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Journal of Difference Equations and Applications

ISSN: 1023-6198 (Print) 1563-5120 (Online) Journal homepage: https://www.tandfonline.com/loi/gdea20

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To cite this article: Pablo Amster & (2007) On multiplicity of solutions of a superlinear equation on time scales , Journal of Difference Equations and Applications, 13:11, 1051-1058, DOI: 10.1080/10236190701414561

To link to this article: https://doi.org/10.1080/10236190701414561

Published online: 19 Oct 2007.

Journal of Difference Equations and Applications,
Vol. 13, No. 11, November 2007, 1051–1058

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On multiplicity of solutions of a superlinear equation on time scales*

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(Received October 2006; revised 9 April 2007; in final form 21 April 2007)

We prove the existence of infinitely many stationary solutions of a nonlinear one-dimensional superlinear equation on time scales using a Lusternik-Schnirelmann type result.

Keywords: Time scale; Boundary value problem; Variational methods; Lusternik-Schnirelmann theory

AMS Subject Classification: 39A12; 39A99

1. Introduction

In this paper, we consider the following superlinear equation on a time scale $\mathbb{T}$:

$$y^{\Delta\Delta} + |y^{\sigma}|^{\gamma} y^{\sigma} = 0 \quad x \in [a, b\,]_{\mathbb{T}}, \tag{1}$$

where γ is a positive constant, under Dirichlet conditions:

$$y(a) = y(\sigma^2(b)) = 0, \tag{2}$$

with $a, b \in \mathbb{T}$, $a < b$. Replacing y by $-y$, we may consider only the case $y^{\Delta}(a) \geq 0$.

Remark 1.1. In particular, equation (1) can be regarded as a time scales version of the stationary one-dimensional case for the nonlinear Schrödinger equation

$$\partial_t y(x,t) = i\left(\partial_x^2 y(x,t) + |y(x,t)|^{\gamma} y(x,t)\right),$$

which has been the subject of great interest in recent years. This equation arises on the study of propagation of electromagnetic waves in a nonlinear medium, or of a laser beam in optical fiber [2,5,9,11,12].

Our main theorem reads as follows.

THEOREM 1.2. *Problem* (1)–(2) *has infinitely many solutions.*

Let us briefly recall that the concept of dynamic equations on time scales was introduced by Hilger in Ref. [8] with the motivation of providing a unified approach to continuous and

*This research was supported under the PIP 5477 (CONICET).
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Journal of Difference Equations and Applications
ISSN 1023-6198 print/ISSN 1563-5120 online © 2007 Taylor & Francis
http://www.tandf.co.uk/journals
DOI: 10.1080/10236190701414561

discrete calculus. The notion of a generalized derivative y^{Δ} was defined, where the domain of the function y is an arbitrary closed non-empty subset $\mathbb{T} \subset \mathbb{R}$. If the time scale $\mathbb{T}$ is a nontrivial interval, then the usual derivative is retrieved, that is $y^{\Delta} = y'$. On the other hand, if $\mathbb{T} = \mathbb{Z}$, then the generalized derivative reduces to the usual forward difference, that is $y^{\Delta} = \Delta y$.

We remark that in the continuous case $\mathbb{T} = \mathbb{R}$, Theorem 1.2 admits a simple proof. Indeed, one may verify for example that, for any $\alpha > 0$, the problem

$$y'' + y^{\gamma+1} = 0, \quad y(0) = y(\alpha) = 0$$

admits a positive solution y_α, which is unique, and symmetric with respect to $t_0 = \alpha/2$. Thus, it suffices to consider $\alpha = (b-a)/N$ for $N \in \mathbb{N}$, and define

$$y(t) = (-1)^j y_\alpha(t - a - j\alpha) \quad \text{if} \ \ a + j\alpha \leq t \leq a + (j+1)\alpha,$$

for $j = 0, \ldots, N-1$. In fact, it is easy to prove that all nontrivial solutions of the problem are constructed in this way. It is worth to observe, however, that the previous argument takes advantage of the self-similarity of the interval $[a, b]$, and for this reason it cannot be generalized for an arbitrary time scale $\mathbb{T}$.

A more general argument, which holds for the equation $y'' + g(y) = p(x)$ where g is *any* superlinear function, relies on the study of the index

$$I(\lambda) := \frac{1}{2\pi} \int_a^b \frac{y'_\lambda(x)^2 - y_\lambda(x) y''_\lambda(x)}{y_\lambda^2(x) + y'_\lambda(x)^2} \mathrm{d}x,$$

where y_λ is the unique solution of the initial value problem

$$y'' + g(y) = p(x), \quad y(0) = 0, \quad y'(0) = \lambda.$$

It is well known that $I(\lambda)$ computes the number of rounds that the curve $\Phi_\lambda : [a, b] \to \mathbb{R}^2$ given by $\Phi_\lambda(x) := (y'_\lambda(x), y_\lambda(x))$ performs around the origin, starting at the point $(\lambda, 0)$; in particular, if $I(\lambda) \in (1/2)\mathbb{Z}$, then y_λ is a solution of (1)–(2). Thus, existence of infinitely many solutions follows from the fact that I is well defined and continuous when λ is large, with $I(\lambda) \to \infty$ as $\lambda \to \infty$. But, once again, it does not seem to be possible to implement this procedure in the context of a general time scale, since its main idea is based on the following equality, which holds by simple integration of the IVP, but has no obvious generalization for time scales:

$$\frac{y'(x)^2}{2} + G(y(x)) = \frac{\lambda^2}{2} + \int_a^x p(s) y'(s) \mathrm{d}s,$$

where $G(u) = \int_0^u g(s)\mathrm{d}s$.

We shall give a different proof for an arbitrary time scale, based on a variant of the Lusternik-Schnirelmann theory given by Clark in Ref. [6]. More precisely, solutions shall be obtained from the critical points of an appropriate even functional.

The paper is organized as follows. In Section 2, we give some preliminary results concerning the Sobolev spaces on time scales.

In Section 3, we introduce an appropriate variational setting for problem (1)–(2) and give a proof of Theorem 1.2.

2. Preliminary results

There exists a vast literature on time scales after the pioneering work [8]. For a general introduction to the theory, we refer the reader to Refs. [3,4].

In order to study problem (1)–(2), by variational methods, let us recall the Lebesgue measure in times scales, firstly defined in Ref. [7], which can be constructed in the following way.

For $a, b \in \mathbb{T}$ with $a < b$, consider $\mathcal{A} \subset \mathcal{P}([a,b)_{\mathbb{T}})$ the completion of σ-algebra generated by the family

$$\{[x_0, x_1)_{\mathbb{T}} : a \leq x_0 < x_1 \leq b, x_0, x_1 \in \mathbb{T}\}.$$

Hence, there is a unique σ-additive measure $\mu_\Delta : \mathcal{A} \to \mathbb{R}^+$ defined over this basis as: $\mu_\Delta([x_0, x_1)_{\mathbb{T}}) = x_1 - x_0$. As mentioned in Ref. [1], it is easy to see that μ_Δ can be characterized as follows:

$$\mu_\Delta = \lambda + \sum_{i \in I} (\sigma(x_i) - x_i)\delta_{x_i},$$

where λ is the Lebesgue measure on $\mathbb{R}$, $\{x_i\}_{i \in I}$ is the (at most countable) set of all right-scattered points of $\mathbb{T}$ and δ_x is the Dirac measure concentrated at x. A function f which is measurable with respect to μ_Δ is called Δ-measurable, and the Lebesgue integral over $[a, b)_{\mathbb{T}}$ is denoted by

$$\int_a^b f(x)\Delta x := \int_{[a,b)_{\mathbb{T}}} f(x) \mathrm{d}\mu_\Delta.$$

Thus, for $1 \leq p < \infty$ the Banach L^p-spaces may be defined in the standard way, namely

$$L^p_\Delta([a,b)_{\mathbb{T}}) := \left\{ \hat{f} : f : [a,b)_{\mathbb{T}} \to \mathbb{R} \text{ is } \Delta - \text{measurable and } \int_a^b |f(x)|^p \Delta x < \infty \right\},$$

where $\hat{f}$ denotes the equivalence class of f, consisting of all Δ-measurable functions on $[a,b)_{\mathbb{T}}$ that coincide with f almost everywhere for the Δ-measure. The norm of this space will be denoted by

$$\|f\|_{L^p_\Delta} := \left(\int_a^b |f(x)|^p \Delta x \right)^{1/p}.$$

Next, we shall introduce as in Ref. [1] the idea of *weak time scale derivative* (for shortness, *weak derivative*). For completeness, let us recall that a function φ is termed to be right-dense continuous on $[a, b]_{\mathbb{T}}$ if φ is continuous at every right-dense point $x \in [a, b]_{\mathbb{T}}$, and $\lim_{x \to x_0^-} \varphi(x)$ exists and is finite at every left-dense point $x_0 \in [a, b]_{\mathbb{T}}$. Further, the space $C^1_{rd}([a, b]_{\mathbb{T}})$ is defined as the set of those functions $\varphi : [a, \sigma(b)]_{\mathbb{T}} \to \mathbb{R}$ that have a right-dense continuous derivative on $[a, b]_{\mathbb{T}}$.

Definition 2.1. *Let* $f \in L^p_\Delta([a,b)_\mathbb{T})$. *A weak derivative of f* (if it exists) is a Δ-measurable function g such that

$$\int_a^b f(x)\varphi^\Delta(x)\Delta x = -\int_a^b g(x)\varphi^\sigma(x)\Delta x$$

for any $\varphi \in C^1_{rd}([a,b]_\mathbb{T})$ such that $\varphi(a) = \varphi(b) = 0$.

Remark 2.2. If $f \in C^1_{rd}([a,b]_\mathbb{T})$, then by the product rule it follows that f^Δ is also a weak derivative of f.

Remark 2.3. Let $g \in C_{rd}([a,b]_\mathbb{T})$, and define $f(x) = \int_0^x g(s)\Delta s$. Then, by the fundamental theorem (see [7]) it follows that g is the derivative of f.

Remark 2.4. It is easy to see that if f has zero weak derivative, then $f \equiv c$ for some constant c. In view of the previous remark, we deduce that if f has a right-dense continuous weak derivative on $[a,b]_\mathbb{T}$, then it belongs to $C^1_{rd}([a,b]_\mathbb{T})$.

Thus, the Sobolev spaces $W^{1,p}_\Delta([a,b)_\mathbb{T})$ may be defined as in the standard case $\mathbb{T} = \mathbb{R}$:

$$W^{1,p}_\Delta([a,b)_\mathbb{T}) := \left\{ f \in L^p_\Delta([a,b)_\mathbb{T}) : f \text{ has a weak derivative } f^\Delta \in L^p_\Delta([a,b)_\mathbb{T}) \right\},$$

equipped with the norm

$$\|f\|_{W^{1,p}_\Delta} := \left(\|f\|^p_{L^p_\Delta} + \|f^\Delta\|^p_{L^p_\Delta} \right)^{1/p}.$$

In particular, for $p = 2$ we shall denote $H^1_\Delta([a,b)_\mathbb{T}) := W^{1,2}_\Delta([a,b)_\mathbb{T})$, and the norm is induced by the inner product given by

$$\langle f, g\rangle_{H^1_\Delta} := \int_a^b \left[f(x)g(x) + f^\Delta(x)g^\Delta(x) \right] \Delta x.$$

Basic properties of Sobolev spaces in time scales can be found in Ref. [1].

3. Proof of Theorem 1.2

In this section, we introduce the variational setting for problem (1)–(2), and give a proof of Theorem 1.2 based on a specialization of the Lusternik-Schnirelmann theory to the case of an even functional of a Banach space.

Without loss of generality we may assume that $a = 0$ and $b = 1$. For convenience, let us set $\beta = \sigma^2(1)$ and the space

$$H = H^1_{0\Delta}([0,\beta)_\mathbb{T}) := \left\{ y \in H^1_\Delta([0,\beta)_\mathbb{T}) : y(0) = y(\beta) = 0 \right\}.$$

From now on, we shall denote by $\|\cdot\|$ the norm of H, and by $\|\cdot\|_{L^p}$ the norm of $L^p([0, \beta)_\mathbb{T})$.

Next, define the functional $J{:}H \to \mathbb{R}$ by

$$J(y) := \left(\int_0^\beta y^\Delta(x)^2 \Delta x \right)^{\gamma+1} - \int_0^\beta |y^\sigma(x)|^{\gamma+2} \Delta x = \|y^\Delta\|^{2(\gamma+1)}_{L^2} - \|y^\sigma\|^{\gamma+2}_{L^{\gamma+2}}.$$

It is clear that if y is a critical point of J, then y is a weak solution of the problem

$$\begin{cases} \|y^{\Delta}\|_{L^2}^{2\gamma} y^{\Delta\Delta}(x) + \frac{\gamma+2}{2(\gamma+1)} |y^{\sigma}(x)|^{\gamma} y^{\sigma}(x) = 0 \\ y(0) = y(\beta) = 0. \end{cases} \tag{3}$$

Indeed, it follows from simple computation that $J \in C^1(H, \mathbb{R})$, with

$$DJ(y)(\varphi) = 2(\gamma+1)\|y^{\Delta}\|_{L^2}^{2\gamma} \int_0^{\beta} y^{\Delta}(x)\varphi^{\Delta}(x)\Delta x \quad - (\gamma+2)\int_0^{\beta} |y^{\sigma}(x)|^{\gamma} y^{\sigma}(x)\varphi^{\sigma}(x)\Delta x.$$

Thus, if $DJ(y) = 0$, we deduce that y is a weak solution of (3).

Remark 3.1. As every element of $W_{\Delta}^{1,p}([a,b)_{\mathbb{T}})$ has an absolutely continuous representative (see [1]), it follows from Remark 2.4 that any weak solution of (3) is in fact classical, in the sense that it admits a continuous (standard) time scale second derivative.

The connection between solutions of (3) and solutions of the original problem is clear from the following lemma.

LEMMA 3.2. *Assume that $y \in H - \{0\}$ is a critical point of J and let*

$$r := \left(\frac{\gamma+2}{2(\gamma+1)}\right)^{1/\gamma} \|y^{\Delta}\|_{L^2}^{-2}.$$

Then $\tilde{y} := ry$ is a solution of (1)–(2). *Moreover, if S is any one-dimensional subspace of H, the number of nontrivial critical points of J belonging to S is at most two.*

Proof. If $y \in H - \{0\}$ is a critical point of J, a straightforward computation shows that $\tilde{y}$ is a solution of (1)–(2). Moreover, if $\phi(\alpha) = J(\alpha y)$, then $\phi'(\alpha) = DJ(\alpha y)(y)$. On the other hand, as $\phi(\alpha) = A|\alpha|^{2(\gamma+1)} - B|\alpha|^{\gamma+2}$ with $A = \|y^{\Delta}\|_{L^2}^{2(\gamma+1)}$, $B = \|y\|_{L^{\gamma+2}}^{\gamma+2}$, it follows that ϕ has exactly two nonzero critical points. Thus, $DJ(\alpha y) = 0$ only for $\alpha = 0, \pm 1$. □

In order to obtain solutions of (3) as critical points of J, let us recall the well-known Palais-Smale condition.

DEFINITION 3.3. *Let E be a Banach space and $J \in C^1(E, \mathbb{R})$. It is said that J* satisfies (*PS*) if any sequence $\{y_n\} \subset E$ such that $|J(y_n)| \leq c$ for some constant c and $DJ(y_n) \to 0$, has a convergent subsequence in E.

DEFINITION 3.4. *Let E be a Banach space and let $A \subseteq E \backslash \{0\}$ be closed and symmetric with respect to* 0. *The genus of A* is defined in the following way: If there exists $f : A \to \mathbb{R}^N \backslash \{0\}$ continuous and odd with N minimum, then $\mathrm{gen}(A) = N$; otherwise, $\mathrm{gen}(A) = \infty$.

The following result is a consequence of a Lusternik-Schnirelmann theorem, due to Clark [6].

THEOREM 3.5. *Let E be a Banach space such that $\dim(E) \geq N$ for some $N \in \mathbb{N}$. Let $J \in C^1(E, \mathbb{R})$ be an even functional satisfying* (*PS*), *such that $J(0) = 0$, and assume that $c \in (-\infty, 0)$, where the constant $c = c(N)$ is defined by*

$$c := \inf_{\mathrm{gen}(A) \geq N} \sup_{y \in A} J(y).$$

Then c is a critical value of J. Furthermore, if $c(M) = c$ *for some* $M > N$, *then* $\mathrm{gen}(J^{-1}(c)) > M - N$.

Proof of Theorem 1.2.

1. J is bounded below and coercive.

   Indeed, for $y \in H$ we may write $y(x) = \int_0^x y^\Delta(s)\Delta s$, and deduce that

$$|y(x)| \leq \beta^{1/2}\|y^\Delta\|_{L^2}.$$

   Hence,

$$\|y\|_{L^2} \leq \beta|y^\Delta\|_{L^2},$$

   and

$$\|y^\sigma\|_{L^{\gamma+2}}^{\gamma+2} \leq \beta^{(\gamma+4)/2}\|y^\Delta\|_{L^2}^{\gamma+2}.$$

   Thus

$$J(y) \geq \|y^\Delta\|_{L^2}^{\gamma+2}\left(\|y^\Delta\|_{L^2}^{\gamma} - \beta^{(\gamma+4)/2}\right),$$

   and the claim follows.

2. J satisfies (PS).

   Assume that $J(y_n)$ is bounded, and that $DJ(y_n) \to 0$. Then $\{y_n\}$ is bounded in H, and taking a subsequence we may suppose that $y_n \to y$ uniformly, and weakly in H for some $y \in H$.

   Furthermore, using the fact that $DJ(y_n)(y_n - y) \to 0$ and that $y_n \to y$ uniformly, we deduce that

$$\|y_n^\Delta\|_{L^2}^{2\gamma}\int_0^\beta y_n^\Delta(x)(y_n - y)^\Delta(x)\Delta x \to 0.$$

   If $\|y_n^\Delta\|_{L^2} \to 0$, then $y_n \to 0$ in H and the claim is proved. Otherwise, taking a subsequence we may assume that

$$\int_0^\beta y_n^\Delta(x)(y_n - y)^\Delta(x)\Delta x \to 0.$$

   On the other hand, as $y_n \to y$ weakly in H, we also have:

$$\int_0^\beta y^\Delta(x)(y_n - y)^\Delta(x)\Delta x \to 0.$$

   Hence

$$\int_0^\beta [(y_n - y)^\Delta]^2(x)\Delta x \to 0,$$

   and we conclude that $y_n \to y$ strongly in H.

3. Application of Theorem 3.5.

   Let $N \in \mathbb{N}$ and $c = c(N)$ be defined as in Theorem 3.5. As J is bounded below, it follows trivially that $c > -\infty$.

Moreover, as the elements of H are continuous, it is clear that if $y \in H$ satisfies $y^\sigma = 0$, then $y = 0$. Indeed, $y(0) = y(\beta) = 0$, and for $x \in (0, \beta)_{\mathbb{T}}$ we have:

- If x is right-dense, then $y(x) = y^\sigma(x) = 0$.
- If x is left-scattered, then $y(x) = y^\sigma(\rho(x)) = 0$.
- If x_0 is right-scattered and left-dense satisfies $y(x_0) \neq 0$, then $y \neq 0$ over a neighborhood of x_0. In particular, $y \neq 0$ over a nonempty interval $I = (x_0 - \delta, x_0)_{\mathbb{T}}$ for some $\delta > 0$. As x_0 is left-dense, if $x \in I$, then $\sigma(x) \in I$. It follows that $y(\sigma(x)) \neq 0$, a contradiction.

Next, let $V \subset H$ be any subspace with $\dim(V) = N$. It follows that the norms defined over V by $\|y^\Delta\|_{L^2}$ and $\|y^\sigma\|_{L^2}$ are equivalent; thus, there exists a constant k such that

$$\|y^\Delta\|_{L^2} \leq k\|y^\sigma\|_{L^2} \quad \forall y \in V.$$

Hence, from the imbedding $L_\Delta^{\gamma+2}([0,\beta)) \hookrightarrow L_\Delta^2([0,\beta))$, if $y \in V$ we obtain:

$$J(y) \leq k^{2(\gamma+1)}\|y^\sigma\|_{L^2}^{2(\gamma+1)} - \|y^\sigma\|_{L^{\gamma+2}}^{\gamma+2} \leq \|y^\sigma\|_{L^2}^{\gamma+2}\left(k^{2(\gamma+1)}\|y^\sigma\|_{L^2}^{\gamma} - a\right)$$

for some positive constant a. Hence, there exists $\varepsilon > 0$ such that $J(y) < 0$ for any $y \in V$ such that $0 < \|y\| < 2\varepsilon$. Thus, if we consider the set

$$A = \{y \in V : \varepsilon \leq \|y\| \leq 2\varepsilon\},$$

it follows by compactness that $\sup_{y\in A} J(y) < 0$. Furthermore, from Borsuk Theorem (see, e.g. [10]) we deduce that $\gamma(A) = N$, and then $c < 0$. From Theorem 3.5, we conclude that c is a critical value of J. If $c(M) = c$ for some $M \neq N$, then $\mathrm{gen}(J^{-1}(c)) > |M - N|$ and $J^{-1}(c)$ is an infinite set of critical points of J. Otherwise, $\{c(N) : N \in \mathbb{N}\}$ is an infinite set of critical values of J. Taking into account Lemma 3.2, the proof is complete. □

Acknowledgement

The author would like to thank the anonymous referees for the careful reading of the original manuscript and their fruitful corrections and remarks.

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