) \quad = \quad \|\tilde{\eta}(0)\|^2 = 0 \tag{4.32}$$

$$\dot{V}_\eta \quad = \quad -KV_\eta + (Lz_{\theta_c}^2)^2 \tag{4.33}$$

$$\dot{V}_{z\theta} \quad = \quad -(e - \hat{\eta})^T(e - \hat{\eta}) + V_\eta. \tag{4.34}$$

The following condition is imposed on the progression of $V_{z\theta}$,

$$\dot{V}_{z\theta}(t) = \begin{cases} \dot{V}_{z\theta}(t) & \text{if } \dot{V}_{z\theta}(t) \leq 0 \\ \\ 0 & \text{otherwise} \end{cases} \tag{4.35}$$

The parameter uncertainty set is updated using the following algorithm:

Algorithm 4.4.1. *beginning at time t_0, the set is adapted according to the following iterative process*

1. **Initialize** $z_{\theta_c} = z_{\theta_0}, \theta_c = \theta_0 + \hat{\delta}(t_0)$

2. *At time t, using equations (4.16) and (4.34) perform the update*

$$(\theta_c, z_{\theta_c}) = \begin{cases} (\hat{\theta}(t), z_\theta(t)) & \text{if } z_\theta(t) \leq z_{\theta_c} - \|\hat{\theta}(t) - \theta_c\| \\ \\ (\theta_c, z_{\theta_c}) & \text{otherwise} \end{cases} \tag{4.36}$$

3. *In the case when the uncertainty set is updated, the following values are reset: $c^T(t) =$*

0 *and* $\hat{\eta}(t) = e(t)$

4. *Return to step two and iterate, incrementing to time* $t + \Delta t$ *where* Δt *is some appropriate time interval*

Lemma 4.4.1. *The algorithm ensures that*

(a) *the set is only updated when updating will yield a contraction, and*

(b) *the projection algorithm is such that the non-exclusion of the true value* $\theta \in \Theta(t)$ *is ensured* $\forall t$ *if* $\theta(t_0) \in \Theta(t_0)$.

Proof. (a) At update instance i, at time t_i, where the previous update occurred at time t_{i-1}, If $\Theta(t_i) \nsubseteq \Theta(t_{i-1})$ then

$$sup_{s\epsilon\Theta_{(t_i)}}\|s - \theta_c(t_{i-1})\| \geq z_\theta(t_{i-1}). \tag{4.37}$$

However, it is guaranteed by the set update algorithm presented, that Θ, at update times, obeys the following

$$
\begin{aligned}
&\sup_{s\epsilon\Theta_{t_i}} \|s - \theta_c(t_{i-1})\| \\
\leq\ &\sup_{s\epsilon\Theta_{t_i}} \|s - \theta_c(t_i)\| + \|\theta_c(t_i) - \theta_c(t_{i-1})\| \\
\leq\ &z_\theta(t_i) + \|\theta_c(t_i) - \theta_c(t_{i-1})\| \leq z_\theta(t_{i-1}).
\end{aligned}
\tag{4.38}
$$

This contradicts (4.37). Therefore, $\Theta(t_i) \subseteq \Theta(t_{i-1})$ at times where Θ is updated.

(b) It is shown in (4.23) that

$$\|\tilde{\delta}(t)\| \leq 2z_{\theta_c}.$$

Since it is easily demonstrable that

$$\tilde{\theta} = (\theta_c + \delta) - (\theta_c + \hat{\delta}) = \tilde{\delta},$$

one can conclude that $\theta \in \Theta(t)$, $\forall t$ if $\theta_0 \in \Theta(t_0)$.

$\square$

Remark 4.4.1. *Further work is required to provide a formal proof of convergence of this scheme in the continuous-time case. However following the proof in the discrete-time case, it is clear that a constructive proof should be rather straight forward. At this point, we can sketch that if the persistency of excitation condition is met, then $\lambda_{\min}[\Sigma(t)]$ will grow unbounded. Since $V_{z\theta}$ is bounded as a result of (4.35), we can take the following limit:*

$$\lim_{t \to \infty} z_\theta^2 = \lim_{t \to \infty} \frac{V_{z\theta}(t)}{4\lambda_{min}[\Sigma(t)]} \tag{4.39}$$

which implies that

$$\lim_{t \to \infty} z_\theta^2 = 0. \tag{4.40}$$

It can therefore be concluded that the parameter estimates must converge to the true values of the parameters since the parameter estimates and the true parameters cannot be excluded from the shrinking set. A formal proof of this result will be provided in future work.

4.5 Simulation Example

Consider the following system representing a chemostat operating under Monod kinetics similar to those described by Monod [24] and Monod [25]

$$\dot{x}_1 = \frac{\theta_1 x_1 x_2}{\theta_2 + x_2} - Dx_1$$
$$\dot{x}_2 = \frac{-\theta_3 x_1 x_2}{\theta_2 + x_2} + S_0 - Dx_2,$$

where the system states have the following units $x(t)[=][\frac{g_{cells}}{L} \ \frac{g_{substrate}}{L}]$. The parameter vector represents, $\theta = [\mu_{max} \ K_s \ \frac{1}{Y_{x/s}}]^T$, with units $\theta[=][h^{-1} \ \frac{g_{substrate}}{L} \ \frac{g_{substrate}}{g_{cells}}]^T$ and values $\theta = [0.33 \ 0.5 \ 0.66]^T$. The control input, the substrate feed rate $S_0 = 5\frac{g_{substrate}}{L \cdot h}$ is kept constant and the input dilution rate, D is oscillated sinusoidally such that $D = 0.1 + [0.05 sign(sin(0.0005t))]h^{-1}$. The upper bound is $L = 0.032$, and the correction factor used is $K = 0.5$.

The results shown in Fig. 4.1 demonstrate that the parameter estimates converge to their true values. The excitation injected to the system results in the prediction error trajectories given in Fig. 4.2. The radius of the parameter uncertainty set is demonstrated to shrink each instance the set is updated in Fig. 4.3. It can also be seen that the magnitude of the δ variable error is always less than the radius of the estimated uncertainty set. Since the δ variable error is equal to the parameter estimation error, it is possible to verify that the true parameters remain within the uncertainty set throughout the simulation. Fig. 4.2 indicates that the magnitude of the state estimation error approaches zero as the identification task is completed. This follows logically from the state estimation error dynamics, and the state prediction dynamics. As the identification task progresses, and the parameter estimates approach the true parameter values, it is expected that the center of the uncertainty set will approach the true parameter values. Although this was not formally proven, the results shown in Fig. 4.3 suggest that the center of the uncertainty set will converge to the true

parameters. Therefore it is possible to conclude that as the parameter estimates approach the true parameter values, the $\tilde{\delta}$, $\hat{\delta}$ and δ values, as well as their rates of change will approach zero. Parameter convergence occurs at a slower rate than in Section 3.5, we do not offer an explanation for this result, it is possible that this is as a result of differences in computational and numerical methods, though it is not verified.

4.6 Conclusion

In this chapter, we demonstrated that the adapted parameter uncertainty set method developed in Chapter 3 is applicable to continuous-time nonlinearly parameterized systems. Though the proof guaranteeing convergence is incomplete, it is reasonable to conclude that based on the convergence result presented in Chapter 3, that such a result can be provided. A guarantee of convergence will be provided in a future work. In a similar manner to the conclusion made in Chapter 3, we can conclude that an output filter that meets the persistence of excitation condition at all times throughout the application of the algorithm is a sufficient condition for successful identification of the unknown parameters. The application of this method is illustrated using a highly nonlinear system representing a chemostat operating under Monod kinetics.
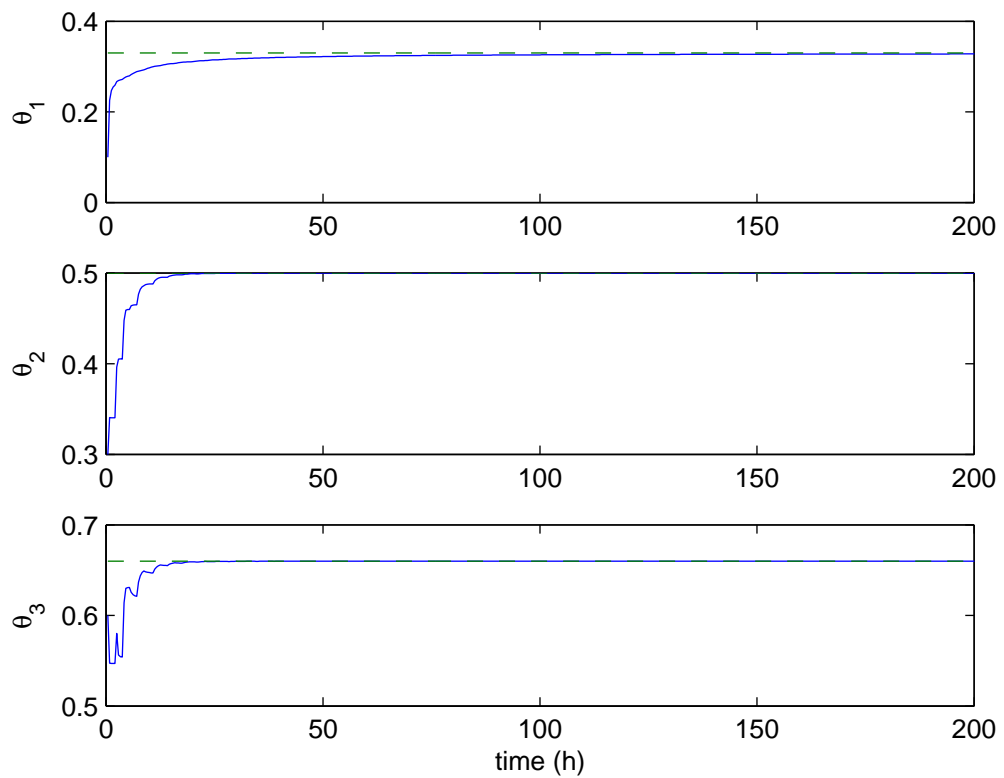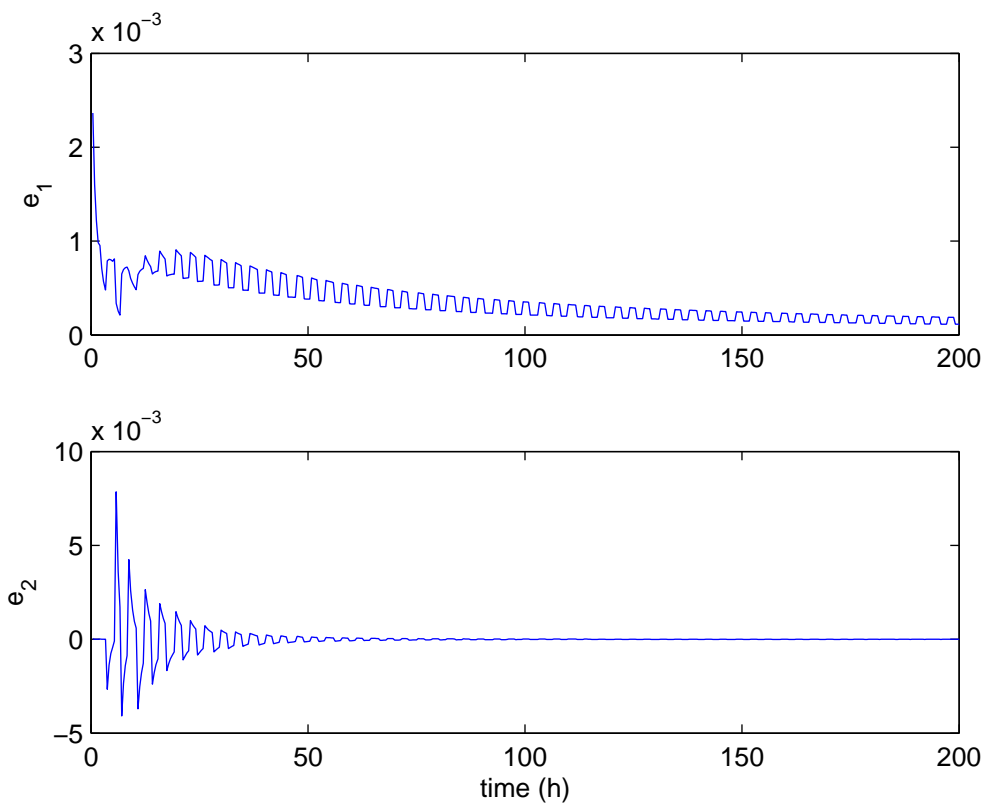
Figure 4.1: Time course plot of the parameter estimates and true values under the parameter uncertainty set algorithm, the dashed lines (- -) represent the true parameter values, the solid lines (–) represent the parameter estimates

Figure 4.2: Time course plot of the state prediction error $e_k = x_k - \hat{x}_k$
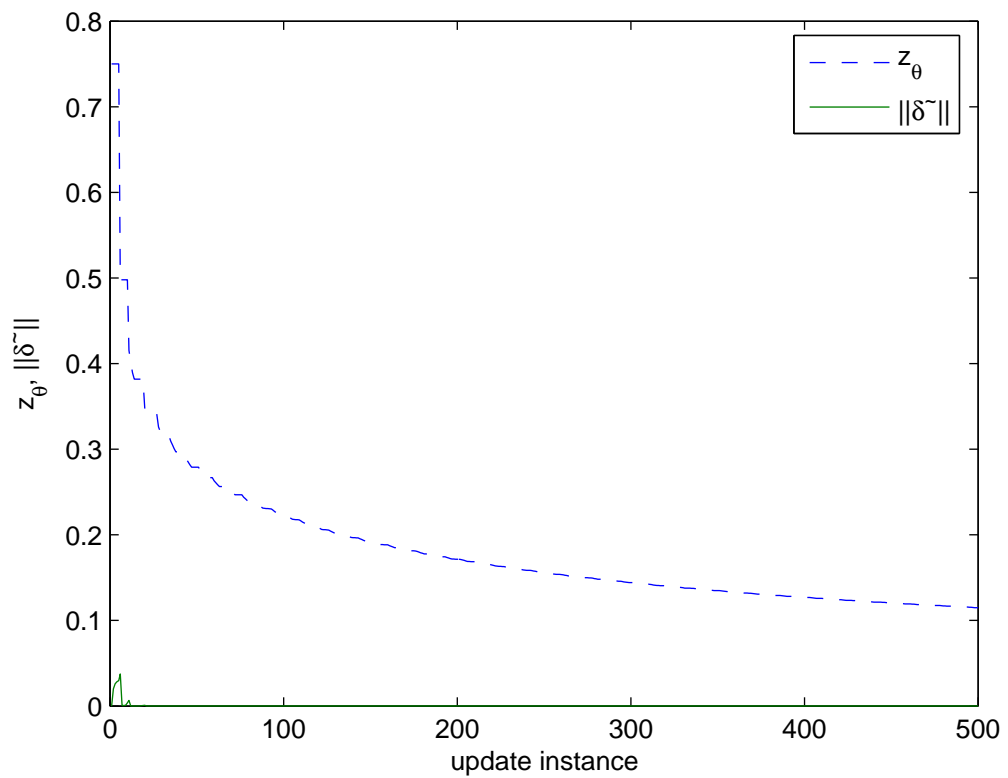
Figure 4.3: The progression of the radius of the parameter uncertainty set and the magnitude of the variable $\tilde{\delta}$

Chapter 5

Conclusions

The identification methods proposed in Chapter 2 provided a set of novel adaptive estimation techniques applicable to a class of discrete-time nonlinear systems. In Chapters 3 and 4, it is shown that the parameter uncertainty set method derived for linearly parameterized systems, can be generalized to generate a suitable technique applicable to a large class of nonlinearly parameterized systems. The proposed technique is shown to be effective for the identification of growth kinetic parameters in a chemostat operating under Monod kinetics in Sections 3.5 and 4.5.

We propose three distinct identification methods. Section 2.1 utilizes a type of identifier matrix that is present throughout the rest of the work. The identifier matrix is used both in the identification algorithm, and in the characterization of the convergence condition. This condition, based on the defined identifier, may be independent from the control input, the benefits of this decoupling are discussed below. The method presented in Section 2.2 introduces a specific online parameter updating law. This method marks an improvement not only by the reduction of computational complexity, but also by providing more accurate parameter estimates as the algorithm progresses. This is a benefit if the scheme is utilized with a control law that depends on parameter estimates. As a result of the nature of the parameter update law, the norm of the parameter estimation error can be shown to be

non-increasing, even if the convergence condition is not yet met. This means that while the identification scheme progresses, the control law will be supplied with parameter estimates of, in the worst case, non-decreasing and, potentially, increasing precision.

The third method, applied in Section 2.3 and in Chapters 3 and 4, is more widely applicable than the previous methods. The parameter uncertainty set method is demonstrated to be applicable to linearly parameterized systems with additive noise, as well as nonlinearly parameterized systems. Provided a convergence condition is met, this method guarantees convergence of the parameter estimates to their true values. Further, the uncertainty set provides a metric for the accuracy of the parameter estimates at any given time by providing an upper bound for the error of the parameter estimates. The algorithm automatically updates this upper bound when it can be guaranteed that an update will yield a contraction of the uncertainty set, while still accurately confining the parameter estimates and true values. The update algorithm provides an indicator as to the progress towards successful identification.

The methods presented herein provide several advantages over previous work. The primary departure from the literature is the development of self-contained identification algorithm. The results in this thesis suggest that it is possible to perform an identification task independent from a desired control task. Further, these result imply that the identification methods presented in this thesis could be used to improve many existing control strategies. The formulation of the parameter uncertainty set method, applied to nonlinearly parameterized systems, greatly expands the applicability of the method to a larger class of systems. Further, in the nonlinearly parameterized case, the construction of the worst case Lyapunov function removes some of the conservativeness from prior applications of this method, it is reasonable to conclude that this leads to a more efficient identification algorithm.

5.1 Future Work and Research Direction

There are several limitations to the work presented here, some of which provide avenues for further research. The primary limitation is the assumption that the system state is available for measurement. Further, all the methods presented require that model structure be well known. These assumptions can limit the application of the algorithm. It will be necessary to employ a technique to guarantee successful implementation of the methods when the system state is not known. A possible solution to this problem is the use of a state observer to recover system state values from measured system outputs. This solution would require the assumption that the system in question is observable at all times or time steps.

With regards to Chapter 4, it is necessary to provide a formal argument that the imposed restriction on the $V_{z\theta}$ value ensures that the true parameter values are not excluded from the set. It also remains to be proven that it is possible to ensure that if the persistence of excitation condition is met, then it will follow that the Σ matrix grows unbounded.

The primary avenue for further research in this area is to properly define the conditions that guarantee that the persistence of excitation conditions are met for each of the algorithms. In the case of the systems without disturbances, it is expected that this may be accomplished by developing a relationship between the output filter and the identifier matrix Q_k. It may be possible to determine what properties the output filter, or the closed-loop trajectories must have to ensure that k_c is reached, and what effect those properties have on the size of k_c. In the cases where disturbances are present, it is likely that it will be necessary to develop a metric for the growth of the covariance matrix Σ_k, and its relation to the output filter, and the closed loop system trajectories. With this metric it may be possible to identify the properties of the output filter signal or the closed loop trajectories that ensure the unbounded growth of the Σ_k matrix.

It is possible that some condition exists with respect to the definition of the correction factor or the output filter that could guarantee that the appropriate convergence condition

is met in each case. It may also be necessary to explore the injection of a dither signal similar to those applied in Lyapunov based adaptive control algorithms such as Guay et al. [15] to produce system excitation, particularly in the case of the nonlinearly parameterized systems. A dither signal may be necessary to guarantee that the convergence conditions are met. At this point, it remains a challenge to develop alternative conditions that can be used to guarantee the existence of state trajectories that meet a persistence of excitation condition. Such techniques could prove extremely useful in the analysis and design of adaptive control systems. The techniques proposed in this work can potentially lead to some suitable alternative conditions that guarantee parameter convergence.

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