

Integrated Real-time Optimization and Model Predictive Control Under Parametric Uncertainties

by

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For Ademi and the boys...

Abstract

The actualization of real-time economically optimal process operation requires proper integration of real-time optimization (RTO) and dynamic control. This dissertation addresses the integration problem and provides a formal design technique that properly integrates RTO and model predictive control (MPC) under parametric uncertainties. The task is posed as an adaptive extremum-seeking control (ESC) problem in which the controller is required to steer the system to an unknown setpoint that optimizes a user-specified objective function.

The integration task is first solved for linear uncertain systems. Then a method of determining appropriate excitation conditions for nonlinear systems with uncertain reference setpoint is provided. Since the identification of the true cost surface is paramount to the success of the integration scheme, novel parameter estimation techniques with better convergence properties are developed. The estimation routine allows exact reconstruction of the system's unknown parameters in finite-time. The applicability of the identifier to improve upon the performance of existing adaptive controllers is demonstrated.

Adaptive nonlinear model predictive controllers are developed for a class of constrained uncertain nonlinear systems. Rather than relying on the inherent robustness of nominal MPC, robustness features are incorporated in the MPC framework to account for the effect of the model uncertainty. The numerical complexity and/or the conservatism of the resulting adaptive controller reduces as more information becomes available and a better uncertainty description is obtained.

Finally, the finite-time identification procedure and the adaptive MPC are combined to achieve the integration task. The proposed design solves the economic optimization and control problem at the same frequency. This eliminates the ensuing interval of “no-feedback” that occurs between economic optimization interval, thereby improving disturbance attenuation.

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Chapter 1

Introduction

Optimization has become a key area in control theory due to the increasing need to optimize plant operations. The optimization criterion could be to reduce operating cost or maximize profit while at the same time meeting product specifications. Consequently, the control engineer is faced with the tasks of designing a good controller, and also making sure the plant operates at an optimum. With the advent of better controllers (that guarantee good plant control), the focus has now shifted to the regulation of processes about conditions that provide maximum profitability. Such a task is usually tackled using a supervisory control technique. One such technique that has received considerable attention in the process industry is real-time optimization (RTO) [1, 2].

In the traditional hierarchical control (Figure 1.1), an upper optimization layer uses a steady-state rigorous model of the plant to compute economic target values that optimize a given objective function at regular intervals, say in a period of a few hours to a few days. The steady-state optimal operational points are passed to a lower layer called advanced process control (APC) for implementation. The APC, usually a predictive controller, uses a dynamic process model to ensure that the plant operates within the desired limits or near optimum efficiency. In the two-

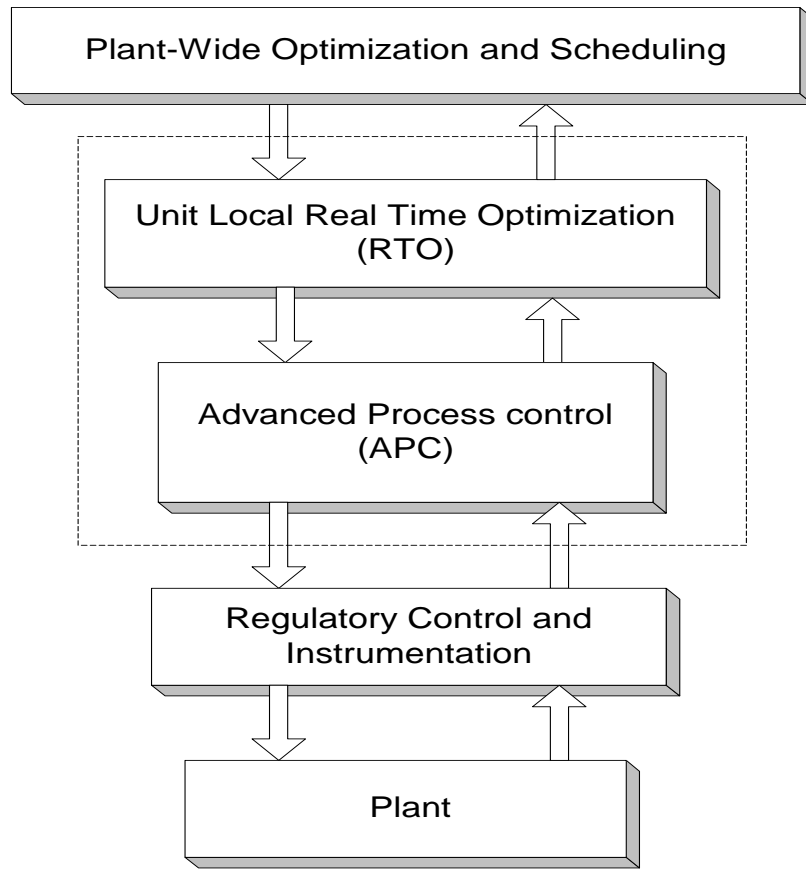


Figure 1.1: Layered structure of control hierarchy

layer approach, the RTO only calculates new set-points when the plant is stabilized. Hence, during the period between the optimizer executions, the presence of varying external disturbances can cause the optimal values to shift thereby rendering the plant operation suboptimal. Also, the fact that the two layers usually employ different process models may generate a conflict between the reference values and the controller predictions.

To avoid these shortcomings and thereby obtain improved economic benefits, the steady-state RTO in the aforementioned hierarchical structure could be replaced with a dynamic real-time optimization [3, 4]. Another solution would be to combine the two separate layers, *i.e.*, to solve the steady-state optimization and control problem together [5, 6]. Despite the fact that the two layers consist of well established tech-

nologies, their full integration remains a non-trivial task. The solution of the resulting optimization problem and the stability analysis of the system in closed-loop becomes complicated.

Since modelling uncertainty inevitably exists in industrial problems, our interest is on the optimization and optimal control of processes under parametric uncertainty. The objectives of this thesis are to

1. provide a formal scheme that integrates real-time optimization and model predictive control by solving an adaptive extremum seeking control problem,
2. develop a parameter adaptation algorithm that guarantees online identification of the true objective function with minimal variability.

Model predictive control (MPC) is the most widely used advanced control strategy in the process industry [7]. Its popularity is largely due to its ability to handle constraints and multivariable interactions. MPC optimizes a given objective function by using process model to predict system's evolution and compute a sequence of optimal control actions. The combination of RTO and MPC results in an extremum seeking control (ESC) algorithm, that adjusts plant parameters so as to optimize some performance criteria.

Extremum-seeking control (ESC) can be viewed as a dual control methodology, having optimum seeking and control features. ESC provides a way of implementing on-line optimization in a control system. The technique is very useful when an uncertain performance function is to be optimized to determine optimal process set-points. The performance function can be the output of a plant or an economic function of the system's variables. This thesis is focused on adaptive (model-based) extremum seeking techniques. In particular, we consider the class of ESC problems introduced in [8] where the structure of the objective function is employed in the design. In contrast to black-box approaches (see [9] for example), no direct measurement of the

objective function is available but must be inferred through the measurements of the state variables and the estimation of model parameters. Examples of this type of problem arise when the objective function reflects profit or cost estimates which are seldom available for measurement. Such a function normally involves economic prices such as operating costs and values of products aside from the system states and unknown parameters.

In ESC, the magnitude of the gradient of the performance function is controlled to a setpoint of zero, which results in a local minimum (or maximum) of the function under the condition that the objective function is locally convex (or concave). However, the uncertainty associated with the function makes it necessary to use some sort of adaptation and perturbation to search for the optimum. Since the efficiency of any ESC algorithm depends on its ability to generate sufficient information about the process during the adaptation procedure, this thesis provides techniques for predicting the optimal operating conditions and simultaneously steering the system to them in an optimum fashion.

1.1 Thesis Outline

This thesis is structured as follows. A review of the current literature that is relevant to the proposed research work is presented in Chapter 2.

Chapter 3 presents a control algorithm that incorporates RTO and MPC techniques to solve an output feedback extremum seeking control problem for a linear unknown system. The development of the controller consist of two steps. First, the optimum setpoint that minimizes a given performance function is obtained via an update law and secondly, the control input that drives the system to the optimum is computed. State estimation filters and a parameter update law are used at each iteration step, to update the unknown states and parameters in the optimization scheme.

The resulting controller is able to drive the system states to the desired unknown optimum by requiring a Lyapunov restriction and a satisfaction of a persistency of excitation condition.

Chapter 4 addresses the problem of parameter convergence in adaptive extremum seeking control design. An alternate version of the popular persistence of excitation condition is proposed for a class of nonlinear systems with parametric uncertainties. The condition is translated to an asymptotic sufficient richness condition on the reference setpoint. Since the desired optimal setpoint is not known *a priori* in ESC problems, the proposed method includes a technique for generating perturbation signal that satisfies this condition in closed-loop. The method guarantees parameter convergence with minimal but sufficient level of perturbation.

Chapter 5 presents a parameter estimation routine that allows exact reconstruction of the unknown parameters in finite-time provided a given excitation condition is satisfied. The robustness of the routine to unknown bounded disturbances or modeling error is also shown. The result is independent of the control and identifier structures employed. The true parameter value is obtained without requiring the measurement or computation of the velocity state vector. Moreover, the technique provides a direct solution to the problem of removing auxiliary perturbation signals when parameter convergence is achieved.

Chapter 6 wraps up the result on closed-loop identification by exploiting the finite-time identification procedure to improve upon the overall performance of existing adaptive control algorithms. This is achieved by modifying standard stabilizing adaptation laws to guarantee exponential stability of closed-loop system provided the PE condition is satisfied. The convergence rate of the algorithm depends linearly on the adaptation gain and a measure of the system's excitation.

Chapter 7 focuses on the control aspect of the proposed integration task. A method is proposed for the adaptive model predictive control of constrained nonlinear

system. Rather than relying on the inherent robustness properties of standard NMPC, the developed techniques explicitly account for the transient effect of parametric estimation error by combining a parameter adjustment mechanism with robust MPC algorithms. The parameter estimation routine employed guarantees non-increase of the estimation error vector. This means that the controller employs a process model which approaches that of the true system over time. These estimates are used to update the parameter uncertainty set at every time step, resulting in a gradual reduction in the conservative and/or computational effects of the incorporated robust features.

The extension of the adaptive MPC design to systems subject to exogenous disturbances, in addition to parametric uncertainties is presented in Chapter 8. To avoid the undesirable behaviours such as poor transient performance and burstiness associated with adaptive controllers under external disturbance input, the parameter estimates are only updated when an improved value is obtained.

Based on the results in Chapters 5 to 7, the full integration procedure for constrained uncertain nonlinear systems is demonstrated in Chapter 9. Both a single-layer and a two-layer approaches are presented. A setpoint update law that converges to the optimum of the constrained RTO problem is developed by the use of interior point barrier functions and Newton's method. Robust adaptive MPC strategies based on finite-time identification are employed for the controller calculations.

Finally, Chapter 10 summarizes the contributions made by this thesis and outlines directions for future research.

Chapter 2

Literature Review

The relevant literature for this thesis is briefly reviewed in this section. Special attention is paid to model predictive control, extremum seeking control and closed-loop identification.

2.1 Real-time Optimization

One of the key challenges in the process industry is how to best operate the plant under different conditions such as feed compositions, production rates, energy availability, feed and product prices that changes all the time. Real-time optimization (RTO), which refers to the online economic optimization of a process plant, is a widely employed technology to meet this challenge. RTO attempts to optimize process performance (usually measured in terms of profit or operating cost) thereby enabling companies to push the profitability of their processes to their true potential as operating conditions change.

The popular RTO is based on the assumption that model and disturbance transients can be neglected if the optimization execution time interval is long enough to allow the process to reach and maintain steady-state. A typical RTO system includes components for steady-state detection, data acquisition and validation, process

model updating, optimization calculations and optimal operating policies transfer to advanced controllers.

2.2 Model Predictive Control

Model predictive control (MPC) or receding horizon control (RHC) is a family of control that utilizes a process model along with cost factors and optimum target operating point to calculate process control moves that drives the plant to the most economic constraints while ensuring stable operation. The control technique has proven to be extremely successful in the process industry. Linear (and nonlinear) model predictive control remains the industry standard with increasing number of reported applications and significant improvements in technical capability [10, 11, 7].

Consider the time-invariant nonlinear system of the form

$$\dot{x}(t) = f(x(t), u(t)) \quad (2.1)$$

subject to the pointwise state and input constraints $x(t) \in \mathbb{X} \subseteq \mathbb{R}^{n_x}$ and $u(t) \in \mathbb{U} \subseteq \mathbb{R}^{n_u}$ respectively. The vector field $f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \rightarrow \mathbb{R}^{n_x}$ satisfies $f(0, 0) = 0$, the set \mathbb{U} is compact, \mathbb{X} is connected and $(0, 0) \in (\mathbb{X}, \mathbb{U})$.

MPC algorithms optimize the future plant behaviour and satisfy the given constraints by solving the following finite horizon open loop optimal control problem:

$$\begin{aligned} \min_{u^p} J &= \int_t^{t+T} L(x^p(\tau), u^p(\tau)) d\tau + W(x^p(t+T)) \\ \text{such that } \dot{x}^p(\tau) &= f(x^p(\tau), u^p(\tau)), \quad x^p(t) = x_t \\ x^p(\tau) &\in \mathbb{X}, \quad u^p(\tau) \in \mathbb{U} \\ x^p(t+T) &\in \mathbb{X}_f \end{aligned} \quad (2.2)$$

where $(.)^p$ denotes the predicted variables (internal to the controller). The stage cost

$L(x^p, u^p)$ is a semi-definite positive function. The terminal penalty $W(x^p(t+T))$ and terminal constraint $x^p(t+T) \in \mathbb{X}_f$ are included for stability considerations.

At each time step, the solution to the optimal control problem (2.2) is found over a certain prediction horizon, T , using the current state of the plant or its estimate as the initial state. The optimization yields an optimal control sequence and the first control action is implemented on the plant until the results of the next update are available.

Model predictive control is part of the multi-level hierarchy of control structure. Using a numerical optimization scheme as an integral part of the structure allows great flexibility, especially concerning the incorporation of constraints. Though such optimization over a finite horizon does not guarantee stability and performance, considerable research has been devoted to address these shortcomings. Linear MPC theory and related issues such as closed-loop stability and online computation have been well studied and characterized [12, 13]. Over the past few years, nonlinear model predictive control (NMPC) schemes with some favorable properties have been developed. The theory related to stability of state and output feedback NMPC have reached a point of relative maturity, see for example [14, 15] and [16, 17] for review.

2.2.1 Closed-loop Stability of NMPC Based on Nominal Model

A general sufficient condition for closed-loop stability of MPC based on nominal models is given below [16].

Criterion 2.1 *The terminal penalty function $W : \mathbb{X}_f \rightarrow \mathbb{R}_{\geq 0}$ and terminal set \mathbb{X}_f are such that there exists a local feedback $k_f : \mathbb{X}_f \rightarrow \mathbb{U}$ satisfying*

1. $0 \in \mathbb{X}_f \subseteq \mathbb{X}$, \mathbb{X}_f closed
2. $W(x)$ is positive semi-definite and continuous with respect to $x \in \mathbb{R}^{n_x}$
3. $k_f(x) \in \mathbb{U}$, $\forall x \in \mathbb{X}_f$

4. \mathbb{X}_f is positively invariant under k_f
5. $L(x, k_f(x)) + \frac{\partial W}{\partial x} f(x, k_f(x)) \leq 0, \forall x \in \mathbb{X}_f.$

The conditions are primarily concerned with the selection of terminal region \mathbb{X}_f and terminal penalty term $W(\cdot)$. Condition 5 requires $W(\cdot)$ to be a control Lyapunov function, over the (local) domain \mathbb{X}_f , and dissipates energy at a rate $L(x, k_f(\cdot))$. This criterion, which is able to re-cast many of the available MPC frameworks with guaranteed stability [18], provides a means of checking whether a given MPC scheme guarantees stability a-priori. Stability is proven by showing strict decrease of the optimal cost function J^* , which is a Lyapunov function for the closed-loop system. The domain of attraction for the controller is the set where the optimization problem is feasible.

2.2.2 Efficient Computational Techniques

Computational efficiency is a critical issue in NMPC design because the real-time implementation of the algorithm requires the completion of a non-convex nonlinear optimization procedure within the time constraints dictated by the sampling period of an application. Increasing research efforts are now directed to developing computationally efficient algorithms for NMPC implementation [19,20,21]. It is not the aim of this thesis to focus on computation issues, however, its importance in practical implementation is recognized.

2.2.3 NMPC for Uncertain Systems

The quality of the model used in MPC is crucial to the performance of the controller. The assumption that the prediction model is identical to the actual model is unrealistic. Although, due to the receding horizon policy, a standard implementation of MPC using a nominal model of the system dynamics exhibits nominal robustness to

sufficiently small disturbances [22, 23], such marginal robustness guarantee may be unacceptable in practical applications. Present model/plant mismatch and disturbances must be accounted for in the computation of the control law to achieve robust stability.

One way to cope with uncertainty in the system model is to employ robust MPC methods, which explicitly account for system’s uncertainties. Since robust controllers (in general) cannot learn changes in the plant, their performance is limited by the quality of the model plus the uncertainty description initially available. On the other hand, adaptive control has the potential to improve system performance as it updates the model online based on measurement data. However, practical applications of adaptive controllers are limited by the conflicting objective of parameter estimation and control. This could lead to a worse transient performance than a non-adaptive controller when poor estimates are used. Moreover, the controller may induce large transient oscillations in an effort to improve the estimation quality.

Robust Model Predictive Control

Robust techniques have been employed in MPC designs to reduce the sensitivity of the controller to uncertainty. Consider the following uncertain system

$$\dot{x} = f(x, u, \vartheta) \tag{2.3}$$

where $\vartheta(t) \in \mathcal{D}$ represents any arbitrary bounded uncertainty or disturbance signal. Many robust MPC techniques have been proposed to stabilize the uncertain system for all possible realization of the disturbance $\vartheta(t) \in \mathcal{D}$. These include approaches based on nominal prediction [24, 22, 25] and those based on min-max or worst-case techniques [26, 27].

The nominal based approach in [22] uses global Lipschitz constants to compute

worst-case upper bound on the distance between a solution of the actual uncertain model and the nominal model. These bounds are then used to redefine the terminal region and constraints in a way that guarantees robust feasibility of the closed-loop system. The controller proposed in [25] is based on the concept of reachable sets. The approach uses a local procedure to approximate the sets that contain the predicted evolution of the uncertain system for all possible uncertainties. Then, a dual mode MPC strategy is proposed to robustly stabilize the system. The methods based on nominal prediction have similar computational complexity with standard NMPC but exhibit a higher level of conservatism.

The min-max approach modifies the online optimal problem (2.2) by minimizing the worst case value of the objective function, where the worst case is taken over $\vartheta \in \mathcal{D}$. In general, when MPC incorporates explicit model uncertainty, the resulting online computation will likely grow significantly. Min-max or worst case MPC approaches can be overly conservative especially when applied in open loop. This is because the algorithm tries to find a single control sequence that works well in open loop for all admissible disturbance realizations. A well embraced method for reducing conservatism in open loop min-max scheme and improve performance is to introduce some form of feedback in the prediction [16, 26]. This can be achieved by parameterizing the control sequence in terms of the system's state. Unfortunately, such optimization with respect to closed-loop strategies is intractable (in most cases) since the problem size grows exponentially with the size of the problem data.

In general, robust MPC is designed to meet control specifications for the “worst case” uncertainty. This approach may not always achieve optimal performance, in particular, if the worst case scenario rarely exists. Other approaches, such as adaptive control may yield a better performance.

Adaptive Model Predictive Control

Adaptive MPC is an attractive way to handle static uncertainties that can be expressed in the form of constant unknown model parameters. While a few results are available for linear adaptive MPC [28, 29], only a small amount of progress has been made in developing adaptive NMPC schemes.

Consider the parameter-affine nonlinear system of the form

$$\dot{x} = f(x, u, \theta) \triangleq f(x, u) + g(x, u)\theta \quad (2.4)$$

$$\triangleq f(x) + g_1(x)u + g_2(x)\theta \quad (2.5)$$

The result in [30], implements a certainty equivalence nominal-model MPC feedback to stabilize (2.4) subject to an input constraint $u \in \mathbb{U}$. Assuming the availability of the state vector \hat{x} , the identifier guarantees parameter convergence when an excitation condition is satisfied. It is only by assumption that the true system trajectory remains bounded during the identification phase. Moreover, there is no mechanism to enhance the satisfaction of the PE condition and thereby decrease the identification period.

In [31], an input-to-state stable control Lyapunov function (iss-clf) was used to develop a MPC scheme that provides robust stabilization for (2.5) in the absence of a parameter estimation algorithm, and ensures asymptotic stability of the closed-loop with parameter adaptation. However, the work only deals with unconstrained nonlinear systems.

In general, the design of adaptive nonlinear MPC schemes is very challenging because the “separation principle assumption” widely employed in linear control theory is not applicable to a general class of nonlinear systems, in particular in the presence of constraints. Moreover, it is difficult to guarantee state constraints satisfaction in the presence of an adaptive mechanism. A true adaptive nonlinear MPC algorithm

must address the issue of robustness to model uncertainty while updating the system's parameters.

Recent work [32] has focused on the use of adaptation to improve upon the performance of robust MPC for constrained nonlinear systems. A set-valued description of the parametric uncertainty is directly adapted online to reduce the conservativeness of the robust MPC solutions, especially with respect to the design of the terminal penalty and constraints. The parameterization of the feedback MPC policy in terms of the uncertainty set and the underlying min-max feedback MPC used in the study make the controller's computation very challenging. The result can be viewed as a conceptual result that focus on performance improvement rather than implementation. The idea of coupling set-based identification with robust control calculations was extended in [33] to a less computationally complex robust-MPC framework. While the full performance of the min-max framework could not be recovered, the work demonstrates that the approach is amenable to any robust-MPC design.

2.3 Integrated Real-Time Optimization and Control

In the conventional layered optimization-control strategy, an upper (or supervisory) real-time optimization (RTO) layer computes optimal operating values and a lower layer (or control) implements them. The RTO is based on a steady state plant model and it is only executed when the plant is stabilized. The control problem is solved apart from the optimization problem at different frequencies and using different models. Since the two layers may not be dealing exactly with the same information, there may be some conflict and the predicted optimal operational point is often sub-optimal.

The integration of RTO and control for an optimal plant operation is still an open

research problem. There have been attempts in the literature to address the integration task in a systematic way. The only results are focused on specific applications. No theoretical foundation such as stability and performance analysis of the integrated scheme exists. The proposed techniques can be classified into two main classes, based on the frequency of RTO execution and the type of model employed.

1. Single level strategy, integration of steady-state RTO and Linear

MPC In this approach, the economic optimization and control problems are solved simultaneously in a single algorithm. model-based predictive control is formulated to perform both functions by adding an economic objective term to standard MPC objective function [5,6]. The resulting problem involves solving a nonlinear objective function subject to dynamic and steady-state constraints. Implementation of the extended controller relies on extensive tuning of the weighting factors for stability and performance. Extensive simulations are needed to select appropriate values for the weighting factors [6] unlike in linear MPC where robust tuning techniques are available. The one-layer approach could respond to changes in the plant optimal conditions faster than the two-layer approach. However, the method is only applicable to plants that have relatively short settling time and can be effectively described with a linear model.

2. Two level strategy, integration of dynamic RTO and nonlinear MPC

This method attempts to account for dynamic nonlinear behaviour of production plants. It involves an upper level dynamic economic optimization and a lower level nonlinear MPC [3,4,34]. The RTO problem need not be solved at each sample time, but based on disturbance dynamics or plant condition. A technique based on disturbance sensitivity analysis is embedded into the two level strategy to trigger the dynamic RTO computation when persistent disturbances have been detected that have high sensitivities [3,4]. This approach

considers dynamical models in the RTO and recomputes feasible set-points only if economic benefits are possible. Since the performance of the approach depends on the interaction between the two layers, the scheme may still suffer some disadvantages due to the different dynamical models used in the two layers approach.

2.4 Extremum Seeking Control

Extremum seeking (also known as peak seeking or self optimizing) control is a class of adaptive control that deals with regulation to unknown set points. The control algorithm finds the operating set-points that maximize or minimize a performance function. The pioneering work in ESC was documented in 1922 [35] and several applications of ESC have been reported since then [36,37,38,39,40,41,42]. However, it was not until the early 2000 that solid theoretical foundations for the stability and performance of ESC for a given scheme were established [8,9,43]. The main difference between the existing ESC algorithms lies in the formulation of the problem. The existing methods can be broadly classified into model-free and model-based methods.

2.4.1 Model-free Methods

In this class of ESC, the objective function is available for online measurement but has an unknown structure. The function is described in terms of unknown parameter(s) and it is assumed that there exists a control law that is capable of stabilizing the system over a range of the parameter(s). The most popular schemes under this category are the methods based on perturbation.

Perturbation method was introduced in [35], and a number of perturbation based techniques and applications are available in the literature [36,38,44]. Since the idea of extremum control is to stay at the optimum of a given objective function, this class of

ESC uses an excitation signal to obtain information about the gradient of the function given the state of a process. A basic form of the extremum seeking scheme [44] is shown in Figure 2.1. A known perturbation signal is added to the input of the controlled system to induce a cyclic response in the performance function. Then, the functional signal response is passed through filter(s) to design an online update law, which determines the sign of the gradient (derivative of the function with respect to the parameter) and the parameter estimate is adjusted in the proper direction. A successful application of the scheme is based on an appropriate choice of the filter parameters, the perturbation signal parameters and the adaptation gain.

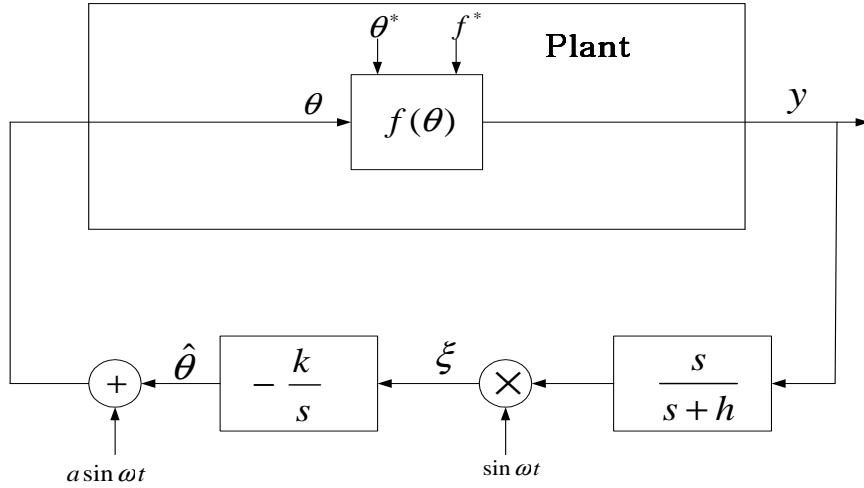


Figure 2.1: A basic model-free extremum seeking scheme

The stability and convergence properties of the perturbation method were presented in [9]. The problem is separated by time scales, assuming that the controlled system (the plant dynamics and the stabilizing controller) is fast with respect to the peak seeking loop (filters and the periodic perturbation). The stability analysis adopts the methods of averaging and singular perturbation. It is shown that the plant output (performance index) converges to a neighborhood of the extremum when the initial plant output error is sufficiently small. An extension of the algorithm to multivariable parameter space was developed in [45] and a generalization of the basic

scheme to slope seeking is presented in [46]. Instead of seeking a point of zero slope as in extremum seeking, the technique seeks a reference slope and allows the system to operate at a point of arbitrary slope.

Another perturbation method was developed and applied to formation flight in [40]. Similar to [38], a perturbation signal is introduced into the loop to facilitate the gradient estimation. The difference is that a modified Kalman filter is used in the peak seeking loop instead of classical frequency domain filters and the actual gradient of the performance function is estimated instead of the direction towards the extremum.

From the above discussion, it is noticed that the knowledge of the system model and the structure of the performance function is not required to develop the feedback scheme. Hence, the design can be regarded as a model-free adaptive control method. However, the assumption that the plant comes complete with a robust tracking control law is restrictive. In most cases, especially for nonlinear systems, a good knowledge of the system model may be required to design such a robust controller. In fact, the two applications mentioned above used the nonlinear dynamical model in developing the control law. A natural question one may ask is: *why couldn't the same model or knowledge of the model be used in the ESC design if it is available?*

2.4.2 Model-based Methods

If a good parametric model of the performance function is found and its parameters are identified, it is possible to perform far more efficient optimization than would be possible with the knowledge of only a gradient estimate. Some researchers have embraced this idea and developed ESC algorithms for nonlinear systems in which the performance function may not be directly measurable for feedback but it has to be a known function of the system's state parameterized by unknown parameters [8].

The performance index is assumed to be smooth and convex for all admissible

range of parameters and an update law is used to determine the parameters of the surface online. Once the parameters are identified, the location of the extremum on the surface is known which makes the method very fast compared with the non-parametric methods.

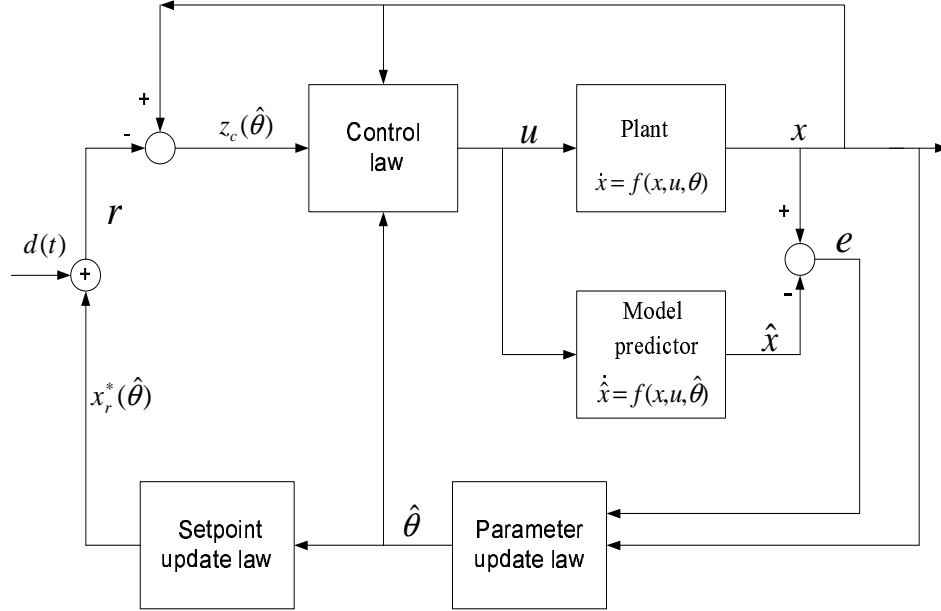


Figure 2.2: A basic model-based extremum seeking scheme

The overall scheme is presented in Figure 2.2. Unlike the techniques reviewed in Section 2.4.1 where the plant is assumed to come complete with a robust stabilizable control law, a feedback control algorithm that is capable of simultaneously seeking the extremum and tracking the dictated setpoint is developed. Lyapunov based techniques are used in the design of the optimum seeking scheme. The resulting controller is able to drive the systems states to a neighborhood of the desired optimum by requiring a satisfaction of a persistence of excitation (PE) condition. This class of ESC has been successfully applied to the maximization of production rate in a continuous stirred tank bioreactor [42]. A variant of the Lyapunov based extremum seeking framework is reported in [47] and an extension of the scheme to constrained nonlinear systems is presented in [48].

Other articles about optimum searching using the structure of the performance index include [49, 50]. The approach presented in [50] approximates the performance index by an assumed function with a finite number of parameters. Specifically a quadratic function or an exponential of a quadratic form is assumed to fit the optimized function. A stable extremum seeking controller is designed based on the affine nature of the gradient and without assuming the time scale separation between the system dynamic and the gradient search.

2.5 Closed-loop Identification and Control (CLIC)

The efficiency of any ESC scheme depends on the convergence properties of the adaptive algorithm utilized. Convergence of the system's output response as well as that of the estimated parameters to their true values are essential. One approach to tackle the problem of parameter convergence in model-based ESC is to develop a technique that is simultaneously capable of identifying the system's model parameters and at the same time tracking the desired reference value. In this section, issues related to persistence of excitation in identification are discussed and control systems that have the elements of duality (identification and control) are reviewed.

2.5.1 Background

Consider an input-output data sequence $\{u(k)\}$ and $\{y(k)\}$. The objective of system identification is to find the best model to fit the data. If we consider the deterministic autoregressive exogenous (DARX) model structure

$$y(k) = \sum_{i=1}^n a_i y(k-i) + \sum_{i=1}^m b_i u(k-i) + w(k) \quad (2.6)$$

where $y(k)$ is the output, $u(k)$ is the input, a_i and b_i are constant unknown parameters while $w(k) = w + \epsilon(k)$, w is a constant disturbance and ϵ is a zero mean white noise with variance σ_ϵ^2 . If we let

$$\theta = [a_1 \dots a_n \ b_1 \dots b_m \ w]^T$$

and

$$\phi(k-1) = [y(k-1) \dots y(k-n) \ u(k-1) \dots u(k-m) \ 1]^T$$

then we can write equation (2.6) in the standard regression form as

$$y(k) = \phi(k-1)^T \theta + \epsilon(k), \quad k = 1, \dots, N \quad (2.7)$$

or in vector form as

$$Y = \Phi^T \theta + \epsilon \quad (2.8)$$

where Y is the vector of output measurements, Φ is the matrix containing past outputs and inputs, θ is the unknown constant parameter vector and ϵ is the error vector. Equation (2.8) can be solved in a least-squares sense. The solution of the various least-squares algorithms requires the inverse of the matrix $\Phi\Phi^T$. For example, a least squares estimate of θ that minimizes the cost function $V(\theta) = \frac{1}{2}\epsilon^T\epsilon$ is given by $\hat{\theta} = (\Phi\Phi^T)^{-1}\Phi Y$. The accuracy of the parameter estimate can be quantified by its covariance estimate

$$\mathbf{P} = E\{(\theta - \hat{\theta})(\theta - \hat{\theta})^T\} = (\Phi\Phi^T)^{-1}\sigma_\epsilon^2 \quad (2.9)$$

where $\hat{\theta}$ is an estimate of the parameter. Looking closely at equation (2.9), it is essential that the information matrix $\Phi\Phi^T$ be non-singular and it is desirable that

the matrix be large enough, since the parameter estimate covariance is inversely proportional to it. This leads to the idea of persistence of excitation (PE). A unique solution of (2.8) is guaranteed if the information matrix $\Phi\Phi^T$ is invertible and bounded [51].

2.5.2 Closed-loop Identification and Control Strategies

CLIC algorithms are designed to cautiously achieve the control goal and at the same time improve parameter estimation. One of the early techniques for CLIC is termed adaptive dual control (ADC). In this method, a measure of estimation accuracy is transmitted online from the estimator to the control design algorithm [52]. The formal solution to the proposed optimal ADC problem involves solving stochastic dynamic programming equations, that are computationally prohibitive due to the growing dimension of the underlying space [53]. To avert this problem, sub-optimal dual controllers have been developed based on the approximation or reformulation of the problem. While the approximation techniques still inherit some level of complexity and computational disadvantages, the methods of reformulation provides a more computationally feasible approach. The reformulated ADC algorithms are mainly based on the simultaneous or sequential minimization of a two-part cost function [54,55,56,53]. The function minimized is of the form

$$\mathcal{J}_k^{EDC} = \mathcal{J}_k^c + \lambda_k \mathcal{J}_k^a \quad (2.10)$$

where $\lambda_k \geq 0$ is a time dependent weighting factor. The first part \mathcal{J}_k^c is a measure of control losses while the second part \mathcal{J}_k^a is a measure of parameter uncertainty. The uncertainty measures suggested in the literature include the the determinant of the parameter covariance matrix [53] and innovation or prediction error cost, that reflects the need to gather as much information as possible about the unknown parameters

[56].

In a similar approach [57], the control is chosen to minimize a one-step ahead output variance subject to the constraint that the trace of the information matrix be positive. An improved dual adaptive technique termed bicriterial design method was presented in [54, 55]. In this scheme, the cost function (2.10) is split in to two, \mathcal{J}_k^c and \mathcal{J}_k^a , and minimized sequentially. The resulting technique provided an intuitive and easy way of selecting the magnitude of the excitation as a function of the uncertainty measure. There is no doubt that these adaptive dual control techniques would improve the adaptation and may even result in parameter convergence. However, convergence to their true values is not guaranteed if the control systems do not ensure the invertibility of the information matrix, which is a sufficient condition for parameter convergence.

Another CLIC approach is iterative feedback control, that emerged in the nineties to bridge the gap between system identification and robust control design for linear systems. The basic idea of the feedback control is to iteratively use operational data to re-design a controller in order to improve closed-loop performance. The design techniques can be viewed either as iterative model-based or model-free controller redesign. The model-based design approach [58] involves successive model and controller iteration. The system is observed under fixed feedback for a period of time, after which identification and control redesign is performed by minimizing a performance criterion that accounts for both the identification and control objectives. Since the controller is fixed during the identification experiment, a persistently exciting external signal is injected to the loop to guarantee closed-loop identifiability. In the model-free method, the iterative identification and control scheme is reformulated as a parameter optimization problem, in which the controller parameter is estimated directly, thereby eliminating the identification step. The optimal controller parameter is obtained by minimizing a control performance criterion via a gradient based

scheme. At each iteration, an estimate of the gradient of the criterion is computed using information from closed-loop experiments with the most recent controller in the loop. A comprehensive overview of this approach termed iterative feedback tuning can be found in [59].

To handle system constraints during closed-loop identification, a technique that combines least squares parameter estimation and linear model predictive control was introduced in [60]. The framework termed “model predictive control and identification” (MPCI) is based on a finite horizon optimization problem that minimizes a quadratic objective function subject to standard linear MPC constraints and PE constraint. The main distinguishing feature between MPCI and MPC is that the online optimization is performed, at each time step, with respect to process inputs that satisfy a PE constraint formulated in terms of the process inputs. The implementation of the algorithm follows the typical MPC fashion. The feasibility of the online optimization problem ensures the dual action of the controller. Even though the work is focused on linear systems, the online computation involved in the solution of the algorithm remains a concern due to the non-convexity of the PE constraint [61]. The advantage of this simultaneous MPC and identification method in comparison to other CLIC techniques is that input and output constraints are handled explicitly and the direct inclusion of the PE constraint results in a closed-loop identification with a minimum loss of regulation performance. An extension of the work to nonlinear system would require a PE condition that is formulated in terms of only the process input or desired reference setpoint.

2.5.3 Persistence of Excitation Requirement for CLIC

In both linear and nonlinear adaptive systems, parameter convergence is related to the satisfaction of persistence of excitation condition, which can be defined in the continuous time as follows.

Definition 2.2 [62, 63]: A vector function $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^{n_\theta}$ is said to be persistently exciting if there are positive constants c_1 , c_2 , and T_0 such that

$$c_1 I \geq \int_t^{t+T_0} \phi(\tau)\phi(\tau)^T d\tau \geq c_2 I, \quad \forall t \geq 0 \quad (2.11)$$

Although the matrix $\phi(\tau)\phi(\tau)^T$ may be singular at every instant τ , the PE condition requires that ϕ span the entire n_θ -dimensional space as τ varies from t to T_0 , that is, the integral of the matrix $\phi(\tau)\phi(\tau)^T$ should attain full rank over any interval of some length T_0 . Though this condition is difficult to check because of its dependency on closed-loop signals, this shortcoming has been remedied for linear systems.

In adaptive linear systems, the PE condition was converted to sufficient richness (SR) condition on the reference input signal. Necessary and sufficient conditions for parameter convergence are then developed in terms of the reference signal. A popular result implies that exponential convergence is achieved whenever the reference signal “contains enough frequencies”, *i.e.*, whenever the spectral density of the signal is nonzero in at least n_θ points, where n_θ is the number of unknown parameters in the adaptive scheme. Otherwise, convergence to a characterizable subspace of the parameter space is achieved [64].

Despite the fact that the theory of parameter convergence for linear systems is well established, very few results are available for nonlinear systems. This is mainly because the familiar tools in linear adaptive control cannot be directly extended to nonlinear systems. In most of the available results, stability and performance properties are proved by assuming that a vector function, which depends on closed-loop signals is persistently exciting. However, the means of verifying this PE condition *a priori* for a given nonlinear system remains an open problem, in general.

The first attempts to relate a closed-loop PE condition to a SR condition on the external reference signal for nonlinear adaptive control systems are presented

in [65, 66]. Though the papers did not provide a generic answer as in linear case, they have some interesting results. In [65], a procedure is provided for determining *a priori* whether or not a specific reference signal is sufficiently rich for a specific output feedback nonlinear system, and hence whether or not parameter estimates will converge. A similar result was obtained in [66] for strict feedback systems. For the strict feedback case, the procedure only involves checking the linear independence of the rows of an appropriately defined constant real matrix. Exponential parameter convergence is guaranteed when the matrix is full rank and a partial convergence result is established when the rows of the matrix are linearly dependent. Neither of the techniques developed are directly applicable to ESC problems because the reference signal in this class of problem is not well known in advance. Nevertheless, the main result in [66, 65] is that the presence of nonlinearities in the plant usually reduces the SR condition requirement on the reference signal and thus enhances parameter convergence.

2.5.4 Excitation Signal Design

The accuracy of an identified model, depends on the choice of input signal. In most cases, the input signal is required to be persistently exciting. The optimal identification signals have been recognized to be either white noise, pseudo-random binary sequences or multi-sine signals. Nevertheless, it is intuitively clear that excitation at some frequencies is likely to be more useful than excitation at other frequencies.

Different optimization criteria have been proposed for generating persistently exciting inputs. In the early work on identification, a matrix of input and output measurements called the Fisher information matrix was defined and a D-optimality criterion which consists in maximizing the logarithm of the determinant of this matrix is used. However, this information criteria does not always accurately measure the amount of information that can be extracted from a system [67]. A criterion based

on the maximization of the smallest eigenvalue of the average information matrix was proposed in [68]. The selection of this criteria was motivated from adaptive control literature, where the rate of parameter convergence in a number of the adaptive estimation schemes is bounded from below by the smallest eigenvalue of the information matrix determined by the regressor vector [69]. Maximizing the smallest eigenvalue will produce a reasonable convergence rate for the adaptive algorithm. Using a first order linear system example, [68] compared the input signal that maximizes the determinant of the information matrix, the minimum eigenvalue and the inverse of the matrix condition number. It was shown that the determinant based criterion leads to a rather poor input design. The main reason for its popular use is probably the simplicity of computation of the associated optimal input. However, the findings that are based on a simple example, must be carefully extrapolated to more general situations.

Perturbation signals have been used in adaptive control to enforce PE condition on the reference input and thereby ensure parameter convergence. The downside of this is that a constant PE deteriorates the tracking of the reference input leading to poor control performance. A common approach is to introduce such PE signal and remove it when the parameters are *assumed* to have converged. In some cases the dither signal (often a multi-sine signal) is modulated with an exponential function to ensure that the perturbation signal dies out gradually. However, there is no formal guideline for selecting the exponential parameter that dictates the rate of disappearance of the dither signal. Furthermore, if the system's unknown parameters change due to disturbances or any other reasons, convergence may not be guaranteed except when the excitation is manually re-activated.

To address the above shortcomings, the concept of intelligent excitation signal design was developed in [70] for model reference adaptive control of linear systems and extended to feedback linearizable systems in [71]. The proposed design attenuates

the PE signal as parameter convergence is achieved and re-activates it only when required. The intelligent excitation algorithms adjust the excitation magnitude online to meet the conflicting objectives of control and identification by exploring the relationship between parameter estimation error and the state variable errors.

Chapter 3

Integration of RTO and Linear Output Feedback MPC

3.1 Introduction

As discussed in Section 2.3, the theoretical issues related to the integration of RTO and control for an optimal plant operation has not been well dealt with. This chapter focuses on the economic optimization and control of uncertain linear systems. Even though most processes are nonlinear, linear process models are developed and used in the control of chemical processes and research has shown that this model is often adequate for control of some nonlinear systems. The result has been published [72] under the title “Adaptive Output Feedback Extremum Seeking Receding Horizon Control of Linear Systems”.

The formulation consists of two-phase optimization problems that are solved at every sample time. Assuming that a suitable functional expression for the plant profit is available, which in some applications, may depend on unknown plant parameters, the first phase (RTO) takes the current value of the parameter estimates and calculates the optimal value which maximizes the economic objective. The second phase

(MPC) solves the dynamic finite horizon optimal control problem that regulates the output to the desired target value computed by the RTO. A basic representation of the proposed scheme is given in Figure 3.1. The design achieves dynamic tracking of the unknown optimum and ensures both transient and asymptotic performance.

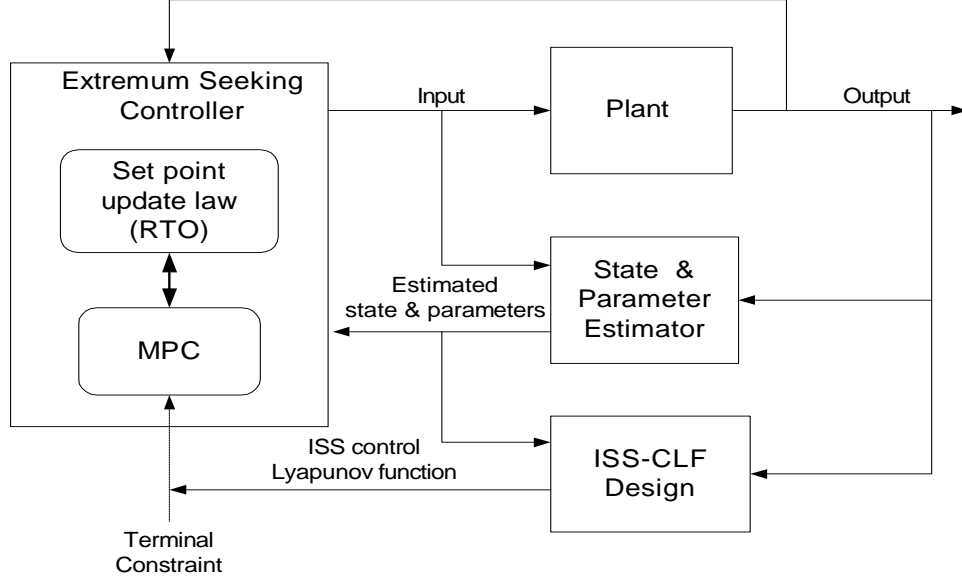


Figure 3.1: Basic structure of the proposed extremum seeking scheme

3.2 Problem Description

Consider an objective (*profit*) function of the form

$$y_p = p(y, \theta_p) \quad (3.1)$$

where $\theta_p \in \mathbb{R}^p$ is a parameter vector that satisfies

$$\theta_p \in \Theta_p = \left\{ \theta_p \in \mathbb{R}^p \mid \frac{\partial^2 p(y, \theta_p)}{\partial y^2} \leq c_0 I < 0, y \in \mathbb{R} \right\} \quad (3.2)$$

The objective function depends on the output of the linear plant

$$y(s) = \frac{b_m s^m + \dots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0} u(s) \quad (3.3)$$

where the a_i 's and b_i 's are unknown constants.

The condition given in (3.2) ensures that the performance function $p(y, \theta_p)$ is strictly concave, which means that the objective function y_p achieves its maximum at a unique point y^* . This study is carried out under the following basic assumptions.

Assumption 3.1 *The plant is minimum phase, the relative degree ρ , an upper bound for the plant order n and the high frequency gain are known.*

Assumption 3.2 *The unknown parameter $\theta_p \in \mathbb{R}^p$ in (3.1) consists of some of the constants $a = [a_{n-1} \dots a_0]^T$ and $\bar{b} = [b_{m-1} \dots b_0]^T$.*

The control objective is to design a controller that finds and tracks the optimum value of the system's state. Such optimal operating conditions must yield the maximum value of the uncertain objective function (3.1).

Remark 3.3 *The convexity assumption is necessary to ensure that the optimum of the performance function is obtained. The function may not satisfy this assumption on a global basis but it can be approximated by some convex functions when the process is sufficiently close to the optimum. The assumptions about the plant are the same as in the traditional adaptive control. The minimum phase assumption is required for the design of the output feedback controller.*

3.3 Design Procedure

Let us re-write (3.3) in the observer canonical form

$$\begin{aligned} \dot{x} &= Ax + \begin{bmatrix} 0_{(\rho-1) \times 1} \\ b \end{bmatrix} u - ya \\ y &= \varepsilon_1^T x \end{aligned} \quad (3.4)$$

where ε_i is a row vector of appropriate dimension with i^{th} entry of one and zero elsewhere.

$$A = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 1 \\ 0 & \cdots & \cdots & 0 \end{bmatrix} \quad b = \begin{bmatrix} b_m \\ \vdots \\ b_0 \end{bmatrix} \quad a = \begin{bmatrix} a_{n-1} \\ \vdots \\ a_0 \end{bmatrix}$$

or

$$\begin{aligned} \dot{x} &= Ax + g(y, u)^T \theta_m \\ y &= \varepsilon_1^T x \end{aligned} \quad (3.5)$$

$$g(y, u)^T = \begin{bmatrix} \begin{bmatrix} 0_{(\rho-1) \times (m+1)} \\ I_{m+1} \end{bmatrix} u, & -I_n y \end{bmatrix} \quad \theta_m = \begin{bmatrix} b \\ a \end{bmatrix}_{(m+n+1) \times 1}$$

3.3.1 State Estimation

Following the procedure in [73], the following state estimation filters are employed for the system.

$$\dot{\zeta} = A_0\zeta + Ly \quad (3.6)$$

$$\dot{\Omega}^T = A_0\Omega^T + g(y, u)^T \quad (3.7)$$

where the vector $L = [l_1, \dots, l_n]^T$ is chosen so that the matrix

$$A_0 = A - L\varepsilon_1^T \quad (3.8)$$

is Hurwitz. The state estimate is defined as

$$\hat{x} = \zeta + \Omega^T\theta_m \quad (3.9)$$

and the dynamics of the state estimation error $e = x - \hat{x}$ becomes

$$\begin{aligned} \dot{e} &= Ax + g(y, u)^T\theta_m - [A_0\zeta + Ly + A_0\Omega^T\theta_m + g(y, u)^T\theta_m] \\ &= A_0[x - (\zeta + \Omega^T\theta_m)] = A_0e \end{aligned} \quad (3.10)$$

This implies that e vanishes exponentially.

To lower the order of the Ω -filter dynamics, we can exploit the structure of the matrix $g(y, u)$ in (3.7). If we denote the first $m + 1$ columns of Ω^T by v_m, \dots, v_0 and the remaining n columns by Ξ , then the dynamics of the Ω -filter are governed by

$$\begin{aligned} \Omega^T &= [v_m, \dots, v_1, v_0, \Xi] \\ \dot{v}_j &= A_0v_j + \varepsilon_{n-j}u \quad j = 0, \dots, m \\ \dot{\Xi} &= A_0\Xi - Iy \end{aligned}$$

Moreover, due to the special structure of A_0 , the filter can be implemented according to [73] as follows.

$$\dot{\eta} = A_0\eta + \varepsilon_n y \quad (\text{output filter}) \quad (3.11a)$$

$$\dot{\lambda} = A_0\lambda + \varepsilon_n u \quad (\text{input filter}) \quad (3.11b)$$

$$\Xi = -[A_0^{n-1}\eta, \dots, A_0\eta, \eta] \quad (3.11c)$$

$$\zeta = -A_0^n \eta \quad (3.11d)$$

$$v_j = -A_0^j \lambda \quad j = 0, \dots, m \quad (3.11e)$$

$$\Omega^T = [v_m, \dots, v_1, v_0, \Xi] \quad (3.11f)$$

3.3.2 ISS Controller Design via Backstepping

The existence of an iss control Lyapunov function ensures that the plant (3.4) is input-to-state stabilizable with respect to parameter estimation error. Such a control Lyapunov function (CLF) will be used to develop a terminal constraint that guarantees the performance of the proposed extremum seeking receding horizon control (ESRHC) algorithm.

The iss controller design begins by considering the first equation in (3.4),

$$\dot{y} = x_2 - a_{n-1}y = x_2 - y\varepsilon_1^T a \quad (3.12)$$

Replacing x_2 by its estimate

$$\hat{x}_2 = \zeta_2 + \Omega_{(2)}^T \theta_m + e_2 \quad (3.13)$$

$$= \zeta_2 + b_m v_{m,2} + [0, v_{m-1,2}, \dots, v_{0,2}, \Xi_{(2)}] \theta_m + e_2 \quad (3.14)$$

results in

$$\dot{y} = \zeta_2 + b_m v_{m,2} + \phi^T \theta + e_2 \quad (3.15)$$

where the regressor vector, ϕ , and the unknown parameter θ are defined as

$$\phi = [v_{m-1,2}, \dots, v_{0,2}, \Xi_{(2)} - y \varepsilon_1^T]^T \quad (3.16)$$

$$\theta = [b_{m-1}, \dots, b_0, a_{1n-1}, \dots, a_{10}]^T. \quad (3.17)$$

Let us choose $v_{m,2}$ as the ‘virtual control’ because both $v_{m,2}$ and the unmeasured state x_2 are separated by only $\rho-1$ integrators from the actual control u . Considering (3.15) and (3.11) for $j = m$, the design system chosen to replace (3.4) is

$$\dot{y} = \zeta_2 + b_m v_{m,2} + \phi^T \theta + e_2 \quad (3.18a)$$

$$\dot{v}_{m,2} = v_{m,3} - l_2 v_{m,1} \quad (3.18b)$$

$$\vdots \quad (3.18c)$$

$$\dot{v}_{m,\rho-1} = v_{m,\rho} - l_{\rho-1} v_{m,1} \quad (3.18d)$$

$$\dot{v}_{m,\rho} = v_{m,\rho+1} - l_\rho v_{m,1} + u \quad (3.18e)$$

Given a constant setpoint, y^r , to be tracked, our goal is to achieve input to state stability (iss) of the tracking error

$$z_1 = y - y^r$$

with respect to the parameter estimation error $\tilde{\theta}$. The dynamics of the tracking error are given as

$$\dot{z}_1 = b_m v_{m,2} + \zeta_2 + \phi^T \theta + e_2.$$

Let

$$z_i = b_m v_{m,i} - \alpha_{i-1}, \quad \text{for } i = 2, \dots, \rho. \quad (3.19)$$

For $z_2 = b_m v_{m,2} - \alpha_1$, we choose

$$\alpha_1 = -c_1 z_1 - g_1 z_1 - \zeta_2 - \phi^T \hat{\theta} - k_1 \phi^T \phi z_1, \quad c_1 > 0, \quad g_1 > 0$$

to obtain

$$\dot{z}_1 = -c_1 z_1 - g_1 z_1 + e_2 + \phi^T \tilde{\theta} - k_1 \phi^T \phi z_1 + z_2 \quad (3.20)$$

which will be asymptotically stabilizing if $\tilde{\theta}$, e_2 and z_2 were zero. Consider a Lyapunov function

$$V_1 = \frac{1}{2} z_1^2,$$

taking the time derivative of V_1 along (3.20) results in

$$\dot{V}_1 = -c_1 z_1^2 - g_1 z_1^2 + e_2 z_1 + \phi^T \tilde{\theta} z_1 - k_1 \phi^T \phi z_1^2 + z_1 z_2. \quad (3.21)$$

Using the fact that

$$-g_1 z_1^2 + e_2 z_1 \leq \frac{1}{4g_1} e_2^2 \quad \text{and} \quad \phi^T \tilde{\theta} z_1 - k_1 \phi^T \phi z_1^2 \leq \frac{1}{4k_1} \|\tilde{\theta}\|^2,$$

equation (3.21) becomes

$$\dot{V}_1 \leq -c_1 z_1^2 + \frac{1}{4g_1} e_2^2 + \frac{1}{4k_1} \|\tilde{\theta}\|^2 + z_1 z_2. \quad (3.22)$$

Note that α_1 is a function of $y, y^r, \eta, \hat{\theta}, v_{m-1,2}, \dots, v_{0,2}$ and in view of (3.11), $v_{i,j}$ can

be expressed as

$$v_{i,j} = [*, \dots, *, 1] \bar{\lambda}_{i+j},$$

where $\bar{\lambda}_k \triangleq [\lambda_1, \dots, \lambda_k]^T$ and $\lambda_k \triangleq 0$ for $k > n$. It is therefore concluded that $\alpha_1 = \varphi(y, y^r, \eta, \hat{\theta}, \bar{\lambda}_{m+1})$. Also, as it will be seen later, $\alpha_i = \varphi(y, y^r, \eta, \hat{\theta}, \bar{\lambda}_{m+i})$.

Remark 3.4 *In the proposed ESRHC development and implementation, the unknown plant parameter vector θ is replaced by its estimated value $\hat{\theta}$ at every time step. This is assumed to be time invariant over the prediction horizon. Since the iss-clf is used as a terminal constraint to guarantee performance of the receding horizon controller, it is sufficient to consider $\hat{\theta} = \bar{\theta}$, with $\dot{\hat{\theta}} = 0$ in the iss-controller design.*

Step 2: Differentiating (3.19) for $i = 2$ along the second equation in (3.18), we obtain

$$\begin{aligned} \dot{z}_2 &= b_m v_{m,3} - \beta_2 - \frac{\partial \alpha_1}{\partial y} \left(\phi_1^T \tilde{\theta}_p + \phi_2^T \tilde{\theta}_2 + e_2 \right) \\ &= z_3 + \alpha_2 - \beta_2 - \frac{\partial \alpha_1}{\partial y} \left(\phi_1^T \tilde{\theta}_p + \phi_2^T \tilde{\theta}_2 + e_2 \right) \quad \text{from (3.19)} \end{aligned}$$

where β_2 is a function of available signals given by

$$\beta_2 = b_m l_2 v_{m,1} + \frac{\partial \alpha_1}{\partial y} \left(\phi_1^T \hat{\theta}_p + \phi_2^T \hat{\theta}_2 + \zeta_2 + b_m v_{m,2} \right) + \frac{\partial \alpha_1}{\partial \eta} \dot{\eta} + \sum_{j=1}^{m+1} \frac{\partial \alpha_1}{\partial \lambda_j} (-k_j \lambda_1 + \lambda_{j+1}).$$

Choosing $V_2 = V_1 + \frac{1}{2} z_2^2$ and

$$\alpha_2 = -c_2 z_2 - g_2 \left(\frac{\partial \alpha_1}{\partial y} \right)^2 z_2 + \beta_2 - z_1 - k_{21} \left(\frac{\partial \alpha_1}{\partial y} \right)^2 \|\phi\|^2 z_2 - k_{22} \left(\frac{\partial \alpha_1}{\partial y} \right)^2 \|\phi_2\|^2 z_2, \quad c_2 > 0, \quad g_2 > 0$$

results in

$$\begin{aligned} \dot{V}_2 &\leq -c_1 z_1^2 + \frac{1}{4g_1} e_2^2 + \frac{1}{4k_{11}} \|\tilde{\theta}_p\|^2 + \frac{1}{4k_{12}} \|\tilde{\theta}_2\|^2 \\ &\quad - c_2 z_2^2 + \frac{1}{4g_2} e_2^2 + \frac{1}{4k_{21}} \|\tilde{\theta}_p\|^2 + \frac{1}{4k_{22}} \|\tilde{\theta}_2\|^2 + z_2 z_3. \end{aligned}$$

Steps 3 ... $\rho - 1$. For $i = 3 \dots \rho - 1$

$$\begin{aligned}
z_i &= b_m v_{m,i} - \alpha_{i-1}, \\
\dot{z}_i &= b_m v_{m,i+1} - \beta_i - \frac{\partial \alpha_{i-1}}{\partial y} (\phi^T \tilde{\theta} + e_2) = z_{i+1} + \alpha_i - \beta_i - \frac{\partial \alpha_{i-1}}{\partial y} (\phi^T \tilde{\theta} + e_2), \\
\beta_i &= b_m l_i v_{m,1} + \frac{\partial \alpha_{i-1}}{\partial y} (\phi^T \hat{\theta} + \zeta_2 + b_m v_{m,2}) + \frac{\partial \alpha_1}{\partial \eta} \dot{\eta} + \sum_{j=1}^{m+i-1} \frac{\partial \alpha_{i-1}}{\partial \lambda_j} (-k_j \lambda_1 + \lambda_{j+1}), \\
V_i &= V_{i-1} + \frac{1}{2} z_i^2, \\
\alpha_i &= -c_i z_i - g_i \left(\frac{\partial \alpha_{i-1}}{\partial y} \right)^2 z_i + \beta_i - z_{i-1} - k_i \left(\frac{\partial \alpha_{i-1}}{\partial y} \right)^2 \|\phi\|^2 z_i, \quad c_i > 0, \quad g_i > 0 \\
\dot{V}_i &\leq \sum_{l=1}^{\rho-1} -c_l z_l^2 + \frac{1}{4g_l} e_2^2 + \phi^T \tilde{\theta} z_1 - k_1 \phi^T \phi z_1^2 - \sum_{j=2}^{\rho-1} \frac{\partial \alpha_{j-1}}{\partial y} z_j \phi^T \tilde{\theta} - \sum_{j=2}^{\rho-1} \left[\frac{\partial \alpha_{j-1}}{\partial y} z_j \right]^2 k_j \|\phi\|^2, \\
&\leq \sum_{l=1}^i -c_l z_l^2 + \frac{1}{4g_l} e_2^2 + \frac{1}{4k_l} \|\tilde{\theta}\|^2 + z_i z_{i+1}.
\end{aligned}$$

step ρ . Now for $i = \rho$, define $z_\rho, \dot{z}_\rho, \beta_\rho, \alpha_\rho$ as in the previous step, $z_{\rho+1} = 0$ and

$$V_\rho = \sum_{j=1}^{\rho} \frac{1}{2} z_j^2 \quad (3.23)$$

Designing the iss control law as

$$u^{iss} = \frac{1}{b_m} [\alpha_\rho - b_m v_{m,\rho+1}], \quad (3.24)$$

and defining $z = [z_1, \dots, z_\rho]^T$, $C = c_i I$, for $i = 1 \dots \rho$, $G = \sum_{l=1}^{\rho} g_l$ and $K = \sum_{l=1}^{\rho} k_l$, we have

$$V^{iss} = \frac{1}{2} z^T z, \quad (3.25)$$

$$\begin{aligned}
\dot{V}^{iss} &\leq -z^T C z + \frac{1}{4G} e_2^2 + \phi^T \tilde{\theta} z_1 - k_1 \phi^T \phi z_1^2 \\
&\quad - \sum_{j=2}^{\rho} \frac{\partial \alpha_{j-1}}{\partial y} z_j \phi^T \tilde{\theta} - \sum_{j=2}^{\rho} \left[\frac{\partial \alpha_{j-1}}{\partial y} z_j \right]^2 k_j \|\phi\|^2,
\end{aligned} \quad (3.26)$$

$$\leq -z^T C z + \frac{1}{4G} e_2^2 + \frac{1}{4K} \|\tilde{\theta}\|^2 \quad (3.27)$$

This implies that the error dynamic z is bounded whenever $\tilde{\theta}$ and e_2 are bounded. Hence, (3.25) is an iss-Lyapunov function candidate for the extremum seeking problem under consideration. Since z_1 and y^r are bounded, y is also bounded. This follows from (3.11) that η is bounded. Following the argument presented in [73], it can be shown that λ is bounded which implies the boundedness of x .

3.4 ESRHC Formulation and Analysis

3.4.1 Formulation

The formulation consists of two-phase design procedure as described below.

First Phase (RTO)

At every time step t , the maximum value of the objective (*profit*) function (3.1) is obtained via an online setpoint update law as follows.

Consider a Lyapunov function candidate

$$V_{sp} = \frac{1}{2} \left(\frac{\partial p(r, \hat{\theta}_p)}{\partial r} \right)^2 \quad (3.28)$$

where r denotes an optimal setpoint for the output y . Taking the time derivative of V_{sp} , we have

$$\dot{V}_{sp} = \frac{\partial p}{\partial r} \left[\frac{\partial^2 p}{\partial r^2} \dot{r} + \frac{\partial^2 p}{\partial r \partial \hat{\theta}_p} \dot{\hat{\theta}}_p \right].$$

Choosing the update law as

$$\dot{r} = - \left(\frac{\partial^2 p}{\partial r^2} \right)^{-1} \left[k_r \frac{\partial p}{\partial r} + \frac{\partial^2 p}{\partial r \partial \hat{\theta}_p} \dot{\hat{\theta}}_p \right], \quad k_r > 0 \quad (3.29)$$

leads to

$$\dot{V}_{sp} \leq -k_r \left(\frac{\partial p}{\partial r} \right)^2. \quad (3.30)$$

Equations (3.28) and (3.30) imply that r approaches the ($\hat{\theta}_p$ -dependent) optimal setpoint y^* exponentially.

To provide some richness condition on the setpoint, we append it with a bounded dither signal $d(t)$ and define an approximate setpoint

$$y^r(t) = r(t) + d(t).$$

In general, $d(t)$ is chosen to contain at least n_θ (n_θ =number of unknown parameters) distinct frequencies, required for parameter convergence. Other specific details will be given later.

Second Phase (MPC)

At this step, a finite horizon optimal control problem is solved subject to the system dynamics and terminal state inequality constraints at every time step with the estimated plant states $x(t)$ as initial condition. The goal of this phase is to minimize a given cost while ensuring that the system output y tracks the reference setpoint y^r dictated by the first phase.

To this end, let us re-write (3.4) as

$$\dot{x} = Ax + \varepsilon_\rho b_m u + \bar{g}(y, u)^T \theta \quad (3.31)$$

where

$$\bar{g}(y, u)^T = \begin{bmatrix} \begin{bmatrix} 0_{\rho \times m} \\ I_m \end{bmatrix} u, & -I y_n \end{bmatrix} \quad (3.32)$$

and define the cost

$$J = \int_t^{t+T_p} z^T(\tau) C z(\tau) + u(\tau)^T R u(\tau) d\tau + z^T(t + T_p) P z(t + T_p) \quad (3.33)$$

where P and R are positive definite weighting matrices, T_p is the length of the prediction horizon and

$$y^r = r + d(t), \quad (3.34a)$$

$$z_1 = x_1 - y^r, \quad (3.34b)$$

$$z_i = x_i - \alpha_{i-1}, \quad i = 2 \dots \rho, \quad (3.34c)$$

$$\alpha_1 = -c_1 z_1 - g_1 z_1 + \varepsilon_{m+1}^T \bar{\theta} x_1, \quad (3.34d)$$

$$\alpha_{i-1} = -(c_{i-1} + g_{i-1}) z_{i-1} - z_{i-2} + \dot{\alpha}_{i-2} + \varepsilon_{m+i}^T \bar{\theta} x_1. \quad (3.34e)$$

The proposed ESRHC scheme is given by:

$$\min_u J(z, \bar{\theta}, u^p) \quad (3.35a)$$

$$\text{s.t} \quad \dot{x} = Ax + \varepsilon_\rho b_m u + \bar{g}(x_1, u)^T \bar{\theta} \quad (3.35b)$$

$$x(t) = \hat{x}(t), \quad \bar{\theta} = \hat{\theta}(t) \quad (3.35c)$$

$$V(t + T_p) \leq V^{iss}(t + T_p). \quad (3.35d)$$

The function V is the value of the CLF resulting from the application of ESRHC and V^{iss} is the value of the CLF that results from the application of the iss controller. Constraint (3.35d) guarantees that the states under the ESRHC are brought within

the level set of the iss-controller at the end of the prediction horizon, thereby ensuring that the state variables under the ESRHC remain bounded. By (3.35c) the optimization problem is initialized by the estimated state, and the unknown parameters $\bar{\theta}$ in (3.35a) and (3.35b) are replaced by the estimated values $\hat{\theta}$. The optimizer computes the required control moves over the horizon. The input $u(t)$ is implemented on the plant at time t . An estimate of the unmeasured state \hat{x} and the unknown parameters $\hat{\theta}(t)$ are obtained via an observer and a parameter update law respectively. The horizon is shifted forward and a new optimization problem is solved at the next time step $t + \delta$ with the new $x = \hat{x}(t + \delta)$ and $\bar{\theta} = \hat{\theta}(t + \delta)$. The control $u(t + \delta)$ is applied at time $t + \delta$ and the process is repeated. In general, it is assumed that the time step length δ can be chosen to be arbitrarily small.

3.4.2 Analysis

The stability and performance of the proposed scheme is demonstrated in the following. Consider the function

$$W(z(t)) = z^T(t + T_p)Pz(t + T_p) + \frac{1}{2} \int_t^{t+T_p} z(\sigma)^T C z(\sigma) d\sigma \quad (3.36)$$

where $P = \frac{1}{2}I$ and $z(\cdot)$ is the error trajectory resulting from the ESRHC. This function is positive definite and it is radially unbounded if the system's CLF $V = \frac{1}{2}z^T P z$ is positive definite and radially unbounded.

For $\tau \in [t, t + \delta]$, eq. (3.36) becomes

$$W(z(\tau)) = z^T(\tau + T_p)Pz(\tau + T_p) + \frac{1}{2} \left[\int_\tau^{t+T_p} z(\sigma)^T C z(\sigma) d\sigma + \int_{t+T_p}^{\tau+T_p} z(\sigma)^T C z(\sigma) d\sigma \right] \quad (3.37)$$

However, from (3.26), we have

$$\dot{V}^{iss} \leq -z^T C z + \frac{1}{4G} e_2^2 + \phi^T \tilde{\theta} z_1 - \sum_{j=2}^{\rho} \frac{\partial \alpha_{j-1}}{\partial y} z_j \phi^T \tilde{\theta} \quad (3.38)$$

which when integrated over $[t + T_p, \tau + T_p]$ results in

$$\frac{1}{2} \int_{t+T_p}^{\tau+T_p} z(\sigma)^T C z(\sigma) d\sigma \leq V(t + T_p) - V(\tau + T_p) + \int_{t+T_p}^{\tau+T_p} \Upsilon(\sigma) d\sigma$$

where

$$\Upsilon = -\frac{1}{2} z^T C z + \frac{1}{4G} e_2^2 + \phi^T \tilde{\theta} z_1 - \sum_{j=2}^{\rho} \frac{\partial \alpha_{j-1}}{\partial y} z_j \phi^T \tilde{\theta}$$

Hence, equation (3.37) becomes

$$W(z(\tau)) \leq W(z(t)) + \int_{t+T_p}^{\tau+T_p} \Upsilon(\sigma) d\sigma$$

dividing both sides by $\tau - t$ and taking the lim sup as τ goes to t results in

$$\dot{W}(z(t)) \leq \Upsilon(t + T_p) \quad (3.39)$$

Closed-loop Analysis

Re-write (3.31) as

$$\dot{x} = Ax + Bu(x)^{RHC} - ya, \quad x(t) = x(t) \quad (3.40)$$

$$\dot{\hat{x}} = A\hat{x} + Bu(\hat{x})^{RHC} - ya, \quad x(t) = \hat{x}(t) \quad (3.41)$$

where $B = f_1 \bar{b} + \varepsilon_\rho b_m$,

$$f_1 = \begin{bmatrix} 0_{\rho \times m} \\ I_m \end{bmatrix} \quad \bar{b} = \begin{bmatrix} b_{m-1} \\ \vdots \\ b_0 \end{bmatrix} \quad a = \begin{bmatrix} a_{n-1} \\ \vdots \\ a_0 \end{bmatrix}$$

The state error dynamic $\tilde{x} = x - \hat{x}$ between (3.40) and (3.41) is

$$\dot{\tilde{x}}(t) = A\tilde{x} + Bu(\tilde{x})^{RHC}, \quad \tilde{x}(t) = x(t) - \hat{x}(t) \quad (3.42)$$

and the solution of (3.42) for $\tau \in [t, t + T_p]$ is

$$\tilde{x}(\tau) = e^{A(\tau-t)}\tilde{x}(t) + \int_t^\tau e^{A(\tau-\sigma)}Bu(\tilde{x}(\sigma))^{RHC}d\sigma \quad (3.43)$$

The solution u^{RHC} resulting from the ESRHC scheme has been shown to be piecewise affine [74], *i.e.*,

$$u(t) = k_i x(t) + m_i \quad (3.44)$$

for $x(t) \in C_i \triangleq [x : H_i s \leq s_i] \quad i = 1, \dots, s$ where $\bigcup_{i=1}^s C_i$ is the set of states for which a feasible solution to the finite horizon optimal control problem (second phase) exists.

Therefore,

$$u(\tilde{x}(t)) = k_i \tilde{x}(t) + \epsilon_i \quad (3.45)$$

If we let $\bar{k} := \max_i \|k_i\|$, $\bar{\epsilon} := \max_i |\epsilon_i|$, Then (3.45) becomes

$$u(\tilde{x}(t)) \leq \bar{k} \|\tilde{x}\|(t) + \bar{\epsilon}. \quad (3.46)$$

When $\|\tilde{x}\| \geq 1$, $|u| \leq (\bar{k} + \bar{\epsilon})\|\tilde{x}\|$. Also, when $0 \leq \|\tilde{x}\| < 1$, and $\bar{\epsilon}$ sufficiently small, there exists $\nu > 0$ such that $|u| \leq (\bar{k} + \nu)\|\tilde{x}\|$. So, without loss of generality, it is assumed that

$$|u(\tilde{x}(t))| \leq L\|\tilde{x}(t)\|, \quad L := \max(\bar{k} + \bar{\epsilon}, \bar{k} + \nu)$$

Then, using Bellman Gronwall Lemma, (3.43) results in

$$\|\tilde{x}(\tau)\| \leq e^{(\tau-t)}\|\tilde{x}(t)\| + \int_t^\tau e^{(\tau-\sigma)}\varpi\|\tilde{x}(\sigma)\|d\sigma \leq \varrho\|\tilde{x}(t)\| \quad (3.47)$$

where $\varpi = L\|B\|$ and $\varrho = \exp(-\varpi + \varpi e^{(\tau-t)})$.

Parameter Estimation

We define the predicted state, \hat{x}_a as

$$\hat{x}_a = \zeta + \Omega^T \hat{\theta}_m$$

Considering (3.6) and (3.7), the predicted state dynamic is given as

$$\dot{\hat{x}}_a = A_0\zeta + Ly + A_0\Omega^T \hat{\theta}_m + g(y, u)^T \hat{\theta}_m \quad (3.48)$$

Noting that $g(y, u)^T \theta_m = \bar{g}(y, u)^T \theta + \varepsilon_\rho b_m u$ and b_m is assumed known, the prediction error $e_a = x - \hat{x}_a$ dynamic results in

$$\dot{e}_a = \bar{g}(y, u)^T \tilde{\theta} + A_0 e_a. \quad (3.49)$$

Consider a Lyapunov function

$$V_1(t) = W(t) + \frac{1}{G} e^T(t) P_0 e(t) + \frac{1}{2} e_a^T(t) Q_0 e_a(t) + \frac{1}{2} \tilde{\theta}^T(t) \Gamma^{-1} \tilde{\theta}(t) \quad (3.50)$$

where $\Gamma = \Gamma^T > 0$, Q_0 and P_0 are real symmetric positive definite matrices that satisfy $P_0 A_0 + A_0^T P_0 = -\varrho^2 I$ and $Q_0 A_0 + A_0^T Q_0 = -I$ respectively. Taking the time derivative of V_1 along the solutions of (3.10) and (3.49) we have

$$\begin{aligned} \dot{V}_1(t) &\leq \Upsilon(t + T_p) - \frac{\varrho^2}{G} e(t)^T e(t) - \dot{\hat{\theta}}^T(t) \Gamma^{-1} \tilde{\theta}(t) - \frac{1}{2} e_a^T(t) e_a(t) + e_a^T(t) Q_0 \bar{g}(y, u)^T \tilde{\theta}(t) \\ &= -\frac{1}{2} z^T(t + T_p) C z(t + T_p) + \frac{1}{4G} e_2^2(t + T_p) + v(t + T_p) \tilde{\theta}(t + T_p) - \frac{\varrho^2}{G} e(t)^T e(t) \\ &\quad - \dot{\hat{\theta}}^T(t) \Gamma^{-1} \tilde{\theta}(t) - \frac{1}{2} e_a^T(t) e_a(t) + e_a^T(t) Q_0 \bar{g}(y, u)^T \tilde{\theta}(t) \end{aligned} \quad (3.51)$$

where

$$v(t + T_p) = \phi^T(t + T_p) z_1(t + T_p) - \sum_{j=2}^{\rho} \frac{\partial \alpha_{j-1}}{\partial y}(t + T_p) \phi^T(t + T_p) z_j(t + T_p).$$

It is deduced from (3.47) that $\|e(t + T_p)\| \leq \varrho \|e(t)\|$ since it is initialized by the state estimates obtained via (3.9) at the beginning of the prediction horizon *i.e.* $\tilde{x}(t) = e(t)$.

It follows that

$$\frac{1}{4G} e_2^2(t + T_p) \leq \frac{\varrho^2}{4G} e_2^2(t)$$

and since $e_2^2(t) \leq e^T(t) e(t)$,

$$\frac{\varrho^2}{4G} e_2^2(t) - \frac{\varrho^2}{G} e(t)^T e(t) \leq -\frac{3\varrho^2}{4G} e(t)^T e(t).$$

Moreover, considering the fact that there is no adaptation along the prediction horizon, we have $\tilde{\theta}(t + T_p) = \tilde{\theta}(t)$. Collecting the terms in equation (3.51) with respect to $\tilde{\theta}$, we obtain

$$\dot{V}_1(t) \leq -\frac{1}{2} z(t + T_p)^T C z(t + T_p) - \frac{3\varrho^2}{4G} e(t)^T e(t) - \frac{1}{2} e_a^T(t) e_a(t) + \left[\psi - \dot{\hat{\theta}}^T \Gamma^{-1} \right] \tilde{\theta} \quad (3.52)$$

where $\psi = e_a^T Q_0 \bar{g}(y, u)^T + v(t + T_p)$.

The parameter adaptation rule is selected to ensure that $[\psi - \dot{\hat{\theta}}^T \Gamma^{-1}] \tilde{\theta} \leq 0$ and that the parameter estimates remain in some given set. This is achieved by using a standard parameter projection law. Refer to [73] for more details.

Defining $\Psi = \Gamma \psi^T$, the update law is given by

$$\dot{\hat{\theta}} = Proj \{ \hat{\theta}, \Psi \} \quad (3.53)$$

$$= \begin{cases} \Psi, & \text{if } \|\hat{\theta}\| < \pi \text{ or } \left(\|\hat{\theta}\| = \pi \text{ and } \nabla \mathcal{P}(\hat{\theta}) \Psi \leq 0 \right) \\ \Psi - \Psi \frac{\gamma \nabla \mathcal{P}(\hat{\theta}) \nabla \mathcal{P}(\hat{\theta})^T}{\|\nabla \mathcal{P}(\hat{\theta})\|_\gamma^2}, & \text{otherwise} \end{cases}$$

where $\mathcal{P}(\hat{\theta}) = \hat{\theta}^T \hat{\theta} - \pi^2 \leq 0$, $\hat{\theta}$ is the vector of parameter estimates, γ is a positive definite symmetric matrix.

The properties of the projection operator ensures that the parameters are bounded and that

$$\dot{V}_1(t) \leq -\frac{1}{2} z(t + T_p)^T C z(t + T_p) - \frac{3}{4G} \varrho^2 e(t)^T e(t) - e_a^T(t) K e_a(t) \quad (3.54)$$

From (3.54), it is concluded that z , e , e_a and $\tilde{\theta}$ are uniformly bounded and that z , e and e_a converge to the origin asymptotically.

Parameter Convergence

From the previous sub-section, it is established that e_a converges to zero, hence, $\int_0^\infty \dot{e}_a(\sigma) d\sigma = -e_a(0)$ exists and is finite. Also, from (3.49), it is known that \dot{e}_a is a function of bounded signals $y, u, \tilde{\theta}$ and \ddot{e}_a is bounded. Hence, \dot{e}_a is uniformly continuous. By Barbalat's lemma [73], it is concluded that $\dot{e}_a \rightarrow 0$ as $t \rightarrow \infty$. This

implies that

$$\lim_{t \rightarrow \infty} \bar{g}(y, u)^T \tilde{\theta} = 0 \quad \text{or} \quad \lim_{t \rightarrow \infty} \tilde{\theta}^T \bar{g}(y, u) \bar{g}(y, u)^T \tilde{\theta} = 0.$$

If $\bar{g}(y, u) \bar{g}(y, u)^T$ is positive definite, then the parameter error $\tilde{\theta}$ converges to zero asymptotically. However, this condition is not always true because $\bar{g}(y, u) \bar{g}(y, u)^T$ can be singular at any given time. The integral of $\bar{g}(y, u) \bar{g}(y, u)^T$ for $t \rightarrow \infty$ is considered. It then follows that over any bounded interval of length $0 < T_0 < \infty$ that

$$\lim_{t \rightarrow \infty} \frac{1}{T_0} \int_t^{t+T_0} \left(\tilde{\theta}(\tau)^T \bar{g}(\tau) \bar{g}(\tau)^T \tilde{\theta}(\tau) \right) d\tau = 0 \quad (3.55)$$

In order to prove the convergence of $\tilde{\theta}$ to zero, a condition on the richness of the dither signal $d(t)$ is required.

Definition 3.5 (Persistence of Excitation) *The closed-loop dynamics exhibit persistency of excitation (PE) if there exists constants $T_0 > 0$, $c_{PE} > 0$ and a sequence $\{t_i\}$ with $t_i \rightarrow \infty$ as $i \rightarrow \infty$ such that the following is true*

$$\frac{1}{T_0} \int_{t_i}^{t_i+T_0} \bar{g}(\tau) \bar{g}(\tau)^T d\tau \geq c_{PE} I \quad (3.56)$$

Lemma 3.6 *Consider the adaptive system, eq.(3.4), with receding horizon controller eqs.(3.35a)-(3.35d), the adaptive law (3.53), the state observer (3.9). If the dither signal $d(t)$ is chosen such that the PE condition (3.56) is satisfied, then the parameter estimation error $\tilde{\theta}$ converges to zero asymptotically.*

The proof of this result is given in Section 3.7.

Theorem 3.7 *Consider the objective function (3.1) subject to the system dynamics (3.4), and satisfying the given assumptions. If the dither signal $d(t)$ satisfies the persistence of excitation condition (3.56), then the ESRHC (3.35a)-(3.35d), the state*

observer (3.9) and the parameter update law (3.53) solves the extremum seeking problem.

Proof:

It follows from the stability analysis that $z_1 \triangleq y - y^r \rightarrow 0$ as $t \rightarrow \infty$. If $d(t)$ is designed to satisfy the PE condition (3.56), then it is concluded by Lemma 3.6 that $\lim_{t \rightarrow \infty} \hat{\theta} = \theta$. Hence y converges to a neighborhood of the optimal setpoint r^* whose size depends on $d(t)$. ■

3.4.3 Transient Performance of the ESRHC

Corollary 3.8 *The tracking error z of the closed-loop dynamical system is bounded by*

$$\|z\|_p \leq 2\sqrt{\gamma V_1(0)}, \quad p = 2 \text{ or } \infty, \quad \gamma = \frac{1}{\lambda_{\min}(C)}$$

Proof: \mathcal{L}_2 Performance:

$$\|z\|_2^2 = \int_0^{T_p} z^T(\tau)z(\tau)d\tau + \int_{T_p}^{\infty} z^T(\tau)z(\tau)d\tau$$

From (3.54) we know that $z(t+T_p)^T z(t+T_p) \leq -2\gamma \dot{V}_1$. Since V_1 is non-increasing,

$$\int_{T_p}^{\infty} z^T(\tau)z(\tau)d\tau \leq 2\gamma [V_1(T_p) - V_1(\infty)] \leq 2\gamma V_1(0) \quad (3.57)$$

Also from (3.36), we have

$$\int_0^{T_p} z^T(\tau)z(\tau)d\tau \leq 2\gamma [W(z(0)) - z^T(T_p)Pz(T_p)] \quad (3.58)$$

Equations (3.57) and (3.58) lead to

$$\|z\|_2^2 \leq 2\gamma [W(z(0)) + V_1(0)] \quad (3.59)$$

Noting from (3.50), that $W(z(0)) \leq V_1(0)$ concludes the \mathcal{L}_2 norm proof. The proof for \mathcal{L}_∞ performance follows from (3.50) and the fact that V_1 is non-increasing. ■

Remark 3.9 *From above, it is clear that the transient performance depends on $\tilde{\theta}$, $e(0)$, $e_a(0)$, $z(0)$, G , P_0 and Γ . We can set $z_1(0)$ to zero by setting $\hat{y}^r(0) = y(0)$ and use the other tuning functions to systematically reduce the bounds.*

3.4.4 Controller Implementation and Tuning

In this subsection, we provide a brief discussion of the design and tuning parameter that are needed for the implementation of the proposed controller.

- Tuning parameters

There are some design parameters that must be selected by the user. These include the prediction horizon length T_p , control horizon length T_c , sampling time and the parameter adaptation rate Γ . The choice of prediction and control horizon length depend on the amount of time available for on-line computation. When more time is available, the horizons can be extended. The sampling time selected depends on the dynamics of the system considered. For example, a plant with fast dynamics will require a small sampling time in order to capture the evolution of the state and output variables. The parameter update rate should be chosen large enough to provide some robustness to the estimation routine. The rate of convergence of the state estimate is determined by the choice of the vector L in equation (3.8) and it must be selected so that the matrix A_0 is Hurwitz.

- Choice of dither signal.

In this study, the dither signal considered is of the form

$$d(t) = \sum_{i=1}^{N_1} a_{1i} \sin(\omega_{1i}t) \exp(-\alpha_1 t) + \sum_{i=1}^{N_2} a_{2i} \cos(\omega_{2i}t) \exp(-\alpha_2 t).$$

The amplitudes a_{1i} , a_{2i} were chosen as random numbers while the frequencies ω_{1i} , ω_{2i} were selected as distinct random numbers. The number of the different frequencies, $N_1 + N_2$ was chosen to be equal or greater than the number of parameters to be estimated. The sinusoidal signal is modulated with an exponential function to ensure that the perturbation signal dies out as $t \rightarrow \infty$. A sufficiently small value of α is required; in general, it is selected as $\alpha \ll \tau$, where τ is the process time constant. For most applications, the choice of the input dither signal that provides sufficient excitation remains difficult to evaluate *a priori*.

3.5 Simulation Examples

3.5.1 Example 1

To illustrate the procedure developed, consider a linear system represented by the transfer function

$$G(s) = \frac{1}{s(s + \theta)}$$

where θ is an unknown constant parameter. The control objective is to maximize the objective function

$$y_p = p(y, \theta) = 1 - \theta y - y^2.$$

It follows that the performance function reaches its maximum at $y = y^* = -\theta/2$ since $\frac{\partial p(y, \theta)}{\partial y} = -\theta - 2y$ and $\frac{\partial^2 p(y, \theta)}{\partial y^2} = -2$.

Following the design procedure, we re-write the plant in the state space form (3.4) as

$$\begin{aligned}\dot{x}_1 &= x_2 - \theta x_1 \\ \dot{x}_2 &= u \\ y &= x_1\end{aligned}$$

The filters are implemented as

$$\begin{aligned}\dot{\eta} &= A_0 \eta + \varepsilon_2 y, \quad \Xi = -[A_0 \eta], \quad v = \lambda, \\ \dot{\lambda} &= A_0 \lambda + \varepsilon_n u, \quad \zeta = -A_0^2 \eta \\ A_0 &= \begin{bmatrix} -l_1 & 1 \\ -l_2 & 0 \end{bmatrix}\end{aligned}$$

and the state observer as

$$\begin{aligned}\hat{x}_1 &= -(l_1^2 - l_2 - \hat{\theta} l_1) \eta_1 + (l_1 - \hat{\theta}) \eta_2 + \lambda_1 \\ \hat{x}_2 &= -(l_2 l_1 - \hat{\theta} l_2) \eta_1 + l_2 \eta_2 + \lambda_2\end{aligned}$$

Defining, z_1 and z_2 as in the design procedure,

$$V^{iss} = \frac{1}{2}(z_1^2 + z_2^2) \tag{3.60}$$

The predicted state is generated by

$$\begin{aligned}\dot{\hat{x}}_{1a} &= \hat{x}_2 - \hat{\theta} y + m_1(x_1 - \hat{x}_{1a}) \\ \dot{\hat{x}}_{2a} &= u + m_2(x_2 - \hat{x}_{2a})\end{aligned}$$

and the adaptive laws are designed as

$$\dot{\hat{\theta}} = \begin{cases} \Gamma\Psi, & \text{if } \|\hat{\theta}\| < 10 \quad \text{or} \\ & \|\hat{\theta}\| = 10 \text{ and } 2\hat{\theta}\Psi \leq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\Psi = \Gamma \left[\phi z_1 - \frac{\partial \alpha_1}{\partial y} \phi z_1 - (x_1 - \hat{x}_{1a})y \right]$$

$$\phi = (l_2 \eta_1 - y), \quad \frac{\partial \alpha_1}{\partial y} = -c_1 - d_1 + \hat{\theta}.$$

The formulation of our optimization is as follows

$$\min_u J = \int_t^{t+T_p} z^T(\tau)z(\tau) + u(\tau)^T R u(\tau) d\tau + z^T(t+T_p)z(t+T_p)$$

subject to

$$\dot{x}_1 = x_2 - \bar{\theta}x_1, \quad x_1(t) = y(t), \quad \bar{\theta} = \hat{\theta}(t)$$

$$\dot{x}_2 = u, \quad x_2(t) = \hat{x}_2(t)$$

$$V(t+T_p) \leq V^{iss}(t+T_p)$$

$$z_1 = x_1 - y^r, \quad z_2 = x_2 - \alpha_1$$

$$\alpha_1 = -c_1 z_1 - g_1 z_1 + \bar{\theta}x_1$$

where the function $V(\cdot) = \frac{1}{2}(z_1^2 + z_2^2)$, the setpoint y^r is obtained from the first optimization step as $y^r = r + d(t)$ with $r = -\hat{\theta}/2$. The setpoint r can also be generated from the dynamical equation $\dot{r} = -\frac{1}{2}(2r + \hat{\theta} + \dot{\hat{\theta}})$.

The parameters used in the simulation are selected as $l_1 = l_2 = m_1 = m_2 = 1$, $c_1 = c_2 = 5$, $d_1 = d_2 = 0.1$, $\Gamma = 0.432$, $x_1(0) = 0.1$, $x_2(0) = 0.5$, $\hat{\theta}(0) = -0.2$ and

The dither signal is chosen to be $d(t) = 0.1 \sin(3t) \exp(-0.2t)$. The exponential term appearing in the dither signal ensures that the excitation signal $d(t)$ disappears as t increases. The prediction and control horizons length are chosen as $T_p = T_c = 1$ and a sampling time of 0.2 is used for the simulation experiment.

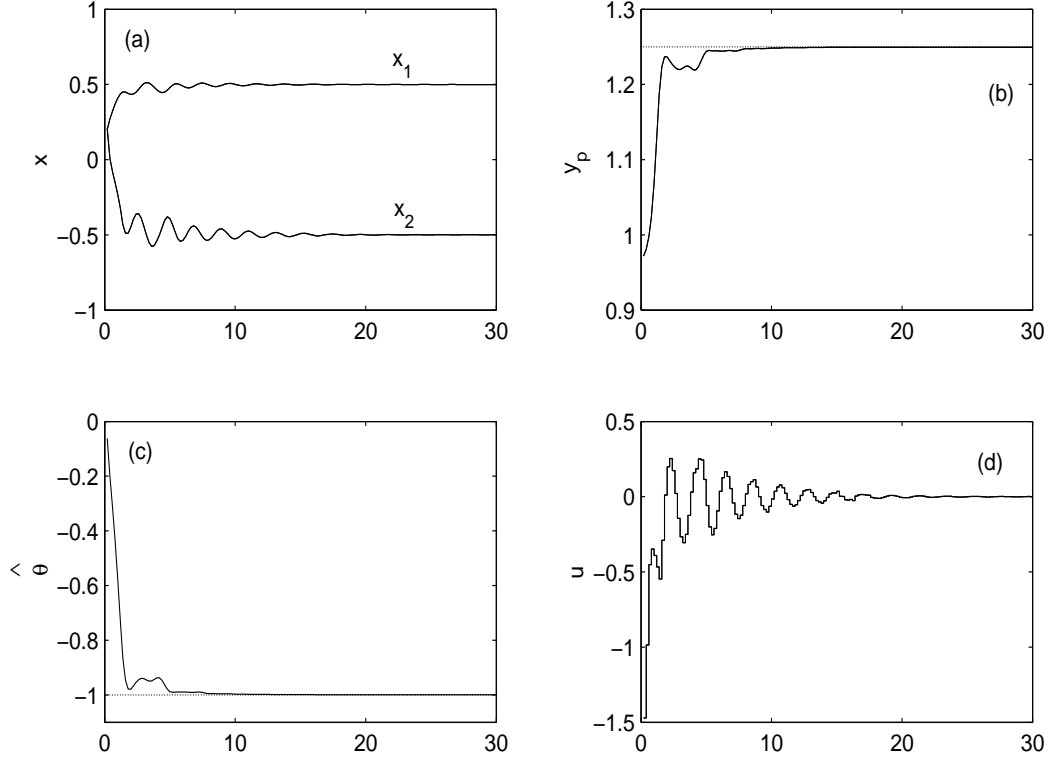


Figure 3.2: System's trajectories under ESRHC. (a) state x_1 , (b) performance function y_p , (c) parameter estimate $\hat{\theta}$, (d) control input

Figure 3.2 shows the states, performance function, parameter estimate and control input from the simulation. From the above discussion, it is clear that the optimum occurs when the state $x_1 = 0.5$ and $x_2 = -0.5$. Figure 3.2(a) shows that the states x_1 and x_2 converge to their optimum values. Also, the parameter estimate converges to the actual parameter value of $\theta = -1$. This suggests that the proposed control action (d) provides sufficient excitation for the system. Moreover, it is seen that the performance function in (b) achieves its maximum value of 1.25.

3.5.2 Example 2 - Reactor Dynamics

Consider the following linearized model for a non-isothermal continuous stirred tank reactor (CSTR) where a first order, irreversible exothermic reaction $\mathbf{A} \rightarrow \mathbf{B}$ is carried out [75, 76]. The dynamic of the reactor is given as

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= [1 \ 0]x \end{aligned} \tag{3.61}$$

where x is a vector of the reactor temperature and concentration, and u is the coolant flow rate. The matrix A and B are as follows:

$$A = \begin{bmatrix} -\frac{g}{V} - \frac{UA}{V\rho C_p} + (-\Delta H_{rxn})\frac{E/(RT^{*2})}{\rho C_p}k(T^*)C_A^* & \frac{-\Delta H_{rxn}k(T^*)}{\rho C_p} \\ -\frac{E}{RT^{*2}}k(T^*)C_A^* & -\frac{g}{V} - k(T^*) \end{bmatrix}$$

$$B = \begin{bmatrix} -2.1 \times 10^5 \frac{T^* - T_{cin}}{V\rho C_p} \\ 0 \end{bmatrix}$$

The expression for the reaction rate is given by $k(T^*) = K_o e^{-(E/RT^*)}$, where K_o is the kinetic constant of the reaction and E/R is the activation energy. We assume that $\theta_p = k(T^*)$ and $\theta_2 = E/RT^*$ are not known and use the proposed scheme to adaptively stabilize the system to the unknown setpoint (T^*, C_A^*) that maximizes the function

$$y_p = -\frac{1}{2}V\theta_p C_A^{*2} + \gamma F C_{A0} C_A^*, \quad \theta_p > 0, \quad \gamma = 0.9$$

This economic function is designed to ensure that C_A stabilizes at a value that guarantees 90 percent conversion of reactant A .

The specific parameters and operating conditions used for the simulation are $F = 1\text{ m}^3/\text{min}$, $V = 1\text{ m}^3$, $T_{cin} = 365\text{K}$, $C_{A0} = 2.0\text{ kmole}/\text{m}^3$, $C_p = 1\text{ cal}/(\text{g K})$, $\rho = 10^6\text{ g}/\text{m}^3$, $-\Delta H_{rxn} = 130 \times 10^6 \text{ cal}/(\text{kmole})$.

The true values of the unknown parameters are chosen as $\theta_1 = 9\text{ min}^{-1}$ and $\theta_2 = 21.07$. The coolant flow is restricted to $13 \leq u \leq 17 \text{ m}^3/\text{min}$. The initial values used are $C_A(0) = 0.084 \text{ kmol}/\text{m}^3$, $T(0) = 385.27 \text{ K}$, $\hat{\theta}_1(0) = 12 \text{ min}^{-1}$ and $\hat{\theta}_2(0) = 10$ while the target optimal values are $C_A^* = 0.2 \text{ kmol}/\text{m}^3$ and $T^* = 395.27\text{K}$. Other design parameters used are prediction horizon length $T_p = 0.8\text{min}$, control horizon length $T_c = 0.5\text{min}$, sampling time = 0.1min , estimation parameters $l_1 = 1$, $l_2 = 0.5$, $\Gamma_{11} = 0.72$, $\Gamma_{22} = 1.72$ and perturbation signal $d(t) = (0.5 \sin 9t + \sin 7t) e^{-0.5t} + 0.7 \cos 6t e^{-0.3t}$.

Figure 3.3 represents the controlled reactor trajectories. The simulation result shows that the adaptive ESRHC stabilizes the system states to the required steady-state values and the economic function reaches the optimum in a reasonable time. The parameter estimates $\hat{\theta}_1$ and $\hat{\theta}_2$ converge to the true values of 9 min^{-1} and 21.07 respectively. The control input also converges to the appropriate steady-state value and satisfies the given bounds.

To access the robustness of the proposed ESRHC algorithm to model uncertainties, the developed controller and estimators based on the linearized model are applied on the original nonlinear reactor dynamics [75]:

$$\begin{aligned} V\rho C_p \frac{dT}{dt} &= \rho C_p F(T_0 - T) - \frac{aF_c^{b+1}}{F_c + \frac{aF_c^b}{2\rho_c C_{pc}}}(T - T_{cin}) + (-\Delta H_{rxn})VK_o e^{-(E/RT)}C_A \\ V \frac{dC_A}{dt} &= F(C_{A0} - C_A) - VK_o e^{-(E/RT)}C_A \end{aligned} \quad (3.62)$$

The additional model parameters are $C_{pc} = 1\text{ cal}/(\text{gK})$, $\rho_c = 10^6\text{ g}/\text{m}^3$, $a = 1.678 \times 10^6(\text{cal}/\text{min})/(\text{K})$, $b = 0.5$ and F_c is the coolant flow rate u . In the simulation study, the RHC uses the linear model (3.61) to predict system evolution and the first control action among the optimal sequence generated is implemented on the nonlinear plant

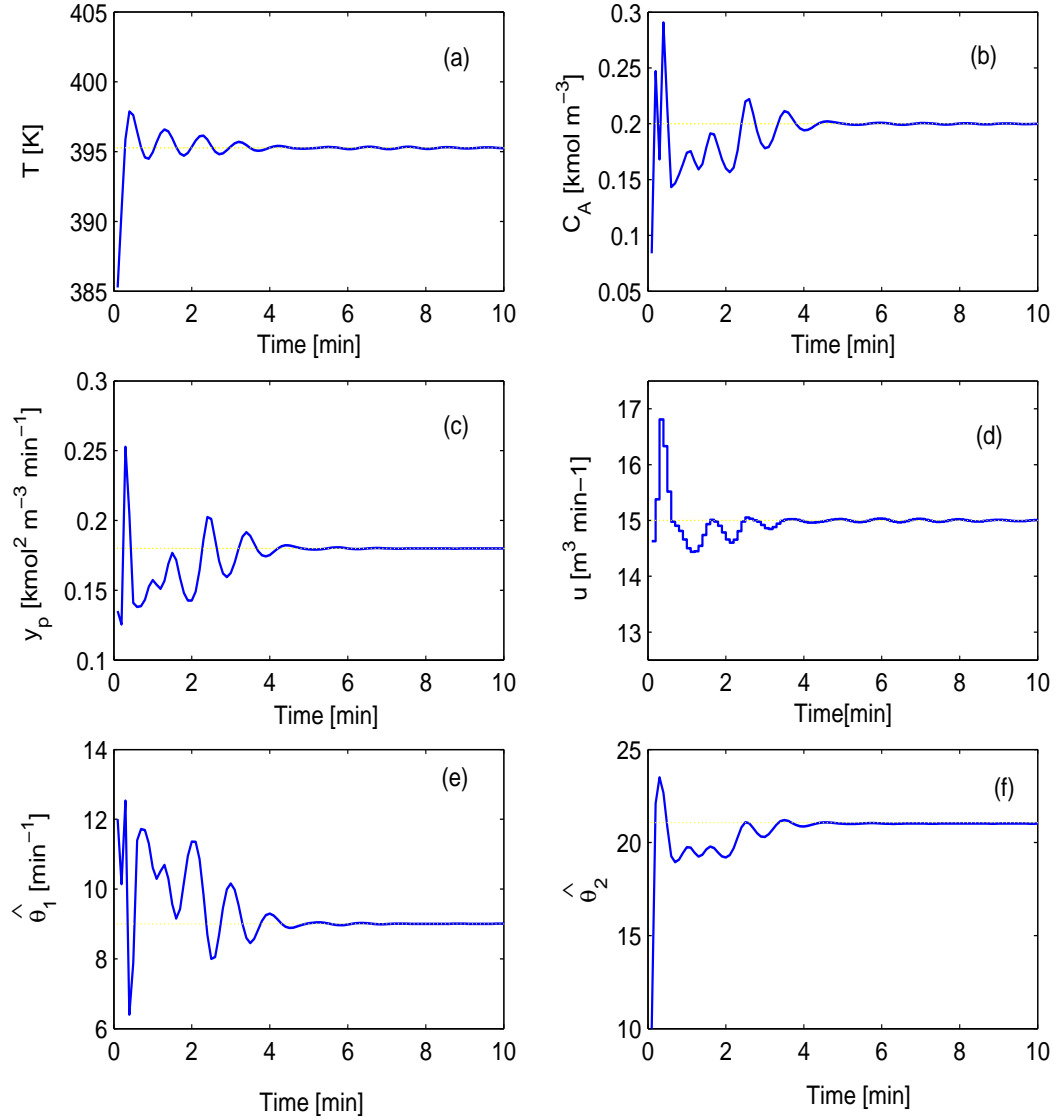


Figure 3.3: Reactor (linearized dynamics) closed-loop trajectories under the proposed ES-RHC (a) temperature T [K], (b) concentration C_A [kmol m^{-3}], (c) performance function y_p [$\text{kmol}^2 \text{m}^{-3} \text{min}^{-1}$], (d) control input u [$\text{m}^3 \text{min}^{-1}$], (e) parameter estimate $\hat{\theta}_1$ [min^{-1}], (g) parameter estimate $\hat{\theta}_2$

(3.62). The initial conditions and tuning parameters used in this study remain the same except for the estimator parameters that are tuned to $l_1 = 10$, $l_2 = 5$, $\Gamma_{11} = 0.046$ and $\Gamma_{22} = 0.29$.

The simulation plots are shown in Figure 3.4. It can be seen that the controller performs very well on the nonlinear system. The trajectories of the closed-loop nonlinear system states are smoother than that of the linear systems shown in Figure 3.3. However, this is achieved at the expense of a more aggressive controller action.

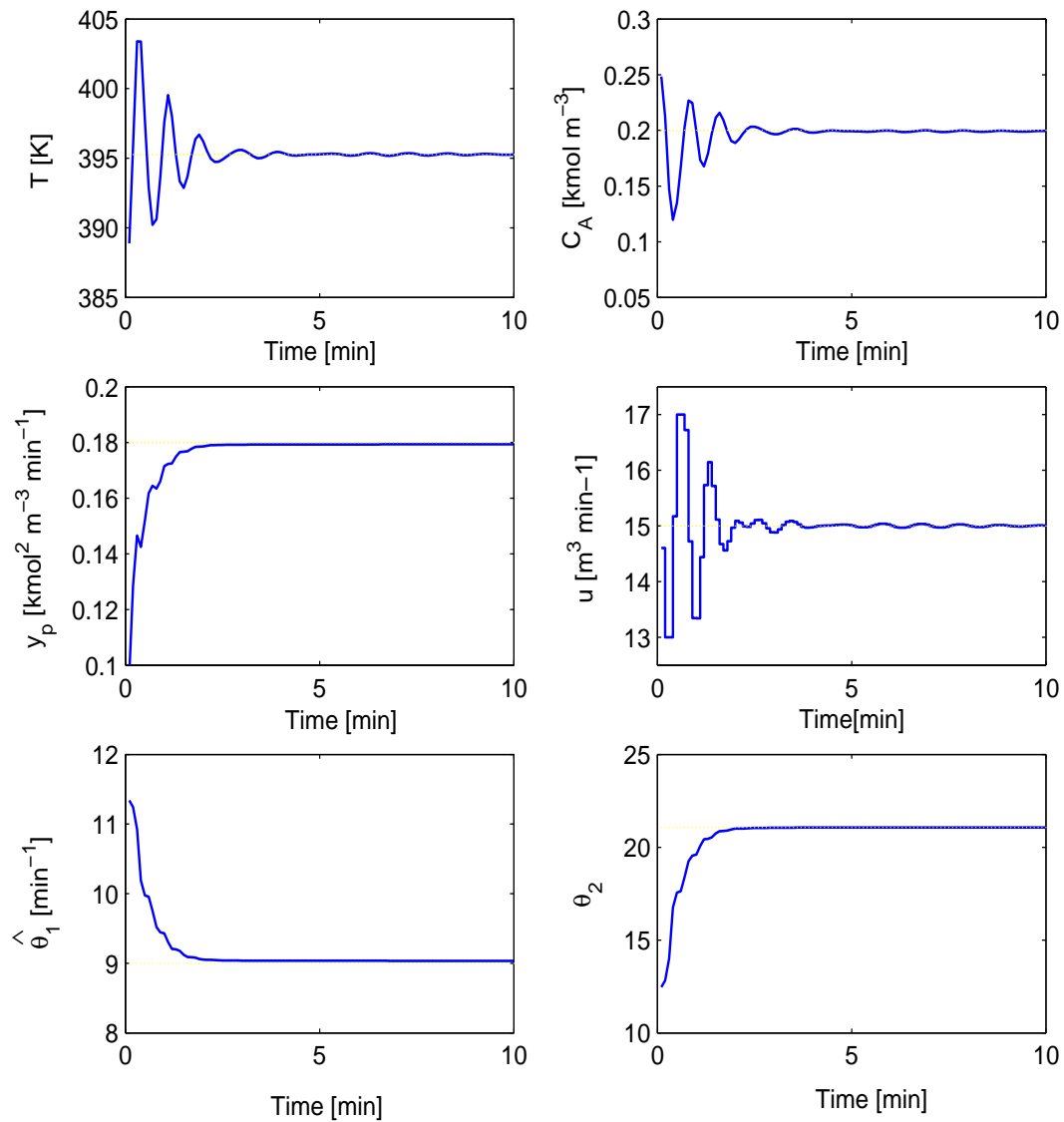


Figure 3.4: Reactor (nonlinear dynamics) closed-loop trajectories under the proposed ES-RHC (a) temperature T [K], (b) concentration C_A [kmol m⁻³], (c) performance function y_p [kmol² m⁻³ min⁻¹], (d) control input u [m³ min⁻¹], (e) parameter estimate $\hat{\theta}_1$ [min⁻¹], (f) parameter estimate $\hat{\theta}_2$

3.6 Conclusions

A method is proposed to solve a class of output feedback extremum seeking control problems for linear uncertain plants. The challenging part of this application is that in addition to the unknown parameters, only the system's output is available for measurement. An observer is employed to estimate the unavailable states and a parameter update law is implemented on the plant to provide estimates of the unknown parameters. These estimates are used, at each iteration step, to update the unknown states and parameters in the economic and control optimization schemes. The desired target value is obtained at every time step via an online optimization procedure and the model predictive controller is designed to track this setpoint. The practical advantage of this two-phase approach, over a one-phase optimization scheme, is that it allows for checks on the designed reference signal. Moreover, since we are dealing with uncertain dynamical systems, closed-loop identifiability can be guaranteed by designing a sufficiently exciting reference input signals. The receding horizon technique employs an iss-control Lyapunov function to ensure stability and performance. It is shown that the performance function converges to a small neighborhood of the optimum provided a PE condition is satisfied. Moreover, the transient performance of the algorithm is analyzed by deriving bounds for the closed-loop tracking error. The simulation results demonstrate the applicability of the proposed integrated scheme.

3.7 Proof of Parameter Convergence Result

Theorem 3.6

Proof: Let $\|z_w\|^2 = W(t)$. From eq.(3.54), it is concluded that $(z, e, e_a) \rightarrow 0$ as $t \rightarrow \infty$. If Lemma 3.6 is true, for every compact neighbourhood of $(z, e, e_a \tilde{\theta}) = 0$, there must exist a finite-time from which the neighbourhood of the origin of the

closed-loop system is positively invariant. Since (3.54) ensures V_1 is non-increasing, the level curves of V_1 are rendered positively invariant. To prove the Lemma, it is therefore sufficient to prove that $(z, e, e_a \tilde{\theta})$ will enter every level curve of V_1 .

The proof will proceed by contradiction, with the contradictory assumption that $\exists \epsilon_V > 0$ such that $V_1 \geq \epsilon_V, \forall t \geq 0$. That is

$$\lim_{t \rightarrow \infty} \left(\|z_w\|^2 + \frac{1}{G} \|e\|_{P_0}^2 + \frac{1}{2} \|e_a\|^2 + \frac{1}{2} \tilde{\theta}^T \Gamma^{-1} \tilde{\theta} \right) \geq \epsilon_V \quad (3.63)$$

However from (3.54), it is clear that $(z, e, e_a) \rightarrow 0$, from which it can be $\exists t_1 = t_{zee_a}^*(\epsilon_V, k) < \infty$ such that $\max(\|z_w\|, \|e\|_{P_0}, \|e_a\|) \leq \sqrt{k\epsilon_V}, \forall t \geq t_1$, for any $0 < k < \frac{1}{\vartheta}$, where $\vartheta = \frac{3G+2}{2G} > 1$. It then follows from (3.63) that

$$\|\tilde{\theta}\| \geq \sqrt{2(1 - \vartheta k) \lambda_{\min}\{\Gamma\} \epsilon_V} \quad \forall t \geq t_1 \quad (3.64)$$

Moreover, from (3.55), it can be concluded that for any $\epsilon > 0$ (independent of ϵ_V), $\exists t_2 = t_{g\tilde{\theta}}^*(\epsilon) < \infty$ such that

$$\frac{1}{T_0} \int_t^{t+T_0} \left(\tilde{\theta}(\tau)^T \bar{g}(\tau) \bar{g}(\tau)^T \tilde{\theta}(\tau) \right) d\tau \leq \epsilon \quad \forall t \geq t_2 \quad (3.65)$$

Substituting $\tilde{\theta}(\tau) = \tilde{\theta}_t + \int_t^\tau \dot{\tilde{\theta}}(\sigma) d\sigma$ in (3.65), for $\tau \in [t, t + T_0]$, where $\tilde{\theta}_t = \tilde{\theta}(t)$ is constant over the interval of integration,

$$\begin{aligned} & \frac{1}{T_0} \tilde{\theta}_t^T \int_t^{t+T_0} \bar{g}(\tau) \bar{g}(\tau)^T d\tau \tilde{\theta}_t + \frac{2}{T_0} \tilde{\theta}_t^T \int_t^{t+T_0} \bar{g}(\tau) \bar{g}(\tau)^T \left(\int_t^\tau \dot{\tilde{\theta}} d\sigma \right) d\tau \\ & + \frac{1}{T_0} \int_t^{t+T_0} \left(\int_t^\tau \dot{\tilde{\theta}} d\sigma \right)^T \bar{g}(\tau) \bar{g}(\tau)^T \left(\int_t^\tau \dot{\tilde{\theta}} d\sigma \right) d\tau \leq \epsilon \quad \forall t \geq t_2 \end{aligned} \quad (3.66)$$

From (3.53) and the properties of the Projection algorithm, it can be deduced that

$$\left\| \int_t^\tau \dot{\tilde{\theta}} d\sigma \right\| = \left\| \int_t^\tau \text{Proj}\{\Gamma \psi^T\} d\tau \right\|$$

$$\leq \sqrt{\frac{\lambda_{max}\{\Gamma\}}{\lambda_{min}\{\Gamma\}}} \left\| \int_t^\tau \Gamma \Pi \chi d\tau \right\|$$

where

$$\Pi = \left[\bar{g}(y, u) \quad \phi \quad \frac{\partial \alpha_1}{\partial y} \phi \quad \dots \quad \frac{\partial \alpha_{\rho-1}}{\partial y} \phi \right] \quad \text{and} \quad \chi = [e_a^T \quad z^T]^T$$

since $\|z\| \leq \|z_w\|$, then, $\|\chi\| \leq \sqrt{k\epsilon_V}$ and since Π is a function of uniformly bounded signals, $f = [y, z, \lambda, \eta, \hat{\theta}]$, we have

$$\sup_{f \in B_f} \|\Gamma \Pi\| = \mathcal{K} < \infty \quad (3.67)$$

where

$$\begin{aligned} B_f &= B_{yz} \times B_{\eta\lambda} \times B_{\hat{\theta}} \\ B_{yz} &= \left\{ y \in \mathbb{R}, z \in \mathbb{R}^\rho \mid \|z\| \leq \sqrt{V_1(0)} \right\} \\ B_{\eta\lambda} &= \left\{ \eta \in \mathbb{R}^n, \lambda \in \mathbb{R}^n \mid \|e\| \leq \sqrt{\frac{GV_1(0)}{\lambda_{min}(P_0)}} \right\} \\ B_{\hat{\theta}} &= \left\{ \hat{\theta} \in \mathbb{R}^{n+m}, \lambda \in \mathbb{R}^n \mid \|\tilde{\theta}\| \leq \sqrt{2V_1(0)\lambda_{max}(\Gamma)} \right\} \end{aligned}$$

Hence,

$$\left\| \int_t^\tau \dot{\tilde{\theta}} d\sigma \right\| \leq T_0 \mathcal{M} \sqrt{k\epsilon_V} \quad \forall t \geq t_1 \quad (3.68)$$

$$\mathcal{M} \triangleq \mathcal{K} \left(\sqrt{\frac{\lambda_{max}\{\Gamma\}}{\lambda_{min}\{\Gamma\}}} \right) \quad (3.69)$$

By the uniform boundedness of all closed-loop dynamics, it follows that there exists a constant $\bar{c}_{PE} < \infty$ such that the PE condition can be rewritten as

$$c_{PE} I \leq \frac{1}{T_0} \int_{t_i}^{t_i+T_0} \bar{g}(\tau) \bar{g}(\tau)^T d\tau \leq \bar{c}_{PE} I \quad (3.70)$$

Furthermore, since $t_i \rightarrow \infty$ and $i \rightarrow \infty$, we define the nonempty set $i^* \triangleq \{i \in \{1, 2, \dots\} \mid t_i \geq \max(t_1, t_2)\}$. Substituting into (3.66), noting the semi-positive definiteness of the third term on the LHS, yields

$$c_{PE} \|\tilde{\theta}(t_i)\|^2 - 2\bar{c}_{PE} T_0 M_{\tilde{\theta}} \sqrt{k\epsilon_V} \|\tilde{\theta}(t_i)\| - \epsilon \leq 0 \quad \forall i \in i^* \quad (3.71)$$

from which it follows that

$$\|\tilde{\theta}(t_i)\| \leq \frac{\bar{c}_{PE}}{c_{PE}} T_0 M_{\tilde{\theta}} \sqrt{k\epsilon_V} + \frac{1}{c_{PE}} \sqrt{\bar{c}_{PE}^2 T_0^2 M_{\tilde{\theta}}^2 k\epsilon_V + \epsilon c_{PE}} \quad \forall i \in i^* \quad (3.72)$$

The constants $k > 0$ and $\epsilon > 0$ may be chosen arbitrarily small, independent of ϵ_V . As $(k, \epsilon) \rightarrow 0$, (3.72) approaches $\|\tilde{\theta}(t_i)\| = 0$, which is a violation of (3.64). ■

Chapter 4

Parameter Convergence in Adaptive Extremum Seeking Control

4.1 Introduction

One of the main challenges with extremum-seeking control (ESC) and most deterministic adaptive control approach is the ability to recover the true unknown values of the parameters. In most approaches, the convergence of parameters to their true values can only be ensured if the closed-loop trajectories provide sufficient excitation for the parameter estimation routine. In standard linear adaptive control approaches, this problem is tractable [77] and can be solved satisfactorily. A dither signal can be introduced momentarily in the control system to achieve the necessary excitation. For nonlinear systems, the problem of determining appropriate excitation conditions remains open. Although some limited persistence of excitation (PE) conditions have been derived, they remain difficult to apply. Such conditions appear naturally in [8] for the solutions of an adaptive extremum-seeking control problem. In fact, the ful-

fillment of such conditions dictates the performance of the optimization routine.

The existing adaptive control techniques that address the problem of parameter convergence in nonlinear systems [65, 66] cannot be applied directly to ESC because of the uncertainty associated with the reference signals. This leads us to the quest for a technique that could generate sufficiently rich optimal reference signals in real-time. Rather than checking parameter convergence on a case-to-case basis [65, 66], a method that guarantees convergence for (at least) a class of nonlinear systems is desirable.

The work presented in this chapter [78] complements the previous works in adaptive or model-based ESC [8, 48, 72] by translating the PE condition, which depends on the nonlinear closed-loop signals, into a sufficient richness (SR) condition on the desired setpoint signals. However, since the desired optimal setpoint is uncertain in this type of problem, the solution presented includes a technique for generating such signal in closed-loop. The design guarantees parameter convergence with a minimum loss of regulation performance.

4.2 Problem Description and Assumptions

Consider the following optimization problem

$$\min_{x_p} p(x_p, \theta) \tag{4.1}$$

subject to the system dynamics

$$\begin{aligned} \dot{x}_p &= f_p(x, u) + g(x_p)\theta \\ \dot{x}_q &= f_q(x) \end{aligned} \tag{4.2}$$

where $x = [x_p^T \ x_q^T]^T \in \mathbb{R}^{n_x}$ are the systems states, $u \in \mathbb{R}^{n_u}$ are the control inputs. The vector $x_p \in \mathbb{R}^m$ represents the system states involved in the objective function, θ represents unknown parameter vector assumed to be uniquely identifiable and to lie in a known, convex set $\theta \in \Theta \subseteq \mathbb{R}^{n_\theta}$. The mappings $f_p(x, u) : \mathbb{R}^{n_x \times n_u} \rightarrow \mathbb{R}^m$, $g(x_p) : \mathbb{R}^m \rightarrow \mathbb{R}^{m \times n_\theta}$ are \mathcal{C}^1 and $g(x_p)$ is bounded for bounded x_p .

Assumption 4.1 *The following assumptions are made about (4.1) and (4.2).*

1. *The function p is \mathcal{C}^2 in its arguments and $\frac{\partial^2 p}{\partial x_p^2} \geq c_0 I > 0$, $\forall (x_p, \theta) \in (\mathbb{R}^m \times \Theta)$.*
2. *The state $x_q \in \mathbb{R}^{n_x - m}$ belongs to a positively invariant set for any bounded x_p .*

Assumption 4.1.1 is a common assumption in any standard adaptive ESC literature. It requires the cost surface to be strictly convex in x_p .

4.3 Setpoint and Controller Design

4.3.1 Setpoint Update Law

Considering the fact that the cost function contains unknown parameters θ , the desired setpoint measurement cannot be obtained off-line. However, if the function $p(x_p, \theta)$ is not complicated, the optimal value can be determined as a function of θ by solving for x_p in $\partial p / \partial x = 0$. When the analytical expression of x_p is not available, the desired setpoint may be obtained online using a Lyapunov method as follows.

Let $x_p^r \in \mathbb{R}^m$ denote a reference setpoint for x_p and $\hat{\theta}$ denote an estimate of the unknown parameter θ . An online update law is designed such that $x_p^r(t)$ approaches the optimum value $x_p^*(\hat{\theta})$ exponentially. To this end, let

$$z_r \triangleq \frac{\partial p(x_p^r, \hat{\theta})}{\partial x_p^r} \tag{4.3}$$

and consider an optimization Lyapunov function candidate

$$V_{sp}(z_r) = \frac{1}{2} \|z_r\|^2 \quad (4.4)$$

Taking the time derivative of V_{sp} , we have

$$\dot{V}_{sp} = z_r^T \dot{z}_r = \frac{\partial p}{\partial x_p^r}{}^T \left[\frac{\partial^2 p}{\partial x_p^r \partial x_p^r} \dot{x}_p^r + \frac{\partial^2 p}{\partial x_p^r \partial \hat{\theta}} \dot{\hat{\theta}} \right]. \quad (4.5)$$

Choosing the update law as

$$\dot{x}_p^r = - \left(\frac{\partial^2 p}{(\partial x_p^r)^2} \right)^{-1} \left[k_r \frac{\partial p}{\partial x_p^r}{}^T + \frac{\partial^2 p}{\partial x_p^r \partial \hat{\theta}} \dot{\hat{\theta}} \right] \quad (4.6)$$

with $k_r > 0$, the dynamic of the system becomes

$$\dot{z}_r = -k_r z_r \quad (4.7)$$

and (4.5) results in

$$\dot{V}_{sp} \leq -k_r \|z_r\|^2 \quad (4.8)$$

Proposition 4.2 *The optimal setpoint $x_p^r(t)$ generated by (4.6) is feasible and converges to $x_p^*(\hat{\theta})$ exponentially.*

Proof: Assuming (for now, it will be shown later) that $(\hat{\theta}, \dot{\hat{\theta}})$ are bounded. This assumption coupled with Assumption 4.1.1 ensure that (4.6) exist and it is finite. It follows from (4.8) that the origin $z_r = 0$ is exponentially stable. Applying the inverse function theorem, it can be seen that the mapping z_r is a diffeomorphism. Hence it concluded that $x_p^r(t)$ converges to $\hat{\theta}$ -dependent optimal setpoint $x_p^*(\hat{\theta})$ exponentially fast. ■

4.3.2 Sufficiently Rich Optimal Setpoint

Since parameter convergence is a vital issue in ESC design, we have to provide some richness condition on the setpoint x_p^r to ensure that $\hat{\theta} \rightarrow \theta$ as $t \rightarrow \infty$. To achieve this, the setpoint is appended with a bounded perturbation signal $d(t) \in \mathbb{R}^m$. The rich setpoint is given by

$$r(t) := x_p^r(t) + d(t) \quad (4.9)$$

where $d(t)$ is a sufficiently smooth and uniformly bounded signal vector. In particular, the signal is parameterized as

$$d(t) := \sum_{k=1}^{\hbar} a_k(t) \sin(\omega_k t) = \mathcal{A}(t)\zeta(t) \quad (4.10)$$

where

$$\mathcal{A}(t) = \begin{bmatrix} a_{11} & \cdots & a_{1\hbar} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{m\hbar} \end{bmatrix} \quad (4.11)$$

is the signal amplitude matrix and

$$\zeta(t) = [\sin \omega_1 t \quad \sin \omega_2 t \quad \dots \quad \sin \omega_{\hbar} t]^T, \quad \omega_i \neq \omega_j \text{ for } i \neq j, \quad (4.12)$$

is the corresponding sinusoidal function vector. A method for generating the coefficients $\mathcal{A}(t)$ is provided in Subsection 4.4.1. The design ensures that $\mathcal{A}(t) \rightarrow \mathcal{A}^*$, the optimal value that satisfies a PE condition, asymptotically.

4.3.3 Adaptive Tracking Controller

Let the tracking and estimation error vectors be defined as $z_c = x_p - r$, $\tilde{\theta} = \theta - \hat{\theta}$ respectively. Since the focus of this chapter is on the identification of the true cost surface $p(x_p, \theta)$, rather than designing a specific adaptive controller for (4.2), we make the following assumption.

Assumption 4.3 *An adaptive controller consisting of a control law $u = \kappa_1(\cdot)$ and a parameter update law $\dot{\hat{\theta}} = \kappa_2(\cdot)$ is available such that*

$$\lim_{t \rightarrow \infty} z_c \rightarrow 0 \Rightarrow \lim_{t \rightarrow \infty} x_p \rightarrow x_p^*(\hat{\theta}) + \mathcal{A}^* \zeta(t) \triangleq r^* \quad (4.13a)$$

$$\lim_{t \rightarrow \infty} g(r^*) \tilde{\theta}(t) = 0 \quad (4.13b)$$

$$\lim_{t \rightarrow \infty} \tilde{\theta}(t) = \bar{\theta}, \quad \text{a constant vector.} \quad (4.13c)$$

In (4.13a), $\lim_{t \rightarrow \infty} z_c \rightarrow 0 \Rightarrow \lim_{t \rightarrow \infty} x_p \rightarrow r^*$ because $x_p^r(t) \rightarrow x_p^*(\hat{\theta})$ exponentially (proposition 4.2) and $d(t) \rightarrow \mathcal{A}^* \zeta(t)$ asymptotically (by assumption, for now).

An adaptive controller that satisfies (4.13) can be constructively designed depending on the structure of the nonlinear function $f_p(x, u)$. For instance, adaptive backstepping technique can be employed for systems in parametric feedback form. For the direct case, where $f_p(x, u)$ is affine in control, *i.e.*, $f_p(x, u)$ can be re-written as $f_{p1}(x) + f_{p2}(x)u$ and where $f_{p2}(x)^{-1}$ exist for all $x \in \mathbb{R}^{n_x}$. An adaptive controller that satisfies (4.13) can be constructed as follows:

Consider the Lyapunov function candidate

$$V_c := \frac{1}{2} \|z_c\|^2 + \frac{1}{2} \tilde{\theta}^T \Gamma^{-1} \tilde{\theta} \quad (4.14)$$

with $\Gamma = \Gamma^T > 0$. Taking the time derivative of V_c along the trajectory of (4.2), we have

$$\dot{V}_c = z_c^T \left(f_{p_1}(x) + f_{p_2}(x)u + g(x_p)\hat{\theta} - \dot{r} \right) - \dot{\hat{\theta}}^T \Gamma^{-1} \tilde{\theta} + z_c^T g(x_p) \tilde{\theta}$$

Considering the control law

$$u = -f_{p_2}(x)^{-1} \left(f_{p_1}(x) + g(x_p)\hat{\theta} - \dot{r} + k_c z_c \right), \quad (4.15)$$

with $k_c > 0$ and the parameter update law

$$\dot{\hat{\theta}} = \Gamma g(x_p)^T z_c, \quad (4.16)$$

it follows from (4.2) and (4.15) that

$$\dot{z}_c = g(x_p)\tilde{\theta} - k_c z_c, \quad (4.17)$$

$$\dot{V}_c \leq -k_c \|z_c\|^2. \quad (4.18)$$

This implies uniform boundedness of z_c and $\tilde{\theta}$ as well as global asymptotic convergence of z_c to zero. Moreover, it can be shown by Barbalat's lemma [73] that $\lim_{t \rightarrow \infty} \dot{z}_c = 0$. Since $z_c, \dot{z}_c \rightarrow 0$ in the limit as $t \rightarrow \infty$, we have $x_p \rightarrow r^*$ and (4.17) becomes

$$\lim_{t \rightarrow \infty} \dot{z}_c(t) = \lim_{t \rightarrow \infty} g(r^*)(t)\tilde{\theta}(t) = 0. \quad (4.19)$$

Also, (4.13c) can be shown by noting from (4.16) that $\tilde{\theta}(t) = \tilde{\theta}(t_0) - \Gamma \int_{t_0}^t g^T(x_p(\sigma)) z_c(\sigma) d\sigma$.

Since g and z_c are bounded signals and $z_c \rightarrow 0$, the integral term exist and it is finite.

Hence, $\lim_{t \rightarrow \infty} \tilde{\theta}(t) = \bar{\theta}$.

4.4 Parameter Convergence

By an argument similar to the one used in traditional adaptive control theory, a sufficient condition for parameter convergence is that the regressor $g(x_p)$ be persistently exciting. That is, there exists positive constants μ_0 and T such that

$$\int_t^{t+T_0} g(\tau)^T g(\tau) d\tau \geq \mu_0 I, \quad \forall t \geq 0. \quad (4.20)$$

The PE condition (4.20) requires that g rotates sufficiently in space that the integral of the matrix $g(\tau)^T g(\tau)$ is uniformly positive definite over any interval of some length T_0 . However, it is difficult to check that g satisfies the PE condition since the solution of the closed-loop trajectories are not known in advance. In the following, an alternative sufficient condition that addresses the above limitations and guarantees parameter convergence is presented.

To this end, we examine the frequency content of the regressor matrix $g(r^*(t))$. The procedure employed is similar to the one presented in [66]. The time-varying signal g is decomposed into a constant matrix and a periodic part by computing the trigonometric (or Fourier) series expansion for each nonlinearity vector in the signal. Let

$$g^T = \begin{bmatrix} \psi_1 & \psi_2 & \dots & \psi_m \end{bmatrix}, \quad (4.21)$$

where ψ_i is the i^{th} column of matrix g . Also, let $\omega_{i1}, \omega_{i2}, \dots, \omega_{iC_i}$ ($0 \leq \omega_{i1} < \omega_{i2} < \dots < \omega_{iC_i}$) and $\nu_{i1}, \nu_{i2}, \dots, \nu_{iS_i}$ ($0 < \nu_{i1} < \nu_{i2} < \dots < \nu_{iS_i}$) denote the distinct frequencies appearing in the cosine terms and the sine terms of the Fourier series expansion respectively. If we let

$$\xi_i(t) = \begin{bmatrix} \cos \omega_{i1} t & \dots & \cos \omega_{iC_i} t & \sin \nu_{i1} t & \dots & \sin \nu_{iS_i} t \end{bmatrix}^T$$

$$\triangleq \left[\xi_{i1}(t) \cdots \xi_{iC_i}(t) \quad \xi_{i(C_i+1)}(t) \cdots \xi_{i(C_i+S_i)}(t) \right]^T, \quad i = 1, \dots, m. \quad (4.22)$$

Then, each nonlinearity vector ψ_i defined in (4.21) can be expressed in the form

$$\psi_i = \Upsilon_i \xi_i(t) = \sum_{j=1}^{C_i+S_i} \Upsilon_{ij} \xi_{ij}(t), \quad i = 1, \dots, m \quad (4.23)$$

where Υ_i are $n_\theta \times (C_i + S_i)$ constant matrices whose elements are the real Fourier coefficients of the corresponding signals, and $\Upsilon_{ij}, j = 1, \dots, (C_i + S_i)$ is the j^{th} column of Υ_i . This decomposition method allows one to judge the richness of the regressor based on a constant matrix only. However, as pointed out in [66], the Fourier series expansion employed in the decomposition may contain an infinite number of terms, when the elements of (4.21) are not polynomial nonlinearities. In this case, the series expansion may be truncated. Combining (4.13b) with equations (4.21) and (4.23), we obtain

$$\lim_{t \rightarrow \infty} \xi_{ij}(t) \Upsilon_{ij}^T \tilde{\theta}(t) = 0, \quad i = 1, \dots, m, \quad j = 1, \dots, C_i + S_i \quad (4.24)$$

and since the scalar functions ξ_{ij} are all of the form $\cos \omega t$ or $\sin \nu t$, equation (4.24) is equivalent to

$$\lim_{t \rightarrow \infty} \Upsilon_{ij}^T \tilde{\theta}(t) = 0, \quad i = 1, \dots, m, \quad j = 1, \dots, C_i + S_i. \quad (4.25)$$

Moreover, defining

$$\Upsilon_1 = \Upsilon_{11}, \Upsilon_{12}, \cdots \Upsilon_{1(C_1+S_1)}$$

$$\Upsilon_2 = \Upsilon_{21}, \Upsilon_{22}, \cdots \Upsilon_{2(C_2+S_2)}$$

$$\Upsilon_3 = \Upsilon_{31}, \Upsilon_{32}, \cdots \Upsilon_{3(C_3+S_3)}$$

etc,

and

$$\Upsilon^T = [\Upsilon_1, \Upsilon_2, \dots, \Upsilon_m]^T$$

equation (4.25) can be written in a more compact form as

$$\lim_{t \rightarrow \infty} \Upsilon^T \tilde{\theta}(t) = 0 \quad \text{or} \quad \lim_{t \rightarrow \infty} \tilde{\theta}^T(t) \Upsilon \Upsilon^T \tilde{\theta}(t) = 0 \quad (4.26)$$

The constant matrix Υ contains the setpoint x_p^* and the elements of \mathcal{A}^* in its entries (in the limit as $t \rightarrow \infty$). Therefore, if the n_θ rows of Υ are linearly independent or if $\mathcal{W} = \Upsilon \Upsilon^T$ is positive definite, then $\tilde{\theta} = 0$ is guaranteed. However, it is not possible to verify this conditions *a priori* for a given dither signal because the matrix depends on unknown reference setpoint (the θ -dependent solution of (4.1)). In the next section, we show how to generate optimal size of some pre-selected sinusoids online.

4.4.1 Dither Signal Design

It has been shown that the presence of nonlinearities in a regressor vector increases the degree of PE of a given reference signal for nonlinear systems with special structure [65, 66]. However, for a general nonlinear system, this may not be the case, the nonlinearities may detract or add to the excitation [79]. In this work, we propose that the dither signal be chosen as a linear combination of sinusoids with at least n_θ distinct frequencies. Since such a choice with constant arbitrary amplitude may not be optimal for nonlinear systems, a method for generating optimal coefficients of the different basis functions (sinusoids) is provided. A quadratic objective function is minimized subject to a constraint that optimizes the size of the selected frequency contents in order to ensure positive definiteness of matrix $\mathcal{W} = \Upsilon \Upsilon^T$. The condition requires all the eigenvalues of \mathcal{W} to be positive. This is true if and only if the determinant of \mathcal{W} (the product of the eigenvalues) is positive since it is a symmetric

positive semidefinite matrix.

Let $\mathcal{H} \leq (\hbar \times m)$ denotes the number of distinct elements in the dither amplitude matrix $\mathcal{A}(t)$ and let $a \in \mathbb{R}^{\mathcal{H}}$ be a vector of these distinct coefficients. The optimum amplitude of the dither signal is obtained from the solution of the following constrained optimization problem:

$$\min_{a \in \mathbb{R}^{\mathcal{H}}} a^T Q a \quad (4.27a)$$

$$\text{such that } \mathcal{W}_d = \det(\mathcal{W}) > 0 \quad (4.27b)$$

with $Q \succ 0$. The optimization problem is tackled using an infeasible interior point technique [80]. Firstly, a slack variable ε is added so that (4.27) becomes

$$\min_{a \in \mathbb{R}^{\mathcal{H}}, \varepsilon \in \mathbb{R}} a^T Q a \quad (4.28a)$$

$$\text{such that } \mathcal{W}_d - \varepsilon = 0, \quad \varepsilon > 0. \quad (4.28b)$$

The constraints are then eliminated by augmenting the objective function with high costs for violating them as follows:

$$\min_{a, \varepsilon} P_a = a^T Q a - \frac{1}{M_1} \log(\sigma - \varepsilon) + M_2 (\mathcal{W}_d - \varepsilon)^2, \quad \sigma > 0 \quad (4.29)$$

with $M_1, M_2 > 0$. By the logarithmic barrier term, the slack variable is required to be greater than a design variable σ at all times. However, the equality constraint ($\mathcal{W}_d - \varepsilon = 0$) can be violated at any instant, its satisfaction is only achieved as the optimum solution is approached. The solution of (4.29) can be shown to converge to that of (4.27) in the limit as the positive constants $M_1, M_2 \rightarrow \infty$.

Since we assume that system (4.2) is fundamentally identifiable at the defining parameter values, feasibility of (4.27) (and hence (4.29)) is guaranteed by including sufficiently large number of sinusoids in (4.10). The unconstrained optimization prob-

lem (4.29) can be solved with gradient techniques. Let $\bar{a}^* = [a^*, \varepsilon^*]$ be the optimizer of (4.29), an update law that ensures $\bar{a} \rightarrow \bar{a}^*$ as $t \rightarrow \infty$ is chosen as

$$\dot{\bar{a}} = \text{Proj} \{-k_{\bar{a}} \mathcal{D} z_{\bar{a}}, \bar{a}\}, \quad \bar{a}(0) = [a_0, \varepsilon_0] \quad (4.30)$$

where $\text{Proj}\{\cdot\}$ is a standard projection algorithm [73] used to ensure that the vector \bar{a} is bounded or remains in some given set. The vector $z_{\bar{a}} = \partial P_{\bar{a}} / \partial \bar{a}$ is the gradient function, $k_{\bar{a}} > 0$ is a design parameter and \mathcal{D} is a positive definite matrix function. Matrix \mathcal{D} can be chosen as in steepest descent method where $\mathcal{D} = I$ (identity matrix) or as in trust region where $\mathcal{D} = \left(\partial^2 P_{\bar{a}} / \partial \bar{a}^2 + (F + \kappa)I \right)^{-1}$ with F =Frobenius matrix norm of $\partial^2 P_{\bar{a}} / \partial \bar{a}^2$ and $\kappa > 0$ is a small design constant parameter. The initial conditions are to be selected such that $\varepsilon_0 > \sigma$ and some elements of a_0 equals zero to avoid excessive initial perturbation of the system.

Theorem 4.4 *Consider the optimization problem (4.1) for system (4.2) and assuming Assumptions 4.1 and 4.13 are satisfied. Then the update laws (4.6) and (4.30) ensure that the system's state $x_p(t)$ converge to an optimal neighborhood of $x_p^*(\theta)$ -the unique minimizer of (4.1).*

Proof: It follows from (4.13a) that

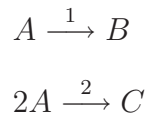
$\lim_{t \rightarrow \infty} \|x_p(t) - x_p^r(t)\| = \lim_{t \rightarrow \infty} \|d(t)\|$ and it is known from proposition (4.2) that $\lim_{t \rightarrow \infty} \|x_p^r(t) - x_p^*(\hat{\theta})\| = 0$. Moreover, (4.30) ensures $\lim_{t \rightarrow \infty} a(t) = a^*$ and hence $\lim_{t \rightarrow \infty} \mathcal{A}(t) = \mathcal{A}^*$. Therefore, the only solution of (4.26) is $\lim_{t \rightarrow \infty} \tilde{\theta}(t) = 0$, which implies $\lim_{t \rightarrow \infty} \|x_p^*(\hat{\theta}) - x_p^*(\theta)\| = 0$. Using the triangle inequality, we conclude that

$$\lim_{t \rightarrow \infty} \|x_p(t) - x_p^*(\theta)\|_2 \leq \|\mathcal{A}^* \zeta(t)\|_2 \leq \sqrt{\hbar} \|\mathcal{A}^*\|_2.$$

■

4.5 Simulation Example

Consider two parallel isothermal stirred-tank reactors [48] in which reagent A forms product B and waste-product C



Material balances for the reactors give

$$\begin{aligned} \frac{dA_i}{dt} &= A_i^{in} \frac{F_i^{in}}{V_i} - A_i \frac{F_i^{out}}{V_i} - k_{i1} A_i - 2k_{i2} A_i^2, \\ \frac{dB_i}{dt} &= -B_i \frac{F_i^{out}}{V_i} + k_{i1} A_i, \\ \frac{dC_i}{dt} &= -C_i \frac{F_i^{out}}{V_i} + k_{i2} A_i^2, \end{aligned}$$

where A_i , B_i , C_i denote concentrations in reactor i , k_{ij} are the reaction kinetic constants, which are only nominally known. The inlet flows F_i^{in} are the control inputs, while the outlet flows F_i^{out} are governed by PI controllers which regulate reactor volume to V_i^0 .

Denoting $x_p = [A_1, A_2]^T$, and $\theta = [k_{11}, k_{12}, k_{21}, k_{22}]^T$, the economic steady state cost function to be optimized is given by

$$p(x_p, \theta) = \sum_{i=1}^2 [(p_{i1} + P_A - P_B)k_{i1}A_iV_i^0 + (p_{i2} + 2P_A)k_{i2}A_i^2V_i^0],$$

where P_A , P_B denote component prices, p_{ij} is the net operating cost of reaction j in reactor i . The function $p(x_p, \theta)$ represents the net expense of operating the process

at steady state. The dynamic of the system can be expressed in the form (4.2) as

$$\dot{x}_p = - \underbrace{\begin{bmatrix} \frac{x_{p1}k_{V1}(x_{q1}-V_1^0+x_{q3})}{x_{q1}} \\ \frac{x_{p2}k_{V2}(x_{q2}-V_2^0+x_{q4})}{x_{q2}} \end{bmatrix}}_{f_{p1}} + \underbrace{\begin{bmatrix} \frac{A_{in}}{x_{q1}} & 0 \\ 0 & \frac{A_{in}}{x_{q2}} \end{bmatrix}}_{f_{p2}} u - \underbrace{\begin{bmatrix} x_{p1} & 2x_{p1}^2 & 0 & 0 \\ 0 & 0 & x_{p2} & 2x_{p2}^2 \end{bmatrix}}_g \theta,$$

where x_{q1}, x_{q2} are the two tank volumes and x_{q3}, x_{q4} are the PI integrators.

Following the design procedure, the optimizing controller, parameter estimates and the setpoint signal x_p^r are generated via equations (4.15), (4.16) and (4.6) respectively. For the simulation, the dither signal is selected as $d_1(t) = d_2(t) = a_1(t)\sin(\omega_1 t) + a_2(t)\sin(\omega_2 t)$, $\omega_1 = 0.3$ and $\omega_2 = 0.18$. The matrix Υ is obtained via the method presented in Section 4.4. In this case, g^T has two columns. Each of the columns is first decomposed into $\psi_i = \Upsilon_i \xi_i(t)$, $i = 1, 2$ where

$$\Upsilon_1 = \begin{bmatrix} x_{p1}^* & 0 & 0 & 0 & 0 & a_1 & a_2 \\ \Upsilon_1^{21} & -a_1^2 & -a_2^2 & 2a_1a_2 & -2a_1a_2 & 4a_1x_{p1}^* & 4a_2x_{p1}^* \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Upsilon_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_{p2}^* & 0 & 0 & 0 & 0 & a_1 & a_2 \\ \Upsilon_2^{41} & -a_1^2 & -a_2^2 & 2a_1a_2 & -2a_1a_2 & 4a_1x_{p2}^* & 4a_2x_{p2}^* \end{bmatrix},$$

$$\Upsilon_1^{21} = 2x_{p1}^{*2} + a_1^2 + a_2^2, \quad \Upsilon_2^{41} = 2x_{p2}^{*2} + a_1^2 + a_2^2 \quad \text{and}$$

$$\xi_1 = \xi_2 = [1, \cos \omega_1 t, \cos \omega_2 t, \cos(\omega_1 - \omega_2)t, \cos(\omega_1 + \omega_2)t, \sin \omega_1 t, \sin \omega_2 t]^T.$$

The matrix $\Upsilon^T = [\Upsilon_1 \ \Upsilon_2]^T$ and $\mathcal{W} = \Upsilon\Upsilon^T$. The optimal value of the dither amplitude that ensures the positive definiteness of \mathcal{W} is obtained via (4.30). In the simulation, the setpoint x_p^* is replaced with its estimate x_p^r at each time t . The results of the simulation are presented in Figure 4.1.

Figure 4.1(a) shows that the estimated cost $p(x_p^r, \hat{\theta})$ and the true cost $p(x_p^r, \theta)$ converge to the unknown optimal $p^*(x_p^*, \theta)$. Fig. 4.1(b) shows that the setpoint signal converges to the optimum value $x_p^*(\theta)$ while the state x_p oscillates about the optimum. The parameter estimates converge to the true values as shown in fig. 4.1(c-d) and the control input, fig. 4.1(e), is implementable. The trajectories of the dither amplitude and the determinant are shown in fig. 4.1(f) for completeness. The figure showed that $a(t)$ converges to the required optimum (vertical-axis labelling on the left) and that the determinant \mathcal{W}_d remains positive (vertical-axis labelling on the right).

4.6 Conclusions

In this chapter, a persistence of excitation condition is developed for extremum seeking control of a class of nonlinear systems. An optimization-based method is then developed for generating sufficiently rich optimum set-points that satisfy this condition online. The proposed design method guarantees parameter convergence and, at the same time, ensures small steady-state error in the cost function. The result shows that the size of the error is proportional to the optimal size \mathcal{A}^* and number of sinusoids \hbar used to perturb the system. However, it is intuitively clear that as the number of distinct sinusoids \hbar increases, the norm of \mathcal{A}^* required to achieve the same level of excitation reduces.

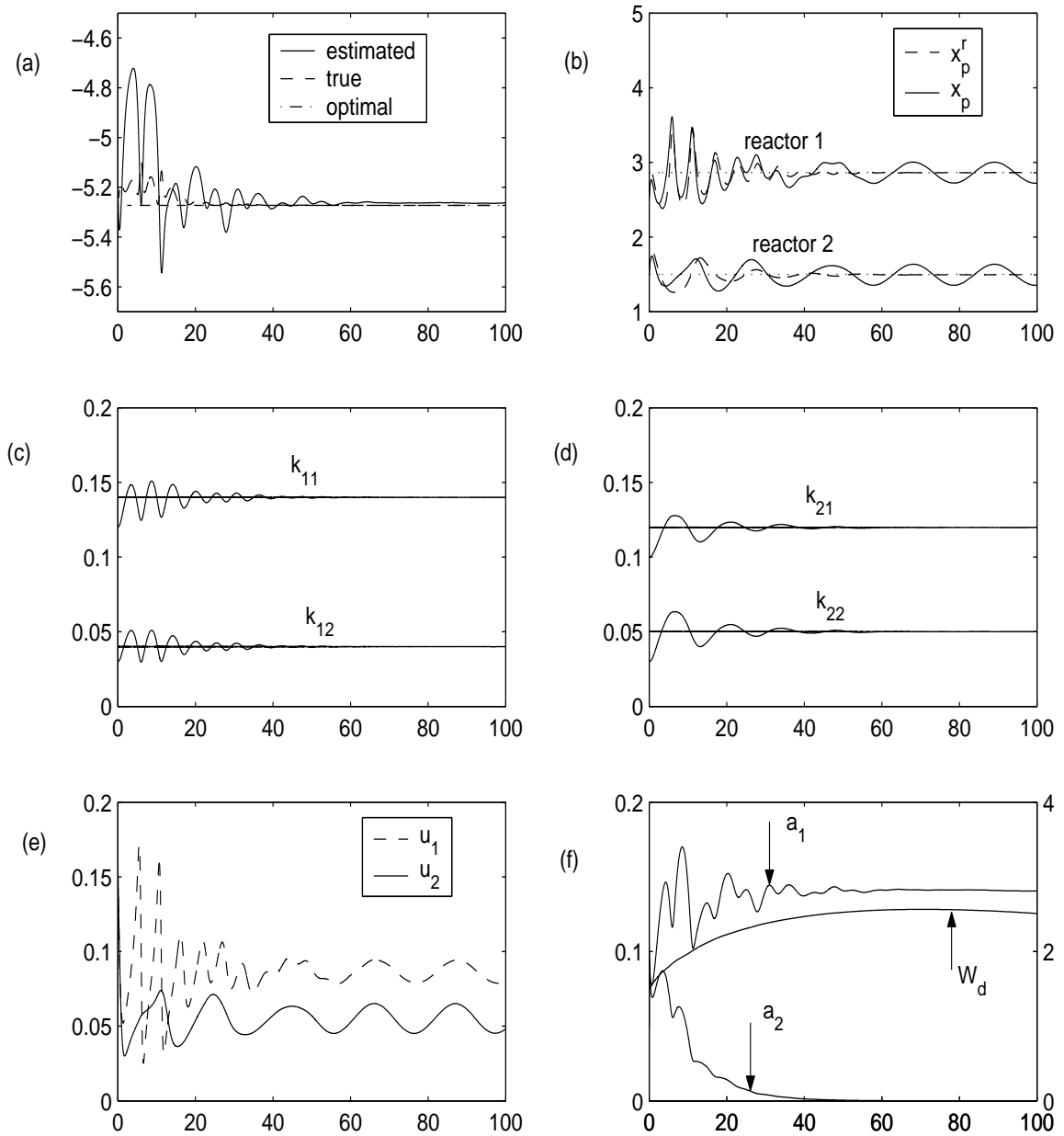


Figure 4.1: Reactor closed-loop trajectories. (a) cost function values, (b) reference setpoint and state, (c, d) unknown parameters and estimates for reactors 1 and 2 respectively, (e) control inputs, (f) dither signal amplitude and determinant of matrix \mathcal{W} .

Chapter 5

Finite-time Parameter Estimation in Adaptive Control

5.1 Introduction

There are two major approaches to online parameter identification of nonlinear systems. The first is the identification of parameters as a part of state observer while the second deals with parameter identification as a part of controller. In the first approach, the observer is designed to provide state derivatives information and the parameters are estimated via estimation methods such as least squares method [81] and dynamic inversion [82]. The second trend of parameter identification is much more widespread, as it allows identification of systems with unstable dynamics. Algorithms in this area include parameter identification methods based on variable structure theory [83,84] and those based on the notion of passivity [85].

In the conventional adaptive control algorithms, the focus is on the tracking of a given reference trajectory and in most cases parameter estimation errors are not guaranteed to converge to zero due to a lack of excitation [77]. Parameter convergence is an important issue as it enhances the overall stability and robustness properties of

the closed-loop adaptive systems [65]. Moreover, there are control problems whereby the reference trajectory is not known *a priori* but depends on the unknown parameters of the system dynamics. For example, in adaptive extremum seeking control problems, the desired target is the operating setpoint that optimizes an uncertain cost function [42, 39].

Assuming the satisfaction of appropriate excitation conditions, asymptotic and exponential parameter convergence results are available for both linear and nonlinear systems. Some lower bounds which depends (nonlinearly) on the adaptation gain and the level of excitation in the system have been provided for some specific control and estimation algorithms [86, 87, 62]. However, it is not always easy to characterize the convergence rate.

Since the performance of any adaptive extremum seeking control is dictated by the efficiency of its parameter adaptation procedure. This chapter presents a parameter estimation scheme that allows exact reconstruction of the unknown parameters in finite-time provided a given persistence of excitation (PE) condition is satisfied. The true parameter estimate is recovered at any time instant the excitation condition is satisfied. This condition requires the integral of a filtered regressor matrix to be invertible. The finite-time (FT) identification procedure assumes the state of the system $x(\cdot)$ is accessible for measurement but does not require the measurement or computation of the velocity state vector $\dot{x}(\cdot)$. The robustness of the estimation routine to bounded unknown disturbances or modeling errors is also examined. It is shown that the parameter estimation error can be rendered arbitrarily small for a sufficiently large filter gain.

A common approach to ensuring a PE condition in adaptive control is to introduce a perturbation signal as the reference input or to add it to the target setpoint or trajectory. The downside of this approach is that a constant PE deteriorates the desired tracking or regulation performance. Aside from the recent results on intelligent

excitation signal design [71,70], the standard approach has been to introduce such PE signal and remove it when the parameters are *assumed* to have converged. The fact that one has perfect knowledge of the convergence time in the proposed framework allows for a direct and immediate removal of the added PE signal. The results in this chapter have been published in [88].

5.2 Problem Description and Assumptions

The system considered is the following nonlinear parameter affine system

$$\dot{x} = f(x, u) + g(x, u)\theta \quad (5.1)$$

where $x \in \mathbb{R}^{n_x}$ is the state and $u \in \mathbb{R}^{n_u}$ is the control input. The vector $\theta \in \mathbb{R}^{n_\theta}$ is the unknown parameter vector whose entries may represent physically meaningful unknown model parameters or could be associated with any finite set of universal basis functions. It is assumed that θ is uniquely identifiable and lie within an initially known compact set Θ^0 . The n_x -dimensional vector $f(x, u)$ and the $(n_x \times n_\theta)$ -dimensional matrix $g(x, u)$ are bounded and continuous in their arguments. System (5.1) encompasses the special class of linear systems,

$$\begin{aligned} f(x, u) &= A_0x + B_0u \\ g(x, u) &= [A_1x + B_1u, A_2x + B_2u, \dots, A_{n_\theta}x + B_{n_\theta}u], \end{aligned}$$

where A_i and B_i for $i = 0 \dots n_\theta$ are known matrices possibly time varying.

Assumption 5.1 *The following assumptions are made about system (5.1).*

1. *The state of the system $x(\cdot)$ is assumed to be accessible for measurement.*
2. *There is a known bounded control law $u = \alpha(\cdot)$ and a bounded parameter update*

law $\dot{\hat{\theta}}$ that achieves a primary control objective.

The control objective can be to (robustly) stabilize the plant and/or to force the output to track a reference signal. Depending on the structure of the system (5.1), adaptive control design methods are available in the literature [73, 89].

For any given bounded control and parameter update law, the aim of this chapter is to provide the true estimates of the plant parameters in finite-time while preserving the properties of the controlled closed-loop system.

5.3 Finite-time Parameter Identification

Let \hat{x} denote the state predictor for (5.1), the dynamics of the state predictor is designed as

$$\dot{\hat{x}} = f(x, u) + g(x, u)\hat{\theta} + k_w(t)e + w\dot{\hat{\theta}}, \quad (5.2)$$

where $\hat{\theta}$ is a parameter estimate generated via any update law $\dot{\hat{\theta}}$, $k_w > 0$ is a design matrix, $e = x - \hat{x}$ is the prediction error and w is the output of the filter

$$\dot{w} = g(x, u) - k_w, \quad w(t_0) = 0. \quad (5.3)$$

Denoting the parameter estimation error as $\tilde{\theta} = \theta - \hat{\theta}$, it follows from (5.1) and (5.2) that

$$\dot{e} = g(x, u)\tilde{\theta} - k_w e - w\dot{\hat{\theta}}. \quad (5.4)$$

The use of the filter matrix w in the above development provides direct information about parameter estimation error $\tilde{\theta}$ without requiring a knowledge of the velocity

vector \hat{x} . This is achieved by defining the auxiliary variable

$$\eta = e - w\tilde{\theta} \quad (5.5)$$

with η , in view of (5.3, 5.4), generated from

$$\dot{\eta} = -k_w \eta, \quad \eta(t_0) = e(t_0). \quad (5.6)$$

Based on the dynamics (5.2), (5.3) and (5.6), the main result is given by the following theorem.

Theorem 5.2 *Let $Q \in \mathbb{R}^{n_\theta \times n_\theta}$ and $C \in \mathbb{R}^{n_\theta}$ be generated from the following dynamics:*

$$\dot{Q} = w^T w, \quad Q(t_0) = 0 \quad (5.7a)$$

$$\dot{C} = w^T(w\hat{\theta} + e - \eta), \quad C(t_0) = 0 \quad (5.7b)$$

Suppose there exists a time t_c and a constant $c_1 > 0$ such that $Q(t_c)$ is invertible i.e.

$$Q(t_c) = \int_{t_0}^{t_c} w^T(\tau)w(\tau) d\tau \succ c_1 I, \quad (5.8)$$

then

$$\theta = Q(t)^{-1}C(t) \quad \text{for all } t \geq t_c. \quad (5.9)$$

Proof: The result can be easily shown by noting that

$$Q(t)\theta = \int_{t_0}^t w^T(\tau)w(\tau) [\hat{\theta}(\tau) + \tilde{\theta}(\tau)] d\tau. \quad (5.10)$$

Using the fact that $w\tilde{\theta} = e - \eta$, it follows from (5.10) that

$$\theta = Q(t)^{-1} \int_{t_0}^t \dot{C}(\tau) d\tau = Q(t)^{-1} C(t) \quad (5.11)$$

and (5.11) holds for all $t \geq t_c$ since $Q(t) \succeq Q(t_c)$. ■

The result in theorem 5.2 is independent of the control u and parameter identifier $\dot{\hat{\theta}}$ structure used for the state prediction (eqn 5.2). Moreover, the result holds if a nominal estimate θ^0 of the unknown parameter (no parameter adaptation) is employed in the estimation routine. In this case, $\hat{\theta}$ is replaced with θ^0 and the last part of the state predictor (5.2) is dropped ($\dot{\hat{\theta}} = 0$).

Let

$$\theta^c \triangleq Q(t_c)^{-1} C(t_c) \quad (5.12)$$

The finite-time (FT) identifier is given by

$$\hat{\theta}^c(t) = \begin{cases} \hat{\theta}(t), & \text{if } t < t_c \\ \theta^c, & \text{if } t \geq t_c. \end{cases} \quad (5.13)$$

The piecewise continuous function (5.13) can be approximated by a smooth approximation using the logistic functions

$$\psi_1 \triangleq \frac{\hat{\theta}(t)}{2} \left(1 - \tanh \nu_1(t - t_c) \right) = \frac{\hat{\theta}(t)}{1 + \exp^{2\nu_1(t-t_c)}} \quad (5.14a)$$

$$\psi_2 \triangleq \frac{\theta^c}{2} \left(1 + \tanh \nu_2(t - t_c) \right) = \frac{\theta^c}{1 + \exp^{-2\nu_2(t-t_c)}} \quad (5.14b)$$

$$\hat{\theta}^{\tilde{c}} = \psi_1 + \psi_2 \quad (5.14c)$$

where larger ν_1, ν_2 correspond to a sharper transition at $t = t_c$ and $\lim_{(\nu_1, \nu_2) \rightarrow \infty} \hat{\theta}^{\tilde{c}} =$

$\hat{\theta}^c$. An example of such approximation is depicted in Figure 5.1 where the function

$$z(t) = \begin{cases} 6 + t^{0.3} & \text{if } t < 5 \\ 4 & \text{otherwise} \end{cases}$$

is approximated by (5.14) with $\nu_1 = \nu_2 = 5$.

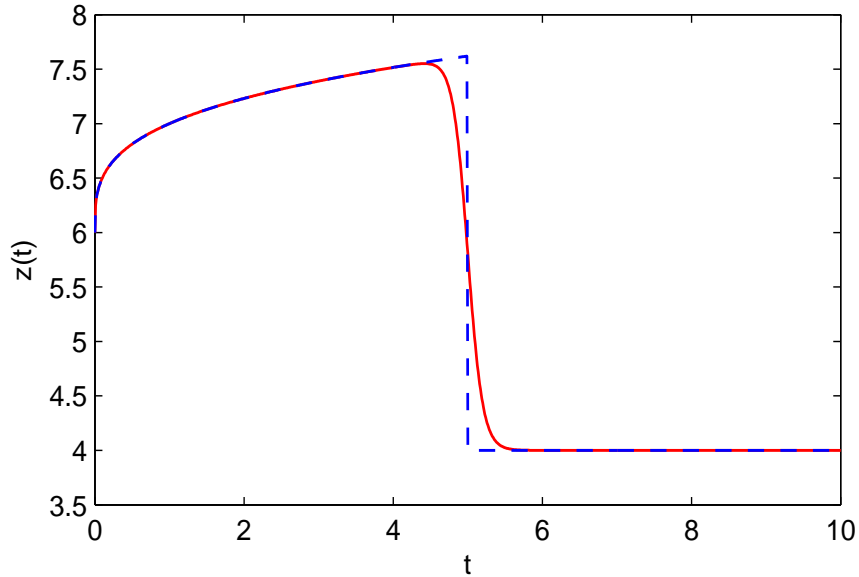


Figure 5.1: Approximation of a piecewise continuous function. The function $z(t)$ is given by the full line. Its approximation is given by the dotted line

The invertibility condition (5.8) is equivalent to the standard persistence of excitation (PE) condition required for parameter convergence in adaptive control. The condition (5.8) is satisfied if the regressor matrix g is PE. To show this, consider the filter dynamic (5.3), from which it follows that

$$w(t) = \int_{t_0}^t \exp^{-k_w(t-\tau)} g(\tau) d\tau = \frac{1}{s + k_w} [g(t)]. \quad (5.15)$$

Since $g(t)$ is PE by assumption and the transfer function $\frac{1}{s+k_w}$ is stable, minimum phase and strictly proper, we know that $w(t)$ is PE [90]. Hence, there exists t_c and

a c_1 for which (5.8) is satisfied. The superiority of the above design lies in the fact that the true parameter value can be computed at any time instant t_c the regressor matrix becomes positive definite and subsequently stop the parameter adaptation mechanism.

The procedure in theorem 7.3 involves solving matrix valued ordinary differential equations (5.3, 5.7) and checking the invertibility of $Q(t)$ online. For computational considerations, the invertibility condition (5.8) can be efficiently tested by checking the determinant of $Q(t)$ online. Theoretically, the matrix is invertible at any time $\det(Q(t))$ becomes positive definite. The determinant of $Q(t)$ (which is a polynomial function) can be queried at pre-scheduled times or by propagating it online starting from a zero initial condition. One way of doing this is to include a scalar differential equation for the derivative of $\det(Q(t))$ as follows [91]:

$$\frac{d}{dt}\det(Q) = \text{Trace}(\text{Adjugate}(Q) w^T w), \quad \det(Q(t_0)) = 0 \quad (5.16)$$

where $\text{Adjugate}(Q)$, admittedly not a light numerical task, is also a polynomial function of the elements of Q .

5.3.1 Absence of PE

If the PE condition (5.8) is not satisfied, a given controller and the corresponding parameter estimation scheme preserve the system established closed-loop properties. When a bounded controller that is robust with respect to input $(\tilde{\theta}, \dot{\hat{\theta}})$ is known, it can be shown that the state prediction error e tends to zero as $t \rightarrow \infty$. An example of such robust controller is an input-to-state stable (iss) controller [73].

Theorem 5.3 *Suppose the design parameter k_w in (5.2) is replaced with $k_w(t) = k_{w_1} + k_{w_2}(t)$, $k_{w_1} > \frac{1}{4}I$ and $k_{w_2}(t) = \frac{1}{4}g\gamma g^T$. Then the state predictor (5.2) and the*

parameter update law

$$\dot{\hat{\theta}} = \gamma(w^T + g^T) e \quad (5.17)$$

with $\gamma = \gamma^T > 0$ a design constant matrix, guarantee that

1. $(e, \eta, \tilde{\theta}) \in \mathcal{L}_\infty$ and $(e, \eta) \rightarrow 0$ as $t \rightarrow \infty$.
2. $\lim_{t \rightarrow \infty} \tilde{\theta}(t) = \bar{\theta}$, a constant.

Proof:

1. Consider a Lyapunov function

$$V = \frac{1}{2} \left(e^T e + \tilde{\theta}^T \gamma^{-1} \tilde{\theta} + \eta^T \eta \right). \quad (5.18)$$

It follows from equations (5.4), (5.5), (5.6) and (5.17) that

$$\dot{V} = -e^T k_{w_1} e - e^T \left(\frac{1}{4} g \gamma g^T + w \gamma w^T + w \gamma g^T \right) e - \tilde{\theta}^T w^T w \tilde{\theta} - \tilde{\theta}^T w^T \eta - \eta^T k_w(t) \eta \quad (5.19)$$

$$\leq -e^T k_{w_1} e - \lambda_{\min}(\gamma) \|w^T e + \frac{1}{2} g^T e\|^2 - \eta^T k_{w_3} \eta - \|w \tilde{\theta} + \frac{1}{2} \eta\|^2 \quad (5.20)$$

$$\leq -(e^T k_{w_1} e + \eta^T k_{w_3} \eta). \quad (5.21)$$

where $k_{w_3} = k_{w_1} - \frac{1}{4}$. This implies uniform boundedness of $(\eta, e, \tilde{\theta})$ as well as global asymptotic convergence of (η, e) to zero. Hence, it follows from (5.5) that $\lim_{t \rightarrow \infty} w \tilde{\theta} = 0$.

2. This can be shown by noting from (5.17) that $\tilde{\theta}(t) = \tilde{\theta}(t_0) - \gamma \int_{t_0}^t (w^T + g^T) e d\sigma$. Since $g(\cdot)$ and e are bounded signals and $e \rightarrow 0$, the integral term exists and it is finite. ■

5.4 Robustness Property

In this section, the robustness of the finite-time identifier to unknown bounded disturbances or modeling errors is demonstrated. Consider a perturbation of (5.1):

$$\dot{x} = f(x, u) + g(x, u)\theta + \vartheta(t, x, \theta) \quad (5.22)$$

where $\vartheta(\cdot)$ is a disturbance or modeling error term that satisfies $\|\vartheta(t)\| \leq M_\vartheta(t) < \infty$. If the PE condition (5.8) is satisfied and the disturbance term is known, the true unknown parameter vector is given by

$$\theta_\vartheta^c \triangleq \theta = Q(t)^{-1} \int_{t_0}^t w^T(\tau) [w(\tau)\hat{\theta}(\tau) + e(\tau) - \eta_\vartheta(\tau)] d\tau, \quad \text{for all } t \geq t_c, \quad (5.23)$$

with $e = x - \hat{x}$ and the signals \hat{x} , w , $\eta_\vartheta = e - w\theta$ generated from (5.2), (5.3) and

$$\dot{\eta}_\vartheta = -k_w \eta_\vartheta + \vartheta(\cdot), \quad \eta_\vartheta(t_0) = e(t_0) \quad (5.24)$$

respectively.

Since $\vartheta(\cdot)$ is unknown, we provide a bound on the parameter identification error $\tilde{\theta}^c = \theta_\vartheta^c - \theta^c$ when (5.6) is used instead of (5.24).

Considering (5.9) and (5.23), it follows that

$$\tilde{\theta}^c = Q(t)^{-1} \int_{t_0}^t w^T(\tau) (-\eta_\vartheta(\tau) + \eta(\tau)) d\tau \quad (5.25)$$

$$= -Q(t)^{-1} \int_{t_0}^t w^T(\tau) \tilde{\eta}(\tau) d\tau. \quad (5.26)$$

where $\tilde{\eta} = \eta_\vartheta - \eta$ is the output of

$$\dot{\tilde{\eta}} = -k_w \tilde{\eta} + \vartheta(\cdot), \quad \tilde{\eta}(t_0) = 0. \quad (5.27)$$

Since $k_w \geq k_{w_1} > 0$, it follows that

$$\|\tilde{\eta}(t)\| \leq \frac{M_\vartheta}{k_{w_1}} \quad (5.28)$$

and hence

$$\|\tilde{\theta}^c(t)\| \leq \|Q(t)^{-1}\| \left\{ \frac{\bar{w}M_\vartheta(t-t_0)}{k_{w_1}} \right\}, \quad \text{for all } t \geq t_c. \quad (5.29)$$

where $\bar{w} = \max_{\sigma \in [t_0, t]} \|w^T(\sigma)\|$.

This implies that the identification error can be rendered arbitrarily small by choosing a sufficiently large filter gain k_{w_1} . In addition, if the disturbance term ϑ and the system satisfies some given properties, then asymptotic convergence can be achieved as stated in the following theorem.

Theorem 5.4 *Suppose $\vartheta \in \mathcal{L}_p$, for $p = 1$ or 2 and $\lim_{t \rightarrow \infty} \lambda_{\min}(Q) = \infty$, then $\tilde{\theta}^c \rightarrow 0$ asymptotically with time.*

To proof this theorem, we need the following lemma

Lemma 5.5 [92]: *Consider the system*

$$\dot{x}(t) = Ax(t) + u(t) \quad (5.30)$$

Suppose the equilibrium state $x_e = 0$ of the homogeneous equation is exponentially stable,

1. *if $u \in L_p$ for $1 < p < \infty$, then $x \in L_p$ and*
2. *if $u \in L_p$ for $p = 1$ or 2 , then $x \rightarrow 0$ as $t \rightarrow \infty$.*

Proof of theorem 5.4. It follows from Lemma 5.5.2 that $\tilde{\eta} \rightarrow 0$ as $t \rightarrow \infty$ and therefore $\lim_{t \rightarrow \infty} \int_{t_0}^t w^T(\tau) \tilde{\eta}(\tau) d\tau$ is finite. So

$$\lim_{t \rightarrow \infty} \tilde{\theta}^c = \lim_{t \rightarrow \infty} \left\{ Q(t)^{-1} \int_{t_0}^t w^T(\tau) \tilde{\eta}(\tau) d\tau \right\} = 0. \quad (5.31)$$

5.5 Dither Signal Design

The problem of tracking a reference signal is usually considered in the study of parameter convergence and in most cases, the reference signal is required to provide sufficient excitation for the closed-loop system. To this end, the reference signal $y_r(t) \in \mathbb{R}^r$ is appended with a bounded excitation signal $d(t)$ as

$$y_{rd}(t) = y_r(t) + d(t) \quad (5.32)$$

where the auxiliary signal $d(t)$ is chosen as a linear combination of sinusoidal functions with \hbar distinct frequencies:

$$d(t) := \sum_{k=1}^{\hbar} \underline{a}_k(t) \sin(\omega_k t) = \mathcal{A}(t) \zeta(t) \quad (5.33)$$

where

$$\mathcal{A}(t) = \begin{bmatrix} a_{11} & \cdots & a_{1\hbar} \\ \vdots & & \vdots \\ a_{r1} & \cdots & a_{r\hbar} \end{bmatrix}$$

is the signal amplitude matrix and

$$\zeta(t) = [\sin \omega_1 t \ \dots \ \sin \omega_{\hbar} t]^T, \quad \omega_i \neq \omega_j \text{ for } i \neq j$$

is the corresponding sinusoidal function vector.

For this approach, it is sufficient to design the perturbation signal such that the regressor matrix g is PE. There are very few results on the design of persistently exciting (PE) input signals for nonlinear systems. By converting the closed-loop PE condition to a sufficient richness (SR) condition on the reference signal, attempts have been made to provide verifiable conditions for parameter convergence in some classes of nonlinear systems [71, 78, 65, 66].

5.5.1 Dither Signal Removal

Let $\mathcal{H} \leq (\hbar \times r)$ denotes the number of distinct elements in the dither amplitude matrix $\mathcal{A}(t)$ and let $a \in \mathbb{R}^r$ be a vector of these distinct coefficients. The amplitude of the excitation signal is specified as

$$a(t) = \begin{cases} a, & \text{if } t < t_c \\ 0, & \text{otherwise} \end{cases} \quad (5.34)$$

or approximated by

$$a(t) \approx \frac{a}{1 + \exp^{2\nu(t-t_c)}} \quad (5.35)$$

where equality holds in the limit as $\nu \rightarrow \infty$.

5.6 Simulation Examples

5.6.1 Example 1

We consider the following nonlinear system in parametric strict feedback form [66]:

$$\begin{aligned}\dot{x}_1 &= x_2 + \theta_1 x_1 \\ \dot{x}_2 &= x_3 + \theta_2 x_1 \\ \dot{x}_3 &= \theta_3 x_1^3 + \theta_4 x_2 + \theta_5 x_3 + (1 + x_1^2)u \\ y &= x_1,\end{aligned}\tag{5.36}$$

where $\theta^T = [\theta_1, \dots, \theta_5]$ are unknown parameters. Using an adaptive backstepping design, the control and parameter update law presented in [66] were used for the simulation. The pair stabilize the plant and ensure that the output y tracks a reference signal $y_r(t)$ asymptotically. For simulation purposes, parameter values are set to $\theta^T = [-1, -2, 1, 2, 3]$ as in [66] and the reference signal is $y_r = 1$, which is sufficiently rich of order one. The simulation results for zero initial conditions are shown in Figure 5.2. Based on the convergence analysis procedure in [66], all the parameter estimates cannot converge to their true values for this choice of constant reference. As confirmed in Fig. 5.2, only θ_1 and θ_2 estimates are accurate. However, following the proposed estimation technique and implementing the FT identifier (5.14), we obtain the exact parameter estimates at $t = 17sec$. This example demonstrates that, with the proposed estimation routine, it is possible to identify parameters using perturbation or reference signals that would otherwise not provide sufficient excitation for standard adaptation methods.

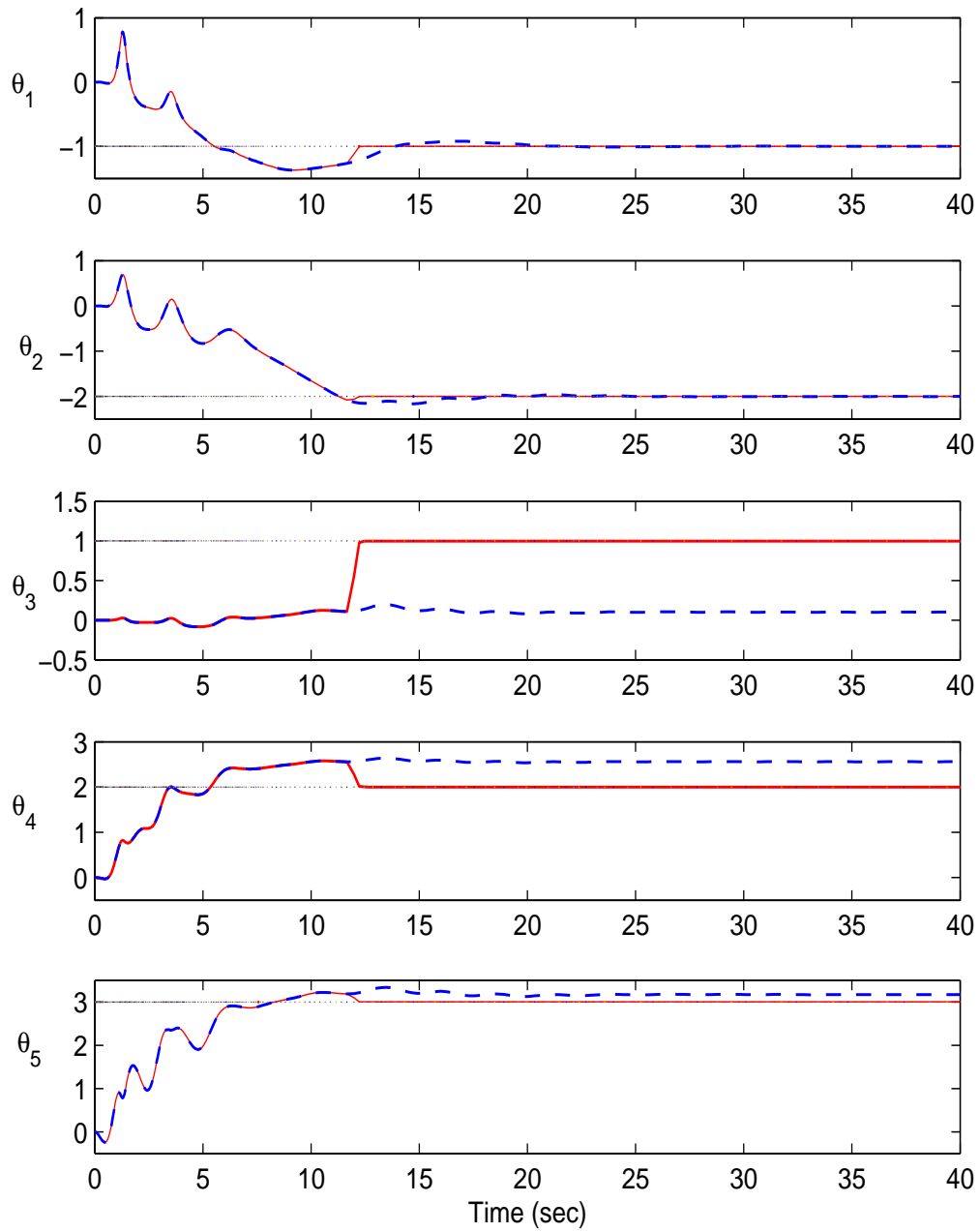


Figure 5.2: Trajectories of parameter estimates. Solid(-): FT estimates $\hat{\theta}^c$; dashed(--): standard estimates $\hat{\theta}$ [66]; dashdot(-.): actual value.

5.6.2 Example 2

To corroborate the superiority of the developed procedure, we demonstrate the robustness of the developed procedure by considering system (5.36) with added exogenous disturbances as follows:

$$\begin{aligned}\dot{x}_1 &= x_2 + \theta_1 x_1 + [1 \ 0]\vartheta \\ \dot{x}_2 &= x_3 + \theta_2 x_1 + [1 \ x_1]\vartheta \\ \dot{x}_3 &= \theta_3 x_1^3 + \theta_4 x_2 + \theta_5 x_3 + (1 + x_1^2)u + [0 \ 1]\vartheta \\ y &= x_1,\end{aligned}\tag{5.37}$$

where $\vartheta = [0.1 \sin(2\pi t/5), 0.2 \cos(\pi t)]^T$ and the tracking signal remains a constant $y_r = 1$.

The simulation result, Figure 5.3, shows convergence of the estimate vector to a small neighbourhood of θ under finite-time identifier with filter gain $k_w = 1$ while no full parameter convergence is achieved with the standard identifier. The parameter estimation error $\tilde{\theta}(t)$ is depicted in Figure 5.4 for different values of the filter gain k_w . The switching time for the simulation is selected as the time for which the condition number of Q becomes less than 20. It is noted that the time at which switching from standard adaptive estimate to FT estimate occurs increases as the filter gain increases. The convergence performance improves as k_w increases, however, no significant improvement is observed as the gain is increased beyond 0.5.

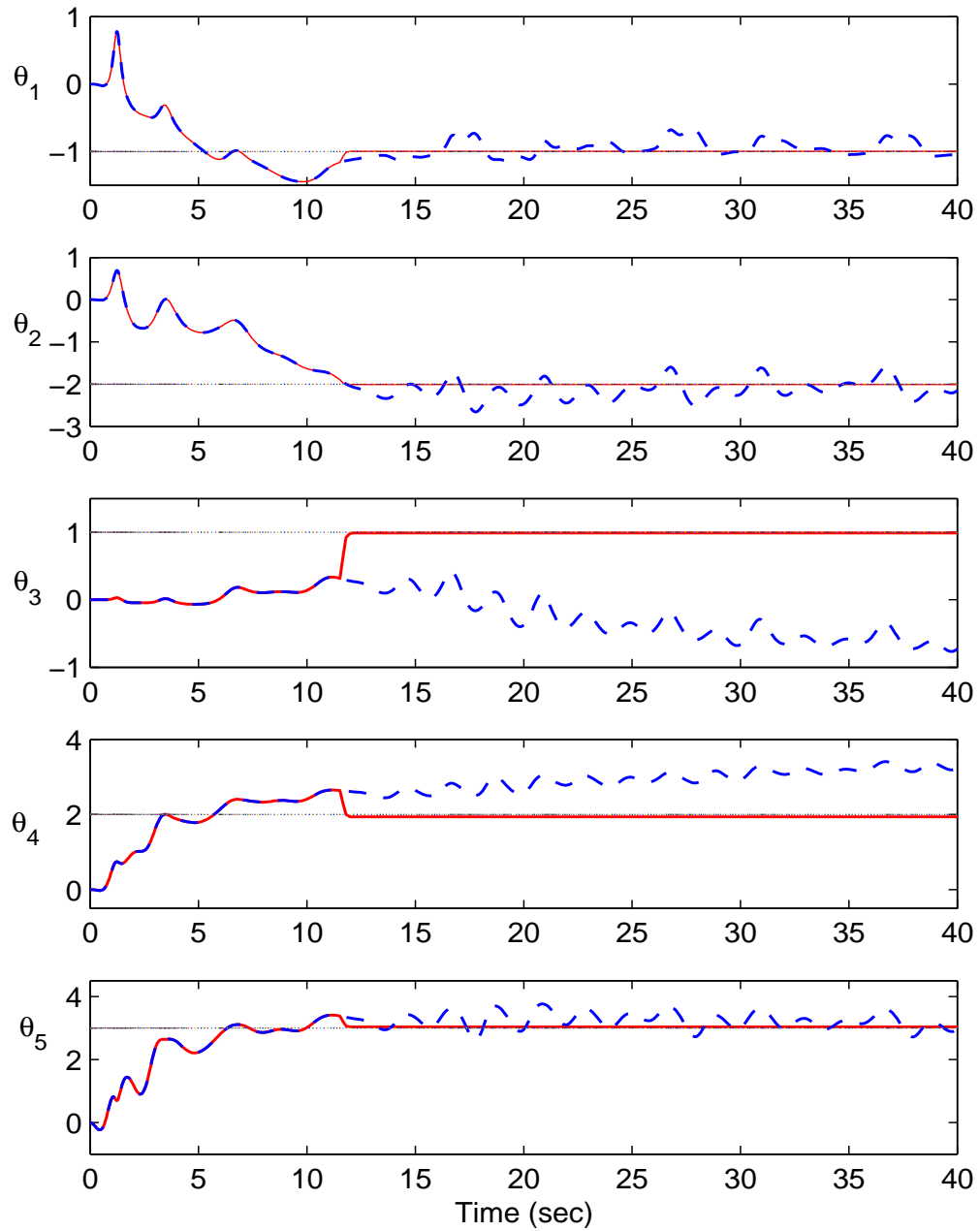


Figure 5.3: Trajectories of parameter estimates. Solid(-): FT estimates for the system with additive disturbance $\hat{\theta}^c$; dashed(--): standard estimates $\hat{\theta}$ [66]; dashdot(-.): actual value.

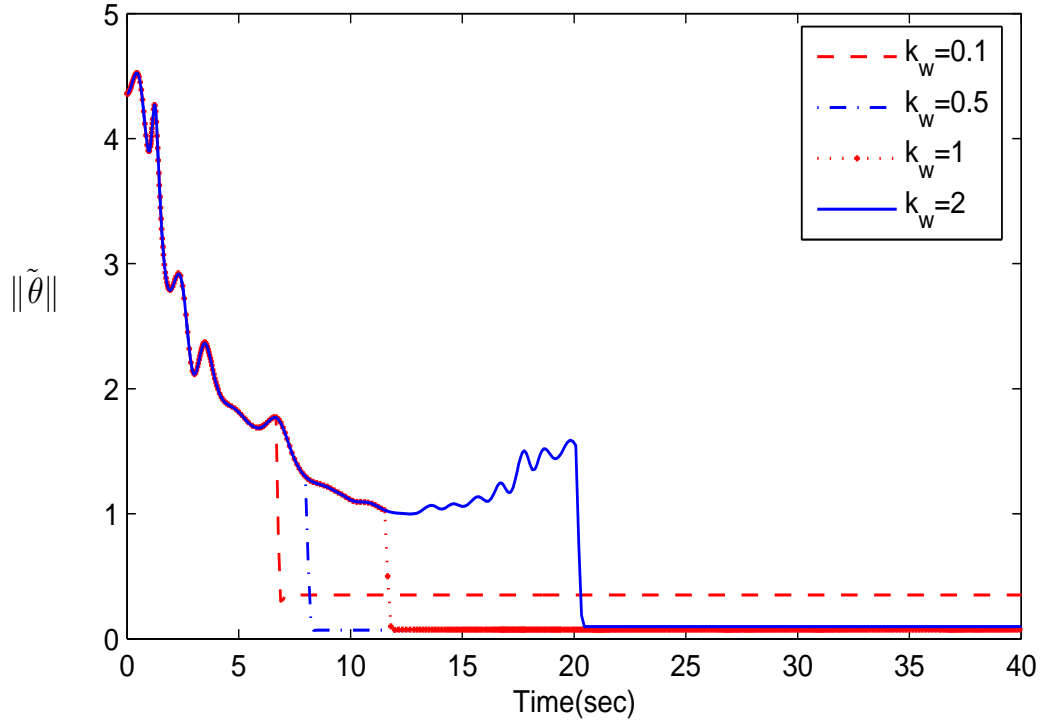


Figure 5.4: Parameter estimation error for different filter gains k_w

5.7 Conclusions

The work presented in this chapter transcends beyond characterizing the parameter convergence rate. A method is presented for computing the exact parameter value at a finite-time selected according to the observed excitation in the system. A smooth transition from a standard estimate to the FT estimate is proposed. In the presence of unknown bounded disturbances, the FT identifier converges to a neighbourhood of the true value whose size is dictated by the choice of the filter gain. Moreover, the procedure preserves the system's established closed-loop properties whenever the required PE condition is not satisfied.

Chapter 6

Performance Improvement in Adaptive Control

6.1 Introduction

The finite-time identification method developed in Chapter 5 has two distinguishing features. First, the true parameter estimate is obtained at any time instant the excitation condition is satisfied, and second, the procedure allows for a direct and immediate removal of any perturbation signal injected in to the closed-loop system to aid in parameter estimation. However, the drawback of the finite-time identification algorithm is the requirement to check the invertibility of a matrix online and compute the inverse matrix when appropriate.

To avoid these concerns and enhance the applicability of the FT method in practical situations, the procedure is hereby exploited to develop a novel adaptive compensator that (almost) recovers the performance of the FT identifier. The compensator guarantees exponential convergence of the parameter estimation error at a rate dictated by the closed-loop system's excitation. It was shown how the adaptive compensator can be used to improve upon existing adaptive controllers. The modification

proposed guarantees exponential stability of the parametric equilibrium provided the given PE condition is satisfied. Otherwise, the original system's closed-loop properties are preserved.

6.2 Adaptive Compensation Design

Consider the nonlinear system 5.1 satisfying assumption 5.1 and the state predictor

$$\dot{\hat{x}} = f(x, u) + g(x, u) \theta^0 + k_w(x - \hat{x}) \quad (6.1)$$

where $k_w > 0$ and θ^0 is the nominal initial estimate of θ . If we define the auxiliary variable

$$\eta = x - \hat{x} - w(\theta - \theta^0) \quad (6.2)$$

and select the filter dynamic as

$$\dot{w} = g(x, u) - k_w w, \quad w(t_0) = 0 \quad (6.3)$$

then η is generated by

$$\dot{\eta} = -k_w \eta, \quad \eta(t_0) = e(t_0). \quad (6.4)$$

Based on (6.1) to (6.4), our novel adaptive compensation result is given in the following theorem.

Theorem 6.1 *Let Q and C be generated from the following dynamics:*

$$\dot{Q} = w^T w, \quad Q(t_0) = 0 \quad (6.5a)$$

$$\dot{C} = w^T (w \theta^0 + x - \hat{x} - \eta), \quad C(t_0) = 0 \quad (6.5b)$$

and let t_c be the time such that $Q(t_c) \succ 0$, then the adaptation law

$$\dot{\hat{\theta}} = \Gamma (C - Q \hat{\theta}), \quad \hat{\theta}(t_0) = \theta^0 \quad (6.6)$$

with $\Gamma = \Gamma^T \succ 0$ guarantees that $\|\tilde{\theta}\| = \|\theta - \hat{\theta}\|$ is non-increasing for $t_0 \leq t \leq t_c$ and converges to zero exponentially fast, starting from t_c . Moreover, the convergence rate is lower bounded by $\mathcal{E}(t) = \lambda_{\min}(\Gamma Q(t))$.

Proof: Consider a Lyapunov function

$$V_{\tilde{\theta}} = \frac{1}{2} \tilde{\theta}^T \tilde{\theta}, \quad (6.7)$$

it follows from (6.6) that

$$\dot{V}_{\tilde{\theta}}(t) = -\tilde{\theta}^T(t) \Gamma (C(t) - Q(t)\hat{\theta}(t)). \quad (6.8)$$

Since $w\theta = w\theta^0 + x - \hat{x} - \eta$ (from (6.2)), then

$$C(t) = \int_{t_0}^t \dot{C}(\tau) d\tau = \int_{t_0}^t w^T(\tau)w(\tau) d\tau \theta = Q(t)\theta \quad (6.9)$$

and equation (6.8) becomes

$$\dot{V}_{\tilde{\theta}}(t) = -\tilde{\theta}^T(t) \Gamma Q(t) \tilde{\theta}(t) \quad (6.10)$$

$$\leq -\mathcal{E}(t) V_{\tilde{\theta}}(t) \quad (6.11)$$

This implies non-increase of $\|\tilde{\theta}\|$ for $t \geq t_0$ and the exponential claim follows from the fact that $\Gamma Q(t) = \Gamma \int_{t_0}^t w(\tau)^T w(\tau) d\tau$ is positive definite for all $t \geq t_c$. The

convergence rate is shown by noting that

$$\dot{V}_{\tilde{\theta}}(t) = -\tilde{\theta}^T(t) \Gamma \left(Q(t_c) + \int_{t_c}^t w(\tau)^T w(\tau) d\tau \right) \tilde{\theta}(t), \quad \forall t \geq t_c \quad (6.12)$$

$$\leq -\tilde{\theta}^T(t) \Gamma Q(t_c) \tilde{\theta}(t) \leq -\mathcal{E}(t_c) V(t) \quad (6.13)$$

which implies

$$\|\tilde{\theta}(t)\| \leq \exp^{-\mathcal{E}(t_c)(t-t_0)} \|\tilde{\theta}(t_0)\|, \quad \forall t \geq t_c \quad (6.14)$$

■

Both the FT identification (5.9) and the adaptive compensator (6.6) use the static relationship developed between the unknown parameter θ and some measurable matrix signals C , *i.e.*, $Q\theta = C$. However, instead of computing the parameter values at a known finite-time by inverting matrix Q , the adaptive compensator is driven by the estimation error $C - Q\hat{\theta} = Q\tilde{\theta}$.

6.3 Incorporating Adaptive Compensator for Performance Improvement

It is assumed that the given control law u and stabilizing update law (herein denoted as $\dot{\hat{\theta}}^s$) result in closed-loop error system

$$\dot{Z} = AZ + \Phi^T \tilde{\theta}^s \quad (6.15a)$$

$$\dot{\tilde{\theta}}^s = -\Gamma \Phi Z \quad (6.15b)$$

where the matrix A is such that $A + A^T < -2k_A I < 0$, Φ is a bounded matrix function of the regressor vectors, $\tilde{\theta}^s = \theta - \hat{\theta}^s$ and $Z = [z_1, z_2, \dots, z_{n_x}]^T$ is a vector function

of the tracking error with $z_1 = y - y^r$. This implies that the adaptive controller guarantees uniform boundedness of the estimation error $\tilde{\theta}^s$ and asymptotic convergence of the tracking error Z dynamics. Such adaptive controllers are very common in the literature. Examples include linearized control laws [89] and controllers designed via backstepping [73, 66].

Given the stabilizing adaptation law $\dot{\hat{\theta}}^s$, we propose the following update law which is a combination of the stabilizing update law (6.15b) and the adaptive compensator (6.6)

$$\dot{\hat{\theta}} = \Gamma \left(\Phi Z + C - Q \hat{\theta} \right). \quad (6.16)$$

Since $C(t) = Q(t) \theta$, the resulting error equations becomes

$$\begin{bmatrix} \dot{Z} \\ \dot{\tilde{\theta}} \end{bmatrix} = \begin{bmatrix} A & \Phi^T \\ -\Gamma\Phi & -\Gamma Q \end{bmatrix} \begin{bmatrix} Z \\ \tilde{\theta} \end{bmatrix}. \quad (6.17)$$

Considering the Lyapunov function $V = \frac{1}{2} \left(z^T z + \tilde{\theta}^T \Gamma^{-1} \tilde{\theta} \right)$, and differentiating along (6.17) we have

$$\dot{V} = \frac{1}{2} z^T (A + A^T) z - \tilde{\theta}^T Q \tilde{\theta} \leq -k_A z^T z - \tilde{\theta}^T Q \tilde{\theta} \quad (6.18)$$

Hence $\tilde{\theta} \rightarrow 0$ exponentially for $t \geq t_c$ and the initial asymptotic convergence of Z is strengthened to exponential convergence.

For feedback linearizable systems

$$\begin{aligned} \dot{x}_i &= x_{i+1} \quad 1 \leq i \leq n-1 \\ \dot{x}_n &= f_1(x) + f_2(x)u + \theta^T g_n(x) \\ y &= x_1 \end{aligned}$$

the PE condition $Q(t_c) \succ 0$ translates to a priori verifiable sufficient condition on the reference setpoint. It requires the rows of the regressor vector $g_n(x)$ to be linearly independent along a desired trajectory $x^r(t)$ on any finite interval $t \in [t_1, t_2)$, $t_1 < t_2 < \infty$. This condition is less restrictive than the one given in [93] for the same class of system. This is because the linear independence requirement herein is only required over a finite interval and it can be satisfied by a non-periodic reference trajectory while the asymptotic stability result in [93] relies on a T-periodic reference setpoint. Moreover exponential, rather than asymptotic stability of the parametric equilibrium is achieved.

6.4 Dither Signal Update

Perturbation signal is usually added to the desired reference setpoint or trajectory to guarantee the convergence of system parameters to their true values. To reduce the variability of the closed-loop system, the added PE signal must be systematically removed in a way that sustains parameter convergence.

Suppose the dither signal $d(t)$ is selected as a linear combination of sinusoidal functions as detailed in Section 5.5. Let a^0 be the vector of the selected dither amplitude and let $T > 0$ be the first instant for which $d(T) = 0$, the amplitude of the excitation signal is updated as follows:

$$a(t) = \begin{cases} a^0, & t \in [0, T) \\ \exp^{-\gamma \varepsilon T} a^0, & t \in [jT, (j+1)T), \quad j \geq 1 \end{cases} \quad (6.19)$$

where the gain $\gamma > 0$ is a design parameter, $a(0) = a^0$ and

$$\begin{aligned}\mathcal{E}(0) &= 0, & \mathcal{E}(\tau) &= \lambda_{\min}(Q(\tau)) \\ \bar{\mathcal{E}} &= \max \{ \mathcal{E}(jT), \mathcal{E}((j-1)T) \}.\end{aligned}$$

It follows from (6.19) that the reference setpoint will be subject to PE with constant amplitude a^0 if $t \in [0, T)$. After which the trajectory of $a(t)$ will be dictated by the filtered regressor matrix Q . The amplitude vector $a(t)$ will start to decay exponentially when $Q(t)$ becomes positive definite. Note that parameter convergence will be achieved regardless of the value of the gain γ selected as the only requirement for convergence is $Q(t) \succ 0$.

Remark 6.2 *The other major approach used in traditional adaptive control is parameter estimation based design. A well designed estimation based adaptive control method achieves modularity of the controller-identifier pair. For nonlinear systems, the controller module must possess strong parametric robustness properties while the identifier module must guarantee certain boundedness properties independent of the control module. Assuming the existence of a bounded controller that is robust with respect to $(\tilde{\theta}, \dot{\hat{\theta}})$, the adaptive compensator (6.6) serves as a suitable identifier for modular adaptive control design.*

6.5 Simulation Example

To demonstrate the effectiveness of the adaptive compensator, we consider the example in Section 5.6 for both the nominal system (5.36) and the system under additive disturbance (5.37). The simulation is performed for the same reference setpoint $y_r = 1$, disturbance vector $\vartheta = [0.1 \sin(2\pi t/5), 0.2 \cos(\pi t)]^T$, parameter values $\theta = [-1, -2, 1, 2, 3]$ and zero initial conditions.

The adaptive controller presented in [66] is also used for the simulation. We modify the given stabilizing update law by adding the adaptive compensator (6.6) to it. The modification significantly improve upon the performance of the standard adaptation mechanism as shown in Figures 6.1 and 6.2. All the parameters converged to their values and we recover the performance of the finite-time identifier (5.14). Figures 6.3 and 6.4 depict the performance of the output and the input trajectories. While the transient behaviour of the output and input trajectories is slightly improved for the nominal adaptive system, a significant improvement is obtained for the system subject to additive disturbances.

6.6 Conclusions

This Chapter demonstrates how the finite-time identification procedure can be used to improve the overall performance (both transient and steady state) of adaptive control systems in a very appealing manner. First, we develop an adaptive compensator which guarantees exponential convergence of the estimation error provided the integral of a filtered regressor matrix is positive definite. The approach does not involve online checking of matrix invertibility and computation of matrix inverse nor switching between parameter estimation methods. The convergence rate of the parameter estimator is directly proportional to the adaptation gain and a measure of the system's excitation. The adaptive compensator is then combined with existing adaptive controllers to guarantee exponential stability of the closed-loop system. The application reported in Section 6.3 is just an example, the adaptive compensator can easily be incorporated into other adaptive control algorithms.

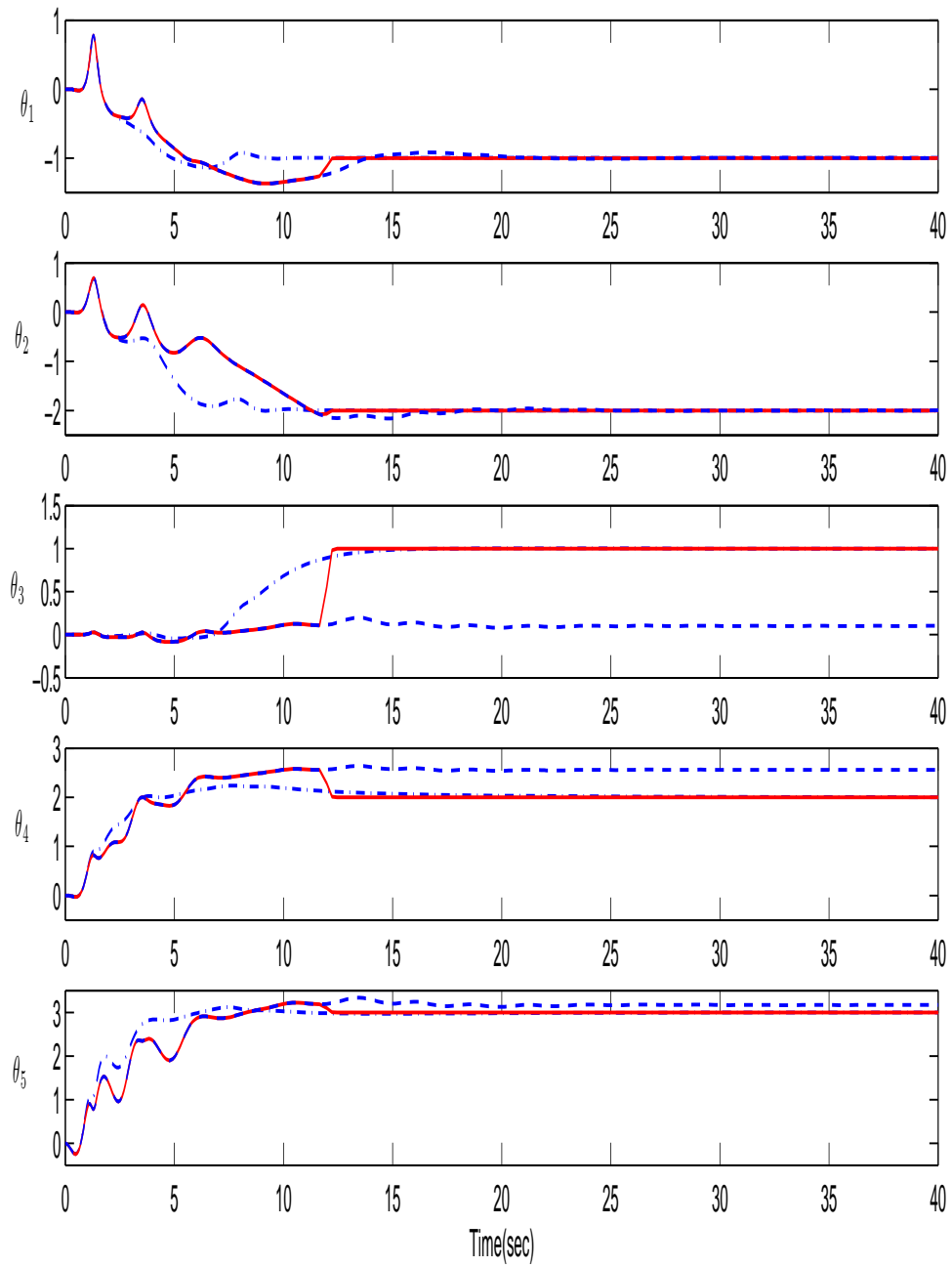


Figure 6.1: Trajectories of parameter estimates. Solid(-): compensated estimates; dashdot(-.): FT estimates; dashed(--): standard estimates [66].

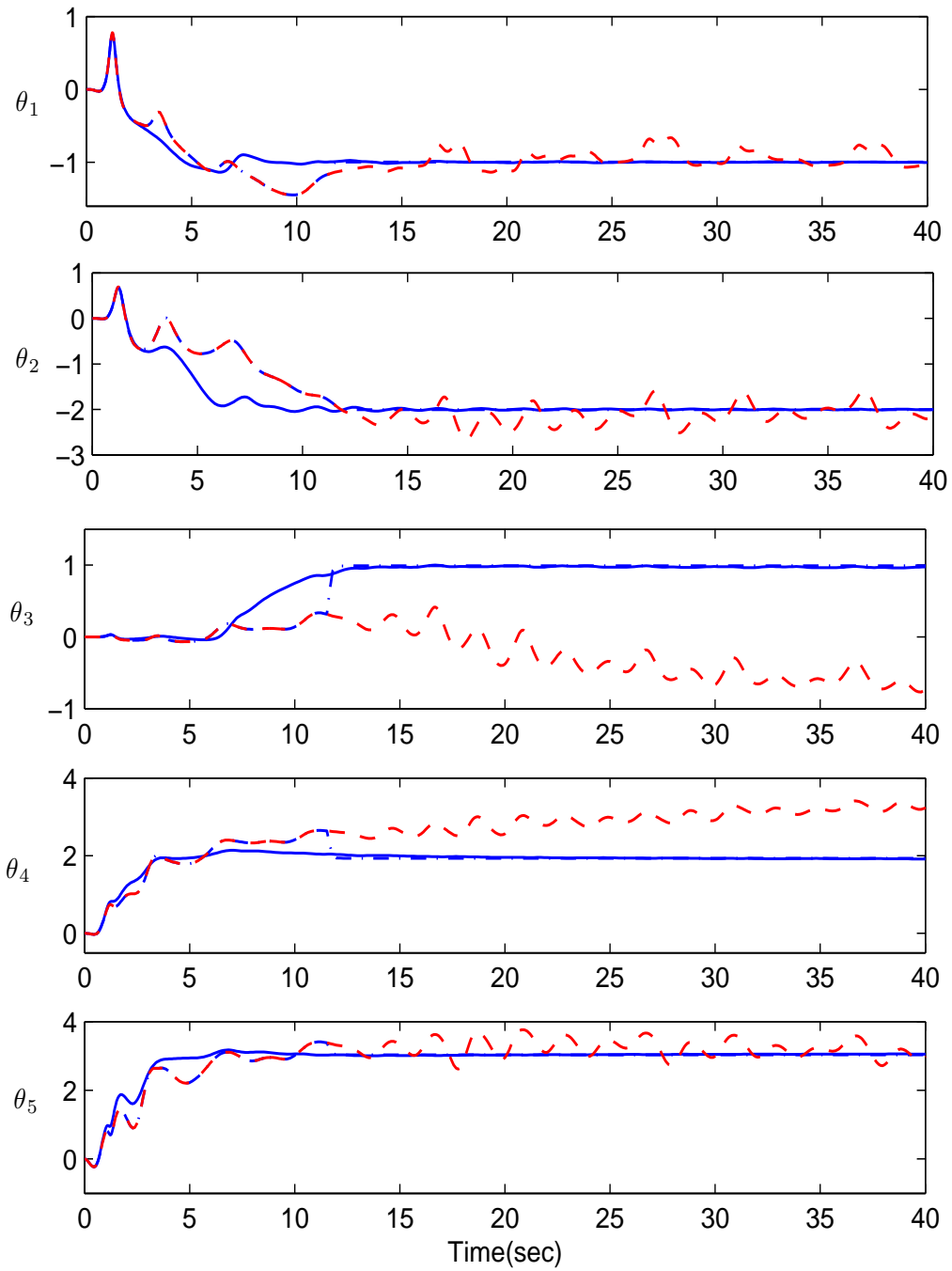


Figure 6.2: Trajectories of parameter estimates under additive disturbances. Solid(-): compensated estimates; dashdot(-.): FT estimates; dashed(--): standard estimates [66].

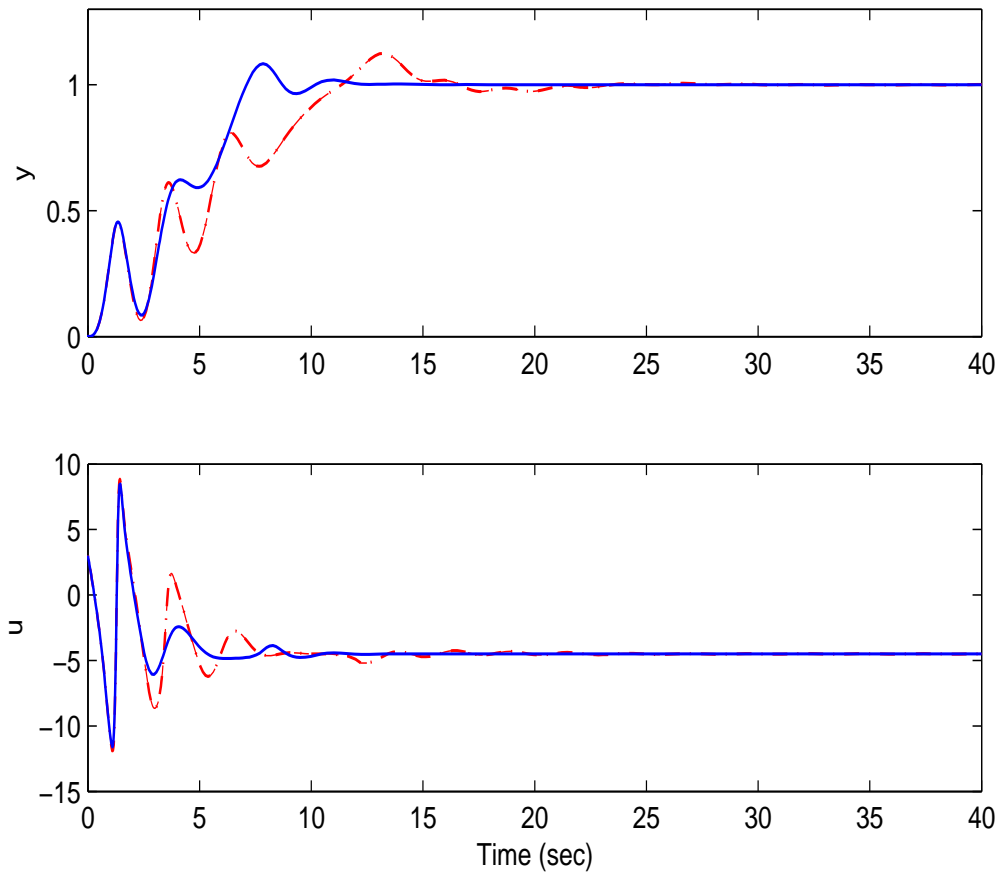


Figure 6.3: Trajectories of system's output and input for different adaptation laws. Solid(-): compensated estimates; dashdot(-.): FT estimates; dashed(--): standard estimates [66].

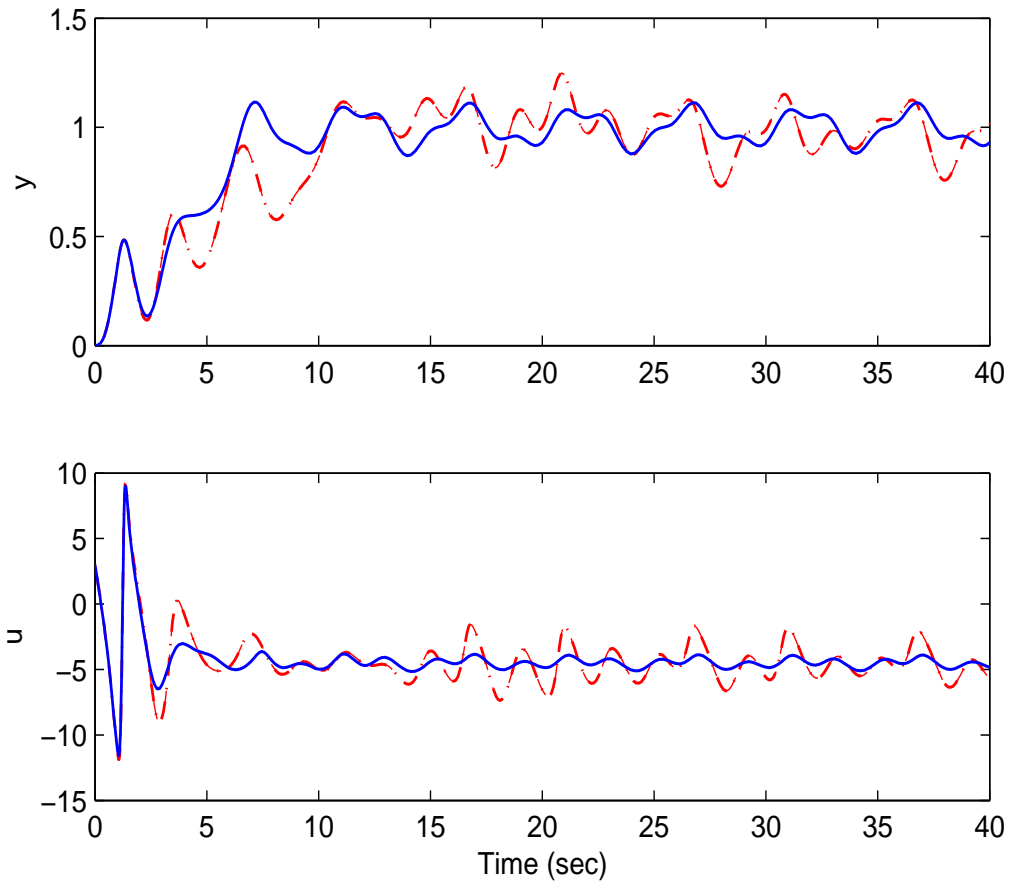


Figure 6.4: Trajectories of system's output and input under additive disturbances for different adaptation laws. Solid(-): compensated estimates; dashdot(-.): FT estimates; dashed(--): standard estimates [66].

Chapter 7

Adaptive Model Predictive Control for Constrained Nonlinear Systems

7.1 Introduction

Most physical systems possess parametric uncertainties or unmeasurable parameters. Examples in chemical engineering include reaction rates, activation energies, fouling factors, and microbial growth rates. Since parametric uncertainty may degrade the performance of MPC, mechanisms to update the unknown or uncertain parameters are desirable in application. One possibility would be to use state measurements to update the model parameters off-line. A more attractive possibility is to apply adaptive extensions of MPC in which parameter estimation and control are performed online.

The literature contains very few results on the design of adaptive nonlinear MPC [31, 30]. Existing design techniques are restricted to systems that are linear in the unknown (constant) parameters and do not involve state constraints. Although MPC exhibits some degree of robustness to uncertainties, in reality, the degree of robustness provided by nominal models or certainty equivalent models may not be sufficient

in practical applications. Parameter estimation error must be accounted for in the computation of the control law.

This chapter is inspired by [32, 33]. While the focus in [32, 33] is on the use of adaptation to reduce the conservatism of robust MPC controller, this study addresses the problem of adaptive MPC and incorporates robust features to guarantee closed-loop stability and constraint satisfaction. Simplicity is achieved here-in by generating a parameter estimator for the unknown parameter vector and parameterizing the control policy in terms of these estimates rather than adapting a parameter uncertainty set directly.

First, a min-max feedback nonlinear MPC scheme is combined with the adaptation mechanism developed in Chapter 6. The parameter estimation routine employed guarantees non-increase of the norm of the estimation error vector and provides exponential parameter convergence when an excitation condition is satisfied. The estimates are used to update the parameter uncertainty set, at every time step, in a manner that guarantees non-expansion of the set leading to a gradual reduction in the conservativeness or computational demands of the algorithms. The min-max formulation explicitly accounts for the effect of future parameter estimation and automatically injects some useful excitation into the closed-loop system to aid in parameter identification.

Second, the technique is extended to a less computationally demanding robust MPC algorithm. The nominal model rather than the unknown bounded system state is controlled, subject to conditions that ensure that given constraints are satisfied for all possible uncertainties. State prediction error bound is determined based on assumed Lipschitz continuity of the model. Using a nominal model prediction, it is impossible to predict the actual future behavior of the parameter estimation error as was possible in the min-max framework. It is shown how the future model improvement over the prediction horizon can be considered by developing a worst-case

upper bound on the future parameter estimation error. The conservativeness of the algorithm reduces as the error bound decreases monotonically over time.

Finally, it is shown how the finite-time identifier developed in Chapter 5 can be incorporated in the proposed adaptive MPC algorithms. The true value of the uncertain parameter vector is recovered in a known finite-time when an excitation condition is satisfied. Subsequently, the adaptive and robustness features of the MPC is eliminated and the complexity of the resultant controller reduces to that of nominal model predictive control.

7.2 Problem Description

The system considered is the following nonlinear parameter affine system

$$\dot{x} = f(x, u) + g(x, u)\theta \triangleq \mathcal{F}(x, u, \theta) \quad (7.1)$$

$\theta \in \mathbb{R}^{n_\theta}$ is the unknown parameter vector whose entries may represent physically meaningful unknown model parameters or could be associated with any finite set of universal basis functions. It is assumed that θ is uniquely identifiable and lie within an initially known compact set $\Theta^0 \triangleq B(\theta^0, z_\theta^0)$, a ball described by an initial nominal estimate θ^0 and associated error bound $z_\theta^0 = \sup_{s \in \Theta^0} \|s - \theta^0\|$. The mapping $\mathcal{F} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_\theta} \rightarrow \mathbb{R}^{n_x}$ is assumed to be locally Lipschitz with respect to its arguments. The state and the control input trajectories are assumed to be subject to pointwise constraints $x \in \mathbb{X} \in \mathbb{R}^{n_x}$ and $u \in \mathbb{U} \in \mathbb{R}^{n_u}$ respectively. The objective of the study is to (robustly) stabilize the plant by means of state feedback adaptive MPC. Optimality of the resulting trajectories are measured with respect to the accumulation of some stage cost $L(x, u) \geq 0$. The cost is assumed to be continuous, $L(0, 0) = 0$,

and $L(x, u) \geq \mu_L(\|x, u\|)$, where μ_L is a \mathcal{K}_∞ function.¹

7.3 Estimation of Uncertainty

7.3.1 Parameter Adaptation

Since parameter convergence is fundamental to the overall goal of this thesis work. The estimation algorithm presented in Chapter 6 is used. The adaptive update law, driven by the parameter estimation error $\tilde{\theta} = \theta - \hat{\theta}$, remains active until parameter convergence is achieved and results in faster convergence than any traditional update laws that depends only on tracking or prediction error.

For ease of reference, the adaptive law is given by

$$\dot{\hat{\theta}} = \Gamma (C - Q \hat{\theta}), \quad \hat{\theta}(t_0) = \theta^0 \quad (7.2)$$

where the adaptive gain $\Gamma = \Gamma^T \succ 0$ and

$$\dot{Q} = w^T w, \quad Q(t_0) = 0 \quad (7.3a)$$

$$\dot{C} = w^T (w \theta^0 + x - \hat{x} - \eta), \quad C(t_0) = 0 \quad (7.3b)$$

$$\dot{\hat{x}} = f(x, u) + g(x, u) \theta^0 + k_w (x - \hat{x}), \quad \hat{x}(t_0) = x(t_0) \quad (7.3c)$$

$$\eta = x - \hat{x} - w(\theta - \theta^0) \quad (7.3d)$$

$$\dot{w} = g(x, u) - k_w w, \quad w(t_0) = 0 \quad (7.3e)$$

$$\dot{\eta} = -k_w \eta, \quad \eta(t_0) = e(t_0). \quad (7.3f)$$

The vector \hat{x} is the adaptive predictor for (7.1), the constant $k_w > 0$ is the filter gain, θ^0 is the nominal initial estimate of θ , w is a first order filter, η is an auxiliary variable defined to provide a direct relationship between the parameter estimation error $\tilde{\theta}$ and

¹A continuous function $\mu : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is of class \mathcal{K}_∞ if $\mu(0) = 0$, $\mu(\cdot)$ is strictly increasing on \mathbb{R}^+ and is radially unbounded.

the prediction error $x - \hat{x}$. As shown in Section 6.2, the parameter estimation error $\|\tilde{\theta}\|$ is non-increasing for $t \leq t_c$ and converges to zero exponentially for $t \geq t_c$ where t_c is the time at which the matrix $Q(t_c) = \int_{t_0}^{t_c} w^T(\tau)w(\tau) d\tau > 0$. This was achieved by defining a Lyapunov function

$$V_{\tilde{\theta}} = \frac{1}{2} \tilde{\theta}^T \tilde{\theta} \quad (7.4)$$

and using the fact that $C(t) = Q(t) \theta$, to show

$$\dot{V}_{\tilde{\theta}}(t) = -\tilde{\theta}^T(t) \Gamma Q(t) \tilde{\theta}(t) \leq -\mathcal{E}(t) V_{\tilde{\theta}}(t) \quad (7.5)$$

where

$$\mathcal{E}(t) = \lambda_{\min}(\Gamma Q(t)).$$

7.3.2 Set Adaptation

The uncertainty set $\Theta \triangleq B(\hat{\theta}, z_\theta)$ is updated online by updating the parameter estimate $\hat{\theta}$ and its associated error bound $z_\theta = \sup_{s \in \Theta} \|s - \hat{\theta}\|$. The vector $\hat{\theta}$ is updated via (7.2) while z_θ is updated based on the observed system's excitation contained in $\mathcal{E}(t)$ according to the following algorithm.

Algorithm 7.1 *Let $\mathcal{E}(\sigma) = \lambda_{\min}(\Gamma Q(\sigma))$, beginning from time $t_{i-1} = t_0$, the parameter and set adaptation is implemented iteratively as follows:*

1. **Initialize** $z_\theta(t_0) = z_\theta^0$, $\Theta(t_0) = B(\hat{\theta}(t_0), z_\theta(t_0))$, $\bar{\mathcal{E}} = \mathcal{E}(t_0) = 0$
2. **Implement** the following adaptation law over the interval $\tau \in [t_{i-1}, t_i)$

$$\dot{z}_\theta(\tau) = -\bar{\mathcal{E}} z_\theta(\tau) \quad (7.6)$$

3. At time t_i , **perform** the updates

$$\bar{\mathcal{E}} = \begin{cases} \mathcal{E}(t_i), & \text{if } \mathcal{E}(t_i) \geq \mathcal{E}(t_{i-1}) \\ \mathcal{E}(t_{i-1}), & \text{otherwise} \end{cases} \quad (7.7)$$

$$(\hat{\theta}, \Theta) = \begin{cases} (\hat{\theta}(t_i), \Theta(t_i)), & \text{if } z_\theta(t_i) \leq z_\theta(t_{i-1}) - \|\hat{\theta}(t_i) - \hat{\theta}(t_{i-1})\| \\ (\hat{\theta}(t_{i-1}), \Theta(t_{i-1})), & \text{otherwise} \end{cases} \quad (7.8)$$

4. **Iterate** back to step 2, **incrementing** $i = i + 1$.

The advantage of updating z_θ according to (7.6) is that contraction of z_θ can be triggered even when the actual parameter estimation error is zero. The uncertainty set Θ when implemented according to algorithm 7.1 contracts in a strictly nested fashion without excluding θ as shown in the following lemma.

Lemma 7.2 *The evolution of $\Theta \triangleq B(\hat{\theta}, z_\theta)$ under (7.2), (7.6) and Algorithm 7.1 is such that*

$$i) \quad \Theta(t_2) \subseteq \Theta(t_1), \quad t_0 \leq t_1 \leq t_2$$

$$ii) \quad \theta \in \Theta(t_0) \Rightarrow \theta \in \Theta(t), \quad \forall t \geq t_0$$

Proof:

i) If $\Theta(t_{i+1}) \not\subseteq \Theta(t_i)$, then

$$\sup_{s \in \Theta(t_{i+1})} \|s - \hat{\theta}(t_i)\| \geq z_\theta(t_i). \quad (7.9)$$

However, it follows from triangle inequality and algorithm 7.1 that Θ , at update times, obeys

$$\begin{aligned} \sup_{s \in \Theta(t_{i+1})} \|s - \hat{\theta}(t_i)\| &\leq \sup_{s \in \Theta(t_{i+1})} \|s - \hat{\theta}(t_{i+1})\| + \|\hat{\theta}(t_{i+1}) - \hat{\theta}(t_i)\| \\ &\leq z_\theta(t_{i+1}) + \|\hat{\theta}(t_{i+1}) - \hat{\theta}(t_i)\| \leq z_\theta(t_i), \end{aligned}$$

which contradicts (7.9). Hence, Θ update guarantees $\Theta(t_{i+1}) \subseteq \Theta(t_i)$ and the strict contraction claim follows from the fact that Θ is held constant over update intervals $\tau \in (t_i, t_{i+1})$.

ii) The θ inclusion claim is proven by showing that

$$\|\tilde{\theta}(t)\| \leq z_\theta(t), \quad \forall t \geq t_0 \quad (7.10)$$

which in turn establish that $\theta \in \Theta(t_0) \Rightarrow \theta \in B(\hat{\theta}(t), z_\theta(t))$, $\forall t \geq t_0$. We know that $\|\tilde{\theta}(t_0)\| \leq z_\theta(t_0)$ (by definition). It follows from (7.5) that

$$\begin{aligned} \dot{V}_{\tilde{\theta}}(\tau) &= -\tilde{\theta}^T(\tau) \Gamma \left(Q(t_i) + \int_{t_i}^{\tau} w(\sigma)^T w(\sigma) d\sigma \right) \tilde{\theta}(\tau), \quad \tau \in [t_i, t_{i+1}) \\ &\leq -\tilde{\theta}^T(\tau) \Gamma Q(t_i) \tilde{\theta}(\tau) \leq -\mathcal{E}(t_i) V_{\tilde{\theta}}(\tau). \end{aligned} \quad (7.11)$$

and using (7.4), (7.6) and (7.11) we have $\|\dot{\tilde{\theta}}(t)\| \leq \dot{z}_\theta(t)$. Hence, by comparison lemma ([63], Lemma 3.4) we have (7.10). ■

7.4 Robust Adaptive MPC - A Min-max Approach

In this section, the concept of min-max robust MPC is employed to provide robustness for the MPC controller during the adaptation phase. The resulting optimization

problem can either be solved in open-loop or closed-loop. In the approach proposed, we choose the least conservative option by performing optimization with respect to closed-loop strategies. As in typical feedback-MPC fashion, the controller chooses input u as a function of the current states. The formulation consists of maximizing a cost function with respect to θ and minimizing over feedback control policies κ .

The receding horizon control law is defined by

$$u = \kappa_{mpc}(x, \hat{\theta}, z_\theta) \triangleq \kappa^*(0, x, \hat{\theta}, z_\theta) \quad (7.12a)$$

$$\kappa^* \triangleq \arg \min_{\kappa(\tau, x^p, \hat{\theta}^p, z_\theta)} J(x, \hat{\theta}, z_\theta, \kappa) \quad (7.12b)$$

where $J(x, \hat{\theta}, z_\theta, \kappa)$ is the (worst-case) cost associated with the optimal control problem:

$$J(x, \hat{\theta}, z_\theta, \kappa) \triangleq \max_{\theta \in \Theta \triangleq B(\hat{\theta}, z_\theta)} \int_0^T L(x^p, u^p) d\tau + W(x^p(T), \tilde{\theta}^p(T)) \quad (7.13a)$$

$$s.t. \quad \forall \tau \in [0, T]$$

$$\dot{x}^p = f(x^p, u^p) + g(x^p, u^p) \theta, \quad x^p(0) = x \quad (7.13b)$$

$$\dot{w}^p = g^T(x^p, u^p) - kw^p, \quad w^p(0) = w \quad (7.13c)$$

$$\dot{Q}^p = w^{pT} w^p, \quad Q^p(0) = Q \quad (7.13d)$$

$$\dot{\hat{\theta}}^p = \Gamma Q^p \tilde{\theta}^p, \quad \tilde{\theta}^p = \theta - \hat{\theta}^p, \quad \hat{\theta}^p(0) = \hat{\theta} \quad (7.13e)$$

$$u^p(\tau) \triangleq \kappa(\tau, x^p(\tau), \hat{\theta}^p(\tau)) \in \mathbb{U} \quad (7.13f)$$

$$x^p(\tau) \in \mathbb{X}, \quad x^p(T) \in \mathbb{X}_f(\tilde{\theta}^p(T)) \quad (7.13g)$$

In the formulation, the effect of future parameter adaptation is accounted for, which results in less conservative worst-case predictions. Also, the conservativeness of the terminal cost is reduced by parameterizing both W and \mathbb{X}_f as functions of $\tilde{\theta}(T)$. Parameterizing the terminal penalty as a function of $\tilde{\theta}$ ensures that the algorithm will seek to reduce the parameter error in the process of optimizing the cost function J .

This may require the algorithm to automatically inject some useful excitation into the closed-loop system.

7.4.1 Implementation Algorithm

Algorithm 7.3 *The min-max MPC algorithm performs as follows: At sampling instant t_i*

1. **Measure** the current state of the plant x and obtain the current values of matrices w and Q from (7.3e) and (7.3a) respectively
2. **Update** the parameter estimates $\hat{\theta}$ and the uncertainty set $\Theta(t) \triangleq B(\hat{\theta}(t), z_\theta(t))$ using (7.2) and Algorithm 7.1.
3. **Solve** the optimization problem (7.12,7.13) and apply the resulting feedback control law to the plant until the next sampling instant
4. **Repeat** the procedure from step 1 for the next sampling instant, incrementing $i = i + 1$.

An alternative to the Θ update (7.8) is to define the uncertainty set

$$\Theta(t) \triangleq \bigcap_{\tau \in [t_0, t]} B(\hat{\theta}(\tau), z_\theta(\tau)), \quad (7.14)$$

and replace step 3 of Algorithm 7.3 with

2. Obtain the current value of parameter estimates $\hat{\theta}$ and uncertainty bound z_θ from equations (7.2) and (7.6). Then update the MPC quantities Θ and $\hat{\theta}$ as

$$\Theta = \Theta(t_i) \triangleq \Theta(t_{i-1}) \bigcap B(\hat{\theta}(t_i), z_\theta(t_i)) \quad (7.15a)$$

$$\hat{\theta} = \bar{\theta} \quad (7.15b)$$

where $\bar{\theta}$ is any point in Θ .

We note that the evolution of Θ when updated according to (7.15) satisfies the main requirement for the MPC performance. The set contracts in a strictly nested fashion without excluding θ . The set contraction follows by definition:

$$\Theta(t_2) = \Theta(t_1) \cap B\left(\hat{\theta}(t_2), z_\theta(t_1)\right) \subseteq \Theta(t_1), \quad \forall t_2 \geq t_1. \quad (7.16)$$

Moreover, since z_θ is such that $\|\tilde{\theta}(t)\| \leq z_\theta(t) \forall t \geq t_0$, the θ inclusion claim follows by noting that $\theta \in \Theta(t_0) \Rightarrow \theta \in B(\hat{\theta}(t), z_\theta(t)), \forall t \geq t_0$. Hence, $\theta \in \Theta(t_j) := \bigcap_{i=0}^j B\left(\hat{\theta}(t_i), z_\theta(t_i)\right)$. The benefit of using (7.15) is that the size of the uncertainty description Θ reduces faster over time but this is achieved at the expense of increased online computation due to the additional task of calculating the intersection of sets.

In the remainder of this section, we drop the explicit constraint (7.13g) by using the convention that if some of the constraints are not satisfied, then the value of J is $+\infty$ *i.e.*

$$L(x, u) = \begin{cases} L(x, u) < \infty & \text{if } (x, u) \in \mathbb{X} \times \mathbb{U} \\ +\infty & \text{otherwise} \end{cases}$$

$$W(x, \tilde{\theta}) = \begin{cases} W(x, \tilde{\theta}) < \infty & \text{if } x \in \mathbb{X}_f(\tilde{\theta}) \\ +\infty & \text{otherwise} \end{cases}$$

7.4.2 Closed-loop Robust Stability

Robust stability is guaranteed if predicted state at terminal time belong to a robustly invariant set for all possible uncertainties. Let $\tilde{\Theta}^0 = \left\{ \tilde{\theta} : \|\tilde{\theta}\| \leq z_\theta^0 \right\}$, a sufficient conditions for the robust MPC (7.12) to guarantee stabilization of the origin is outlined below:

Criterion 7.4 *The terminal penalty function $W : \mathbb{X}_f \times \tilde{\Theta}^0 \rightarrow [0, +\infty]$ and the ter-*

minimal constraint function $\mathbb{X}_f : \tilde{\Theta}^0 \rightarrow \mathbb{X}$ are such that for each $(\theta, \hat{\theta}, \tilde{\theta}) \in (\Theta^0 \times \Theta^0 \times \tilde{\Theta}^0)$, there exists a feedback $k_f(., \hat{\theta}) : \mathbb{X}_f \rightarrow \mathbb{U}$ satisfying

1. $0 \in \mathbb{X}_f(\tilde{\theta}) \subseteq \mathbb{X}$, $\mathbb{X}_f(\tilde{\theta})$ closed
2. $k_f(x, \hat{\theta}) \in \mathbb{U}$, $\forall x \in \mathbb{X}_f(\tilde{\theta})$
3. $W(x, \tilde{\theta})$ is continuous with respect to $x \in \mathbb{R}^{n_x}$
4. $\forall x \in \mathbb{X}_f(\tilde{\theta})$, $\mathbb{X}_f(\tilde{\theta})$ is strongly positively invariant under $k_f(x, \hat{\theta})$ with respect to the differential inclusion $\dot{x} \in f(x, k_f(x, \hat{\theta})) + g(x, k_f(x, \hat{\theta}))\Theta$
5. $W(x(t + \delta), \tilde{\theta}(t)) - W(x(t), \tilde{\theta}(t)) \leq -\int_t^{t+\delta} L(x, k_f(x, \hat{\theta}))d\tau$, $\forall x \in \mathbb{X}_f(\tilde{\theta})$.

In addition to criterion (7.4), the $\tilde{\theta}$ dependence of W and \mathbb{X}_f is required to satisfy the following:

Criterion 7.5 For any $\tilde{\theta}_1, \tilde{\theta}_2 \in \tilde{\Theta}^0$ s.t. $\|\tilde{\theta}_2\| \leq \|\tilde{\theta}_1\|$,

1. $W(x, \tilde{\theta}_2) \leq W(x, \tilde{\theta}_1)$, $\forall x \in \mathbb{X}_f(\tilde{\theta}_1)$
2. $\mathbb{X}_f(\tilde{\theta}_2) \supseteq \mathbb{X}_f(\tilde{\theta}_1)$

Note that criterion (7.4) requires only the existence, not knowledge, of $k_f(x, \hat{\theta})$ and the stability condition requires the terminal penalty function $W(x, \tilde{\theta})$ to be a robust-CLF on the domain $\mathbb{X}_f(\tilde{\theta})$. Criterion (7.5) requires W to decrease and the domain \mathbb{X}_f to enlarge with decreased parametric uncertainty as expected.

Theorem 7.6 Let $X_0 \triangleq X_0(\Theta^0) \subseteq \mathbb{X}$ denote the set of initial states with uncertainty Θ^0 for which (7.12) has a solution. Assuming criteria 7.4 and 7.5 are satisfied, then the closed-loop system state x , given by (7.1,7.2,7.6,7.12), originating from any $x_0 \in X_0$ feasibly approaches the origin as $t \rightarrow +\infty$.

The proof of the theorem is given in Section 7.9.

7.5 Robust Adaptive MPC - A Lipschitz-based Approach

Due to the computational complexity associated with (feedback) min-max optimization problem for nonlinear systems, it is (sometimes) more practical to use more conservative but computationally efficient methods. Examples of such approaches include Lipschitz-based methods [22, 24] and those based on the concept of reachable sets [25].

In this section, we present a Lipschitz-based method whereby the nominal model rather than the unknown bounded system state is controlled, subject to conditions that ensure that given constraints are satisfied for all possible uncertainties. State prediction error bound is determined based on the Lipschitz continuity of the model. A knowledge of appropriate Lipschitz bounds for the x -dependence of the dynamics $f(x, u)$ and $g(x, u)$ are assumed as follows:

Assumption 7.7 *A set of functions $\mathcal{L}_j : \mathbb{X} \times \mathbb{U} \rightarrow \mathbb{R}^+$, $j \in \{f, g\}$ are known which satisfy*

$$\mathcal{L}_j(\mathbb{X}, u) \geq \min \left\{ \mathcal{L}_j \mid \sup_{x_1, x_2 \in \mathbb{X}} \left(\|j(x_1, u) - j(x_2, u)\| - \mathcal{L}_j \|x_1 - x_2\| \right) \leq 0 \right\},$$

where for $j \equiv g$ is interpreted as an induced norm since $g(x, u)$ is a matrix.

7.5.1 Prediction of State Error Bound

In order to consider the effect of the uncertainty $\tilde{\theta}$ in the controller synthesis, we have to compute a bound on the difference between the nominal state trajectory and the solution of the actual system. To this end, consider the actual system

$$\dot{x} = f(x, u) + g(x, u) \theta, \tag{7.18}$$

and the nominal model controlled by the same input u

$$\dot{x}^p = f(x^p, u) + g(x^p, u) \hat{\theta}, \quad (7.19)$$

it follows that

$$\begin{aligned} \|\dot{x} - \dot{x}^p\| &\leq \|f(x, u) - f(x^p, u)\| + \|g(x, u)\theta - g(x^p, u)\theta\| + \|g(x^p, u)\theta - g(x^p, u)\theta^p\| \\ &\leq \mathcal{L}_f \|x - x^p\| + \mathcal{L}_g \|\theta\| \|x - x^p\| + \|g(x^p, u)\| \|\theta - \hat{\theta}\|. \end{aligned}$$

Therefore, a worst-case deviation $z_x^p \geq \max_{\theta \in \Theta} \|x - x^p\|$ can be generated from

$$\dot{z}_x^p = (\mathcal{L}_f + \mathcal{L}_g \Pi) z_x^p + \|g(x^p, u)\| z_\theta, \quad z_x^p(t_0) = 0 \quad (7.20)$$

where $\Pi = z_\theta + \|\hat{\theta}\|$.

7.5.2 Lipschitz-based Finite Horizon optimal Control Problem

The model predictive feedback is defined as

$$u = \kappa_{mpc}(x, \hat{\theta}, z_\theta) = u^*(0) \quad (7.21a)$$

$$u^*(\cdot) \triangleq \arg \min_{u_{[0,T]}^p} J(x, \hat{\theta}, z_\theta, u^p) \quad (7.21b)$$

where $J(x, \hat{\theta}, z_\theta, u^p)$ is given by the optimal control problem:

$$\begin{aligned} J(x, \hat{\theta}, z_\theta, u^p) &= \int_0^T L(x^p, u^p) d\tau + W(x^p(T), z_\theta^p(T)) \\ &s.t. \quad \forall \tau \in [0, T] \end{aligned} \quad (7.22a)$$

$$\dot{x}^p = f(x^p, u^p) + g(x^p, u^p) \hat{\theta}, \quad x^p(0) = x \quad (7.22b)$$

$$\dot{z}_x^p = (\mathcal{L}_f + \mathcal{L}_g \Pi) z_x^p + \|g(x^p, u^p)\| z_\theta, \quad z_x^p(0) = 0 \quad (7.22c)$$

$$X^p(\tau) \triangleq B(x^p(\tau), z_x^p(\tau)) \subseteq \mathbb{X}, \quad u^p(\tau) \in \mathbb{U} \quad (7.22d)$$

$$X^p(T) \subseteq \mathbb{X}_f(z_\theta^p(T)) \quad (7.22e)$$

In the proposed formulation, the parameter estimate $\hat{\theta}$ and the uncertainty radius z_θ which appears in (7.22b) and (7.22c) are updated at every sampling instant and held constant over the prediction horizon. However, the effect of the future model improvement along the prediction horizon is incorporated in the formulation by parameterizing the terminal expressions in (7.22a) and (7.22e) as a function of $z_\theta(T)$. This enlarges the terminal domain and hence reduces the conservatism of the robust MPC. Using a nominal model prediction, it is impossible to predict the actual future behavior of the parameter estimation error as was possible in the min-max framework. However, based upon the excitation of the real system at sampling instants t_i , one can generate an upper bound on the future parameter estimation error according to Algorithm 7.1, equation (7.6), that is

$$z_\theta^p(\tau) = \exp^{-\bar{\mathcal{E}}(\tau-t_i)} z_\theta(t_i) \quad \tau \in [t_i, t_i + T) \quad (7.23)$$

where

$$\bar{\mathcal{E}} \geq \mathcal{E}(t_i) = \lambda_{\min}(\Gamma Q(t_i))$$

7.5.3 Implementation Algorithm

Algorithm 7.8 *The Lipschitz-based MPC algorithm is implemented as follows: At sampling instant t_i*

1. **Measure** the current state of the plant $x = x(t_i)$
2. **Update** the parameter estimates $\hat{\theta} = \hat{\theta}(t_i)$ and uncertainty bounds $z_\theta = z_\theta(t_i)$ and $z_\theta^p(T) = z_\theta^p(t_i + T)$ via (7.2), (7.6) and (7.23) respectively.

3. **Solve** the optimization problem (7.21,7.22) and apply the resulting feedback control law to the plant until the next sampling instant
4. **Repeat** the procedure from step 1 for the next sampling instant, incrementing $i = i + 1$.

The conservatism of the Lipschitz-based approach is mainly due to the computation of the uncertainty cone $B(x^p, z_x^p)$ around the nominal trajectory. The rate at which the cone expands over the prediction horizon reduces at each sampling instant as z_θ reduces. When z_θ is zero, the effect of parameter uncertainty on the state prediction can be totally eliminated from the adaptive framework by replacing the error dynamic (7.22c) with $\dot{z}_x^p = 0$ when $z_\theta \approx 0$.

Theorem 7.9 *Let $X'_0 \triangleq X'_0(\Theta^0) \subseteq \mathbb{X}$ denote the set of initial states for which (7.21) has a solution. Assuming Assumption 7.7 and criteria 7.4 and 7.5 are satisfied, then the origin of the closed-loop system given by (7.1,7.2,7.6,7.21) is feasibly asymptotically stabilized from any $x_0 \in X'_0$.*

The proof can be found in Section 7.9.

7.6 Incorporating Finite-time Identifier

The performance and computational demand of the adaptive MPC schemes developed depend on the performance of the parameter and set adaptation mechanism employed. An identification mechanism that provides faster convergence of $\tilde{\theta}$ to zero (in a known time) is beneficial. In this section, we employ the finite-time identifier, presented in Chapter 5, in developing an adaptive predictive control structure that reduces to a nominal MPC problem when exact parameter estimates are obtained.

Let the parameter estimate $\hat{\theta}$, matrices Q and C be generated from (7.2), (7.3a) and (7.3b) respectively. Also, let t_c be a time such that $Q(t_c)$ is invertible, the finite-time identifier (FTI) is given by

$$\hat{\theta}^c(t) = \begin{cases} \hat{\theta}(t), & \text{if } t < t_c \\ Q(t_c)^{-1} C(t_c), & \text{if } t \geq t_c. \end{cases} \quad (7.24)$$

The revised algorithm based on the FTI is given in the following.

7.6.1 FTI-based Min-max Approach

Let the filter (7.13c) and excitation dynamics (7.13d) be replaced by

$$\dot{w}^p = \beta (g^T(x^p, u^p) - kw^p), \quad w^p(0) = w \quad (7.25)$$

$$\dot{Q}^p = \beta(w^{pT} w^p), \quad Q^p(0) = Q \quad (7.26)$$

with $\beta \in \{0, 1\}$ a design parameter. The proposed FTI based algorithm is as follows:

Algorithm 7.10 *Finite-time min-max MPC algorithm: At sampling instant t_i*

1. *Measure the current state of the plant x*
2. **Obtain** *the current value of matrices Q and C from (7.3a) and (7.3b) respectively*
3. **If** *$\det(Q) = 0$ or $\text{cond}(Q)$ is not satisfactory update the parameter estimates $\hat{\theta}$ and the uncertainty set $\Theta(t) \triangleq B(\hat{\theta}(t), z_\theta(t))$ according to Algorithm 7.1*
Else if *$\det(Q) > 0$ and $\text{cond}(Q)$ is satisfactory, set $\beta = 0$ and update*

$$\hat{\theta} = Q^{-1}(t_i)C(t_i), \quad z_\theta = 0$$

End

4. Solve the optimization problem (7.12,7.13) and apply the resulting feedback control law to the plant until the next sampling instant
5. **Increment** $i = i + 1$. **If** $z_\theta > 0$, repeat the procedure from step 1 for the next sampling instant. **Otherwise**, repeat only steps 1 and 4 for the next sampling instant.

Implementing the adaptive MPC controller according to Algorithm 7.10 guarantees that the uncertainty ball $\Theta \triangleq B(\hat{\theta}, z_\theta)$ is contained in the previous one, that is, $\Theta(t_i) \subseteq \Theta(t_{i-1})$. Hence, a successive reduction in the computational requirement of (7.12) is ensured. Moreover, when the parameter estimate θ^c becomes available, the uncertainty set Θ reduces to a single point with $\tilde{\theta} = 0$ and the predictive robust control structure becomes that of a nominal MPC:

$$u = \kappa_{mpc}(x) \triangleq \kappa^*(0, x) \quad (7.27a)$$

$$\kappa^* \triangleq \arg \min_{\kappa(\cdot, \cdot)} J(x, \kappa) \triangleq \int_0^T L(x^p, u^p) d\tau + W(x^p(T)) \quad (7.27b)$$

$$s.t. \quad \forall \tau \in [0, T]$$

$$\dot{x}^p = f(x^p, u^p) + g(x^p, u^p)\theta^c, \quad x^p(0) = x \quad (7.27c)$$

$$u^p(\tau) \triangleq \kappa(\tau, x^p(\tau)) \in \mathbb{U}, \quad x^p(\tau) \in \mathbb{X}, \quad x^p(T) \in \mathbb{X}_f \quad (7.27d)$$

7.6.2 FTI-based Lipschitz-bound Approach

For the Lipschitz based approach, the error bound dynamic (7.22c) is replaced by

$$\dot{z}_x^p = \beta(\mathcal{L}_f + \mathcal{L}_g\Pi)z_x^p + \|g^p\|z_\theta, \quad z_x^p(0) = 0, \quad (7.28)$$

with $\beta \in \{0, 1\}$ and the controller is implemented according to the following algorithm.

Algorithm 7.11 *Finite-time Lipschitz-based MPC algorithm:* At sampling instant t_i

1. **Measure** the current state of the plant x
2. **Obtain** the current value of matrices Q and C from (7.3a) and (7.3b) respectively
3. **If** $\det(Q) = 0$ or $\text{cond}(Q)$ is not satisfactory, set $\beta = 1$ and update the parameter estimates $\hat{\theta} = \hat{\theta}(t_i)$ and uncertainty bounds $z_\theta = z_\theta(t_i)$ and $z_\theta^p(T) = z_\theta^p(t_i + T)$ via (7.2), (7.6) and (7.23) respectively.

Else if $\det(Q) > 0$ and $\text{cond}(Q)$ is satisfactory, set $\beta = 0$ and update

$$\hat{\theta} = Q^{-1}(t_i)C(t_i), \quad z_\theta = 0$$

End

4. **Solve** the optimization problem (7.21,7.22) and apply the resulting feedback control law to the plant until the next sampling instant
5. **Increment** $i = i + 1$. **If** $z_\theta > 0$, repeat the procedure from step 1 for the next sampling instant. **Otherwise**, repeat only steps 1 and 4 for the next sampling instant.

Implementing Algorithm (7.11) ensures that the size of the uncertainty cone around the nominal state trajectory reduces as z_θ shrinks and when exact parameter estimate vector θ^c is obtained, $z_x^p = 0$, which implies that the problem becomes that of a nominal MPC (7.27).

7.7 Simulation Example

Consider the regulation of a continuous stirred tank reactor where a first order, irreversible exothermic reaction $\mathbf{A} \rightarrow \mathbf{B}$ is carried out. Assuming constant liquid level,

the reaction is described by the following dynamic model [94]:

$$\begin{aligned}\dot{C}_A &= \frac{q}{V} (C_{Ain} - C_A) - k_0 \exp\left(\frac{-E}{RT_r}\right) C_A \\ \dot{T}_r &= \frac{q}{V} (T_{in} - T_r) - \frac{\Delta H}{\rho c_p} k_0 \exp\left(\frac{-E}{RT_r}\right) C_A + \frac{UA}{\rho c_p V} (T_c - T_r)\end{aligned}$$

The states C_A and T_r are the concentrations of components A and the reactor temperature respectively. The manipulated variable T_c is temperature of the coolant stream.

It is assumed that reaction kinetic constant k_0 and heat of reaction ΔH are only nominally known and parameterized as $k_0 = \theta_1 \times 10^{10} \text{ min}^{-1}$ and $\Delta H k_0 = -\theta_2 \times 10^{15} \text{ J/mol min}$ with the parameters satisfying $0.1 \leq \theta_1 \leq 10$ and $0.1 \leq \theta_2 \leq 10$. The objective is to adaptively regulate the unstable equilibrium $C_A^{eq} = 0.5 \text{ mol/l}$, $T_r^{eq} = 350 \text{ K}$, $T_c^{eq} = 300 \text{ K}$ while satisfying the constraints $0 \leq C_A \leq 1$, $280 \leq T_r \leq 370$ and $280 \leq T_c \leq 370$. The nominal operating conditions, which corresponds to the given unstable equilibrium are taken from [94]: $q = 100 \text{ l/min}$, $V = 100 \text{ l}$, $\rho = 1000 \text{ g/l}$, $c_p = 0.239 \text{ J/g K}$, $E/R = 8750 \text{ K}$, $UA = 5 \times 10^4 \text{ J/min K}$, $C_{Ain} = 1 \text{ mol/l}$ and $T_{in} = 350 \text{ K}$.

Defining $x = [\frac{C_A - C_A^{eq}}{0.5}, \frac{T_r - T_r^{eq}}{20}]'$, $u = \frac{T_c - T_c^{eq}}{20}$, the stage cost $L(x, u)$ was selected as a quadratic function of its arguments:

$$L(x, u) = x^T Q_x x + u^T R_u u \quad (7.29a)$$

$$Q_x = \begin{bmatrix} 0.5 & 0 \\ 0 & 1.1429 \end{bmatrix} \quad R_u = 1.333. \quad (7.29b)$$

The terminal penalty function used is a quadratic parameter-dependent Lyapunov function $W(x, \theta) = x^T P(\theta) x$ for the linearized system. Denoting the closed-loop system under a local robust stabilizing controller $u = k_f(\theta) x$ as $\dot{x} = A_{cl}(\theta) x$. The matrix $P(\theta) := P_0 + \theta_1 P_1 + \theta_2 P_2 + \dots + \theta_{n\theta} P_{n\theta}$ was selected to satisfy the Lyapunov

system of LMIs

$$\begin{aligned}
 P(\theta) &> 0 \\
 A_{cl}(\theta)^T P(\theta) + P(\theta) A_{cl}(\theta) &< 0
 \end{aligned}$$

for all admissible values of θ . Since θ lie between known extrema values, the task of finding $P(\theta)$ reduces to solving a finite set of linear matrix inequalities by introducing additional constraints [95]. For the initial nominal estimate $\theta^0 = 5.05$ and $z_\theta^0 = 4.95$, the matrix $P(\theta^0)$ obtained is

$$P(\theta^0) = \begin{bmatrix} 0.6089 & 0.1134 \\ 0.1134 & 4.9122 \end{bmatrix} \quad (7.30)$$

and the corresponding terminal region is

$$\mathbb{X}_f = \{x : x^T P(\theta^0) x \leq 0.25\}. \quad (7.31)$$

For simulation purposes, the true values of the unknown parameters were chosen as $k_0 = 7.2 \times 10^{10} \text{min}^{-1}$ and $\Delta H = -5.0 \times 10^4 \text{ J/mol}$ which implies $\theta_1 = 7.2$ and $\theta_2 = 3.6$. The Lipschitz-based approach was used for the controller calculations and the result was implemented according to Algorithm 7.8. Since the regressor matrix for this reactor model is diagonal, we define uncertainty bound z_θ for each parameter estimate and adapt the pairs $(\hat{\theta}_1, z_{\theta_1})$ and $(\hat{\theta}_2, z_{\theta_2})$ separately.

The system was simulated from three different initial states $(C_A(0), T_r(0)) = (0.3, 335)$, $(0.6, 335)$ and $(0.3, 363)$. The closed-loop trajectories are reported in Figures 7.1 to 7.4. The results demonstrate that the adaptive MPC regulates the system states to the open loop unstable equilibrium values and satisfies the imposed state and input constraints. The parameter estimates converge to the true values and the uncertainty bound z_θ reduces over time.

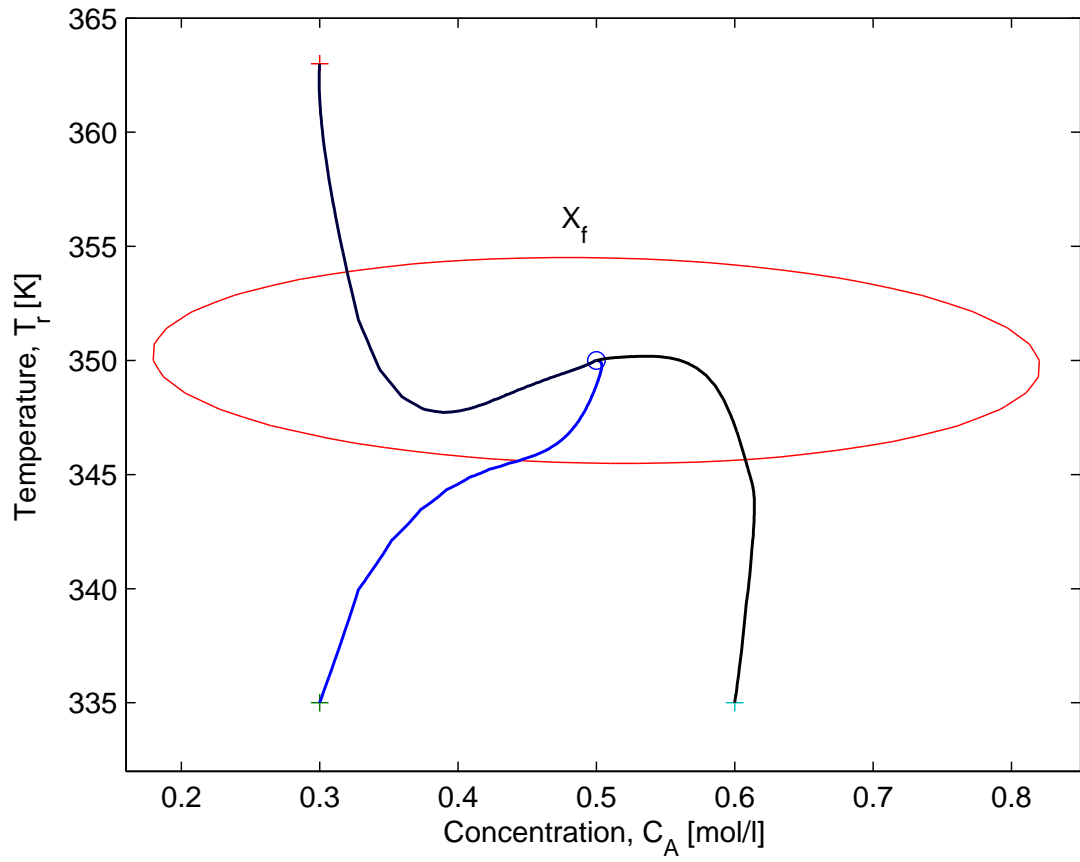


Figure 7.1: Closed-loop reactor state trajectories

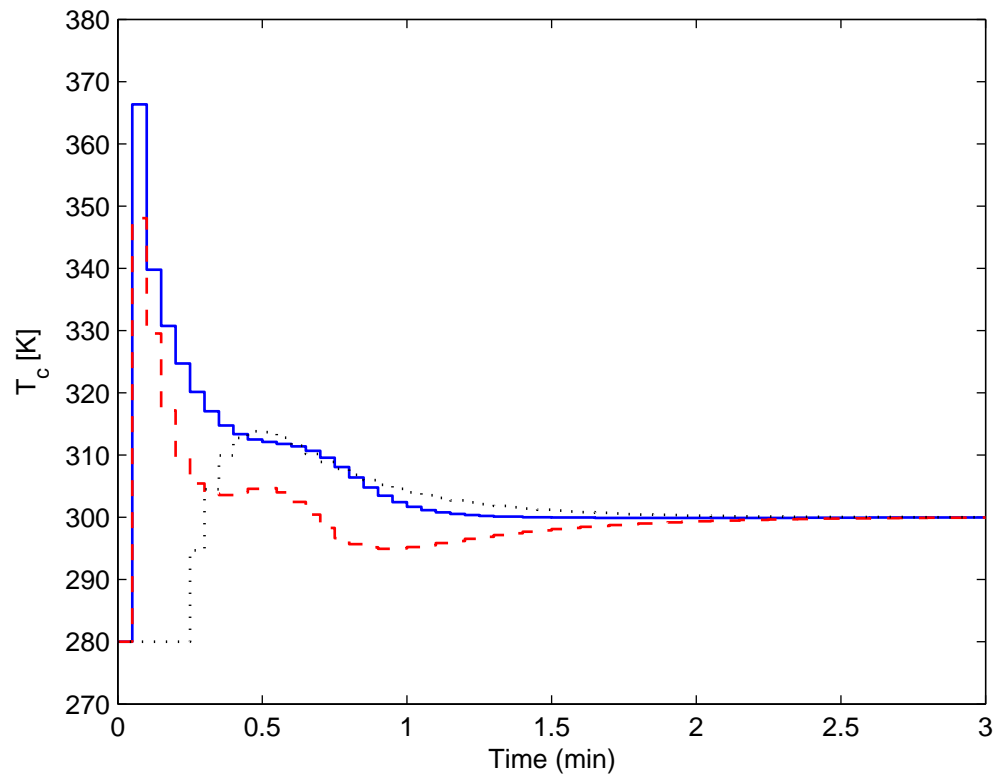


Figure 7.2: Closed-loop reactor input profiles for states starting at different initial conditions $(C_A(0), T_r(0))$: $(0.3, 335)$ is solid line, $(0.6, 335)$ is dashed line and $(0.3, 363)$ is the dotted line

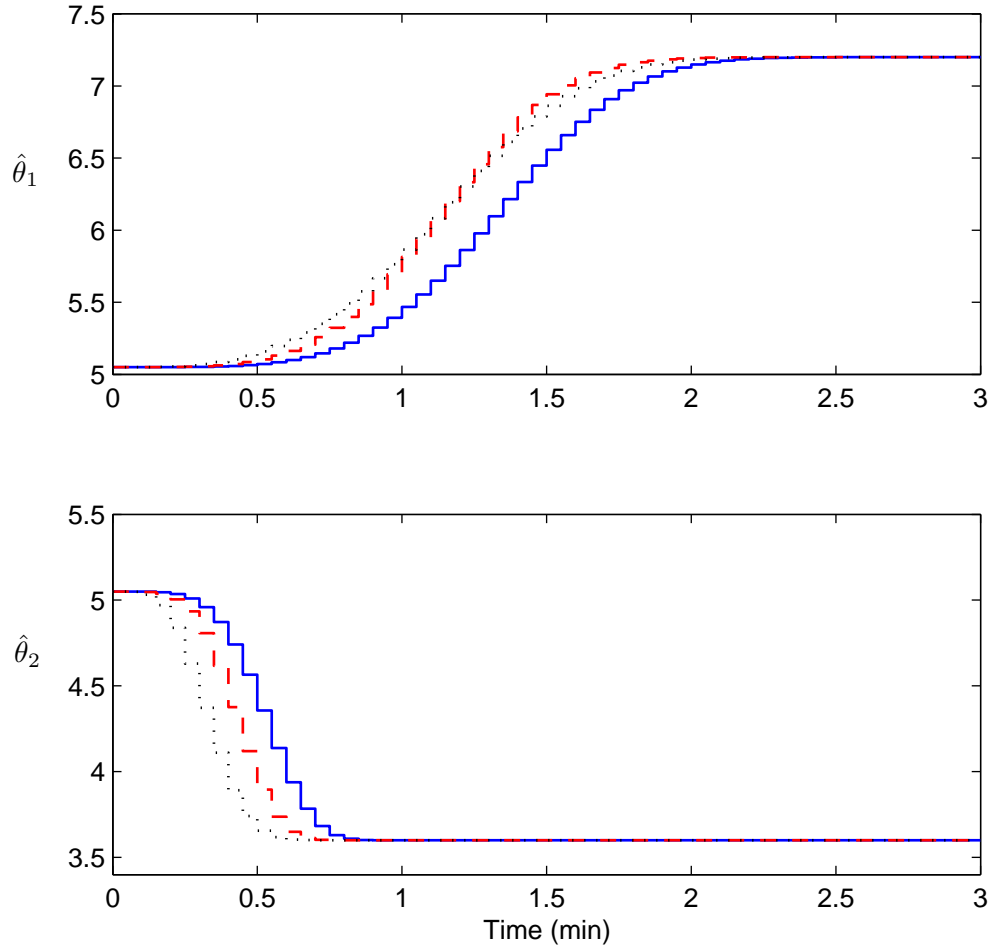


Figure 7.3: Closed-loop parameter estimates profile for states starting at different initial conditions $(C_A(0), T_r(0))$: (0.3, 335) is solid line, (0.6, 335) is dashed line and (0.3, 363) is the dotted line

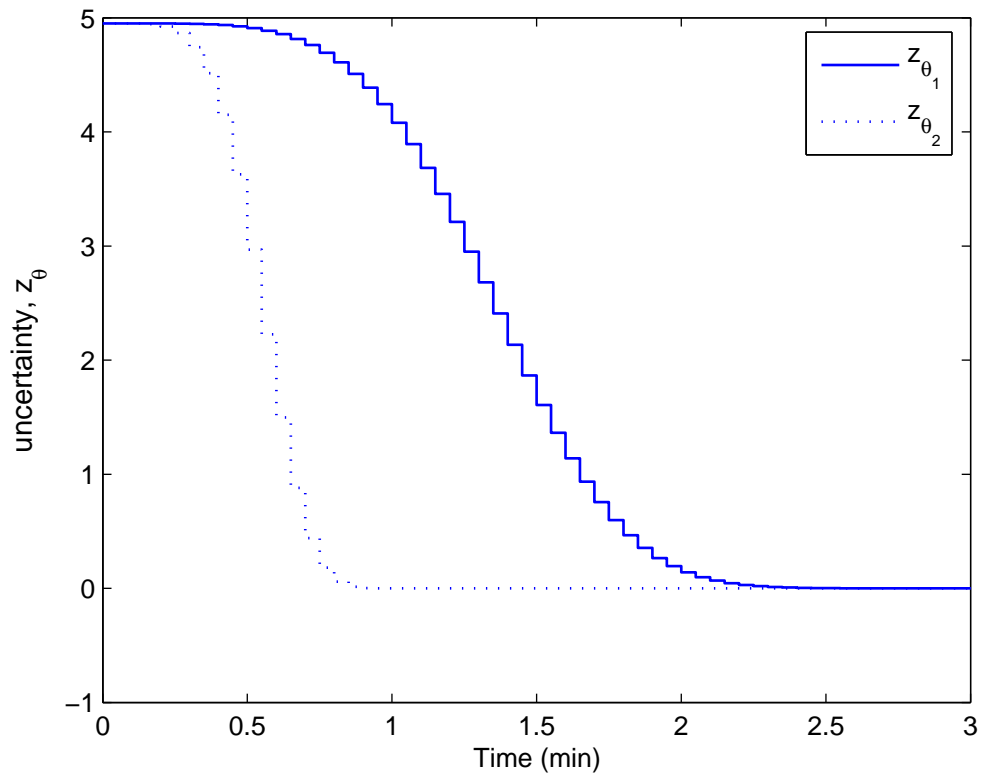


Figure 7.4: Closed-loop uncertainty bound trajectories for initial condition $(C_A, T_r) = (0.3, 335)$

7.8 Conclusions

In this chapter, we presented an adaptive MPC design technique for constrained nonlinear systems with parametric uncertainties. The system's performance is improved over time as the adaptive control updates the model online. The controller parameters is updated only when an improved parameter estimate is obtained. Robustly stabilizing MPC schemes are incorporated to ensure robustness of the algorithm to parameter estimation error during the adaptation phase. The two robust approaches, min-max and Lipschitz-based method, presented provides a tradeoff between computational complexity and conservatism of the solutions. In both cases, the controller is designed in such a way that the computational requirement/conservativeness of the robust adaptive MPC reduces with reduction in parameter uncertainty. Moreover, the complexity of the resultant controller reduces to that of nominal model predictive control when a finite-time identifier is employed and an excitation condition is satisfied.

7.9 Proofs of Main Results

7.9.1 Proof of Theorem 7.6

Feasibility: The closed-loop stability is based upon the feasibility of the control action at each sample time. Assuming, at time t , that an optimal solution $u_{[0,T]}^p$ to the optimization problem (7.12) exist and is found. Let Θ^p denote the estimated uncertainty set at time t and Θ^v denote the set at time $t + \delta$ that would result with the feedback implementation of $u_{[t,t+\delta]} = u_{[0,\delta]}^p$. Also, let x^p represents the worst case state trajectory originating from $x^p(0) = x(t)$ and x^v represents the trajectory originating from $x^v(0) = x + \delta v$ under the same feasible control input $u_{[\delta,T]}^v = u_{[\delta,T]}^p$. Moreover, let $X_{\Theta^b}^a \triangleq \{x^a \mid \dot{x}^a \in \mathcal{F}(x^a, u^p, \Theta^b) \triangleq f(x^a, u^p) + g(x^a, u^p)\Theta^b\}$.

Since the $u_{[0,T]}^p$ is optimal with respect to the worst case uncertainty scenario, it suffice to say that $u_{[0,T]}^p$ drives any trajectory $x^p \in X_{\Theta^p}^p$ into the terminal region \mathbb{X}_f^p . Since Θ is non-expanding over time, we have $\Theta^v \subseteq \Theta^p$ implying $x^v \in X_{\Theta^v}^p \subseteq X_{\Theta^p}^p$. The terminal region \mathbb{X}_f^p is strongly positively invariant for the nonlinear system (7.1) under the feedback $k_f(.,.)$, the input constraint is satisfied in \mathbb{X}_f^p and $\mathbb{X}_f^v \supseteq \mathbb{X}_f^p$ by criteria 2.2, 2.4 and 3.2 respectively. Hence, the input $u = [u_{[\delta,T]}^p, k_{f[T,T+\delta]}]$ is a feasible solution of (7.12) at time $t + \delta$ and by induction, the optimization problem is feasible for all $t \geq 0$.

Stability: The stability of the closed-loop system is established by proving strict decrease of the optimal cost $J^*(x, \hat{\theta}, z_\theta) \triangleq J(x, \hat{\theta}, z_\theta, \kappa^*)$. Let the trajectories $(x^p, \hat{\theta}^p, \tilde{\theta}^p, z_\theta^p)$ and control u^p correspond to any worst case minimizing solution of $J^*(x, \hat{\theta}, z_\theta)$. If $x_{[0,T]}^p$ were extended to $\tau \in [0, T + \delta]$ by implementing the feedback $u(\tau) = k_f(x^p(\tau), \hat{\theta}^p(\tau))$ on $\tau \in [T, T + \delta]$, then criterion 7.4(5) guarantees the inequality

$$\int_T^{T+\delta} L(x^p, k_f(x^p, \hat{\theta}^p)) d\tau + W(x_{T+\delta}^p, \tilde{\theta}_T^p) - W(x_T^p, \tilde{\theta}_T^p) \leq 0 \quad (7.32)$$

where in (7.32) and in the remainder of the proof, $x_\sigma^p \triangleq x^p(\sigma)$, $\tilde{\theta}_\sigma^p \triangleq \tilde{\theta}^p(\sigma)$, for $\sigma = T, T + \delta$.

The optimal cost $J^*(x, \hat{\theta}, z_\theta)$

$$\begin{aligned} &= \int_0^T L(x^p, u^p) d\tau + W(x_T^p, \tilde{\theta}_T^p) \\ &\geq \int_0^T L(x^p, u^p) d\tau + W(x_T^p, \tilde{\theta}_T^p) + \int_T^{T+\delta} L(x^p, k_f(x^p, \hat{\theta}^p)) d\tau + W(x_{T+\delta}^p, \tilde{\theta}_T^p) - W(x_T^p, \tilde{\theta}_T^p) \end{aligned} \quad (7.33)$$

$$\geq \int_0^\delta L(x^p, u^p) d\tau + \int_\delta^T L(x^p, u^p) d\tau + \int_T^{T+\delta} L(x^p, k_f(x^p, \hat{\theta}^p)) d\tau + W(x_{T+\delta}^p, \tilde{\theta}_{T+\delta}^p) \quad (7.34)$$

$$\geq \int_0^\delta L(x^p, u^p) d\tau + J^*(x(\delta), \hat{\theta}(\delta), z_\theta(\delta)) \quad (7.35)$$

Then, it follows from (7.35) that

$$\begin{aligned} J^*(x(\delta), \hat{\theta}(\delta), z_\theta(\delta)) - J^*(x, \hat{\theta}, z_\theta) &\leq - \int_0^\delta L(x^p, u^p) d\tau \\ &\leq - \int_0^\delta \mu_L(\|x, u\|) d\tau. \end{aligned} \quad (7.36)$$

where μ_L is a class \mathcal{K}_∞ function. Hence $x(t) \rightarrow 0$ asymptotically.

Remark 7.12 *In the above proof,*

- (7.33) is obtained using inequality (7.32)
- (7.34) follows from criterion 7.5.1 and the fact that $\|\tilde{\theta}\|$ is non-increasing
- (7.35) follows by noting that the last 3 terms in (7.34) is a (potentially) suboptimal cost on the interval $[\delta, T + \delta]$ starting from the point $(x^p(\delta), \hat{\theta}^p(\delta))$ with associated uncertainty set $B(\hat{\theta}^p(\delta), z_\theta^p(\delta))$.

7.9.2 Proof of Theorem 7.9

Feasibility: Let $u_{[0,T]}^p$ denotes the initial optimal or feasible solution of (7.21) and let $X^p \triangleq B(x^p, z_x^p)$ denotes the corresponding predicted ball of possible trajectories that starts at $(x^p, z_x^p, z_\theta^p)|_{\tau=0} = (x(t), 0, z_\theta(t))$. Similarly, let $X^v \triangleq B(x^v, z_x^v)$ denote the resulting cone originating from $(x^v, z_x^v, z_\theta^v)|_{\tau=\delta} = (x + \delta v, 0, z_\theta(t + \delta))$, under the same feasible control input $u_{[\delta,T]}^v = u_{[\delta,T]}^p$.

At time $\tau = 0$, it follows from (7.22b) and (7.22c) that z_x^p provides an upper bound on $\|\dot{x}^p - \dot{x}\|$, for $\dot{x} \in \mathcal{F}(x, u^p, \Theta)$. Thus the continuity of the trajectories $x^p(\tau)$ and $z_x^p(\tau)$ ensures that for a small enough $\delta > 0$, $x + \delta v \in X^p(\delta)$. Therefore, we assume, without loss of generality, that $x^v(\delta) \in X^p(\delta)$. Since $z_x^v(\delta) = 0$, it follows that $X^v(\delta) \subseteq X^p(\delta)$ and $\|x^p(\delta) - x^v(\delta)\| \leq z_x^p(\delta) - z_x^v(\delta)$.

Next, we establish that the inclusion $X^v(\tau) \subseteq X^p(\tau)$ holds for all $\tau \in [\delta, T]$ by showing that $\|x^p(\tau) - x^v(\tau)\| \leq z_x^p(\tau) - z_x^v(\tau)$, $\forall \tau \in [\delta, T]$. Defining the variable

$e_z \triangleq z_x^p - z_x^v - \|x^p - x^v\|$, the dynamics of e_z satisfy

$$\begin{aligned}
\dot{e}_z &= \dot{z}_x^p - \dot{z}_x^v - \|\dot{x}^p - \dot{x}^v\| \\
&= (\mathcal{L}_f + \mathcal{L}_g \Pi) z_x^p - (\mathcal{L}_f + \mathcal{L}_g \Pi) z_x^v - \|f(x^p, u) - f(x^v, u) + (g(x^p, u) - g(x^v, u)) \hat{\theta}\| \\
&\geq (\mathcal{L}_f + \mathcal{L}_g \Pi) (z_x^p - z_x^v) - \left(\mathcal{L}_f + \mathcal{L}_g \|\hat{\theta}\| \right) \|x^p - x^v\| \\
&\geq (\mathcal{L}_f + \mathcal{L}_g \Pi) e_z,
\end{aligned}$$

from which the initial condition $e_z(\delta) \geq 0$ guarantees that $e_z(\tau) \geq 0, \forall \tau \in [\delta, T]$. This implies that $X^v(\tau) \subseteq X^p(\tau) \subseteq \mathbb{X}$ and $X^v(T) \subseteq X^p(T) \subseteq \mathbb{X}_f^p$. Moreover, from the non-increase of the uncertainty bound z_θ , we have $z_\theta^v \leq z_\theta^p$ which implies that $\mathbb{X}_f^p \subseteq \mathbb{X}_f^v$. Therefore, the input $u = [u_{[\delta, T]}^p, k_{f[T, T+\delta]}]$ serves as a feasible solution for (7.21) at time $t + \delta$ and the feasibility result can be achieved by induction.

Stability: closed-loop stability is established by showing that the optimal value function is non-increasing. The proof is similar to that of theorem 7.6.

Chapter 8

Robust Adaptive MPC for Systems with Exogeneous Disturbances

8.1 Introduction

In general, modelling error consists of parametric and non-parametric uncertainties and the system dynamics can be influenced by exogeneous disturbances as well. In this chapter, we extend the adaptive MPC framework presented in Chapter 7 to nonlinear systems with both constant parametric uncertainty and additive exogenous disturbances.

Intuitively, an adaptive controller should lead to controller with better robustness properties than their non-adaptive counterpart since they use more information on the systems uncertainties. However, this is not generally the case. Under external disturbance input, adaptive controllers can lead to inferior transient behaviour, infinite parameter drift and burstiness in the closed-loop system. To address these problems, parameter projection [73] is used to ensure the estimate remains in a convex set and the parameter estimates are updated only when an improved estimate is obtained. The formulation provides robustness to parameter estimation error and

bounded disturbances $\vartheta \in \mathcal{D}$. While the disturbance set \mathcal{D} remains unchanged over time, the parametric uncertainty set Θ is adapted in such a way that guarantees its contraction.

8.2 Revised Problem Set-up

Consider the uncertain nonlinear system

$$\dot{x} = f(x, u) + g(x, u)\theta + \vartheta \triangleq \mathcal{F}(x, u, \theta, \vartheta) \quad (8.1)$$

where the disturbance $\vartheta \in \mathcal{D} \subset \mathbb{R}^{n_d}$ is assumed to satisfy a known upper bound $\|\vartheta(t)\| \leq M_\vartheta < \infty$. The objective of the study is to (robustly) stabilize the plant to some target set $\Xi \subset \mathbb{R}^{n_x}$ while satisfying the pointwise constraints $x \in \mathbb{X} \subset \mathbb{R}^{n_x}$ and $u \in \mathbb{U} \subset \mathbb{R}^{n_u}$. The target set is a compact set, contains the origin and is robustly invariant under no control.

8.3 Parameter and Uncertainty Set Estimation

8.3.1 Preamble

Consider the dynamical system (8.1) and assume we use the same adaptive compensator (7.2) and (7.3). Since ϑ is not known, the true η dynamic is

$$\dot{\eta}_\vartheta = -k_w \eta_\vartheta + \vartheta, \quad \eta_\vartheta(t_0) = e(t_0) \quad (8.2)$$

which results in the estimation error $\tilde{\eta} = \eta_\vartheta - \eta$ and dynamic

$$\dot{\tilde{\eta}} = -k_w \tilde{\eta} + \vartheta, \quad \tilde{\eta}(t_0) = 0. \quad (8.3)$$

Considering the Lyapunov function

$$V_{\tilde{\theta}} = \frac{1}{2} \tilde{\theta}^T \tilde{\theta} \quad (8.4)$$

Since $w\theta = w\theta^0 + x - \hat{x} - \eta + \tilde{\eta}$ in this case, we have

$$C(t) = Q(t)\theta + \int_{t_0}^t w^T(\sigma)\tilde{\eta}(\sigma) d\sigma \quad (8.5)$$

hence, it follows that

$$\dot{V}_{\tilde{\theta}}(t) = -\tilde{\theta}^T(t)\Gamma Q(t)\tilde{\theta}(t) - \tilde{\theta}^T(t)\Gamma \int_{t_0}^t w^T(\sigma)\tilde{\eta}(\sigma) d\sigma, \quad (8.6)$$

which guarantees boundedness of the parameter estimation error. To compute z_θ , the upper bound on the estimation error that must depend on measurable signal, we replace (8.6) with

$$\dot{V}_{\tilde{\theta}}(t) \leq -\mathcal{E}(t)V_{\tilde{\theta}}(t) + k_d \sqrt{V_{\tilde{\theta}}(t)} \int_{t_0}^t \|w^T(\sigma)\| d\sigma \quad (8.7)$$

where

$$\mathcal{E}(t) = \lambda_{\min}(\Gamma Q(t)) \quad \text{and} \quad k_d = \lambda_{\max}(\Gamma) \frac{M_\vartheta}{k_w}.$$

Though the adaptive compensator gives a stronger convergence result for systems subject to uncertainties, its usefulness in developing robust adaptive MPC for systems subject to disturbances is limited. Updating the uncertainty bound z_θ based on (8.7) would result in a very conservative design, mainly because of the integral in the positive term. To obtain a tighter parameter estimation error bound, we present an alternative update law that is based on the closed-loop system states and M_ϑ .

8.3.2 Parameter Adaptation

Let the estimator model for (8.1) be selected as

$$\dot{\hat{x}} = f(x, u) + g(x, u)\hat{\theta} + k_w e + w\dot{\hat{\theta}}, \quad k_w > 0 \quad (8.8)$$

$$\dot{w} = g(x, u) - k_w w, \quad w(t_0) = 0. \quad (8.9)$$

resulting in state prediction error $e = x - \hat{x}$ and auxiliary variable $\eta = e - w\tilde{\theta}$ dynamics:

$$\dot{e} = g(x, u)\tilde{\theta} - k_w e - w\dot{\hat{\theta}} + \vartheta \quad e(t_0) = x(t_0) - \hat{x}(t_0) \quad (8.10)$$

$$\dot{\eta} = -k_w \eta + \vartheta, \quad \eta(t_0) = e(t_0). \quad (8.11)$$

Since ϑ is not known, an estimate of η is generated from

$$\dot{\hat{\eta}} = -k_w \hat{\eta}, \quad \hat{\eta}(t_0) = e(t_0). \quad (8.12)$$

with resulting estimation error $\tilde{\eta} = \eta - \hat{\eta}$ dynamics

$$\dot{\tilde{\eta}} = -k_w \tilde{\eta} + \vartheta, \quad \tilde{\eta}(t_0) = 0. \quad (8.13)$$

Let $\Sigma \in \mathbb{R}^{n_\theta \times n_\theta}$ be generated from

$$\dot{\Sigma} = w^T w, \quad \Sigma(t_0) = \alpha I \succ 0, \quad (8.14)$$

based on equations (8.8), (8.9) and (8.12), our preferred parameter update law is given by

$$\dot{\Sigma}^{-1} = -\Sigma^{-1} w^T w \Sigma^{-1}, \quad \Sigma^{-1}(t_0) = \frac{1}{\alpha} I \quad (8.15a)$$

$$\dot{\hat{\theta}} = \text{Proj} \left\{ \gamma \Sigma^{-1} w^T (e - \hat{\eta}), \hat{\theta} \right\}, \quad \hat{\theta}(t_0) = \theta^0 \in \Theta^0 \quad (8.15b)$$

where $\gamma = \gamma^T > 0$ and $\text{Proj}\{\phi, \hat{\theta}\}$ denotes a Lipschitz projection operator such that

$$- \text{Proj}\{\phi, \hat{\theta}\}^T \tilde{\theta} \leq -\phi^T \tilde{\theta}, \quad (8.16)$$

$$\hat{\theta}(t_0) \in \Theta^0 \Rightarrow \hat{\theta}(t) \in \Theta_\epsilon^0, \quad \forall t \geq t_0. \quad (8.17)$$

where $\Theta_\epsilon^0 \triangleq B(\theta^0, z_\theta^0 + \epsilon)$, $\epsilon > 0$. More details on parameter projection can be found in [73].

Lemma 8.1 *The identifier (8.15) is such that the estimation error $\tilde{\theta} = \theta - \hat{\theta}$ is bounded. Moreover, if*

$$\vartheta \in \mathcal{L}_2 \quad \text{or} \quad \int_{t_0}^{\infty} \left[\|\tilde{\eta}\|^2 - \underline{\gamma} \|e - \hat{\eta}\|^2 \right] d\tau < +\infty \quad (8.18)$$

with $\underline{\gamma} = \lambda_{\min}(\gamma)$ and the strong condition

$$\lim_{t \rightarrow \infty} \lambda_{\min}(\Sigma) = \infty \quad (8.19)$$

is satisfied, then $\tilde{\theta}$ converges to zero asymptotically.

Proof: Let $V_{\tilde{\theta}} = \tilde{\theta}^T \Sigma \tilde{\theta}$, it follows from (8.15) and the relationship $w\tilde{\theta} = e - \hat{\eta} - \tilde{\eta}$ that

$$\begin{aligned} \dot{V}_{\tilde{\theta}} &\leq -2\underline{\gamma} \tilde{\theta}^T w^T (e - \hat{\eta}) + \tilde{\theta}^T w^T w \tilde{\theta} \\ &= -\underline{\gamma} (e - \hat{\eta})^T (e - \hat{\eta}) + \|\tilde{\eta}\|^2, \end{aligned} \quad (8.20)$$

implying that $\tilde{\theta}$ is bounded. Moreover, it follows from (8.20) that

$$V_{\tilde{\theta}}(t) = V_{\tilde{\theta}}(t_0) + \int_{t_0}^t \dot{V}_{\tilde{\theta}}(\tau) d\tau \quad (8.21)$$

$$\leq V_{\hat{\theta}}(t_0) - \underline{\gamma} \int_{t_0}^t \|e - \hat{\eta}\|^2 d\tau + \int_{t_0}^t \|\tilde{\eta}\|^2 d\tau \quad (8.22)$$

Considering the dynamics of (8.13), if $\vartheta \in \mathcal{L}_2$, then $\tilde{\eta} \in \mathcal{L}_2$ (Lemma 5.5). Hence, the right hand side of (8.22) is finite in view of (8.18), and by (8.19) we have

$$\lim_{t \rightarrow \infty} \tilde{\theta}(t) = 0 \quad \blacksquare$$

8.3.3 Set Adaptation

An update law that measures the worst-case progress of the parameter identifier in the presence of disturbance is given by:

$$z_\theta = \sqrt{\frac{V_{z\theta}}{\lambda_{\min}(\Sigma)}} \quad (8.23a)$$

$$V_{z\theta}(t_0) = \lambda_{\max}(\Sigma(t_0)) (z_\theta^0)^2 \quad (8.23b)$$

$$\dot{V}_{z\theta} = -\underline{\gamma} (e - \hat{\eta})^T (e - \hat{\eta}) + \left(\frac{M_\vartheta}{k_w}\right)^2. \quad (8.23c)$$

Using the parameter estimator (8.15) and its error bound z_θ (8.23), the uncertain ball $\Theta \triangleq B(\hat{\theta}, z_\theta)$ is adapted online according to the following algorithm:

Algorithm 8.2 *Beginning from time $t_{i-1} = t_0$, the parameter and set adaptation is implemented iteratively as follows:*

1 **Initialize** $z_\theta(t_{i-1}) = z_\theta^0$, $\hat{\theta}(t_{i-1}) = \hat{\theta}^0$ and $\Theta(t_{i-1}) = B(\hat{\theta}(t_{i-1}), z_\theta(t_{i-1}))$.

2 At time t_i , using equations (8.15) and (8.23) **perform** the update

$$(\hat{\theta}, \Theta) = \begin{cases} (\hat{\theta}(t_i), \Theta(t_i)), & \text{if } z_\theta(t_i) \leq z_\theta(t_{i-1}) - \|\hat{\theta}(t_i) - \hat{\theta}(t_{i-1})\| \\ (\hat{\theta}(t_{i-1}), \Theta(t_{i-1})), & \text{otherwise} \end{cases} \quad (8.24)$$

3 **Iterate back** to step 2, **incrementing** $i = i + 1$.

The algorithm ensure that Θ is only updated when z_θ value has decreased by an amount which guarantees a contraction of the set. Moreover z_θ evolution as given in (8.23) ensures non-exclusion of θ as shown below.

Lemma 8.3 *The evolution of $\Theta = B(\hat{\theta}, z_\theta)$ under (8.15), (8.23) and algorithm 8.2 is such that*

- i) $\Theta(t_2) \subseteq \Theta(t_1)$, $t_0 \leq t_1 \leq t_2$
- ii) $\theta \in \Theta(t_0) \Rightarrow \theta \in \Theta(t)$, $\forall t \geq t_0$

Proof:

- i) The proof of the first claim is the same as that of Lemma 7.2i)
- ii) We know that $V_{\hat{\theta}}(t_0) \leq V_{z_\theta}(t_0)$ (by definition) and it follows from (8.20) and (8.23c) that $\dot{V}_{\hat{\theta}}(t) \leq \dot{V}_{z_\theta}(t)$. Hence, by the comparison lemma, we have

$$V_{\hat{\theta}}(t) \leq V_{z_\theta}(t), \quad \forall t \geq t_0. \quad (8.25)$$

and since $V_{\hat{\theta}} = \tilde{\theta}^T \Sigma \tilde{\theta}$, it follows that

$$\|\tilde{\theta}(t)\|^2 \leq \frac{V_{z_\theta}(t)}{\lambda_{\min}(\Sigma(t))} = z_\theta^2(t), \quad \forall t \geq t_0. \quad (8.26)$$

Hence, if $\theta \in \Theta(t_0)$, then $\theta \in B(\hat{\theta}(t), z_\theta(t))$, $\forall t \geq t_0$.

■

8.4 Robust Adaptive MPC

8.4.1 A Min-max Approach

The formulation of the min-max MPC consists of maximizing a cost function with respect to $\theta \in \Theta$, $\vartheta \in \mathcal{D}$ and minimizing over feedback control policies κ . The robust

receding horizon control law is

$$u = \kappa_{mpc}(x, \hat{\theta}, z_\theta) \triangleq \kappa^*(0, x, \hat{\theta}, z_\theta) \quad (8.27a)$$

$$\kappa^* \triangleq \arg \min_{\kappa(\cdot, \cdot, \cdot)} J(x, \hat{\theta}, z_\theta, \kappa) \quad (8.27b)$$

where

$$J(x, \hat{\theta}, z_\theta, \kappa) \triangleq \max_{\theta \in \Theta, \vartheta \in \mathcal{D}} \int_0^T L(x^p, u^p) d\tau + W(x^p(T), \tilde{\theta}^p(T)) \quad (8.28a)$$

$$s.t. \quad \forall \tau \in [0, T]$$

$$\dot{x}^p = f(x^p, u^p) + g(x^p, u^p)\theta + \vartheta, \quad x^p(0) = x \quad (8.28b)$$

$$\dot{w}^p = g^T(x^p, u^p) - k_w w^p, \quad w^p(0) = w \quad (8.28c)$$

$$(\dot{\Sigma}^{-1})^p = -(\Sigma^{-1})^p w^T w (\Sigma^{-1})^p, \quad (\Sigma^{-1})^p(0) = \Sigma^{-1} \quad (8.28d)$$

$$\dot{\hat{\theta}}^p = \text{Proj} \left\{ \gamma (\Sigma^{-1})^p w^T (e - \hat{\eta}), \hat{\theta} \right\} \quad \tilde{\theta}^p = \theta - \hat{\theta}^p, \quad \hat{\theta}^p(0) = \hat{\theta} \quad (8.28e)$$

$$u^p(\tau) \triangleq \kappa(\tau, x^p(\tau), \hat{\theta}^p(\tau)) \in \mathbb{U} \quad (8.28f)$$

$$x^p(\tau) \in \mathbb{X}, \quad x^p(T) \in \mathbb{X}_f(\tilde{\theta}^p(T)) \quad (8.28g)$$

The effect of future parameter adaptation is also accounted for in this formulation. The conservativeness of the algorithm is reduced by parameterizing both W and \mathbb{X}_f as functions of $\tilde{\theta}(T)$. While it is possible for the set Θ to contract upon θ over time, the robustness feature due to $\vartheta \in \mathcal{D}$ will still remain.

Algorithm 8.4 *The MPC algorithm performs as follows: At sampling instant t_i*

1. **Measure** the current state of the plant $x(t)$ and obtain the current value of matrices w and Σ^{-1} from equations (8.9) and (8.15a) respectively
2. Obtain the current value of parameter estimates $\hat{\theta}$ and uncertainty bound z_θ from (8.15b) and (8.23) respectively

$$\mathbf{If} \quad z_\theta(t_i) \leq z_\theta(t_{i-1}) - \|\hat{\theta}(t_i) - \hat{\theta}(t_{i-1})\|$$

$$\hat{\theta} = \hat{\theta}(t_i), \quad z_\theta = z_\theta(t_i)$$

Else

$$\hat{\theta} = \hat{\theta}(t_{i-1}), \quad z_\theta = z_\theta(t_{i-1})$$

End

3. **Solve** the optimization problem (8.27) and apply the resulting feedback control law to the plant until the next sampling instant
4. **Increment** $i = i + 1$. Repeat the procedure from step 1 for the next sampling instant.

8.4.2 Lipschitz-based Approach

Assuming a knowledge of the Lipschitz bounds for the x -dependence of the dynamics $f(x, u)$ and $g(x, u)$ as given in Assumption 7.7 and let $\Pi = z_\theta + \|\hat{\theta}\|$, a worst-case deviation $z_x^p \geq \max_{\theta \in \Theta} \|x - x^p\|$ can be generated from

$$\dot{z}_x^p = (\mathcal{L}_f + \mathcal{L}_g \Pi) z_x^p + \|g(x^p, u)\| z_\theta + M_\theta, \quad z_x^p(t_0) = 0. \quad (8.29)$$

Using this error bound, the robust Lipschitz-based MPC is given by

$$u = \kappa_{mpc}(x, \hat{\theta}, z_\theta) = u^*(0) \quad (8.30a)$$

$$u^*(\cdot) \triangleq \arg \min_{u^p_{[0, T]}} J(x, \hat{\theta}, z_\theta, u^p) \quad (8.30b)$$

where

$$J(x, \hat{\theta}, z_\theta, u^p) = \int_0^T L(x^p, u^p) d\tau + W(x^p(T), z_\theta^p) \quad (8.31a)$$

$$s.t. \quad \forall \tau \in [0, T]$$

$$\dot{x}^p = f(x^p, u^p) + g(x^p, u^p) \hat{\theta}, \quad x^p(0) = x \quad (8.31b)$$

$$\dot{z}_x^p = (\mathcal{L}_f + \mathcal{L}_g \Pi) z_x^p + \|g^p\| z_\theta + M_\theta, \quad z_x^p(0) = 0 \quad (8.31c)$$

$$X^p(\tau) \triangleq B(x^p(\tau), z_x^p(\tau)) \subseteq \mathbb{X}, \quad u^p(\tau) \in \mathbb{U} \quad (8.31d)$$

$$X^p(T) \subseteq \mathbb{X}_f(z_\theta^p) \quad (8.31e)$$

The effect of the disturbance is built into the uncertainty cone $B(x^p(\tau), z_x^p(\tau))$ via (8.31c). Since the uncertainty bound is no more monotonically decreasing in this case, the uncertainty radius z_θ which appears in (8.31c) and in the terminal expressions of (8.31a) and (8.31e) are held constant over the prediction horizon. However, the fact that they are updated at sampling instants when z_θ shrinks reduces the conservatism of the robust MPC and enlarges the terminal domain that would otherwise have been designed based on a large initial uncertainty $z_\theta(t_0)$.

Algorithm 8.5 *The Lipschitz-based MPC algorithm performs as follows: At sampling instant t_i*

1. **Measure** the current state of the plant $x = x(t_i)$
2. **Obtain** the current value of the parameter estimates $\hat{\theta}$ and uncertainty bound z_θ from equations (8.15) and (8.23) respectively,

If $z_\theta(t_i) \leq z_\theta(t_{i-1})$

$$\hat{\theta} = \hat{\theta}(t_i), \quad z_\theta = z_\theta(t_i)$$

Else

$$\hat{\theta} = \hat{\theta}(t_{i-1}), \quad z_\theta = z_\theta(t_{i-1})$$

End

3. **Solve** the optimization problem (8.30) and apply the resulting feedback control law to the plant until the next sampling instant
4. **Increment** $i:=i+1$; repeat the procedure from step 1 for the next sampling instant.

8.5 Closed-loop Robust Stability

Robust stabilization to the target set Ξ is guaranteed by appropriate selection of the design parameters W and X_f . The robust stability conditions require the satisfaction of criteria 7.4 and 7.5, with criteria 7.4.4 strengthened to account for the effect of the disturbance $\vartheta \in \mathcal{D}$. The criteria are given below for ease of reference.

Criterion 8.6 *The terminal penalty function $W : \mathbb{X}_f \times \tilde{\Theta}^0 \rightarrow [0, +\infty]$ and the terminal constraint function $\mathbb{X}_f : \tilde{\Theta}^0 \rightarrow \mathbb{X}$ are such that for each $(\theta, \hat{\theta}, \tilde{\theta}) \in (\Theta^0 \times \Theta^0 \times \tilde{\Theta}^0)$, there exists a feedback $k_f(\cdot, \hat{\theta}) : \mathbb{X}_f \rightarrow \mathbb{U}$ satisfying*

1. $0 \in \mathbb{X}_f(\tilde{\theta}) \subseteq \mathbb{X}$, $\mathbb{X}_f(\tilde{\theta})$ closed
2. $k_f(x, \hat{\theta}) \in \mathbb{U}$, $\forall x \in \mathbb{X}_f(\tilde{\theta})$
3. $W(x, \tilde{\theta})$ is continuous with respect to $x \in \mathbb{R}^{n_x}$
4. $\forall x \in \mathbb{X}_f(\tilde{\theta}) \setminus \Xi$, $\mathbb{X}_f(\tilde{\theta})$ is strongly positively invariant under $k_f(x, \hat{\theta})$ with respect to $\dot{x} \in f(x, k_f(x, \hat{\theta})) + g(x, k_f(x, \hat{\theta}))\Theta + \mathcal{D}$
5. $L(x, k_f(x, \hat{\theta})) + \frac{\partial W}{\partial x} \mathcal{F}(x, k_f(x, \hat{\theta}), \theta, \vartheta) \leq 0$, $\forall x \in \mathbb{X}_f(\tilde{\theta}) \setminus \Xi$.

Criterion 8.7 *For any $\tilde{\theta}_1, \tilde{\theta}_2 \in \tilde{\Theta}^0$ s.t. $\|\tilde{\theta}_2\| \leq \|\tilde{\theta}_1\|$,*

1. $W(x, \tilde{\theta}_2) \leq W(x, \tilde{\theta}_1)$, $\forall x \in \mathbb{X}_f(\tilde{\theta}_1)$
2. $\mathbb{X}_f(\tilde{\theta}_2) \supseteq \mathbb{X}_f(\tilde{\theta}_1)$

The revised condition 8.6.5 require W to be a local robust CLF for the uncertain system 8.1 with respect to $\theta \in \Theta$ and $\vartheta \in \mathcal{D}$.

8.5.1 Main Results

Theorem 8.8 *Let $X_{d0} \triangleq X_{d0}(\Theta^0) \subseteq \mathbb{X}$ denote the set of initial states with uncertainty Θ^0 for which (8.27) has a solution. Assuming criteria 8.6 and 8.7 are satisfied, then the closed-loop system state x , given by (8.1,8.15,8.23,8.27), originating from any $x_0 \in X_{d0}$ feasibly approaches the target set Ξ as $t \rightarrow +\infty$.*

Proof: The closed-loop stability is established by the feasibility of the control action at each sample time and the strict decrease of the optimal cost J^* . The proof follows from that of theorem 7.6 since the control law is optimal with respect to the worst case uncertainty $(\theta, \vartheta) \in (\Theta, \mathcal{D})$ scenario and the terminal region \mathbb{X}_f^p is strongly positively invariant for (8.1) under the (local) feedback $k_f(\cdot, \cdot)$. ■

Theorem 8.9 *Let $X'_{d0} \triangleq X'_{d0}(\Theta^0) \subseteq \mathbb{X}$ denote the set of initial states for which (8.30) has a solution. Assuming Assumption 7.7 and Criteria 8.6 and 8.7 are satisfied, then the origin of the closed-loop system given by (8.1,8.15,8.23,8.30) is feasibly asymptotically stabilized from any $x_0 \in X'_{d0}$ to the target set Ξ .*

The proof of the Lipschitz-based control law follows from that of theorem 7.9.

8.6 Simulation Example

To illustrate the effectiveness of the proposed design, we consider the regulation of the CSTR in example 7.7, subject to an additional disturbance on the temperature dynamic:

$$\begin{aligned}\dot{C}_A &= \frac{q}{V} (C_{Ain} - C_A) - k_0 \exp\left(\frac{-E}{RT_r}\right) C_A \\ \dot{T}_r &= \frac{q}{V} (T_{in} - T_r) - \frac{\Delta H}{\rho c_p} k_0 \exp\left(\frac{-E}{RT_r}\right) C_A + \frac{UA}{\rho c_p V} (T_c - T_r) + \vartheta\end{aligned}$$

where $\vartheta(t)$ is an unknown function of time. We also assume that the reaction kinetic constant k_0 and ΔH are only nominally known. The operating conditions and system constraints are as detailed in Section 7.7. The control objective is to robustly regulate the reactor temperature and concentration to the (open loop) unstable equilibrium $C_A^{eq} = 0.5$ mol/l, $T_r^{eq} = 350$ K, $T_c^{eq} = 300$ K by manipulating the temperature of the coolant stream T_c .

For simulation purposes, the disturbance is selected as a fluctuation of the inlet temperature $\vartheta(t) = 0.01 T_{in} \sin(3t)$ and the true values of the unknown parameters were also chosen as $k_0 = 7.2 \times 10^{10} \text{min}^{-1}$ and $\Delta H = -5.0 \times 10^4$ J/mol. The stage cost (7.29), terminal penalty (7.30) and terminal region (7.31) were used. The Lipschitz-based approach was used for the controller calculations and the result was implemented according to Algorithm 8.5. As depicted in Figures 8.1 to 8.3, the robust adaptive MPC drives the system to a neighborhood of the equilibrium while satisfying the imposed constraints and achieves parameter convergence. Figure 8.4 shows that the uncertainty bound z_θ also reduces over time, although at much more conservative rate compared to Figure 7.4 obtained for systems with no disturbances.

8.7 Conclusions

The adaptive MPC design technique is extended to constrained nonlinear systems with both parametric and time varying disturbances. The proposed robust controller updates the plant model online when model improvement is guaranteed. The embedded adaptation mechanism enables us to construct less conservative terminal design parameters based upon subsets of the original parametric uncertainty. While the introduced conservatism/computation complexity due to the parametric uncertainty reduces over time, the portion due to the disturbance $\vartheta \in \mathcal{D}$ remains active for all time.

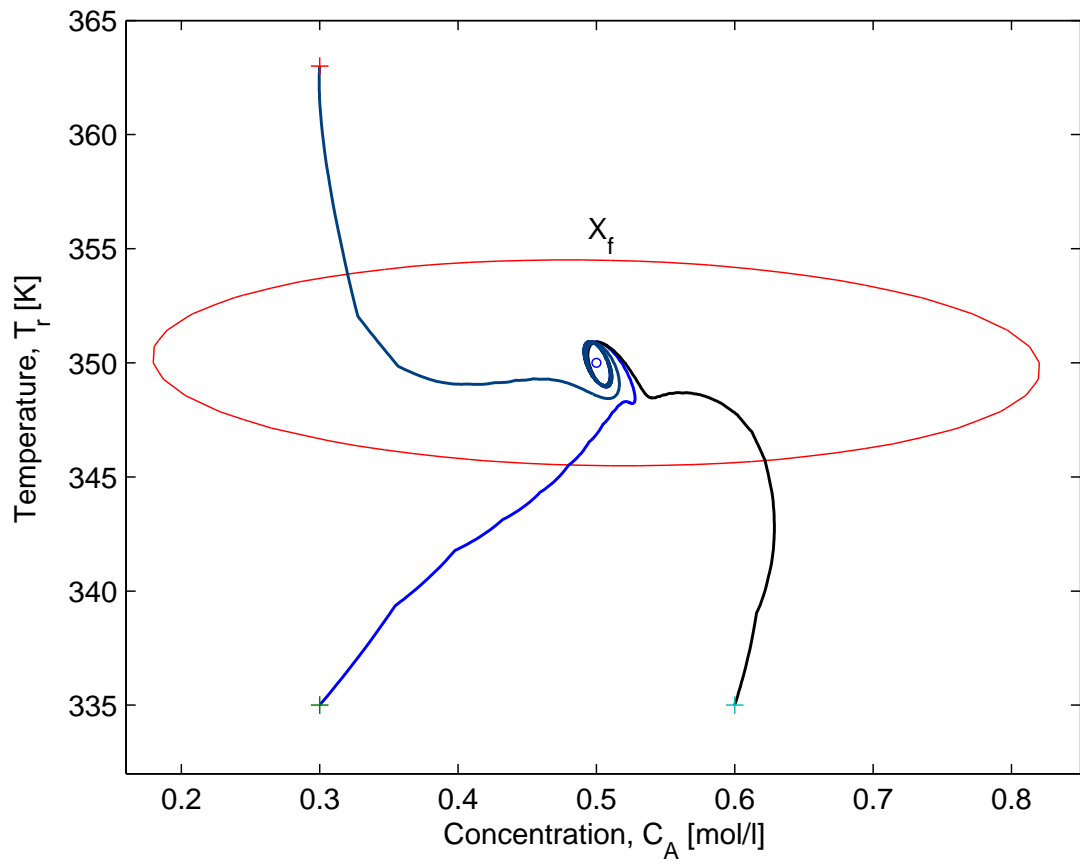


Figure 8.1: Closed-loop reactor trajectories under additive disturbance $\vartheta(t)$

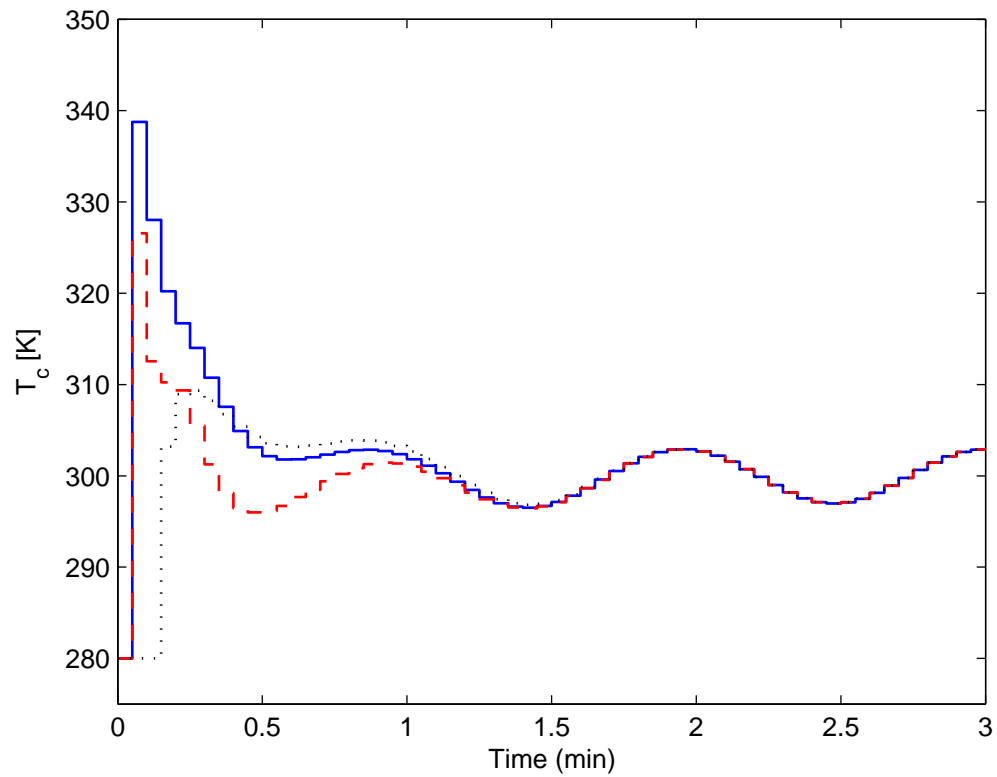


Figure 8.2: Closed-loop input profiles for states starting at different initial conditions $(C_A(0), T_r(0))$: $(0.3, 335)$ is solid line, $(0.6, 335)$ is dashed line and $(0.3, 363)$ is the dotted line

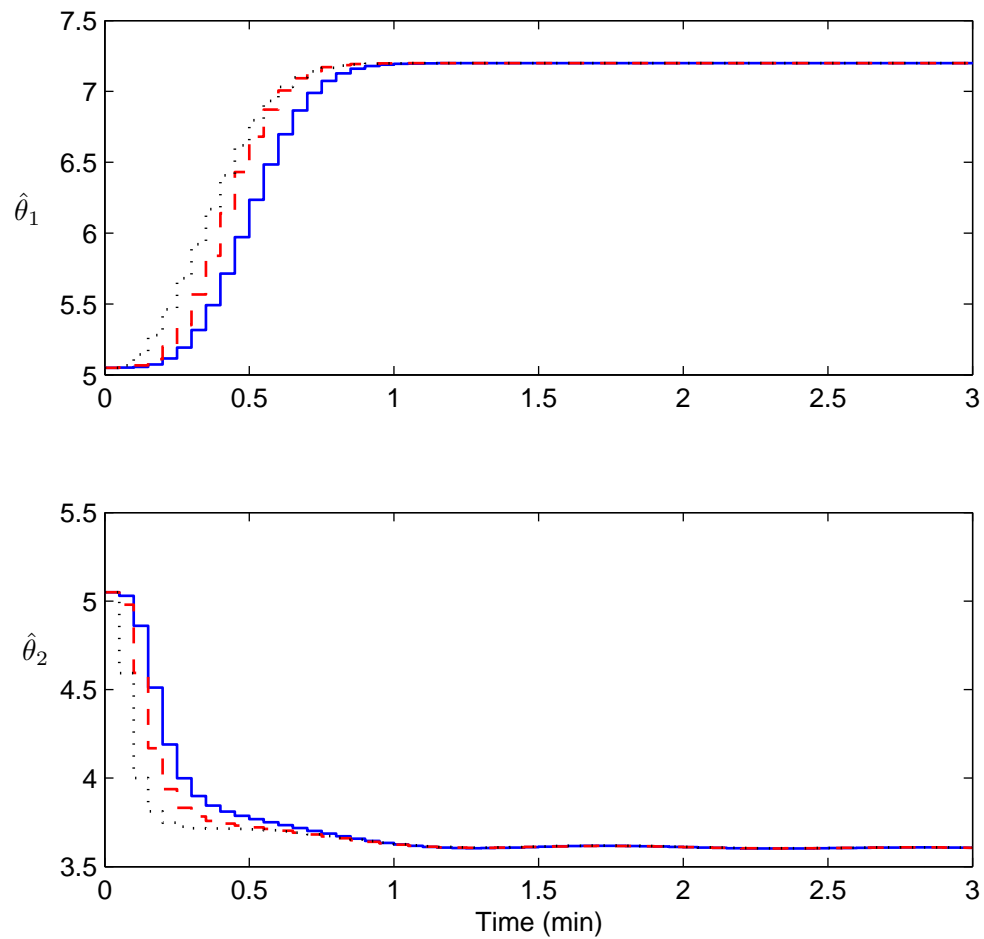


Figure 8.3: Closed-loop parameter estimates profile for states starting at different initial conditions ($C_A(0)$, $T_r(0)$): (0.3, 335) is solid line, (0.6, 335) is dashed line and (0.3, 363) is the dotted line

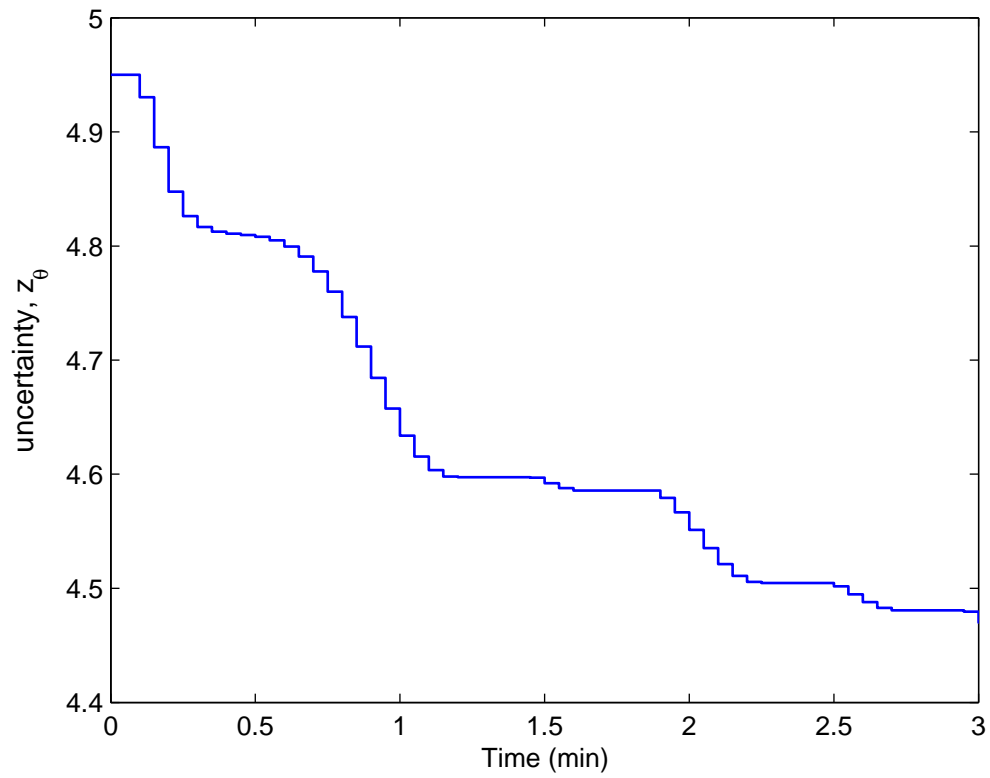


Figure 8.4: Closed-loop uncertainty bound trajectories for initial condition $(C_A, T_r) = (0.3, 335)$

Chapter 9

Integration of RTO and MPC for Constrained Uncertain Nonlinear Systems

In this chapter, we provide a formal design technique that integrates RTO and MPC for constrained uncertain nonlinear systems. The framework considered assumes the economic function is a known function of constrained system's states, parameterized by unknown parameters. The objective and constraint functions may explicitly depend on time, which means that our proposed method is applicable to both dynamic and steady state economic optimization. The control objective is to simultaneously identify and regulate the system to the operating point that optimizes the economic function. The control input may also be required to satisfy some constraints.

The method proposed solves the control and optimization problem at the same frequency. This eliminates the ensuing interval of “no-feedback” that occurs between economic optimization and thereby improving disturbance attenuation. The RTO layer is tackled via a computational efficient approach. The constrained economic optimization problem is converted to an unconstrained problem and Newton based

optimization method is used to develop an update law for the optimum value. The integrated design distinguishes between the extremum seeking and the adaptive tracking of the reference trajectory.

While many advances have been made in nonlinear systems for the stabilization of one fixed operating point, few attempts have been made to address the stabilization problem for time-varying or non-fixed setpoints. In [96], a stabilizing nonlinear MPC algorithm was developed for asymptotically constant reference signals. By selecting a prediction horizon that is longer than the time the reference setpoint is assumed to have converged, the constant pre-programmed value is used to design the stabilizing controller parameters, *i.e.*, the terminal stability constraint \mathbb{X}_f and terminal penalty W . The result is limited to reference signals that converge to *a-priori* known constant setpoint. The method proposed in [97], combines a pseudo-linearization technique with a nonlinear MPC strategy to stabilize a family of (known and constant) setpoints. The linearization technique was used to obtain a constant linearization family that is independent of the setpoint value. Fixed controller parameters were then developed to achieve stability for the whole setpoint family. While the method provides a possible solution for tracking changing setpoints, such pseudo-linearization transformation and feedback is in general difficult to obtain and involve cumbersome computation.

A major challenge of extremum seeking control is that the optimum reference setpoint is not known in advance. The cost function and the dynamical system depend on unknown parameters and hence the optimal operating point can only be determined in real-time. The future behaviour of the setpoint trajectory, which is crucial for calculating model predictive control law is unavailable. Therefore, a single terminal penalty function and constraint set is insufficient to guarantee stability of the reference trajectory. Our stability result anchors on the design of time-varying stability parameters that depend on the current state measurement, setpoint trajectory and parameter estimation error.

9.1 Problem Description

Consider a constrained optimization problem of the form

$$\min_{x \in \mathbb{R}^{n_x}} p(t, x, \theta) \quad (9.1a)$$

$$s.t. \quad c_j(x) \leq 0 \quad j = 1 \dots m_c \quad (9.1b)$$

with θ representing unknown parameters, assumed to be uniquely identifiable and lie within an initially known convex set $\Theta^0 \triangleq B(\theta^0, z_\theta^0)$. The functions p and c_j are assumed to be C^2 in all of their arguments (with locally Lipschitz second derivatives), uniformly for $t \in [0, \infty)$. The constraint $c_j \leq 0$ must be satisfied along the system's state trajectory $x(t)$.

Assumption 9.1 *The following assumptions are made about (9.1).*

1. *There exists $\varepsilon_0 > 0$ such that $\frac{\partial^2 p}{\partial x^2} \geq \varepsilon_0 I$ and $\frac{\partial^2 c}{\partial x^2} \geq 0$ for all $(t, x, \theta) \in (\mathbb{R}^+ \times \mathbb{R}^{n_x} \times \Theta^\epsilon)$, where Θ^ϵ is an ϵ neighborhood of Θ .*
2. *The feasible set*

$$\mathbb{X} = \{x \in \mathbb{R}^{n_x} \mid \max_j c_j(x) \leq 0\},$$

has a nonempty interior.

Assumption 9.1 states that the cost surface is strictly convex in x and \mathbb{X} is a non-empty convex set. Standard nonlinear optimization results guarantee the existence of a unique minimizer $x^*(t, x, \theta) \in \mathbb{X}$ to problem 9.1. In the case of non-convex cost surface, only local attraction to an extremum could be guaranteed. The control

objective is to stabilize the nonlinear system

$$\dot{x} = f(x, \xi, u) + g(x, \xi, u)\theta \triangleq \mathcal{F}(x, \xi, u, \theta) \quad (9.2a)$$

$$\dot{\xi} = f_{\xi}(x, \xi) \quad (9.2b)$$

to the optimum operating point or trajectory given by the solution of (9.1) while obeying the input constraint $u \in \mathbb{U} \in \mathbb{R}^{n_u}$ in addition to the state constraint $x \in \mathbb{X} \in \mathbb{R}^{n_x}$. The dynamics of the state ξ is assumed to satisfy the following input to state stability condition with respect to x .

Assumption 9.2 *If x is bounded by a compact set $B_x \subseteq \mathbb{X}$, then there exists a compact set $B_{\xi} \subseteq \mathbb{R}^{n_{\xi}}$ such that $\xi \in B_{\xi}$ is positively invariant under 9.2.*

9.2 Extremum Seeking Setpoint Design

9.2.1 Constraint Removal

An interior point barrier function method is used to enforce the inequality constraint.

The state constraint is incorporated by augmenting the cost function p as follows:

$$p_a(t, x, \theta) \triangleq p(t, x, \theta) - \frac{1}{\eta_c} \sum_{j=1}^{m_c} \ln(-c_j(x)) \quad (9.3)$$

with $\eta_c > 0$, a fixed constant. The augmented cost function (9.3) is strictly convex in x and the unconstrained minimization of p_a therefore has a unique minimizer in $\text{int}\{\mathbb{X}\}$ which converges to that of (9.1) in the limit as $\eta_c \rightarrow \infty$ [98].

9.2.2 Setpoint Update Law

Let $x_r \in \mathbb{R}^{n_x}$ denote a reference setpoint to be tracked by x and $\hat{\theta}$ denote an estimate of the unknown parameter θ . A setpoint update law \dot{x}_r can be designed based on newton's method, such that $x_r(t)$ converges exponentially to the (unknown) $\hat{\theta}$ dependent optimum value of (9.3). To this end, consider an optimization Lyapunov function candidate

$$V_r = \frac{1}{2} \left\| \frac{\partial p_a}{\partial x}(t, x_r, \hat{\theta}) \right\|^2 \triangleq \frac{1}{2} \|z_r\|^2 \quad (9.4)$$

For the remainder of this section, omitted arguments of p_a and its derivatives are evaluated at $(t, x_r, \hat{\theta})$. Differentiating (9.4) yields

$$\dot{V}_r = \frac{\partial p_a}{\partial x} \left(\frac{\partial^2 p_a}{\partial x \partial t} + \frac{\partial^2 p_a}{\partial x^2} \dot{x}_r + \frac{\partial^2 p_a}{\partial x \partial \theta} \dot{\hat{\theta}} \right). \quad (9.5)$$

Using the update law

$$\dot{x}_r = - \left(\frac{\partial^2 p_a}{\partial x^2} \right)^{-1} \left[\frac{\partial^2 p_a}{\partial x \partial t} + \frac{\partial^2 p_a}{\partial x \partial \theta} \dot{\hat{\theta}} + k_r \frac{\partial p_a^T}{\partial x} \right] \triangleq f_r(t, x_r, \hat{\theta}) \quad (9.6)$$

with $k_r > 0$ and $r(0) = r_0 \in \text{int} \{ \mathbb{X} \}$ results in

$$\dot{V}_r \leq -k_r \|z_r\|^2, \quad (9.7)$$

which implies that the gradient function z_r converges exponentially to the origin.

Lemma 9.3 *Suppose $(\theta, \hat{\theta})$ is bounded, the optimal setpoint $x_r(t)$ generated by (9.6) is feasible and converges to $x_{pa}^*(\hat{\theta})$, the minimizer of (9.3) exponentially.*

Proof: Feasibility follows from the boundedness of $(\theta, \hat{\theta})$ and Assumption 9.1.1 while convergence follows from (9.7) and the fact that z_r is a diffeomorphism. ■

9.3 One-layer Integration Approach

Since the true optimal setpoint depends on θ , the actual desired trajectory $x_r^*(t, \theta)$ is not available in advance. However, $x_r(t, \hat{\theta})$ can be generated from the setpoint update law (9.6) and the corresponding reference input $u_r(x_r)$ can be computed on-line.

Assumption 9.4 $x_r(t, \hat{\theta})$ is such that there exists $u_r(x_r)$ satisfying

$$0 = f(x_r, u_r, \hat{\theta}) \quad (9.8)$$

The design objective is to design a model predictive control law such that the true plant state x tracks the reference trajectory $x_r(t, \hat{\theta})$. Given the desired time varying trajectory (x_r, u_r) , an attractive approach is to transform the tracking problem for a time-invariant system into a regulation problem for an associated time varying control system in terms of the state error $x_e = x - x_r$ and stabilize the $x_e = 0$ state. The formulation requires the MPC controller to drive the tracking error x_e into the terminal set $\mathbb{X}_{e_f}(\tilde{\theta})$ at the end of the horizon. Since the system's dynamics is uncertain, we use the finite-time identifier (7.24) for online parameter adaptation and incorporate robust features in to the adaptive controller formulation to account for the impact of the parameter estimation error $\tilde{\theta}$ in the design.

9.3.1 Min-max Adaptive MPC

Feedback min-max robust MPC is employed to provide robustness for the MPC controller during the adaptation phase. The controller maximizes a cost function with respect to θ and minimizes it over feedback control policies κ .

The integrated controller is given as

$$u = \kappa_{mpc}(t, x_e, \hat{\theta}) \triangleq \kappa^*(0, x_e, \hat{\theta}) \quad (9.9a)$$

$$\kappa^* \triangleq \arg \min_{\kappa(\cdot, \cdot, \cdot)} J(t, x_e, \hat{\theta}, \kappa) \quad (9.9b)$$

where $J(t, x_e, \hat{\theta}, \kappa)$ is the (worst-case) cost associated with the optimal control problem:

$$J(t, x_e, \hat{\theta}, \kappa) \triangleq \max_{\theta \in \Theta} \int_0^T L(\tau, x_e^p, u^p, u_r) d\tau + W(\tau, x_e^p(T), \tilde{\theta}^p(T)) \quad (9.10a)$$

$$s.t. \quad \forall \tau \in [0, T]$$

$$\dot{x}^p = f(x^p, \xi, u^p) + g(x^p, \xi, u^p) \theta, \quad x^p(0) = x \quad (9.10b)$$

$$\dot{\xi} = f(x^p, \xi), \quad \xi^p(0) = \xi \quad (9.10c)$$

$$\dot{x}_r^p = f_r(t, x_r, \theta), \quad x_r^p(0) = x_r \quad (9.10d)$$

$$x_e^p = x^p - x_r^p \quad (9.10e)$$

$$\dot{w}^p = \beta(g^T(x^p, u^p) - k_w w^p), \quad w^p(0) = w \quad (9.10f)$$

$$\dot{Q}^p = \beta(w^{pT} w^p), \quad Q^p(0) = Q \quad (9.10g)$$

$$\dot{\hat{\theta}}^p = \Gamma Q^p \tilde{\theta}^p, \quad \tilde{\theta}^p = \theta - \hat{\theta}^p, \quad \hat{\theta}^p(0) = \hat{\theta} \quad (9.10h)$$

$$u^p(\tau) \triangleq \kappa(\tau, x_e^p(\tau), \hat{\theta}^p(\tau)) \in \mathbb{U} \quad (9.10i)$$

$$x_e^p(\tau) \in \mathbb{X}_e, \quad x_e^p(T) \in \mathbb{X}_{e_f}(\tilde{\theta}^p(T)) \quad (9.10j)$$

where $\mathbb{X}_e = \{x_e^p : x^p \in \mathbb{X}\}$, \mathbb{X}_{e_f} is the terminal constraint and $\beta \in \{0, 1\}$. The effect of the future parameter adaptation is incorporated in the controller design via (9.10a) and (9.10j), which results in less conservative worst-case predictions and terminal conditions.

9.3.2 Implementation Algorithm

Algorithm 9.5 *The finite-time min-max MPC algorithm performs as follows: At sampling instant t_i*

1. **Measure** the current states of the plant $x = x(t_i)$, $\xi = \xi(t_i)$ and obtain the

current value of the desired setpoint $x_r = x_r(t_i)$ via the update law (9.6)

2. **Obtain** the current value of matrices w , Q and C from (7.3e), (7.3a) and (7.3b) respectively

3. **If** $\det(Q) = 0$ or $\text{cond}(Q)$ is not satisfactory update the parameter estimates $\hat{\theta}$ and the uncertainty set $\Theta(t) \triangleq B(\hat{\theta}(t), z_\theta(t))$ according to Algorithm 7.1

Else if $\det(Q) > 0$ and $\text{cond}(Q)$ is satisfactory, set $\beta = 0$ and update

$$\hat{\theta} = Q^{-1}(t_i)C(t_i), \quad z_\theta = 0$$

End

4. Solve the optimization problem (9.9,9.10) and apply the resulting feedback control law to the plant until the next sampling instant

5. **Increment** $i = i + 1$. **If** $z_\theta > 0$, repeat the procedure from step 1 for the next sampling instant. **Otherwise**, repeat only steps 1 and 4 for the next sampling instant.

Since the algorithm is such that the uncertainty set Θ contracts over time, the conservatism introduced by the robustness feature in terms of constraint satisfaction and controller performance reduces over time and when Θ contracts upon θ , the min-max adaptive framework becomes that of a nominal MPC. The drawback of the finite-time identifier is attenuated in this application since the matrix invertibility condition is checked only at sampling instants. The benefit of the identifier, however, is that it allows an earlier and immediate elimination of the robustness feature.

9.3.3 Lipschitz-based Adaptive MPC

While the min-max approach provides the tightest uncertainty cone around the actual system's trajectory, its application is limited by the enormous computation required to obtain the solution of the min-max MPC algorithm. To address this concern,

the robust tracking problem is re-posed as the minimization of a nominal objective function subject to “robust constraints”.

The model predictive feedback is defined as

$$u = \kappa_{mpc}(t, x_e, \hat{\theta}, z_\theta) = u^*(0) \quad (9.11a)$$

$$u^*(\cdot) \triangleq \arg \min_{u_{[0,T]}^p} J(t, x_e, \hat{\theta}, z_\theta, u^p, u_r) \quad (9.11b)$$

where $J(t, x_e, \hat{\theta}, z_\theta, u^p, u_r)$ is given by the optimal control problem:

$$J(t, x_e, \hat{\theta}, z_\theta, u^p, u_r) = \int_0^T L(t, x_e^p, u^p, u_r) d\tau + W(x_e^p(T), z_\theta^p(T)) \quad (9.12a)$$

$$s.t. \quad \forall \tau \in [0, T]$$

$$\dot{x}^p = f(x^p, u^p) + g(x^p, u^p)\hat{\theta}, \quad x^p(0) = x \quad (9.12b)$$

$$\dot{\xi}^p = f(\xi^p, x^p), \quad \xi^p(0) = \xi \quad (9.12c)$$

$$\dot{x}_r^p = f_r(t, x_r, \hat{\theta}), \quad x_r^p(0) = x_r \quad (9.12d)$$

$$x_e^p = x^p - x_r \quad (9.12e)$$

$$\dot{z}_e^p = \beta(\mathcal{L}_f + \mathcal{L}_g\Pi)z_e^p + \|g(x^p, \xi^p, u^p)\|z_\theta, \quad z_x^p(0) = 0 \quad (9.12f)$$

$$X_e^p(\tau) \triangleq B(x_e^p(\tau), z_e^p(\tau)) \subseteq \mathbb{X}_e, \quad u^p(\tau) \in \mathbb{U} \quad (9.12g)$$

$$X_e^p(T) \subseteq \mathbb{X}_{e_f}(z_\theta^p(T)) \quad (9.12h)$$

Since the Lipschitz-based robust controller is implemented in open-loop, there is no setpoint trajectory $x_r(\hat{\theta})$ feedback during the inter-sample implementation. Therefore, the worst-case deviation $z_e^p \geq \max_{\theta \in \Theta} \|x_e - x_e^p\| = \max_{\theta \in \Theta} \|x - x^p\|$. Hence z_e^p given in (9.12f) follows from (7.20), assuming an appropriate knowledge of Lipschitz bounds as follows:

Assumption 9.6 *A set of functions $\mathcal{L}_j : \mathbb{X} \times \mathbb{R}^{n_\epsilon} \times \mathbb{U} \rightarrow \mathbb{R}^+$, $j \in \{f, g\}$ are known which satisfy*

$$\mathcal{L}_j(\mathbb{X}, \xi, u) \geq \min \left\{ \mathcal{L}_j \mid \sup_{x_1, x_2 \in \mathbb{X}} \left(\|j(x_1, \xi, u) - j(x_2, \xi, u)\| - \mathcal{L}_j \|x_1 - x_2\| \right) \leq 0 \right\},$$

9.3.4 Implementation Algorithm

Algorithm 9.7 *The finite-time Lipschitz based MPC algorithm performs as follows:*

At sampling instant t_i

1. **Measure** the current states of the plant $x = x(t_i)$, $\xi = \xi(t_i)$ and obtain the current value of the desired setpoint $x_r = x_r(t_i)$ via the update law (9.6)
2. **Obtain** the current value of matrices w , Q and C from (7.3e), (7.3a) and (7.3b) respectively
3. **If** $\det(Q) = 0$ or $\text{cond}(Q)$ is not satisfactory, set $\beta = 1$ and update the parameter estimates $\hat{\theta} = \hat{\theta}(t_i)$ and uncertainty bounds $z_\theta = z_\theta(t_i)$ and $z_\theta^p(T) = z_\theta^p(t_i + T)$ via (7.2), (7.6) and (7.23) respectively.

Else if $\det(Q) > 0$ and $\text{cond}(Q)$ is satisfactory, set $\beta = 0$ and update

$$\hat{\theta} = Q^{-1}(t_i)C(t_i), \quad z_\theta = 0$$

End

4. **Solve** the optimization problem (9.11,9.12) and apply the resulting feedback control law to the plant until the next sampling instant
5. **Increment** $i = i + 1$. **If** $z_\theta > 0$, repeat the procedure from step 1 for the next sampling instant. **Otherwise**, repeat only steps 1 and 4 for the next sampling instant.

Implementing the adaptive MPC control law according to Algorithm 9.7 ensures that the uncertainty bound z_θ reduces over time and hence, the error margin z_x^p imposed on the predicted state also reduces over time and shrinks to zero when the actual parameter estimate is constructed in finite-time.

9.3.5 Robust Stability

Robust stability is guaranteed under the standard assumptions that $\mathbb{X}_{e_f} \subseteq \mathbb{X}_e$ is an invariant set, W is a local robust CLF for the resulting time varying system and the decay rate of W is greater than the stage cost L within the terminal set \mathbb{X}_{e_f} in conjunction with the requirement for W to decrease and \mathbb{X}_f to enlarge with decreased parametric uncertainty.

9.3.6 Enhancing Parameter Convergence

In min-max adaptive formulation, the terminal penalty is parameterized as a function of $\tilde{\theta}$. This ensures that the algorithm will seek to reduce the parameter error in the process of optimizing the cost function and will automatically inject some excitation in the closed-loop system, when necessary, to enhance parameter convergence. However, this is not the case in the Lipschitz-based approach since the control calculation only uses nominal model. To improve the quality of excitation in the closed-loop and thereby achieve parameter convergence in a minimum time, the invertibility condition (Lemma 5.2, condition (5.8)) can be incorporated in the robust optimization problem by adding an excitation term to the cost function J . The proposed excitation cost is

$$J_{\mathcal{E}} = \frac{\beta}{1 + \mathcal{E}_{\tilde{\theta}}^p(T)} \quad (9.14)$$

where

$$\mathcal{E}_{\tilde{\theta}}^p(\tau) = \lambda_{\min}\{Q^p(\tau)\} \quad \text{or} \quad \mathcal{E}_{\tilde{\theta}}^p(\tau) = \nu^T Q^p(\tau) \nu \quad (9.15)$$

with $\nu \in \mathbb{R}^{n_{\theta}}$ a unit vector. Note that any reduction in the cost function due to $J_{\mathcal{E}}$ implies an improvement in the rank of Q^p . Though, the predicted regressor matrix Q^p differs from the actual matrix Q , a sufficient condition for $Q > 0$ is for $Q^p > z_Q \geq \|Q - Q^p\|$.

9.4 Two-layer Integration Method

The integration task can also be posed as a two degree of freedom paradigm where the problem is divided into two phases. The first phase deals with generating a state trajectory that optimizes a given objective function while respecting the system's dynamics and constraints, and the second phase deals with the design of a controller that would regulate the system around the trajectory. At each sampling instant, the optimum solution to the economic optimization problem obtained via (9.6) is passed down to the tracking controller for implementation. The approach can be viewed as sampling and zero-hold implementation of the setpoint update law.

The MPC controller design follows that of (9.9) and (9.11). The only difference is that rather than solving the setpoint differential equation (9.6) inside the MPC loop, the measurement of x_r obtained at sampling instants is used as the desired setpoint to be tracked, that is, equations (9.10d) and (9.12d) are replaced by

$$\dot{x}_r^p = 0, \quad x_r^p(0) = x_r. \quad (9.16)$$

The adaptive controllers are implemented according to Algorithms 9.5 and 9.7.

9.5 Main Result

The integration result is provided in the following:

Theorem 9.8 *Consider problem (9.1) subject to system dynamics (9.2), and satisfying Assumption 9.1. Let the controller be (9.9) or (9.11) with setpoint update law (9.6) and parameter identifier (7.24). If the invertibility condition (equation 5.8) is satisfied, then for any $\varrho > 0$, there exists constant η_c such that $\lim_{t \rightarrow \infty} \|x(t) - x^*(t, \theta)\| \leq \varrho$, with $x^*(t, \theta)$ the unique minimizer of (9.1). In addition $x \in \mathbb{X}$, $u \in \mathbb{U}$ for all $t \geq 0$.*

Proof: We know from triangle inequality that

$$\|x - x^*(\theta)\| \leq \|x - x_r(\hat{\theta})\| + \|x_r(\hat{\theta}) - x_{pa}^*(\hat{\theta})\| + \|x_{pa}^*(\hat{\theta}) - x^*(\hat{\theta})\| + \|x^*(\hat{\theta}) - x^*(\theta)\| \quad (9.17)$$

where $x_{pa}^*(\hat{\theta})$ denotes the unique minimizer of the unconstrained problem (9.3) for $\theta \equiv \hat{\theta}$. Since the MPC controller guarantees asymptotic convergence of x_e to the origin, we have $\lim_{t \rightarrow \infty} \|x - x_r(\hat{\theta})\| = 0$. Also, it follows from Lemma 9.3, that $\|x_r(\hat{\theta}) - x_{pa}^*(\hat{\theta})\|$ converges exponentially to the origin. Moreover, it is well established that $x_{pa}^*(\hat{\theta})$ converges continuously to $x^*(\hat{\theta})$ as $\eta_c \rightarrow \infty$ [98, Proposition 4.1.1]. Therefore there exists a class \mathcal{K} function ¹ $\alpha_c(\cdot)$ such that

$$\lim_{t \rightarrow \infty} \|x_{pa}^*(\hat{\theta}) - x^*(\hat{\theta})\| \leq \alpha_c\left(\frac{1}{\eta_c}\right). \quad (9.18)$$

The finite-time identification procedure employed ensures that $\hat{\theta} = \theta$ for all $t \geq t_c$, with $t_c < \infty$ and thus $\lim_{t \rightarrow \infty} \|x^*(\hat{\theta}) - x^*(\theta)\| = 0$.

Finally, we have

$$\lim_{t \rightarrow \infty} \|x(t) - x^*(t, \theta)\| \leq \alpha_c\left(\frac{1}{\eta_c}\right) \quad (9.19)$$

and the result follows for sufficiently large η_c . The constraint satisfaction claim follows from the feasibility of the adaptive model predictive controllers. ■

¹A continuous function $\mu : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is of class \mathcal{K} if it is strictly increasing and $\mu(0) = 0$.

9.6 Simulation Example

Consider the parallel isothermal stirred-tank reactor in which reagent A forms product B and waste-product C [48]. The reactors dynamics are given by

$$\begin{aligned}\frac{dA_i}{dt} &= A_i^{in} \frac{F_i^{in}}{V_i} - A_i \frac{F_i^{out}}{V_i} - k_{i1} A_i - 2k_{i2} A_i^2, \\ \frac{dB_i}{dt} &= -B_i \frac{F_i^{out}}{V_i} + k_{i1} A_i, \\ \frac{dC_i}{dt} &= -C_i \frac{F_i^{out}}{V_i} + k_{i2} A_i^2,\end{aligned}$$

where A_i , B_i , C_i denote concentrations in reactor i , k_{ij} are the reaction kinetic constants, which are only nominally known. The inlet flows F_i^{in} are the control inputs, while the outlet flows F_i^{out} are governed by PI controllers which regulate reactor volume to V_i^0 .

The economic cost function is the net expense of operating the process at steady state.

$$p(A_i, s, \theta) = \sum_{i=1}^2 [(p_{i1} s_i + P_A - P_B) k_{i1} A_i V_i^0 + (p_{i2} s_i + 2P_A) k_{i2} A_i^2 V_i^0] \quad (9.20)$$

where P_A , P_B denote component prices, p_{ij} is the net operating cost of reaction j in reactor i . Disturbances s_1 , s_2 reflect changes in the operating cost (utilities, etc) of each reactor. The control objective is to robustly regulate the process to the optimal operating point that optimizes the economic cost (9.20) while satisfying the following state constraints $0 \leq A_i \leq 3$, $c_v = A_1^2 V_1^0 + A_2^2 V_2^0 - 15 \leq 0$ and input constraint $0.01 \leq F_i^{in} \leq 0.2$. The reaction kinetics are assumed to satisfy $0.01 \leq k_i \leq 0.2$.

The two-layer approach was used for the simulation. The setpoint value available at sampling instant is passed down to the MPC controller for implementation. The robustness of the adaptive controller is guaranteed via the Lipschitz bound method. The stage cost is selected as a quadratic cost $L(x_e, u_e) = x_e^T Q_x x_e + u_e^T R_u u_e$, with

$Q_x > 0$ and $R_u \geq 0$.

Terminal Penalty and Terminal Set Design

Let $x = [A_1, A_2]^T$, $\theta = [k_{11}, k_{12}, k_{21}, k_{22}]^T$ and $u = [F_1^{in}, F_2^{in}]^T$, the dynamics of the system can be expressed in the form:

$$\dot{x} = - \underbrace{\begin{bmatrix} \frac{x_1 k_{V1}(\xi_1 - V_1^0 + \xi_3)}{\xi_1} \\ \frac{x_2 k_{V2}(\xi_2 - V_2^0 + \xi_4)}{\xi_2} \end{bmatrix}}_{f_{p1}} + \underbrace{\begin{bmatrix} \frac{A_{in}}{\xi_1} & 0 \\ 0 & \frac{A_{in}}{\xi_2} \end{bmatrix}}_{f_{p2}} u - \underbrace{\begin{bmatrix} x_1 & 2x_1^2 & 0 & 0 \\ 0 & 0 & x_2 & 2x_2^2 \end{bmatrix}}_g \theta,$$

where ξ_1, ξ_2 are the two tank volumes and ξ_3, ξ_4 are the PI integrators. The system parameters are $V_1^0 = 0.9$, $V_2^0 = 1.5$, $k_{v_1} = k_{v_2} = 1$, $P_A = 5$, $P_B = 26$, $p_{11} = p_{21} = 3$ and $p_{12} = p_{22} = 1$.

A Lyapunov function for the terminal penalty is defined as the input to state stabilizing control Lyapunov function (iss-clf):

$$W(x_e) = \frac{1}{2} x_e^T x_e \quad (9.21)$$

Choosing a terminal controller

$$u = k_f(x_e) = -f_{p2}^{-1} \left(-f_{p1} + k_1 x_e + k_2 g g^T x_e \right), \quad (9.22)$$

with design constants $k_1, k_2 > 0$, the time derivative of (9.21) becomes

$$\dot{W}(x_e) = -k_1 x_e^T x_e - x_e^T g \theta - k_2 x_e^T g g^T x_e \quad (9.23)$$

$$\leq -k_1 \|x_e\|^2 + \frac{1}{4k_2} \|\theta\|^2 \quad (9.24)$$

Since the stability condition requires $\dot{W}(x_e(T)) + L(T) \leq 0$. We choose the weighting

matrices of L as $Q = 0.5I$ and $R = 0$. The terminal state region is selected as

$$\mathbb{X}_{e_f} = \{x_e : W(x_e) \leq \alpha_e\} \quad (9.25)$$

such that

$$k_f(x_e) \in \mathbb{U}, \quad \dot{W}(T) + L(T) \leq 0, \quad \forall(\theta, x_e) \in (\Theta, \mathbb{X}_{e_f}) \quad (9.26)$$

Since the given constraints requires the reaction kinetic θ and concentration x to be positive, it follows that

$$\dot{W} + L = -(k_1 - 0.5)\|x_e\|^2 - x_e^T g \theta - k_2 x_e^T g g^T x_e \leq 0 \quad (9.27)$$

for all $k_1 > 0.5$ and $x_e > 0$. Moreover, for $x_e < 0$, the constants k_1 and k_2 can always be selected such that (9.27) is satisfied $\forall \theta \in \Theta$. The task of computing the terminal set is then reduced to finding the largest possible α_e such that for $k_f(\cdot) \in \mathbb{U}$ for all $x \in \mathbb{X}_{e_f}$.

The terminal cost (9.21) is used for this simulation and the terminal set is recomputed at every sampling instant using the current setpoint value. The system was simulated subject to a ramping measured economic disturbance in s_2 from $t = 6$ to 10. The simulation results are presented in Figures 9.1 to 9.5. The phase trajectories displayed in Figure 9.1 shows that the reactor states obeys the imposed constraints while Figure 9.2 shows that the actual, unknown setpoint cost $p(t, x_r, \theta)$ converges to the optimal, unknown $p^*(t, x^*, \theta)$. Figure 9.3 confirms the effectiveness of the adaptive MPC in tracking the desired setpoint while Figure 9.4 shows the convergence of the parameter estimates to the true values. The controls action shown in Figure 9.5 is implementable and satisfies the given constraints.

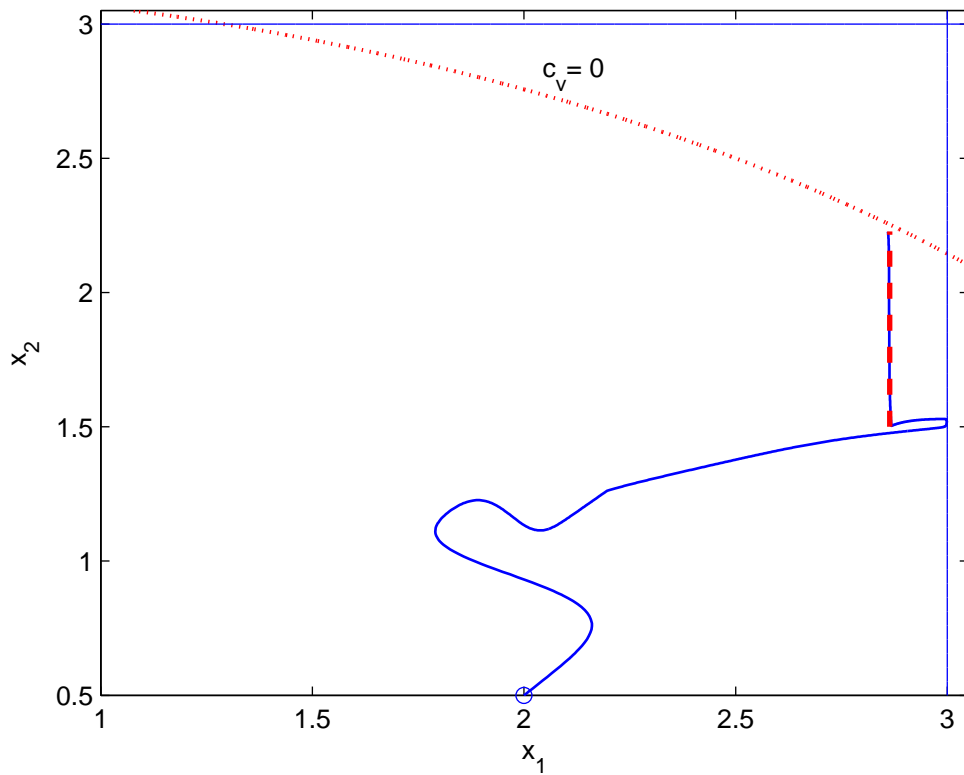


Figure 9.1: Phase diagram and feasible state region

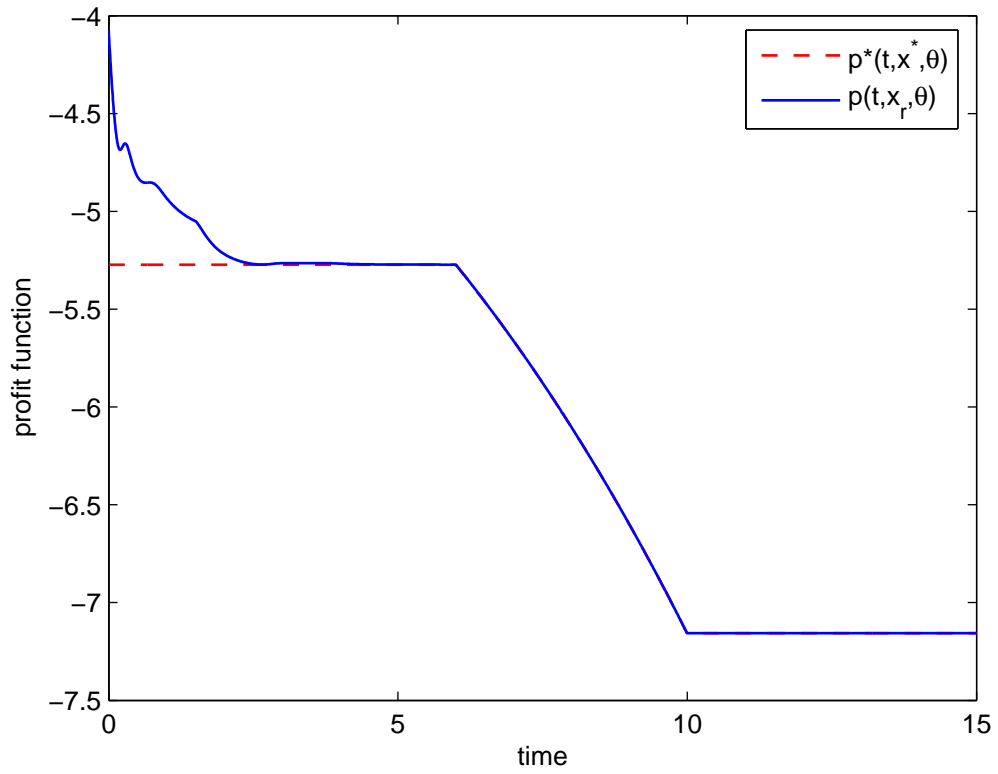


Figure 9.2: Optimal and actual profit functions

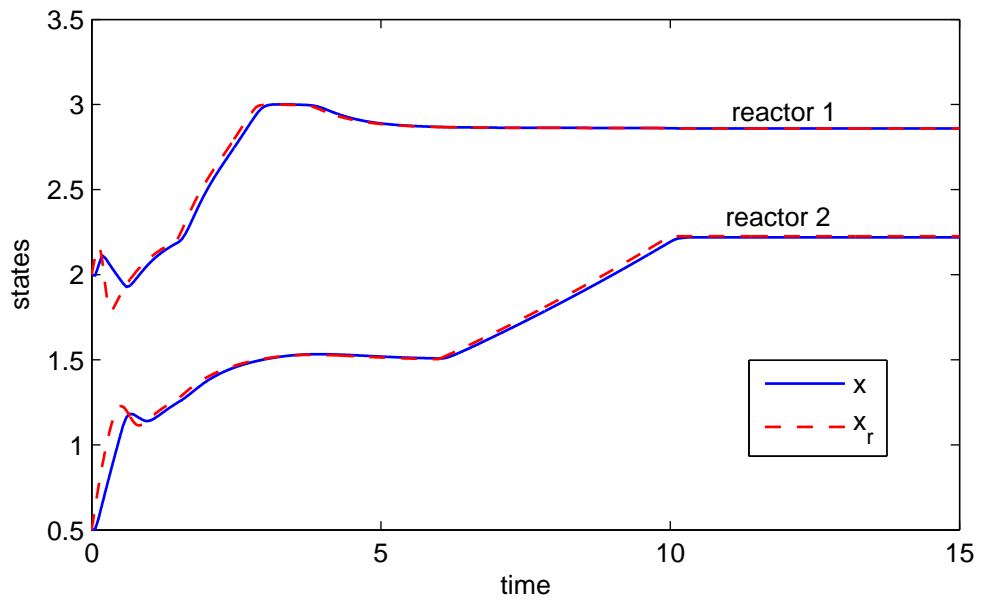


Figure 9.3: Reference trajectories and closed-loop states

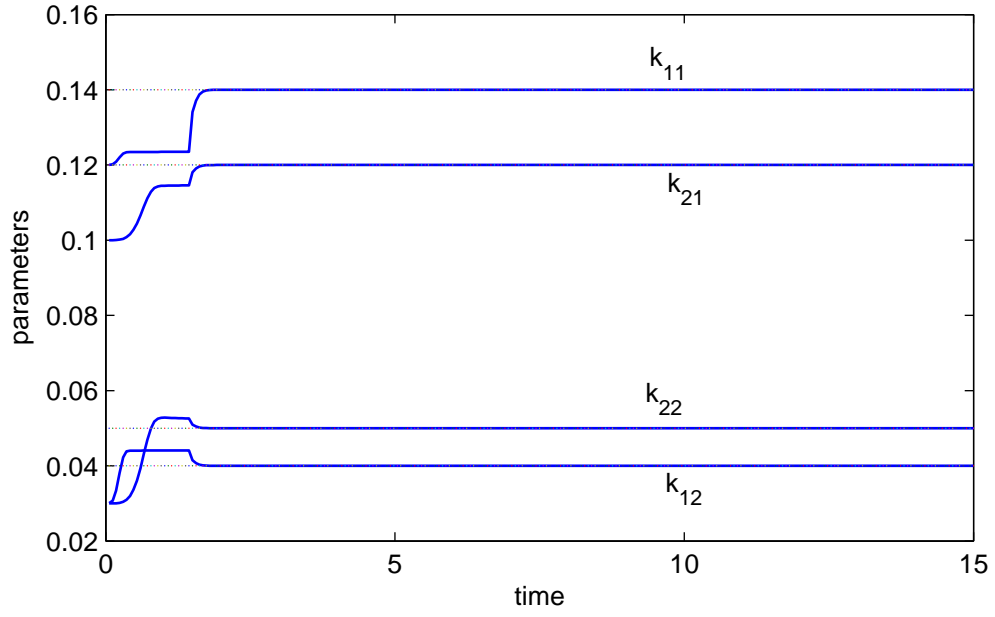


Figure 9.4: Unknown parameters and estimates

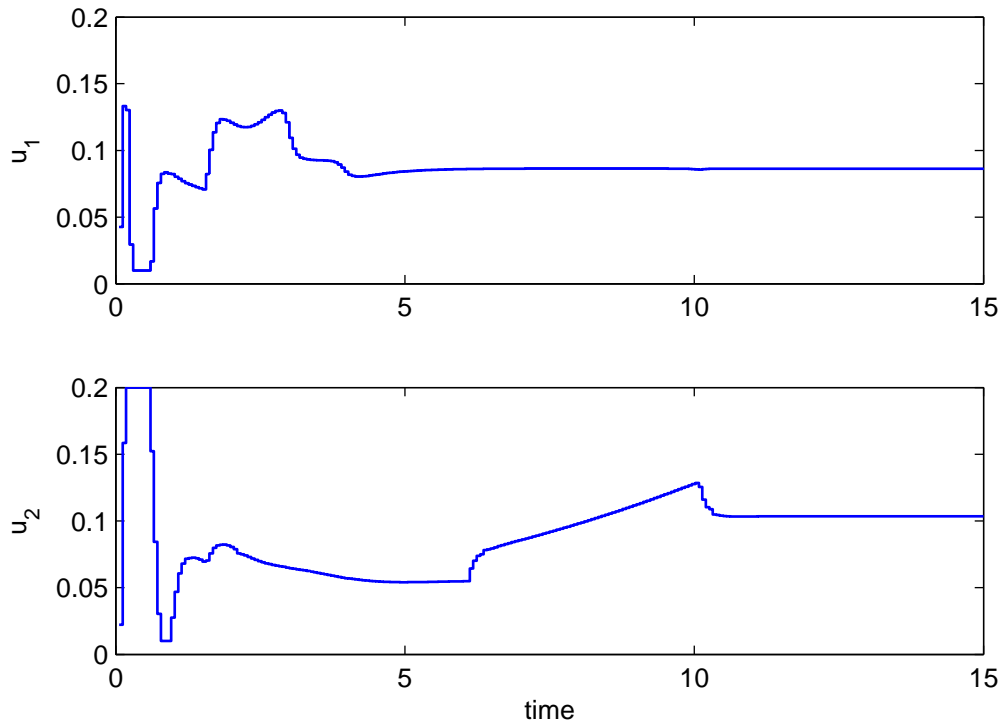


Figure 9.5: Closed-loop system's inputs

9.7 Conclusions

This chapter provides a formal design technique for integrating RTO and MPC for constrained nonlinear uncertain systems. The solution is based upon the tools and strategies developed in the previous chapters. A single layer and two-layer approaches are presented. The constrained RTO problem is solved by the use of interior point barrier functions and Newton's method. Adaptive nonlinear MPC based on min-max framework and Lipschitz bounds proposed drive the system's state to the minimizer of the uncertain objective function.

Chapter 10

Concluding Remarks

The integration of model predictive control (MPC) and real-time optimization (RTO) for an optimal plant operation remains an open field of research. The achievable economic benefit of the integrated module depends, to some extent, on the accuracy of model updates. A better performance can be achieved if the model is updated online as more information becomes available. One way to accomplish this is to pose the problem as an adaptive extremum seeking control problem. This thesis presents a formal scheme that integrates RTO and MPC by solving this class of optimization problems.

10.1 Summary of Contribution

The integration task was addressed in Chapter 3 for linear uncertain systems with only output measurement. The layered control approach was modified and employed for designing the ESC. Lyapunov-based techniques are used to develop a scheme that achieves the integrated task when it can be shown that a persistency of excitation (PE) condition is satisfied. The satisfaction of such a condition ensures that the closed-loop trajectories provide sufficient excitation for the identification of the true cost that is to be optimized. For linear systems, there are established theories that

provide guidelines for selecting or designing a perturbation input signal required to achieve the necessary excitation. However, this is not the case for nonlinear systems.

The problem of determining appropriate excitation conditions for nonlinear systems with uncertain reference setpoint was addressed in Chapter 4. In contrast to standard approaches that determine sufficient dither signals off-line, the method proposed generates perturbation signals that satisfy the excitation condition in closed-loop. The benefit of this approach is that parameter convergence is guaranteed with minimal but sufficient level of perturbation.

In Chapter 5, a finite-time identification procedure that allows exact reconstruction of the unknown parameters in finite-time is developed provided a given excitation condition is satisfied. To enhance the applicability of the finite-time (FT) procedure in practical situations, a novel adaptive compensator that (almost) recovers the performance of the FT identifier is proposed in Chapter 6. The compensator guarantees exponential convergence of the parameter estimation error at a rate dictated by the closed-loop system's excitation. It is shown that the adaptive compensator can be used to improve upon any existing adaptive controllers. The modification provided guarantees exponential stability of the parametric equilibrium when the excitation condition is satisfied. Otherwise, the algorithm preserves the original system's closed-loop properties. The identification techniques are well suited for most adaptive mechanisms and do not require the availability of the velocity state vector. It is demonstrated, via simulation examples, that the identification procedures guarantee parameter convergence in situation where existing methods, driven by tracking or state prediction error, fail.

In general, perturbation signals are usually injected into closed-loop system to provide the sufficient excitation necessary for parameter convergence. However, sustained perturbation adversely affects the closed-loop performance of adaptive controllers. The problem of removing auxiliary perturbation signals when convergence

is achieved is addressed in both Chapters 5 and 6. While the result in Chapter 5 provides a direct solution to the problem, an algorithm is developed in Chapter 6 for the dither signal removal.

Chapters 7 and 8 focus on the control of constrained uncertain nonlinear systems using MPC. Since certainty equivalence approaches are not applicable to nonlinear systems (in general), robust techniques are incorporated in the controller design to account for the transient impact of parameter estimation error. The benefits of incorporating the robustness feature include robust constraint satisfaction and the possibility of obtaining optimal system's excitation without injecting any auxiliary perturbation signal. Another important benefit of the proposed controller framework is that it allows for gradual reduction in the conservatism and numerical burden introduced by the robust features as the model uncertainty reduces. Moreover, a constructive means of recovering the underlying nominal MPC framework when parameters are identified is provided. The MPC scheme is based on generalized stabilizing criteria, which can be satisfied by a series of NMPC strategies such as Lyapunov based approaches and contractive constraints methods. Chapter 8 provides an extension of the adaptive MPC scheme to parametric nonlinear systems subject to additional time-varying disturbances. To avoid the undesirable phenomena such as inferior transient behaviour that plagues adaptive control of systems under external disturbances, a procedure is developed to update the system's model only when an improved estimate is obtained.

Chapter 9 employs the finite-time identification procedure and the adaptive MPC strategy developed in Chapters 5 and 7 respectively to provide a formal integration scheme for constrained uncertain nonlinear systems. A single-layer and two-layer approaches are presented. The methods proposed solve the control and optimization problem at the same frequency, thereby improving disturbance attenuation. The main limitation of the approach is the numerical computations involved especially with respect to the robust control calculations.

10.2 Future Research Directions

While the theoretical development of the FT identification is relatively complete, the result depends on the availability of full state measurement. The procedure in Chapters 5 and 6 can be potentially extended to an output feedback problem and the state estimation error can be treated as disturbances.

The theoretical issues related to the results in Chapters 7 to 9 have not yet been exhausted. From the MPC part, one issue that needs to be further addressed is the design of the stabilizing robust MPC parameters; the terminal cost and terminal constraint set. For nonlinear uncertain systems, the design of robust CLF and more importantly its corresponding invariant set remains an open research problem.

So far the approach has only been tested in simulation, future work could focus on its application to physical problems. While the Lipschitz-based robust method presents a minimal computational limitation, the min-max strategy offers less conservative results. However, its application relies on the availability of efficient min-max solvers.

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